TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 8, Pages 3369–3390
S 0002-9947(04)03416-6
Article electronically published on January 29, 2004

THE FLAT MODEL STRUCTURE ON $\mathbf{Ch}(R)$

JAMES GILLESPIE

ABSTRACT. Given a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ with enough $\mathcal{A}$ objects and enough $\mathcal{B}$ objects, we define two cotorsion pairs in the category $\mathbf{Ch}(\mathcal{C})$ of unbounded chain complexes. We see that these two cotorsion pairs are related in a nice way when $(\mathcal{A}, \mathcal{B})$ is hereditary. We then show that both of these induced cotorsion pairs are complete when $(\mathcal{A}, \mathcal{B})$ is the "flat" cotorsion pair of R-modules. This proves the flat cover conjecture for (possibly unbounded) chain complexes and also gives us a new "flat" model category structure on $\mathbf{Ch}(R)$. In the last section we use the theory of model categories to show that we can define $\mathrm{Ext}^n_R(M, N)$ using a flat resolution of M and a cotorsion coresolution of N.

1. Introduction

The derived category of an abelian category $\mathcal{C}$ is the category $\mathcal{D}(\mathcal{C})$ obtained by formally inverting the homology isomorphisms (H_*-isomorphisms) in the category $\mathbf{Ch}(\mathcal{C})$ of unbounded chain complexes. From the definition alone one immediately finds difficulty proving anything about $\mathcal{D}(\mathcal{C})$. For example, for objects $X, Y \in \mathcal{C}$, is the class of morphisms $\mathcal{D}(\mathcal{C})(X, Y)$ even a set? One application of Quillen's notion of a model category is that in some cases we can get a handle on the derived category. After all, a model category $\mathcal{M}$ comes equipped with three classes of maps: cofibrations, fibrations, and weak equivalences, as well as an associated homotopy category $\mathbf{Ho}(\mathcal{M})$ in which the weak equivalences have been made isomorphisms. So if we can put a model structure $\mathcal{M}$ on $\mathbf{Ch}(\mathcal{C})$ such that the weak equivalences are H_*-isomorphisms, then we can identify $\mathbf{Ho}(\mathcal{M})$ with $\mathcal{D}(\mathcal{C})$. The rich model structure now lets us describe the class of maps between objects X and Y as the set of homotopy classes of maps between a cofibrant replacement of X and a fibrant replacement of Y.

For a commutative ring R with 1, the category $\mathbf{Ch}(R)$ of unbounded chain complexes has two well-known model category structures with weak equivalences being the H_*-isomorphisms. The "projective" model structure is characterized by having the fibrations being all epimorphisms and the cofibrations being the monomorphisms with dg-projective cokernels. The dual "injective" model structure has the cofibrations being the monomorphisms and the fibrations the epimorphisms with dg-injective kernels. Here we will see that there is also a "flat" model structure with cofibrations being the monomorphisms with dg-flat cokernels and the fibrations being the epimorphisms with dg-cotorsion kernels. We also show that this "flat"

Received by the editors October 1, 2002 and, in revised form, May 13, 2003.
2000 *Mathematics Subject Classification.* Primary 55U35, 18G35, 18G15.
The author thanks Mark Hovey of Wesleyan University.

model structure is monoidal; that is, it interacts properly with the tensor product on $\mathbf{Ch}(R)$.

It is a folk theorem of Joyal [Joy84] that there always is an injective model structure on $\mathbf{Ch}(\mathcal{C})$ whenever $\mathcal{C}$ is a Grothendieck category. However, in many important cases, such as when $\mathcal{C}$ is the category of sheaves on a ringed space or quasi-coherent sheaves on a ringed space, we also have a tensor product. Unfortunately the injective model structure is not monoidal and a desired projective model structure does not exist since these categories do not have enough projectives. The methods used here could perhaps be used to construct a "flat" model structure which *is* monoidal. The author hopes to follow this up in future work.

To obtain the flat model structure on $\mathbf{Ch}(R)$ we use a theorem of Mark Hovey (see [Hov00]) which relates complete cotorsion pairs on $\mathbf{Ch}(R)$ to model structures on $\mathbf{Ch}(R)$. (A cotorsion pair is a pair of classes $(\mathcal{A}, \mathcal{B})$ in an abelian category which are orthogonal with respect to Ext. A precise definition is given in the next section.) Section 3 shows in a general way that any hereditary cotorsion pair in **Rmod** induces cotorsion pairs in $\mathbf{Ch}(R)$ for which Hovey's theorem can apply *if* we know the induced cotorsion pairs are complete.

Next, section 4 looks at the two cotorsion pairs of complexes induced by the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ and we show these *are* complete. The method follows Enochs' approach of cogenerating the cotorsion pairs by a set. It follows immediately from this work that every (possibly unbounded) chain complex has a flat cover, injective envelope, dg-flat cover, and dg-cotorsion envelope.

Lastly, in section 5 we will see that we can compute $\mathrm{Ext}^n(M, N)$ for R-modules M and N in terms of a flat resolution of M and a cotorsion coresolution of N. Again we hope this method can be used to give a convenient way to compute Ext^n in more general categories which may not have enough projectives.

2. Preliminaries

Let $\mathcal{C}$ be an abelian category. A *cotorsion pair* (also called a cotorsion theory) is a pair of classes of objects $(\mathcal{A}, \mathcal{B})$ of $\mathcal{C}$ such that $\mathcal{A}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$. Here $\mathcal{A}^{\perp}$ is the class of objects $C \in \mathcal{C}$ such that $\mathrm{Ext}^1(A, C) = 0$ for all $A \in \mathcal{A}$, and similarly ${}^{\perp}\mathcal{B}$ is the class of objects $C \in \mathcal{C}$ such that $\mathrm{Ext}^1(C, B) = 0$ for all $B \in \mathcal{B}$. We will mainly be considering cotorsion pairs in the category **Rmod** of R-modules and the category $\mathbf{Ch}(R)$ of chain complexes of R-modules. Two simple examples of cotorsion theories in **Rmod** are $(\mathcal{P}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{I})$, where $\mathcal{P}$ is the class of projectives, $\mathcal{I}$ is the class of injectives and $\mathcal{A}$ is the class of all R-modules.

The cotorsion pair is said to have *enough projectives* if for any $C \in \mathcal{C}$ there is a short exact sequence $0 \to B \to A \to C \to 0$, where $B \in \mathcal{B}$ and $A \in \mathcal{A}$. We say it has *enough injectives* if it satisfies the dual statement. These two statements are in fact equivalent for a cotorsion pair as long as the *category* $\mathcal{C}$ has enough projectives and injectives. (In the language we just described this means that the cotorsion theory $(\mathcal{P}, \mathcal{A})$ has enough projectives and $(\mathcal{A}, \mathcal{I})$ has enough injectives, where $\mathcal{P}$ is the class of projective objects, $\mathcal{I}$ is the class of injective objects, and $\mathcal{A}$ is the class of all objects in $\mathcal{C}$.) We say that the cotorsion pair is *complete* if it has enough projectives and injectives. For a good reference on cotorsion pairs see [EJ01]. The equivalence of the statements above, although not difficult, is proved as Proposition 7.1.7 in [EJ01] for R-modules, but clearly holds for any abelian category $\mathcal{C}$ with enough projectives and injectives.

Another example of a cotorsion theory in **Rmod** is $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}$ is the so-called cotorsion module. Proving that this cotorsion pair is complete is nontrivial and two different proofs were recently given by the three authors of [BBE00]. For a reference on cotorsion modules see [Xu96] and [EJ01]. Another cotorsion pair that the author has in mind is the pair $(\mathcal{F}', \mathcal{C}')$, the "flat" cotorsion pair on $\mathbf{Sh}(\mathcal{O})$, the category of sheaves of $\mathcal{O}$-modules, where $\mathcal{O}$ is a ringed space on X. This cotorsion pair is also complete as follows from [EO01].

