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Abstract

Let $(a_1, \ldots, a_m, b_1, \ldots, b_n)$ be a random permutation of 1, 2, . . ., m + n. Let *P* be a partial order on the *a*'s and *b*'s involving *only* inequalities of the form $a_i < a_j$ or $b_i < b_j$, and let P' be an extension of *P* to include inequalities of the form $a_i < b_j$; i.e, P' = P ∪ P", where P" involves *only* inequalities of the form $a_i < b_j$. We prove the natural conjecture of R. L. Graham, A. C. Yao, and F. F. Yao [SIAM J. Alg. Discr. Meth. 1 (1980), pp. 251–258] that in particular (*) $\Pr(a_1 < b_1|P') \geq \Pr(a_1 < b_1|P)$. We give a simple example to show that the more general inequality (*) where *P* is allowed to contain inequalities of the form $a_i < b_j$ is false. This is surprising because as Graham, Yao, and Yao proved, the general inequality (*) does hold if *P* totally orders both the *a*'s and the *b*'s separately. We give a new proof of the latter result. Our proofs are based on the FKG inequality.

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THE FKG INEQUALITY AND SOME MONOTONICITY PROPERTIES OF PARTIAL ORDERS*

L. A. SHEPP†

Abstract Let $(a_1, \cdots, a_m, b_1, \cdots, b_n)$ be a random permutation of $1, 2, \cdots, m+n$. Let P be a partial order on the a's and b's involving *only* inequalities of the form $a_i < a_j$ or $b_i < b_j$, and let P' be an extension of P to include inequalities of the form $a_i < b_j$; i.e., $P' = P \cup P''$, where P'' involves *only* inequalities of the form $a_i < b_j$. We prove the natural conjecture of R. L. Graham, A. C. Yao, and F. F. Yao [SIAM J. Alg. Discr. Meth. 1(1980), pp. 251–258] that in particular (*) $\Pr(a_1 < b_1 \mid P') \geqq \Pr(a_1 < b_1 \mid P)$. We give a simple example to show that the more general inequality (*) where P is allowed to contain inequalities of the form $a_i < b_j$ is false. This is surprising because as Graham, Yao, and Yao proved, the general inequality (*) does hold if P totally orders both the a's and the b's separately. We give a new proof of the latter result. Our proofs are based on the FKG inequality.

1. Introduction. Suppose $(a_1, a_2, \cdots, a_m, b_1, \cdots, b_n)$ is a random (uniformly distributed) permutation of $1, 2, \cdots, m+n$. Following [GYY], we might think of the permutation as the actual ranking of the tennis skill of players $a_1, \cdots, a_m, b_1, \cdots, b_n$. Here player x always loses to player y in a match if $x < y$. In a contest between two teams $A = \{a_1, \cdots, a_m\}$ and $B = \{b_1, \cdots, b_n\}$, suppose first that the teams have never met before but the players of each team have played some matches among themselves. Thus there is a partial order P between certain a's and certain b's, e.g., $a_1 < a_2$, $a_1 < a_3$, $b_2 < b_1, \cdots$, but there is no *direct* information about the relative ranking of a's vs. b's. Denote by $\Pr(a_1 < b_1 \mid P)$ the conditional probability that a_1 loses to b_1, given the partial order P.

After some matches between a's and b's have taken place, in which we shall suppose that the a's have lost each match to the b's so far, we have a new partial ordering $P' = P \cup P''$, where P'' contains inequalities of the form $a_i < b_j$; e.g. $P'' = \{a_3 < b_4, a_5 < b_2, \cdots\}$. Note that there are two ways to think about P: if P is thought of as a partial order on $\{a_1, \cdots, a_m, b_1, \cdots, b_n\}$, then the *union* $P \cup P'' = P'$ is the *larger* partial order based on the additional information in P''. However, we shall think of P as a subset of permutations *defined* by the partial order P so that the *intersection* $P \cap P'' = P'$ is the *smaller* subset of permutations based on the additional information in P''. Denote by $\Pr(a_1 < b_1 \mid P')$ the conditional probability that a_1 loses to b_1 given P'. It is tempting to conjecture that, in particular,

$$\Pr(a_1 < b_1 \mid P') \geqq \Pr(a_1 < b_1 \mid P). \tag{1.1}$$

The additional knowledge with P' that a's have lost to b's prompts the belief (prejudice?) that a's are inferior to b's, and seems to make it more likely under P' than under P that a_1 loses to b_1. This conjecture of R. L. Graham, A. C. Yao, and F. F. Yao [GYY] is true, as we show. However the same intuition makes it even more tempting to conclude that (1.1) holds even if P contains inequalities of the form $a_i < b_j$, because the prejudice under P that a's are inferior to b's is apparently further reinforced by the new inequalities in P'. Nevertheless we give a simple example to show this is false. Indeed let $m = n = 2$ and

$$\begin{aligned} P &= \{a_1 < b_2, a_2 < b_1\}, \\ P' &= \{a_2 < b_2\} \cap P. \end{aligned} \tag{1.2}$$

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It is easy to check that $\Pr(P) = \frac{1}{4}$, $\Pr(P') = \Pr(a_1 < b_1,\ P) = \frac{5}{24}$, $\Pr(a_1 < b_1, P') = \frac{1}{6}$ so that (1.1) asserts that $\frac{4}{5} \geqq \frac{5}{6}$, which is of course false. An even simpler example was found by a referee: $m = n = 2$, $P = \{a_2 < b_1\}$, $P' = \{a_2 < b_2\} \cap P$, $\Pr(a_1 < b_1 \mid P') = \frac{5}{8} < \frac{2}{3} = \Pr(a_1 < b_1 \mid P)$.

The example (1.2) is especially surprising because (1.1) is valid even when P contains inequalities of the form $a_i < b_j$, *provided that* P also contains inequalities which give a *total ordering* of each of A and B separately. This was proved by Graham, Yao, and Yao [GYY], and we give a new proof here.

We next give a more general formulation of the two results to be proved in § 2, and discuss the FKG inequality which we will use in their proofs. Let P_0 be the subset of permutations for which A and B have the complete order:

$$P_0 = \{a_1 < a_2 < \cdots < a_m\} \cap \{b_1 < b_2 < \cdots < b_n\}. \tag{1.3}$$

Suppose P_1, P_2, P_3 are each subsets of permutations which are intersections of subsets of the form $\{a_i < b_j\}$. Then Graham, Yao, and Yao [GYY] proved:

Theorem 1. (Graham, Yao, Yao, [GYY]).

$$\Pr(P_1 \mid P_3 \cap P_2 \cap P_0) \geqq \Pr(P_1 \mid P_2 \cap P_0). \tag{1.4}$$

Note that this is the result stated in the preceding paragraph if P_1 is specialized to a single inequality $\{a_i < b_j\}$.

Let Q_0 be a subset of permutations defined by intersections of subsets of the form $\{a_i < a_j\}$ and $\{b_i < b_j\}$ *but not of the form* $\{a_i < b_j\}$ or $\{a_i > b_j\}$,

$$Q_0 = \{a_{i_1} < a_{j_1}, \cdots, a_{i_r} < a_{j_r}\} \cap \{b_{k_1} < b_{l_1}, \cdots, b_{k_s} < b_{l_s}\}, \tag{1.5}$$

and let P_1, P_2 be as in Theorem 1. Then Graham, Yao, and Yao [GYY] conjectured:

Theorem 2.

$$\Pr(P_1 \mid P_2 \cap Q_0) \geqq \Pr(P_1 \mid Q_0). \tag{1.6}$$

The FKG (Fortuin, Kasteleyn, Ginibre) inequality was discovered [FKG] in proving "intuitively obvious" conjectures about correlations in a statistical mechanics model. Although as shown in [FKG], the FKG hypothesis (1.7)–(1.10) is only sufficient for the conclusion (1.11), in the present case I found the simple counterexample (1.2) by looking for the simplest case of the general conjecture (1.1) for which the FKG technique does not easily apply. Other applications of the FKG inequality to prove known inequalities in combinatorics have been given in [SW]. D. J. Kleitman and J. B. Schearer [KS] also give an example where (1.1) fails if P is allowed to contain $a_i < b_j$ inequalities, and give a different FKG proof for Theorem 1, but do not obtain Theorem 2.