We always assume our ring R is commutative with 1. A chain complex $\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$ will be denoted by (X, d) or simply X. We say X is *bounded below (above)* if $X_n = 0$ for $n < k$ $(n > k)$ for some $k \in \mathbb{Z}$. We say it is *bounded* if it is bounded above and below. The *nth cycle module* is defined as $\ker d_n$ and is denoted $Z_n X$. The *nth boundary module* is $\mathrm{Im} d_{n+1}$ and is denoted $B_n X$. The *nth homology module* is defined to be $Z_n X / B_n X$ and is denoted $H_n X$. Given an R-module M, we let $S^n(M)$ denote the chain complex with all entries 0 except M in degree n. We let $D^n(M)$ denote the chain complex X with $X_n = X_{n-1} = M$ and all other entries 0. All maps are 0 except $d_n = 1_M$. Given X, the *suspension of* X, denoted ΣX, is the complex given by $(\Sigma X)_n = X_{n-1}$ and $(d_{\Sigma X})_n = -d_n$. The complex $\Sigma(\Sigma X)$ is denoted $\Sigma^2 X$ and inductively we define $\Sigma^n X$ for all $n \in \mathbb{Z}$. Finally, a complex C is *finitely generated* if it is bounded and each C_n is a finitely generated R-module. Similarly a complex D is *finitely presented* if it is bounded and each D_n is a finitely presented module. It can be shown that a complex D is finitely presented iff there exists a short exact sequence $0 \to K \to C \to D \to 0$ with C, K finitely generated and C_n free for each n. For example, see Lemma 4.1.1 in [GR99].

Given two chain complexes X and Y we define $\mathit{Hom}(X, Y)$ to be the complex

$$\cdots \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \mathrm{Hom}(X_k, Y_{k+n-1}) \to \cdots,$$

where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. Note that the entries are indeed R-modules since R is commutative. We leave it to the reader to verify that δ_n is R-linear and that $\delta_n \delta_{n+1} = 0$. Also the functor $\mathit{Hom}(X, -)\colon \mathbf{Ch}(R) \to \mathbf{Ch}(R)$ is left exact, and exact if X_n is projective for all n. Similarly the contravariant functor $\mathit{Hom}(-, Y)$ sends right exact sequences to left exact sequences and is exact if Y_n is injective for all n. We also note that if $0 \to I \to X \to Y \to 0$ is exact and the I_n are injective, then $0 \to \mathit{Hom}(Z, I) \to \mathit{Hom}(Z, X) \to \mathit{Hom}(Z, Y) \to 0$ is exact. Indeed in degree n we have the exact sequence

$$\prod_{k \in \mathbb{Z}} \mathrm{Hom}(Z_k, X_{k+n}) \to \prod_{k \in \mathbb{Z}} \mathrm{Hom}(Z_k, Y_{k+n}) \to \prod_{k \in \mathbb{Z}} \mathrm{Ext}^1(Z_k, I_{k+n}) = 0.$$

Of course the dual statement holds for a short exact sequence $0 \to X \to Y \to P \to 0$ with each P_n projective.

Recall that $\mathrm{Ext}^1_{\mathbf{Ch}(R)}(Y, X)$ is the group of (equivalence classes) of short exact sequences $0 \to X \to Z \to Y \to 0$. We let $\mathrm{Ext}^1_{dw}(Y, X)$ be the subgroup of $\mathrm{Ext}^1_{\mathbf{Ch}(R)}(Y, X)$ consisting of those short exact sequences which are split in each dimension. The next lemma is standard and we will not prove it.

Lemma 2.1. *For chain complexes X and Y, we have*

$$\mathrm{Ext}^1_{dw}(X, \Sigma^{(-n-1)} Y) \cong H_n \mathit{Hom}(X, Y) = Ch(R)(X, \Sigma^{-n} Y)/\sim,$$

where $\sim$ is chain homotopy.

In particular, for chain complexes X and Y, $Hom(X,Y)$ is exact iff for any $n \in \mathbb{Z}$, any $f\colon \Sigma^n X \to Y$ is homotopic to 0 (or iff any $f\colon X \to \Sigma^n Y$ is homotopic to 0). Also note that there is nothing special about **Rmod** and $\mathbf{Ch}(R)$ in the lemma. The definitions of Hom and Ext^1_{dw} easily carry over to any chain complex category $\mathbf{Ch}(\mathcal{C})$, where $\mathcal{C}$ is an abelian category, and so does the proof of Lemma 2.1.

Given two chain complexes, X and Y, their tensor product $X \otimes Y$ is defined by $(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$ in degree n. The boundary map δ_n is defined on the generators by $\delta_n(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$, where $|x|$ is the degree of the element x. One can easily check that $\delta^2 = 0$ (and this would not be true if we did not introduce the sign $(-1)^{|x|}$. The tensor product and the above Hom functor make $\mathbf{Ch}(R)$ a closed symmetric monoidal category. In fact, we have the "enriched" adjointness $Hom(X \otimes Y, Z) \cong Hom(X, Hom(Y,Z))$.)

3. Induced cotorsion pairs in chain complex categories

In this section we let $\mathcal{C}$ be any abelian category and $(\mathcal{A}, \mathcal{B})$ denote a cotorsion pair on $\mathcal{C}$. If every object of $\mathcal{C}$ is a quotient of an object in $\mathcal{A}$, then we will say that $\mathcal{C}$ has enough $\mathcal{A}$ objects. If every object of $\mathcal{C}$ is a subobject of an object in $\mathcal{B}$, we will say that $\mathcal{C}$ has enough $\mathcal{B}$ objects. We will see that whenever $\mathcal{C}$ has enough $\mathcal{A}$ objects and enough $\mathcal{B}$ objects, then the cotorsion pair induces two cotorsion pairs on $\mathbf{Ch}(\mathcal{C})$, the category of chain complexes on $\mathcal{C}$. In particular, this holds for a Grothendieck category with enough $\mathcal{A}$ objects and for any abelian category with enough projectives and injectives. As an example, note that the category $\mathbf{Sh}(\mathcal{O})$ does not have enough projectives, but it is a Grothendieck category in which every object is a quotient of a flat sheaf. So our results will apply in this category along with the flat cotorsion pair. The main example to keep in mind though is the flat cotorsion pair of R-modules. It will play a central role in the rest of the paper.

In general, there are a few common and useful adjointness relationships between an abelian category $\mathcal{C}$ and its chain complex category $\mathbf{Ch}(\mathcal{C})$. We start by listing a few easy ones in the following lemma.

Lemma 3.1. *Let $\mathcal{C}$ be an abelian category and let $\mathbf{Ch}(\mathcal{C})$ be the category of chain complexes on $\mathcal{C}$. Then for an object $C \in \mathcal{C}$ and $X, Y \in \mathbf{Ch}(\mathcal{C})$, we have the following natural isomorphisms:*

(1) $\mathcal{C}(C, Y_n) \cong \mathrm{Hom}_{Ch(\mathcal{C})}(D^n(C), Y)$,
(2) $\mathcal{C}(X_{n-1}, C) \cong \mathrm{Hom}_{Ch(\mathcal{C})}(X, D^n(C))$,
(3) $\mathcal{C}(C, Z_n Y) \cong \mathrm{Hom}_{Ch(\mathcal{C})}(S^n(C), Y)$,
(4) $\mathcal{C}(X_n/B_n X, C) \cong \mathrm{Hom}_{Ch(\mathcal{C})}(X, S^n(C))$,
(5) $\mathrm{Ext}^1_{\mathcal{C}}(C, Y_n) \cong \mathrm{Ext}^1_{Ch(\mathcal{C})}(D^n C, Y)$,
(6) $\mathrm{Ext}^1_{\mathcal{C}}(X_n, C) \cong \mathrm{Ext}^1_{Ch(\mathcal{C})}(X, D^{n+1} C)$,
(7) $\mathrm{Ext}^1_{\mathcal{C}}(C, Z_n Y) \cong \mathrm{Ext}^1_{Ch(\mathcal{C})}(S^n C, Y)$,
(8) $\mathrm{Ext}^1_{\mathcal{C}}(Z_n X, C) \cong \mathrm{Ext}^1_{Ch(\mathcal{C})}(X, S^n C)$.