The setting for the FKG inequality is as follows: Let Γ be a finite lattice; i.e., Γ is a finite set $\Gamma = \{x, y, z, \cdots\}$ with a partial order $x < y$ for which each pair $x, y \in \Gamma$ has a unique least upper bound $x \vee y$ and a unique greatest lower bound $x \wedge y$,

$$x \vee y \in \Gamma, \qquad x \wedge y \in \Gamma. \tag{1.7}$$

Further, Γ is assumed distributive; i.e. for all $x, y, z \in \Gamma$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{1.8}$$

or equivalently, for all $x, y, z \in \Gamma$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Suppose μ, f, g are real-valued functions on Γ for which for all $x, y \in \Gamma$,

$$\mu(x) \geqq 0, \qquad \mu(x)\mu(y) \leqq \mu(x \wedge y)\mu(x \vee y), \tag{1.9}$$

and f and g are monotonic in the same direction so that either

$$
(1.10)\qquad \begin{aligned} &f(x) \leqq f(y), \quad g(x) \leqq g(y), \text{ for all } x \text{ and } y, \text{ or} \\ &f(x) \geqq f(y), \quad g(x) \geqq g(y), \text{ for all } x < y. \end{aligned}
$$

The FKG inequality [FKG] then asserts

$$
(1.11)\qquad \sum_{x\in\Gamma} f(x)g(x)\mu(x) \sum_{y\in\Gamma} \mu(y) \geqq \sum_{x\in\Gamma} f(x)\mu(x) \sum_{y\in\Gamma} g(y)\mu(y).
$$

2. Proofs of Theorems 1 and 2. Let Γ be the set of all $\binom{m+n}{m}$ subsets of $\{1, 2, \cdots, m+n\}$ with m elements. For $x = \{x_1 < x_2 < \cdots < x_m\}$, $y = \{y_1 < \cdots < y_m\} \in \Gamma$, say that $x < y$ if $x_i \leqq y_i$, $i = 1, \cdots, m$. Thus the elements of $x \wedge y = \{(x\wedge y)_1 < \cdots < (x \wedge y)_m\}$ and $x \vee y = \{(x \vee y)_1 < \cdots < (x \vee y)_m\}$ are given for $i = 1, \cdots, m$ by

$$
(2.1)\qquad (x \wedge y)_i = \min(x_i, y_i), \quad (x \vee y)_i = \max(x_i, y_i).
$$

Since $x \wedge y, x \vee y \in \Gamma$, (1.7) holds for $\Gamma, <$.

Examining all orderings of any three real numbers α, β, γ shows that

$$
(2.2)\qquad \min(\alpha, \max(\beta, \gamma)) = \max(\min(\alpha, \beta), \min(\alpha, \gamma)).
$$

From (2.1) and (2.2) we see that (1.8) holds so that $\Gamma, <$ is also distributive.

Let $\bar{P}_1, \bar{P}_2, \bar{P}_3$ each be intersections of subsets of Γ of the form $\{x_i \leqq k\}$, $i = 1, \cdots, m$, $k = 1, \cdots, m+n$. Let μ, f, g be defined by

$$
(2.3)\qquad \begin{aligned} &\mu(x) = \begin{cases} 1, & \text{if } x \in \Gamma \cap \bar{P}_2, \\ 0, & \text{else,} \end{cases} \\ &f(x) = \begin{cases} 1, & \text{if } x \in \Gamma \cap \bar{P}_1, \\ 0, & \text{else,} \end{cases}; \qquad g(x) = \begin{cases} 1, & \text{if } x \in \Gamma \cap \bar{P}_3, \\ 0, & \text{else.} \end{cases} \end{aligned}
$$

Since $x_i \leqq k$, $y_i \leqq k$ implies that $\min(x_i, y_i) \leqq \max(x_i, y_i) \leqq k$, we see that $\mu(x) = \mu(y) = 1$ implies that $\mu(x \wedge y) = \mu(x \vee y) = 1$; thus (1.9) holds with equality. If $x \leqq y$ and $f(y) = 1$ then $y \in \bar{P}_1$. But if $y_i \leqq k$ then $x_i \leqq k$, so that $x \in \bar{P}_1$ as well, and $f(x) = 1$. Thus f is decreasing, and similarly so is g. Thus (1.10) holds and the hypothesis of the FKG inequality is satisfied. By (1.11), it follows that

$$
(2.4)\qquad \#(\Gamma \cap \bar{P}_1 \cap \bar{P}_2 \cap \bar{P}_3)\#(\Gamma \cap \bar{P}_2) \geqq \#(\Gamma \cap \bar{P}_1 \cap \bar{P}_2)\#(\Gamma \cap \bar{P}_2 \cap \bar{P}_3),
$$

where $\#(A)$ is the cardinality of A.

Consider the one-one correspondence $\phi : \Gamma \leftrightarrow P_0$ in (1.3); here $\phi(x) = (a_1, \cdots, a_m, b_1, \cdots, b_n)$, the permutation of $(1, 2, \cdots, m+n)$ which has $a_i(x) = x_i$, $i = 1, \cdots, m$, and $b_j(x) = j$th element of the complement of x in $\{1, 2, \cdots, m+n\}$. Because the a's and b's are totally ordered by (1.3) in P_0, we have

LEMMA 2.5. *If* $(a_1, \cdots, a_m, b_1, \cdots, b_n) \in P_0$, *then* $a_i < b_j$ *if and only if* $a_i \leqq i + j - 1$.

It follows from (2.5) that for subsets P_1, P_2, P_3 as in Theorem 1 which are each intersections of subsets $\{a_i < b_j\}$, $\bar{P}_k = \phi^{-1}(P_i)$, $i = 1, 2, 3$, are each of the form $\{x_i \leqq k\}$; so that (2.4) holds. Since ϕ is one-one we have upon dividing by $((m+n)!)^2$,

$$
(2.6)\qquad \begin{aligned} &\Pr(P_0 \cap P_1 \cap P_2 \cap P_3) \Pr(P_0 \cap P_2) \\ &\qquad \geqq \Pr(P_0 \cap P_1 \cap P_2) \Pr(P_0 \cap P_2 \cap P_3), \end{aligned}
$$

which is the same as (1.4). Theorem 1 is thus proved.

We next prove Theorem 2. For $N = 1, 2, \cdots$, let Γ_N be the set of N^{m+n} integer-valued vectors $x = (a_1, \cdots, a_m, b_1, \cdots, b_n)$ where each a_i and $b_j \in \{1, 2, \cdots, N\}$. Denote

$$x_i = a_i = a_i(x), \quad i = 1, \cdots, m; \qquad x_{m+j} = b_j = b_j(x), \quad j = 1, \cdots, n. \tag{2.7}$$

For $x, y \in \Gamma_N$ say that $x < y$ if $x_i = a_i(x) \leqq a_i(y) = y_i$ but $x_{m+j} = b_j(x) \geqq b_j(y) = y_{m+j}, j = 1, \cdots, n$. The components of $x \wedge y$ and $x \vee y$ are

$$(x \wedge y)_i = \min(x_i, y_i), \qquad (x \wedge y)_{m+j} = \max(x_{m+j}, y_{m+j}),$$
$$(x \vee y)_i = \max(x_i, y_i), \qquad (x \vee y)_{m+j} = \min(x_{m+j}, y_{m+j}). \tag{2.8}$$

Since $x \wedge y, x \vee y \in \Gamma_N$, (1.7) holds for Γ_N.

Because of (2.2), we again have (1.8) so $\Gamma_N, <$ is also a finite distributive lattice.

Let Q_0^* be a subset of Γ_N defined by intersections of subsets of the form $\{x : a_i(x) < a_j(x)\}$ and $\{x : b_i(x) < b_j(x)\}$, so that

$$Q_0^* = \{x : a_{i_1}(x) < a_{j_1}(x), \cdots, a_{i_r}(x) < a_{j_r}(x); \; b_{k_1}(x) < b_{l_1}(x), \cdots, b_{k_s}(x) < b_{l_s}(x)\}, \tag{2.9}$$

and let P_1^* and P_2^* be subsets of Γ_N defined by intersections of the form $\{x : a_i(x) < b_j(x)\}$. Let μ, f, g be defined for $x \in \Gamma_N$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in Q_0^*, \\ 0, & \text{else,} \end{cases}$$
$$f(x) = \begin{cases} 1, & \text{if } x \in P_1^*, \\ 0, & \text{else,} \end{cases}; \qquad g(x) = \begin{cases} 1, & \text{if } x \in P_2^*, \\ 0, & \text{else.} \end{cases} \tag{2.10}$$