Proof. The first batch is very straightforward to prove. For (5), define a map

$$\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(D^n C, Y) \to \mathrm{Ext}^1_{\mathcal{C}}(C, Y_n)$$

by sending a short exact sequence

$$0 \to Y \to Z \to D^n C \to 0$$

to the short exact sequence

$$0 \to Y_n \to Z_n \to C \to 0.$$

This is clearly well defined and the inverse map works by taking a short exact sequence

$$0 \to Y_n \xrightarrow{f} Z_n \xrightarrow{g} C \to 0$$

and forming the pushout, P, of the arrows f and $Y_n \to Y_{n-1}$. The crucial part is to observe that any extension

$$0 \to Y \to Z \to D^n C \to 0$$

must necessarily be a pushout square in degrees n and $n-1$. This follows from the universal property of a pushout. The proof of (6) is dual. For (7), define a map

$$\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^n C, Y) \to \mathrm{Ext}^1_{\mathcal{C}}(C, Z_n Y)$$

by sending short exact sequence

$$0 \to Y \to Z \to S^n C \to 0$$

to the short exact sequence

$$0 \to Z_n Y \to Z_n Z \to C \to 0.$$

In the other direction, consider a short exact sequence

$$0 \to Z_n Y \xrightarrow{f} Z \to C \to 0.$$

Form the commutative diagram below, where the rows are exact and the left square is a pushout:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_n Y & \xrightarrow{f} & Z & \longrightarrow & C & \longrightarrow & 0 \\
 & & {\scriptstyle i}\big\downarrow & & {\scriptstyle i'}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & Y_n & \xrightarrow{f'} & P & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Using the definition of pushout, the maps $d_n \colon Y_n \to Y_{n-1}$ along with the zero map $Z \to Y_{n-1}$ induce a map $\delta \colon P \to Y_{n-1}$ such that $\delta f' = d_n$ and $\delta i' = 0$. Thus we can form the exact sequence of complexes $0 \to Y \to \overline{Z} \to S^n(C) \to 0$, a portion of which is shown below:

$$\begin{array}{ccccc}
Y_{n+1} & = & Y_{n+1} & \longrightarrow & 0 \\
{\scriptstyle d}\big\downarrow & & {\scriptstyle f'd}\big\downarrow & & \big\downarrow \\
Y_n & \xrightarrow{f'} & P & \longrightarrow & C \\
{\scriptstyle d_n}\big\downarrow & & {\scriptstyle \delta}\big\downarrow & & \big\downarrow \\
Y_{n-1} & = & Y_{n-1} & \longrightarrow & 0
\end{array}$$

The proof for (8) is dual to (7). □

Notice that (1) implies that $D^n(P)$ is projective whenever P is a projective R-module. Dually, (2) implies that $D^n(I)$ is injective whenever I is injective in **Rmod**.

Proposition 3.2. $\mathbf{Ch}(\mathcal{C})$ *has enough projectives and enough injectives whenever* $\mathcal{C}$ *has enough projectives and injectives. I.e., given* $X \in \mathbf{Ch}(\mathcal{C})$*, there exists a projective chain complex* P *and an epimorphism* $P \to X$*. Dually, there exists an injective complex* I *and a monomorphism* $X \to I$*.*

Proof. Let $X \in \mathbf{Ch}(\mathcal{C})$ be given. We can find a surjection $f_n\colon P_n \to X_n$, where P_n is projective. By the first adjoint relationship above we have a chain map $D^n(f_n)\colon D^n(P_n) \to X$, which is surjective in degree n. Thus

$$\bigoplus_{n\in\mathbb{Z}} D^n(f_n)\colon \bigoplus_{n\in\mathbb{Z}} D^n(P_n) \to X$$

is surjective with $\bigoplus_{n\in\mathbb{Z}} D^n(P_n)$ projective. Similarly we can show that $\mathbf{Ch}(\mathcal{C})$ has enough injectives. □

In particular, $\mathbf{Ch}(R)$ has enough projectives and injectives.

Definition 3.3. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathcal{C}$. Let X be a chain complex.

(1) X is called an $\mathcal{A}$ *complex* if it is exact and $Z_nX \in \mathcal{A}$ for all n.
(2) X is called a $\mathcal{B}$ *complex* if it is exact and $Z_nX \in \mathcal{B}$ for all n.
(3) X is called a *dg-*$\mathcal{A}$ *complex* if $X_n \in \mathcal{A}$ for each n, and $Hom(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.
(4) X is called a *dg-*$\mathcal{B}$ *complex* if $X_n \in \mathcal{B}$ for each n, and $Hom(A, X)$ is exact whenever A is a $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\widetilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $dg\widetilde{\mathcal{A}}$. Similarly, the $\mathcal{B}$ complexes are denoted by $\widetilde{\mathcal{B}}$ and the class of dg-$\mathcal{B}$ complexes are denoted by $dg\widetilde{\mathcal{B}}$.

Note that if X is an $\mathcal{A}$ complex (resp. $\mathcal{B}$ complex), then $X_n \in \mathcal{A}$ (resp. $\mathcal{B}$) since $\mathcal{A}$ (resp. $\mathcal{B}$) is closed under extensions.

One should be aware that in the case of the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathbf{Ch}(R)$, our definitions for "dg-cotorsion" and "cotorsion" are reversed from what appears in the literature, much of which is gathered in [GR99]. Obviously, the author feels that these complexes were named incorrectly: An adjective with the prefix "dg" should stand for a sort of weaker "degreewise" notion rather than the adjective without the "dg", which should be a stronger categorical notion. The projective complexes we define here from $(\mathcal{P}, \mathcal{A})$ are indeed the categorical projectives and the flat complexes obtained from $(\mathcal{F}, \mathcal{C})$ can be described as a colimit of projectives. See [GR99].

Lemma 3.4.

(1) *Bounded below complexes with entries in* $\mathcal{A}$ *are dg-*$\mathcal{A}$ *complexes.*
(2) *Bounded above complexes with entries in* $\mathcal{B}$ *are dg-*$\mathcal{B}$ *complexes.*

Proof. First we prove (1). Let (X, d) be a bounded below complex with entries in $\mathcal{A}$. We need to show that $Hom(X, B)$ is exact whenever B is a $\mathcal{B}$ complex. This will follow from showing that any $f\colon X \to B$ is homotopic to zero, where B is a $\mathcal{B}$ complex. So let such a map $f\colon X \to B$ be given and without loss of generality assume $X_n = 0$ for $n < 0$. Now

$$(\dagger) \qquad 0 \to \ker d_1 \to B_1 \to \mathrm{Im} d_1 \to 0$$

is exact and each object in the sequence belongs to $\mathcal{B}$. Furthermore, $f_0\colon X_0 \to B_0$ lands in $\mathrm{Im} d_1$ and so may be thought of as an element of $\mathrm{Hom}_{\mathcal{C}}(X_0, \mathrm{Im} d_1)$. Now mapping X_0 into (†) yields a short exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X_0, \ker d_1) \to \mathrm{Hom}_{\mathcal{C}}(X_0, B_1) \to \mathrm{Hom}_{\mathcal{C}}(X_0, \mathrm{Im} d_1) \to 0.$$

Thus f_0 lifts to a map $D_1\colon X_0 \to B_1$. I.e., there exists D_1 such that $d_1 D_1 = f_0$. Now set $g_1 = f_1 - D_1 d_1$ and note $d_1 g_1 = d_1 f_1 - d_1 D_1 d_1 = d_1 f_1 - f_0 d_1 = 0$ so that $g_1\colon X_1 \to B_1$ lands in $\mathrm{Im} d_2$. As above, we can lift g_1 to a map $D_2\colon X_1 \to B_2$ such that $d_2 D_2 = g_1 = f_1 - D_1 d_1$. Therefore, $d_2 D_2 + D_1 d_1 = f_1$. Continuing in this way we can construct a homotopy $\{D_k\}$ such that $d_k D_k + D_{k-1} d_{k-1} = f_{k-1}$.