If $x, y \in \Gamma_N$ and $\mu(x)\mu(y) = 1$, then $x, y \in Q_0^*$ so that for $z = x$ or y, $a_{i_t}(z) < a_{j_t}(z)$, $t = 1, \cdots, r$, and $b_{k_t}(z) < b_{l_t}(z)$, $t = 1, \cdots, s$. But then

$$\min(a_{i_t}(x), a_{i_t}(y)) < \min(a_{j_t}(x), a_{j_t}(y)), \qquad t = 1, \cdots, r,$$
$$\max(b_{k_t}(x), b_{k_t}(y)) < \max(b_{l_t}(x), b_{l_t}(y)), \qquad t = 1, \cdots, s, \tag{2.11}$$

so that by (2.7) and (2.8), $x \wedge y \in Q_0^*$. Similarly, $x \vee y \in Q_0^*$, so that $\mu(x \wedge y)\mu(x \vee y) = 1$. Thus (1.9) holds. Note that (1.9) would fail if Q_0^* were allowed to contain inequalities $\{a_i < b_j\}$.

If $x < y$ and $f(y) = 1$, then $y \in P_1^*$; so that $a_i(x) \leqq a_i(y) \leqq b_j(y) \leqq b_j(x)$ if $\{a_i < b_j\}$ is any one of the inequalities involved in P_1^*. Thus $x \in P_1^*$, and so $f(x) = 1$. Thus $f(x)$ is decreasing and so is g. Thus (1.10) holds and the hypothesis of the FKG inequality is satisfied. By (1.11) it follows that

$$\#(Q_0^* \cap P_1^* \cap P_2^*)\#(Q_0^*) \geqq \#(Q_0^* \cap P_1^*)\#(Q_0^* \cap P_2^*). \tag{2.12}$$

Now consider the subset Γ_N^* of Γ_N, for which all a_i's and b_j's are distinct. Since $\#(\Gamma_N^*) = N(N-1) \cdots (N-n-m+1)$, we see that

$$\lim_{N\to\infty} \frac{\#(\Gamma_N^*)}{\#(\Gamma_N)} = 1. \tag{2.13}$$

In Γ_N^* however, $a_1(x), \cdots, a_m(x), b_1(x), \cdots, b_n(x)$ are all distinct, and for each $x \in \Gamma_N^*$ a unique ordering of $a_1, \cdots, a_m, b_1, \cdots, b_n$ is obtained by letting the ordering of $a_i(x), b_j(x)$ determine the ordering of a_i, b_j. For $N \geqq n + m$, the fraction of

Γ_N^* corresponding to each ordering of $a_1, \cdots, a_m, b_1, \cdots, b_n$ is $1/(m+n)!$; so that for $N \geqq m + n$,

$$\frac{\#(A^* \cap \Gamma_N^*)}{\#(\Gamma_N^*)} = \Pr(A). \tag{2.14}$$

Here $A = Q_0, Q_0 \cap P_i, Q_0 \cap P_1 \cap P_2$ given in Theorem 2 and in (1.5), and the corresponding $A^* = Q_0^*, Q_0^* \cap P_i^*, Q_0^* \cap P_1^* \cap P_2^*$. Since for any A and the corresponding A^* we have

$$\#(A^* \cap (\Gamma_N - \Gamma_N^*)) \leqq \#(\Gamma_N - \Gamma_N^*) = o(\#(\Gamma_N^*)), \tag{2.15}$$

as $N \to \infty$, it follows that also

$$\lim_{N \to \infty} \frac{\#(A^*)}{\#(\Gamma_N^*)} = \Pr(A), \tag{2.16}$$

for $A = Q_0, Q_0 \cap P_i, Q_0 \cap P_1 \cap P_2$, and $A^* = Q_0^*, Q_0^* \cap P_i^*, Q_0^* \cap P_1^* \cap P_2^*$, respectively. Thus from (2.12) and (2.16) we obtain, letting $N \to \infty$,

$$\Pr(Q_0 \cap P_1 \cap P_2) \Pr(Q_0) \geqq \Pr(Q_0 \cap P_1) \Pr(Q_0 \cap P_2), \tag{2.17}$$

which is the same as (1.6). Theorem 2 is thus proved.

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