Statement (2) is dual to (1). Indeed suppose we are given a complex (X, d) which is bounded above and such that each X_n is in $\mathcal{B}$. We assume that $X_n = 0$ for $n > 0$. Let (A, d) be an $\mathcal{A}$ complex and $f\colon A \to X$ be a chain map. We want to show f is homotopic to 0. First let $D_n\colon A_{n-1} \to X_n$ be zero for $n > 0$. We first construct $D_0\colon A_{-1} \to X_0$ such that $D_0 d_0 = f_0$. To do this note that $\mathrm{Im} d_1 = \ker d_0$ is in $\mathcal{A}$ and is contained in $\ker f_0$. Thus we have a map $\hat{f}_0\colon \ker d_{-1} \to X_0$ such that $\hat{f}_0 d_0 = f_0$. Since the short exact sequence $0 \to \ker d_{-1} \to A_{-1} \to \mathrm{Im} d_{-1} \to 0$ has entries in $\mathcal{A}$ and $X_0 \in \mathcal{B}$, there exists $D_0\colon A_{-1} \to X_0$ which equals $\hat{f}_0$ when restricted to $\ker d_{-1}$.Thus $D_0 d_0 = \hat{f}_0 d_0 = f_0$. Now define $g_{-1} = f_{-1} - d_0 D_0$. Then we have $g_{-1} d_0 = f_{-1} d_0 - d_0 D_0 d_0 = f_{-1} d_0 - d_0 f_0 = 0$. So $\mathrm{Im} d_0 = \ker d_{-1} \subseteq \ker g_{-1}$. This means we have a map $\hat{g}_{-1}\colon \ker d_{-2} \to X_1$ such that $\hat{g}_{-1} d_{-1} = g_{-1}$. As above, $\hat{g}_{-1}$ extends to a map $D_{-1}\colon A_{-2} \to X_{-1}$. Thus $d_0 D_0 + D_{-1} d_{-1} = (f_{-1} - g_{-1}) + (\hat{g}_{-1} d_{-1}) = (f_{-1} - g_{-1}) + g_{-1} = f_{-1}$. We can continue this way to build a homotopy $\{D_k\}$ from f to 0. □

Now suppose $\mathcal{C}$ has enough $\mathcal{A}$ objects and enough $\mathcal{B}$ objects as defined in the beginning of this section. Then the argument used in Proposition 3.2 will also show that for any chain complex X there exists a surjective map $\overline{A} \to X$ with $\overline{A}$ an $\mathcal{A}$-complex. The dual argument works, too. So we have the following lemma.

Lemma 3.5. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in an abelian category $\mathcal{C}$ with enough $\mathcal{A}$ objects and enough $\mathcal{B}$ objects, then $\mathbf{Ch}(\mathcal{C})$ has enough $\widetilde{\mathcal{A}}$ objects and enough $\widetilde{\mathcal{B}}$ objects.*

Of course if $\mathcal{C}$ does have enough projectives (respectively, injectives), then $\mathbf{Ch}(\mathcal{C})$ has enough $\widetilde{\mathcal{A}}$ objects (respectively, $\widetilde{\mathcal{B}}$ objects).

Proposition 3.6. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$ with enough $\mathcal{A}$ objects and enough $\mathcal{B}$ objects. Then $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ and $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are cotorsion pairs in $\mathbf{Ch}(\mathcal{C})$.*

Proof. First we consider the cotorsion pair $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$. Let $X \in (dg\widetilde{\mathcal{A}})^{\perp}$. We will argue that $X \in \widetilde{\mathcal{B}}$. First we want to show that $Z_n X \in \mathcal{B}$. But for $A \in \mathcal{A}$, adjointness property (5) from Lemma 3.1 tells us $\mathrm{Ext}^1_{\mathcal{C}}(A, Z_n X) \cong \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(S^n A, X) = 0$. So indeed $Z_n X \in \mathcal{B}$. Next we show X is exact. Let $f_n\colon A' \to Z_n X$ be a surjection with $A' \in \mathcal{A}$. It induces a map $f\colon S^n(A') \to X$ which by hypothesis is homotopic to zero since $S^n(A') \in dg\widetilde{\mathcal{A}}$. If $\{D_n\}$ is the homotopy, then $d_{n+1} D_n = f_n$ and thus $B_n X = Z_n X$. Thus $X \in \widetilde{\mathcal{B}}$. Therefore $(dg\widetilde{\mathcal{A}})^{\perp} \subseteq \widetilde{\mathcal{B}}$.

It is easy to see that $(dg\widetilde{\mathcal{A}})^{\perp} \supseteq \widetilde{\mathcal{B}}$: Let $X \in \widetilde{\mathcal{B}}$ and say $A \in dg\widetilde{\mathcal{A}}$. It is enough to show that $\mathrm{Ext}^1_{dw}(A, X) = 0$ and by Lemma 2.1 this follows from the fact that $Hom(A, X)$ is exact.

Now we show that $dg\widetilde{\mathcal{A}} = {}^{\perp}\widetilde{\mathcal{B}}$. Let $X \in {}^{\perp}\widetilde{\mathcal{B}}$. Then $\mathrm{Ext}^1(X,B) = 0$ whenever B is a $\mathcal{B}$ complex. In particular, $\mathrm{Ext}^1_{dw}(X,B) = 0$ and so by Lemma 2.1 we have $Hom(X,B)$ is exact whenever B is a $\mathcal{B}$ complex. It remains to show $X_n \in \mathcal{A}$. So let $B \in \mathcal{B}$. Then using adjointness property (6) from Lemma 3.1 we see

$$\mathrm{Ext}^1_{\mathcal{C}}(X_n, B) = \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{C})}(X, D^{n+1}B) = 0$$

since $D^{n+1}B \in \widetilde{\mathcal{B}}$. So $X_n \in \mathcal{A}$. This shows $dg\widetilde{\mathcal{A}} \supseteq {}^{\perp}\widetilde{\mathcal{B}}$ and $dg\widetilde{\mathcal{A}} \subseteq {}^{\perp}\widetilde{\mathcal{B}}$ is easy to show (just use Lemma 2.1 again).

The method used to show $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ is a cotorsion pair is dual. □

Definition 3.7. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{C}$. Whenever $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ and $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are indeed cotorsion pairs, we will call them the *induced cotorsion pairs (of chain complexes).* We say that the induced cotorsion pairs are *compatible* if $\widetilde{\mathcal{A}} = dg\widetilde{\mathcal{A}} \cap \mathcal{E}$ and $\widetilde{\mathcal{B}} = dg\widetilde{\mathcal{B}} \cap \mathcal{E}$, where $\mathcal{E}$ is the class of exact complexes.

Corollary 3.8. *If $\mathcal{C}$ is any abelian category with enough projectives and injectives or if $\mathcal{C}$ is Grothendieck with enough $\mathcal{A}$ objects, then for any cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$ we have the induced cotorsion pairs $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ and $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ of chain complexes.*

Next we want to investigate when the induced cotorsion pairs are compatible. This is important when one wishes to apply Hovey's Theorem 1.2 from [Hov00] to get a model structure on $\mathbf{Ch}(\mathcal{C})$.

Lemma 3.9. *Every chain map from an $\mathcal{A}$ complex to a $\mathcal{B}$ complex is homotopic to* 0.

Proof. Let X be an $\mathcal{A}$-complex and Y a $\mathcal{B}$-complex, and let $f\colon X \to Y$ be a chain map. The proof is in two stages. First we show that we can replace f with a homotopic map g which satisfies $d_n g_n = 0$ and $g_n d_{n+1} = 0$. Then we show that any map $g\colon X \to Y$ with this property is homotopic to 0.

The map $f_n\colon X_n \to Y_n$ restricts to $\hat{f}_n\colon Z_n X \to Z_n Y$, and

$$0 \to Z_{n+1}Y \to Y_{n+1} \to Z_n Y \to 0$$

is an exact sequence of objects in $\mathcal{B}$. So

$$0 \to \mathrm{Hom}(Z_n X, Z_{n+1}Y) \to \mathrm{Hom}(Z_n X, Y_{n+1}) \to \mathrm{Hom}(Z_n X, Z_n Y) \to 0$$

is a short exact sequence. Therefore we have $\alpha_n\colon Z_n X \to Y_{n+1}$ such that $d_{n+1}\alpha_n = \hat{f}_n$.

Now also $0 \to Z_n X \to X_n \to Z_{n-1}X \to 0$ is an exact sequence of objects in $\mathcal{A}$. So $0 \to \mathrm{Hom}(Z_{n-1}X, Y_{n+1}) \to \mathrm{Hom}(X_n, Y_{n+1}) \to \mathrm{Hom}(Z_n X, Y_{n+1}) \to 0$ is exact and there exists $\beta_n\colon X_n \to Y_{n+1}$ which equals α_n when restricted to $Z_n X$.

Now set $g_n = f_n - (d_{n+1}\beta_n + \beta_{n-1}d_n)$. It is easy to see that $g = \{g_n\}_{n\in\mathbb{Z}}$ is a chain map. It is homotopic to f since $f_n - g_n = d_{n+1}\beta_n + \beta_{n-1}d_n$. Furthermore, a straightforward computation shows it satisfies $d_n g_n = 0$ and $g_n d_{n+1} = 0$.

The remainder of the proof shows that whenever we have a chain map g such that $dg = 0 = gd$, then g is homotopic to 0. Indeed we know that $\mathrm{Im} g_n \subset \ker d_n$ and $\mathrm{Im} d_{n+1} = \ker d_n \subset \ker g_n$. This allows us to define a map $\bar{g_n}\colon X_n/Z_n X \to Z_n Y$

which makes the following diagram commute:

$$\begin{array}{ccccc}
X_n & = & X_n & = & X_n \\
{\scriptstyle d_n}\downarrow & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g_n} \\
Z_{n-1}X & \xleftarrow[\bar{d}_n]{\cong} & X_n/Z_nX & \xrightarrow[\bar{g}_n]{} & Z_nY
\end{array}$$

If we set $\hat{g}_n\colon = \bar{g}_n\bar{d}_n^{-1}$, then $\hat{g}_n\colon Z_{n-1}X \to Z_nY$ and $\hat{g}_n d_n = g_n$. Now (using the argument as above to obtain the maps α_n) there exists a map $\delta_n\colon Z_{n-1}X \to Y_{n+1}$ such that $d_{n+1}\delta_n = \hat{g}_n$. One can easily check that the maps $\delta_n d_n\colon X_n \to Y_{n+1}$ are a homotopy from g to 0. □

Lemma 3.10. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category. Then $\widetilde{\mathcal{A}} \subset dg\widetilde{\mathcal{A}} \cap \mathcal{E}$ and $\widetilde{\mathcal{B}} \subset dg\widetilde{\mathcal{B}} \cap \mathcal{E}$.*

Proof. This follows immediately from the last lemma and the definitions. □

Lemma 3.10 tells us that the critical question in the notion of being compatible is whether or not the containments $\widetilde{\mathcal{A}} \supset dg\widetilde{\mathcal{A}} \cap \mathcal{E}$ and $\widetilde{\mathcal{B}} \supset dg\widetilde{\mathcal{B}} \cap \mathcal{E}$ hold. We now see that this is directly linked to whether or not $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair.

Definition 3.11. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category is called *hereditary* if one of the following hold:

(1) $\mathcal{A}$ is resolving. That is, $\mathcal{A}$ is closed under taking kernels of epis.
(2) $\mathcal{B}$ is coresolving. That is, $\mathcal{B}$ is closed under taking cokernels of monics.
(3) $\mathrm{Ext}^i(A, B) = 0$ for any R-modules $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and $i \geq 1$.

See [GR99] for a proof that these are equivalent.

Theorem 3.12. *Suppose $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair in an abelian category $\mathcal{C}$. If $\mathcal{C}$ has enough projectives, then $dg\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$ and if $\mathcal{C}$ has enough injectives, then $dg\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$. In particular, if $\mathcal{C}$ has enough projectives and injectives, then the induced cotorsion pairs on $\mathcal{C}$ are compatible.*

Proof. We show $dg\widetilde{\mathcal{B}} \cap \mathcal{E} \subset \widetilde{\mathcal{B}}$ when $\mathcal{C}$ has enough projectives. The second statement is dual.

Let X be an exact dg-$\mathcal{B}$ complex. We need to show $\mathrm{Ext}^1(A, Z_nX) = 0$ for all $A \in \mathcal{A}$. Let $P_\circ$ be an augmented projective resolution of $A :\ P_\circ = \cdots P_2 \to P_1 \to A \to 0$. Since $(\mathcal{A}, \mathcal{B})$ is hereditary, $P_\circ \in \widetilde{\mathcal{A}}$. Now $0 \to Z_nX \to X_n \to Z_{n-1}X \to 0$ is exact and $\mathrm{Ext}^1(A, X_n) = 0$, so we will be done if we can show that any $f\colon A \to Z_{n-1}X$ factors through X_n. But any $f\colon A \to Z_{n-1}X$ induces a chain map $\Sigma^{n-1}P_\circ \to X$. (This is easy to check: use the fact that X is exact and build lifts inductively.) Since X is a dg-$\mathcal{B}$ complex and $\Sigma^{n-1}P_\circ$ is an $\mathcal{A}$ complex, this map is homotopic to 0. A chain homotopy $\{D_n\}$ will give the desired lift $f = d_nD_{n-1}$. □

Corollary 3.13. *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in an abelian category with enough projectives and injectives and let $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}}), (dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ be the induced cotorsion pairs of chain complexes. TFAE:*

(1) *$(\mathcal{A}, \mathcal{B})$ is hereditary.*
(2) *$(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ is hereditary.*
(3) *$(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is hereditary.*

(4) $\widetilde{\mathcal{A}} = dg\widetilde{\mathcal{A}} \cap \mathcal{E}$.
(5) $\widetilde{\mathcal{B}} = dg\widetilde{\mathcal{B}} \cap \mathcal{E}$.
(6) *The induced pairs* $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}}), (dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are compatible and hereditary.*

Proof. Clearly (6) implies (2), (3), (4), and (5). The plan is to show (2), (3), (4), and (5) each imply (1) and finally that (1) implies (6).

For (2) $\Rightarrow$ (1) let $0 \to B' \to B \to B'' \to 0$ be a short exact sequence with $B', B \in \mathcal{B}$. Then $0 \to S^\circ B' \to S^\circ B \to S^\circ B'' \to 0$ is exact and $S^\circ B', S^\circ B \in dg\widetilde{\mathcal{B}}$. Thus $S^\circ B'' \in dg\widetilde{\mathcal{B}}$, so $B'' \in \mathcal{B}$. For (3) $\Rightarrow$ (1) we do the analogous thing. Show $\mathcal{A}$ is resolving.

For (4) $\Rightarrow$ (1) we show $\mathcal{A}$ is resolving. Let $0 \to A' \to A \to A'' \to 0$ be exact with $A, A'' \in \mathcal{A}$. Extend this sequence to a resolution $P_\circ = \cdots \to P_2 \to P_1 \to A \to A'' \to 0$, where the P_i's are projective. Then $P_\circ$ is a dg-$\mathcal{A}$ complex since it is bounded below. It is also exact (by construction). The hypothesis implies $P_\circ$ is an $\mathcal{A}$ complex, so $\ker(A \to A'') = A' \in \mathcal{A}$. In a similar way (5) $\Rightarrow$ (1). Show $\mathcal{B}$ is coresolving by dualizing the argument above.

It is left to show (1) $\Rightarrow$ (6). Theorem 3.12 shows that the cotorsion pairs induced from $(\mathcal{A}, \mathcal{B})$ are compatible. So it suffices to show that $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ are hereditary. The statements are in fact dual and so we are done after we show that $dg\widetilde{\mathcal{A}}$ is resolving. So let $0 \to A' \to A \to A'' \to 0$ be exact with $A, A'' \in dg\widetilde{\mathcal{A}}$. Clearly, $A'_n \in \mathcal{A}$ since $(\mathcal{A}, \mathcal{B})$ is hereditary. Now let X be a $\mathcal{B}$ complex. Then $0 \to Hom(A'', X) \to Hom(A, X) \to Hom(A', X) \to 0$ must be exact. (Just check in each degree). It follows by the fundamental lemma of homological algebra that $Hom(A', X)$ is exact. Thus A' is a dg-$\mathcal{A}$ complex and so $dg\widetilde{\mathcal{A}}$ is resolving. $\square$

The examples of cotorsion pairs given in section 2 are all hereditary. For each example, the induced cotorsion pairs exist. Note however that for the flat cotorsion pair $(\mathcal{F}', \mathcal{C}')$ on the category $\mathbf{Sh}(\mathcal{O})$, Theorem 3.12 only allows us to conclude that $\widetilde{\mathcal{F}'} = dg\widetilde{\mathcal{F}'} \cap \mathcal{E}$, since $\mathbf{Sh}(\mathcal{O})$ does not have enough projectives. Nevertheless, the author has been able to show that in this case the induced cotorsion pairs are still compatible. He has not however found a general proof of Theorem 3.12 assuming only the existence of $\mathcal{A}$ objects and $\mathcal{B}$ objects. The next lemma shows how one can deal with "almost compatible" cotorsion pairs such as the sheaf situation just mentioned.

Lemma 3.14. *Suppose $\mathcal{C}$ is an abelian category and $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair.*

(1) *If $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ is a cotorsion pair with enough injectives and $dg\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$, then $dg\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$.*
(2) *If $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ is a cotorsion pair with enough projectives and $dg\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$, then $dg\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$.*

Proof. The two statement are dual. We will prove the first one. By Lemma 3.10 we just need to show $dg\widetilde{\mathcal{B}} \cap \mathcal{E} \subset \widetilde{\mathcal{B}}$. So let X be an exact dg-$\mathcal{B}$ complex. Since $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ has enough injectives we have a short exact sequence

$$0 \to X \to B \to A \to 0$$

with $B \in \widetilde{\mathcal{B}}$ and $A \in dg\widetilde{\mathcal{A}}$. Since X and B are each exact, so is A. But then $A \in \widetilde{\mathcal{A}}$ by hypothesis. So the sequence must split, forcing X to be a direct summand of B. Since $\widetilde{\mathcal{B}}$ is closed under direct summands we have $X \in \widetilde{\mathcal{B}}$. $\square$

The question of whether or not the induced cotorsion pairs are complete when the original cotorsion pair is complete is open.

4. The flat case

Again let $\mathcal{F}$ be the class of flat modules and let $\mathcal{C}$ be the class of cotorsion modules. The goal of this section is to show that the cotorsion theories on $\mathbf{Ch}(R)$ induced by $(\mathcal{F}, \mathcal{C})$ are both complete. Two different proofs were given in [BBE00] that $(\mathcal{F}, \mathcal{C})$ is complete. One was given by L. Bican and R. El Bashir and the second by E. Enochs. Our method of proof is analogous to Enochs'.

Complexes belonging to the class $\widetilde{\mathcal{F}}$ are called flat. That is, flat complexes are exact with all cycle modules flat. Unfortunately the usual tensor product on $\mathbf{Ch}(R)$ does not characterize flatness as it does in **Rmod**. In particular we may have a chain complex X for which $X \otimes -$ is exact and yet X is not flat. Indeed $X \otimes -$ is exact even if we just have X_n flat for all n. However, there is a different closed symmetric monoidal structure on $\mathbf{Ch}(R)$ introduced by Enochs and Rozas which behaves properly with the flat complexes. This was studied in [EGR97]. Here we will briefly discuss the "new" tensor product and hom functor and their important properties. The advantage is that many analogues and proof methods from **Rmod** will carry over to $\mathbf{Ch}(R)$.

Given chain complexes X and Y, $X \overline{\otimes} Y$ is the chain complex with nth entry $(X \overline{\otimes} Y)_n = (X \otimes Y)_n / B_n(X \otimes Y)$ and boundary map

$$\frac{(X \otimes Y)_n}{B_n(X \otimes Y)} \to \frac{(X \otimes Y)_{n-1}}{B_{n-1}(X \otimes Y)}$$

given by $(\overline{x \otimes y}) = \overline{dx \otimes y}$. This gives us a bifunctor

$$-\overline{\otimes}-\colon \mathbf{Ch}(R) \times \mathbf{Ch}(R) \to \mathbf{Ch}(R).$$

For a complex X, the functor $X \overline{\otimes} -$ is right exact. The following is due to Enochs and García-Rozas. Consult [GR99] for a proof.

Theorem 4.1. *A chain complex $F \in \mathbf{Ch}(R)$ is flat iff $F \overline{\otimes} -$ is exact.*

$\overline{Hom}(X, Y)$ is the chain complex defined by

$$\overline{Hom}(X, Y)_n = Z_n Hom(X, Y)$$

with

$$\lambda_n \colon \overline{Hom}(X, Y)_n \to \overline{Hom}(X, Y)_{n-1}$$

the map $(\lambda f)_k = (-1)^n d_{k+n} f_k$. This makes $\overline{Hom}(X, Y)$ a chain complex with nth degree just equal to $\mathrm{Hom}_{\mathbf{Ch}(R)}(X, \Sigma^{-n} Y)$. In this way we have a functor $\overline{Hom}(X, -)\colon \mathbf{Ch}(R) \to \mathbf{Ch}(R)$ and a contravariant functor $\overline{Hom}(-, Y)\colon \mathbf{Ch}(R) \to \mathbf{Ch}(R)$. Both of these are left exact since the functors $\mathrm{Hom}_{Ch(R)}(X, -)$ and $\mathrm{Hom}_{Ch(R)}(-, Y)$ are left exact. (For a contravariant functor, take left exact to mean that it takes right exact sequences to left exact sequences.) Furthermore, if $I \in \mathbf{Ch}(R)$ is injective, then $\overline{Hom}(-, I)$ is exact because each I_n is injective, and similarly if P is projective, then $\overline{Hom}(P, -)$ is exact.

The next theorem appears to be due to Enochs and García-Rozas. Rather than repeat the (long) proofs here we refer the reader to [GR99], pp. 89–96.

Proposition 4.2. *Let X, Y, Z be chain complexes. We have the following natural isomorphisms:*

(1) $\overline{Hom}(X \overline{\otimes} Y, Z) \cong \overline{Hom}(X, \overline{Hom}(Y, Z))$ *(Therefore,* $(\operatorname{colim}_{i \in I} X_i) \overline{\otimes} Y \cong \operatorname{colim}_{i \in I}(X_i \overline{\otimes} Y)$ *for a directed family* $\{ X_i \}$ *of chain complexes.)*
(2) $X \overline{\otimes} Y \cong Y \overline{\otimes} X$.
(3) $X \overline{\otimes} (Y \overline{\otimes} Z) \cong (X \overline{\otimes} Y) \overline{\otimes} Z$.
(4) *For an R-module M,* $D^n(M) \overline{\otimes} X \cong M \otimes_R \Sigma^n X$.

The following definition will not be a surprise.

Definition 4.3. Let $0 \to P \to X \to X/P \to 0$ be a short exact sequence of chain complexes. We say the sequence is *pure* if for any Y, the sequence $0 \to Y \overline{\otimes} P \to Y \overline{\otimes} X \to Y \overline{\otimes} X/P \to 0$ is exact.

Using Theorem 4.1 and Proposition 4.2 one can prove the following characterizations of purity. As noted in [GR99], the proofs are analogous to the corresponding results for R-modules. See for example [Wis91], pp. 286–288.

Proposition 4.4. *Let $0 \to P \to X \to X/P \to 0$ be a short exact sequence of chain complexes. TFAE:*

(1) *For any $Y \in \mathbf{Ch}(R)$, the sequence $0 \to Y \overline{\otimes} P \to Y \overline{\otimes} X \to Y \overline{\otimes} X/P \to 0$ is exact.*
(2) *The sequence*

$$0 \to \overline{Hom}_{\mathbb{Z}}(X/P\,,\, D(\mathbb{Q}/\mathbb{Z})) \to \overline{Hom}_{\mathbb{Z}}(X\,,\, D(\mathbb{Q}/\mathbb{Z})) \to \overline{Hom}_{\mathbb{Z}}(P\,,\, D(\mathbb{Q}/\mathbb{Z})) \to 0$$

is still exact after applying $Y \overline{\otimes} -$ for any $Y \in \mathbf{Ch}(R)$.
(3) $0 \to \overline{Hom}(W, P) \to \overline{Hom}(W, X) \to \overline{Hom}(W, X/P) \to 0$ *is exact for any finitely presented complex W.*
(4) *For every commutative diagram*

$$\begin{array}{ccccc} & & F & \xrightarrow{g} & G \\ & & {\scriptstyle f}\downarrow & & \downarrow \\ 0 & \longrightarrow & P & \longrightarrow & X \end{array}$$

with F, G finitely generated, projective and with each F_n, G_n free, there exists $\beta \colon G \to P$ with $\beta g = f$.
(5) $0 \to P \to X \to X/P \to 0$ *is a direct limit of split exact sequences* $0 \to A_i \to B_i \to C_i \to 0\,(i \in I)$ *with C_i finitely presented for all $i \in I$.*

Next we generalize a lemma of Eklof and Trlifaj regarding transfinite extensions and cotorsion pairs. The author learned this from [Hov00]. First some definitions: A *transfinite composition* in an abelian category $\mathcal{C}$ is a map of the form $X_0 \xrightarrow{f} \operatorname{colim} X_\alpha$, where $X \colon \lambda \to \mathcal{C}$ is a colimit-preserving functor and λ is an ordinal. We refer to f as the transfinite composition of the maps $X_\alpha \to X_{\alpha+1}$. If in addition, the maps $X_\alpha \to X_{\alpha+1}$ are all monomorphisms with cokernel in some class $\mathcal{D}$, then we refer to $f \colon X_0 \to \operatorname{colim} X_\alpha$ as a *transfinite extension of* X_0 *by* $\mathcal{D}$. If $X_0 \in \mathcal{D}$ as well, we just refer to $\operatorname{colim} X_\alpha$ as a *transfinite extension of* $\mathcal{D}$. Notice that this generalizes the usual notion of extension ($\lambda = 2$) and finite extensions ($\lambda = n$).

By way of example, let λ be a limit ordinal and let $(M_\alpha)_{\alpha < \lambda}$ be a family of submodules of a module M. We call the family a *continuous chain* of submodules if $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \lambda$ and if $M_\beta = \bigcup_{\alpha < \beta} M_\beta$ whenever $\beta < \lambda$ is a limit ordinal. Clearly, M is the union of a continuous chain of submodules $(M_\alpha)_{\alpha<\lambda}$ iff $M_0 \subseteq M$ is a transfinite composition of the maps $M_\alpha \subseteq M_{\alpha+1}$. If $M_0 \in \mathcal{D}$ and $M_{\alpha+1}/M_\alpha \in \mathcal{D}$, where $\mathcal{D}$ is some class of modules, then M is a transfinite extension of $\mathcal{D}$. The same ideas apply to chain complexes as well.

Lemma 4.5. *Let $\mathcal{C}$ be a bicomplete abelian category. Given $Y \in \mathcal{C}$ the class of all objects X for which $\operatorname{Ext}^1_{\mathcal{C}}(X,Y)=0$ is closed under transfinite extensions.*

Notice that for a cotorsion pair $(\mathcal{A},\mathcal{B})$, Lemma 4.5 shows that $\mathcal{A}$ is closed under transfinite extensions.

Proof. Let λ be a limit ordinal and let $X\colon \lambda \to \mathcal{C}$ be a colimit-preserving functor such that $\operatorname{Ext}^1_{\mathcal{C}}(X_0,Y)=0$, $X_\alpha \to X_{\alpha+1}$ is a monomorphism for all $\alpha<\lambda$, and $\operatorname{Ext}^1_{\mathcal{C}}(X_{\alpha+1}/X_\alpha,Y)=0$ for all $\alpha<\lambda$. We will show that $\operatorname{Ext}^1_{\mathcal{C}}(X_\beta,Y)=0$ for all $\beta\le\lambda$ by transfinite induction, where we take $X_\lambda=\operatorname{colim}_{\alpha<\lambda}X_\alpha$. The initial step and the successor ordinal step of the induction are easy.

For the limit ordinal step, suppose $\beta\le\lambda$ is a limit ordinal and $\operatorname{Ext}^1_{\mathcal{C}}(X_\alpha,Y)=0$ for all $\alpha<\beta$. An element of $\operatorname{Ext}^1_{\mathcal{C}}(X_\beta,Y)$ is represented by a short exact sequence

$$0 \to Y \xrightarrow{f} N \xrightarrow{p} X_\beta \to 0.$$

By pulling this short exact sequence back through the map $X_\alpha \xrightarrow{i_\alpha} X_{\alpha+1}$ for each $\alpha\le\beta$, we get an enormous commutative diagram as implied by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{f} & N & \xrightarrow{p} & X_\beta & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle j_\alpha} & & \downarrow{\scriptstyle i_\alpha} & & \\
0 & \longrightarrow & Y & \xrightarrow{f_{\alpha+1}} & N_{\alpha+1} & \xrightarrow{p_{\alpha+1}} & X_{\alpha+1} & \longrightarrow & 0.
\end{array}$$

We will construct splittings $s_\alpha\colon X_\alpha \to N_\alpha$ of p_α such that $j_\alpha s_\alpha = s_{\alpha+1}i_\alpha$ by transfinite induction on α. Then since X_β is the colimit of the X_α, the s_α give rise to a unique map $s\colon X_\beta\to N$. The uniqueness of the map will show that indeed $ps=1_{X_\beta}$. Therefore we will get that $\operatorname{Ext}^1_{\mathcal{C}}(X_\beta,Y)=0$ as required to complete the transfinite induction.

By the inductive hypothesis we obviously can choose a splitting $t_\alpha\colon X_\alpha\to N_\alpha$ of p_α. The transfinite induction will consist of modifying the t_α to construct a compatible collection s_α. Begin by setting $s_0=t_0$. This time the limit ordinal case is easy: for a limit ordinal γ we take $s_\gamma\colon X_\gamma=\operatorname{colim}_{\alpha<\gamma}X_\alpha\to N_\gamma$ to be the map induced by the "colimit impostor" $\{k_\alpha s_\alpha\}$, where k_α is the obvious map $N_\alpha\to N_\gamma$. Now we consider the successor ordinal step. So suppose α is an ordinal and we have constructed compatible s_α's. We now construct $s_{\alpha+1}$ such that $j_\alpha s_\alpha = s_{\alpha+1}i_\alpha$. Note that $p_{\alpha+1}(j_\alpha s_\alpha - t_{\alpha+1}i_\alpha)=0$. Since $\operatorname{Ext}^1_{\mathcal{C}}(X_\alpha,Y)=0$, one can find a map $h\colon X_\alpha\to Y$ such that $f_{\alpha+1}h=j_\alpha s_\alpha - t_{\alpha+1}i_\alpha$. Similarly, since

$$\operatorname{Hom}_{\mathcal{C}}(X_{\alpha+1},Y)\xrightarrow{i_\alpha^*}\operatorname{Hom}_{\mathcal{C}}(X_\alpha,Y)\to\operatorname{Ext}_{\mathcal{C}}(X_{\alpha+1}/X_\alpha,Y)=0$$

is exact we have a map $g\colon X_{\alpha+1} \to Y$ such that $gi_\alpha = h$. Now set $s_{\alpha+1} = t_{\alpha+1} + f_{\alpha+1}g$. Then $p_{\alpha+1}s_{\alpha+1} = p_{\alpha+1}t_{\alpha+1} + p_{\alpha+1}f_{\alpha+1}g = p_{\alpha+1}t_{\alpha+1} + 0 = 1_{X_{\alpha+1}}$. So $s_{\alpha+1}$ is a splitting of $p_{\alpha+1}$. Also $s_{\alpha+1}$ is compatible with the other s_α's for $s_{\alpha+1}i_\alpha = (t_{\alpha+1}+f_{\alpha+1}g)i_\alpha = t_{\alpha+1}i_\alpha + f_{\alpha+1}h = t_{\alpha+1}i_\alpha + (j_\alpha s_\alpha - t_{\alpha+1}i_\alpha) = j_\alpha s_\alpha$. □

For a chain complex X, we define its cardinality to be $|\coprod_{n\in\mathbb{Z}} X_n|$. The author learned the next lemma from [GR99].

Lemma 4.6. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Say $X \in \mathbf{Ch}(R)$ and we are given $x \in X$ (by this we mean $x \in X_n$ for some n). Then there exists a pure $P \subseteq X$ with $x \in P$ and $|P| \leq \kappa$.*

Proof. If $x \in X_n$, let $S_n = Rx$ and $S_{n-1} = d_n(Rx)$. Then $S_0 = \cdots \to 0 \to S_n \to S_{n-1} \to 0 \to \cdots$ is a subcomplex of X and $|S_0| \leq \kappa$. Denote $S = S_0$ and consider the class of quadruples (Y, Z, ϕ, ψ), where Y and Z are finitely generated projective complexes with each entry free and $\phi\colon Y \to Z$ and $\psi\colon Y \to S_0$ are maps of complexes with the property that *there exists* a map $Z \to X$ making the diagram below commute:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ {\scriptstyle\psi}\Big\downarrow & & \Big\downarrow \\ S_0 & \longrightarrow & X \end{array}$$

Let $T_0 = \{(Y_i, Z_i, \phi_i, \psi_i,)\}_{i\in I_0}$ be a set of representatives of this class (indexed by a set I_0). Thus for any (Y, Z, ϕ, ψ) with the above property there exists a $k \in I_0$ and isomorphisms $Y_k \xrightarrow{\cong} Y$, $Z_k \xrightarrow{\cong} Z$ such that the diagrams below commute:

$$\begin{array}{ccc} Y_k & \xrightarrow{\phi_k} & Z_k \\ {\scriptstyle\cong}\Big\downarrow & & {\scriptstyle\cong}\Big\downarrow \\ Y & \xrightarrow{\phi} & Z \end{array}$$

$$\begin{array}{ccc} Y_k & \xrightarrow{\cong} & Y \\ {\scriptstyle\psi_k}\Big\downarrow & & \Big\downarrow{\scriptstyle\psi} \\ S_0 & = & S_0 \end{array}$$

Then we have $|T_0| \leq \kappa$. (To see this just note that T_0 is a subset of the set of ALL quadruples (Y, Z, ϕ, ψ) (up to isomorphic representatives), and this set also has cardinality less than or equal to κ, by a simple counting argument). For each $(Y_i, Z_i, \phi_i, \psi_i) \in T_0$ pick an extension $\overline{\psi}_i\colon Z_i \to X$ of ψ. Then set $S_1 = S_0 + \sum_{i\in I_0} \overline{\psi}_i(Z_i)$. Then we have $|S_1| \leq |S_0| + |\sum_{i\in I_0} \overline{\psi}_i(Z_i)| \leq |S_0| + |T_0| \cdot |R| \leq \kappa$.

Now continue inductively. After constructing $x \in S_o \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1}$ with $|S_i| \leq \kappa$, consider the class of quadruples (Y, Z, ϕ, ψ) for which Y and Z are finitely generated projective complexes with each degree free and there exists a map making the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ {\scriptstyle\psi}\Big\downarrow & & \Big\downarrow \\ S_{n-1} & \longrightarrow & X \end{array}$$

commute. Let $T_n = \{(Y_i, Z_i, \phi_i, \psi_i)\}_{i\in I_n}$ be a set of representatives as above. Again $|T_n| \leq \kappa$ since $|S_{n-1}| \leq \kappa$. Now for each $i \in I_n$ pick one such extension, $\overline{\psi}_i$, and set $S_n = S_{n-1} + \sum_{i\in I_n} \overline{\psi}_i(Z_i)$. Then $S_{n-1} \subseteq S_n$ and $|S_n| \leq |S_{n-1}| + |T_n| \cdot |R| \leq \kappa$.

Now set $P = \bigcup S_n$. Then $|P| \leq \kappa$. We show that $P \subseteq X$ is pure. Suppose we are given a commutative diagram

$$\begin{array}{ccccc} & & F & \xrightarrow{g} & G \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & P & \longrightarrow & X \end{array}$$

where F and G are finitely generated projective complexes which are free in each degree. To show that $P \subseteq X$ is pure we want a map $G \to P$ making the upper left triangle commute. But since F is finitely generated, $f(F) \subseteq S_n$ for some n. As a result (F, G, g, f) is isomorphic to an element $(Y_k, Z_k, \phi_k, \psi_k) \in S_n$. By construction, $\psi_k(Y_k) \subseteq S_{n+1}$, and therefore we may complete the diagram as desired. □

Since we have a tensor product which characterizes flatness and purity, the next two lemmas have proofs exactly like the analogous lemmas in **Rmod**.

Lemma 4.7. *Let F be a chain complex. If F is flat and $P \subseteq F$ is pure, then*
(1) *F/P is flat.*
(2) *P is flat.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence. Since $0 \to P \to F \to F/P \to 0$ is a pure sequence we get the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & X \overline{\otimes} P & \longrightarrow & X \overline{\otimes} F & \longrightarrow & X \overline{\otimes} F/P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y \overline{\otimes} P & \longrightarrow & Y \overline{\otimes} F & \longrightarrow & Y \overline{\otimes} F/P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z \overline{\otimes} P & \longrightarrow & Z \overline{\otimes} F & \longrightarrow & Z \overline{\otimes} F/P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The rows are exact because P is pure and the center column is exact because F is flat. Applying the snake lemma tells us that

$$0 \to X \overline{\otimes} P \to Y \overline{\otimes} P \to Z \overline{\otimes} P$$

and

$$0 \to X \overline{\otimes} F/P \to Y \overline{\otimes} F/P \to X \overline{\otimes} F/P$$

are both exact. □

Lemma 4.8. *If $0 \to X \to Y \to F \to 0$ is a short exact sequence of chain complexes and F is flat, then the sequence is pure.*

Proof. Let Z be arbitrary. By Proposition 3.2, $\mathbf{Ch}(R)$ has enough projectives, so we have a short exact sequence $0 \to K \to P \to Z \to 0$, where P is projective. We have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & X \overline{\otimes} K & \longrightarrow & Y \overline{\otimes} K & \longrightarrow & F \overline{\otimes} K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \overline{\otimes} P & \longrightarrow & Y \overline{\otimes} P & \longrightarrow & F \overline{\otimes} P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X \overline{\otimes} Z & \longrightarrow & Y \overline{\otimes} Z & \longrightarrow & F \overline{\otimes} Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The 0 on the left is because any projective complex is flat. The snake lemma now tells us that

$$0 \to X \overline{\otimes} Z \to Y \overline{\otimes} Z \to F \overline{\otimes} Z \to 0$$

is exact. □

Note that the direct limit of pure exact sequences is pure exact since $\overline{\otimes}$ commutes with direct limits. In particular, an increasing union of pure subcomplexes is again a pure subcomplex. Also if $P \subseteq X$ is pure and $P \subseteq S \subseteq X$, then $P \subseteq S$ is pure: For given any Z, the composite $Z \overline{\otimes} P \to Z \overline{\otimes} S \to Z \overline{\otimes} X$ must be injective (because P is pure) and this implies $Z \overline{\otimes} P \to Z \overline{\otimes} S$ is injective.

We are now ready to prove that $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ is complete. We use a generalized version of a well-known theorem of Eklof and Trlifaj [ET99] which says that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in **Rmod** is complete when it is cogenerated by a set. (This result actually holds in any Grothendieck category with enough projectives, as Hovey proved in [Hov00].) We say that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *cogenerated* by a set G if $G^{\perp} = \mathcal{B}$. The next proposition provides us with a set which cogenerates $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$.

Proposition 4.9. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Let $\mathcal{G}$ be the set of all flat complexes $F \in \mathbf{Ch}(R)$ for which $|F| \leq \kappa$ (take one representative for each isomorphism class). Then any flat complex $F \in \mathbf{Ch}(R)$ is (isomorphic to) a transfinite extension of $\mathcal{G}$.*

Proof. Let $F \in \mathbf{Ch}(R)$. We will show that F is equal to the union of a continuous chain $(P_\alpha)_{\alpha<\lambda}$ of pure subcomplexes of F with $|P_0| \leq \kappa$ and $|P_{\alpha+1}/P_\alpha| \leq \kappa$ for all α. By the above lemmas, it follows that F is a transfinite extension of $\mathcal{G}$.

Let $T = \coprod_{n\in\mathbb{Z}} F_n$. We may well order the set T so that for some ordinal λ,

$$T = \{\, x_0, x_1, x_2, \ldots, x_\alpha, \ldots\}_{\alpha\in\lambda}.$$

For x_0, use Lemma 4.6 to find a pure $P_1 \subseteq F$ containing x_0 with $|P_1| \leq \kappa$. Then F/P_1 is flat by Lemma 4.7. Now $\overline{x}_1 \in F/P_1$. Therefore we can find a pure $P_2/P_1 \subseteq F/P_1$ containing $\overline{x}_1$ such that $|P_2/P_1| \leq \kappa$. Then $(F/P_1)/(P_2/P_1) \cong F/P_2$ is flat. It follows by Lemma 4.8 that P_2 is pure. Note that $P_1 \subseteq P_2$ and $x_0, x_1 \in P_2$.

In general, given any ordinal α, and having constructed pure subcomplexes $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\alpha$ where $x_\gamma \in P_\alpha$ for all $\gamma < \alpha$, we find a pure subcomplex $P_{\alpha+1} \subseteq F$ as follows: $\overline{x}_\alpha \in F/P_\alpha$, so by Lemma 4.6 we can find a pure subcomplex $P_{\alpha+1}/P_\alpha \subseteq F/P_\alpha$ containing $\overline{x}_\alpha$ such that $|P_{\alpha+1}/P_\alpha| \leq \kappa$. Thus $(F/P_\alpha)/(P_{\alpha+1}/P_\alpha) \cong F/P_{\alpha+1}$ is flat, whence $P_{\alpha+1}$ is pure. We now have $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\alpha \subseteq P_{\alpha+1}$ and $x_0, x_1, \ldots, x_\alpha \in P_{\alpha+1}$.

For the case when α is a limit ordinal we just define $P_\alpha = \bigcup_{\gamma<\alpha} P_\gamma$. Then as we noted above, P_α is pure, and $x_\gamma \in P_\alpha$ for all $\gamma < \alpha$. This construction gives us the desired continuous chain $(P_\alpha)_{\alpha<\lambda}$. $\square$

Corollary 4.10. *Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair on* **Rmod**. *Then the induced cotorsion pair $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ on $\mathbf{Ch}(R)$ is complete.*

Proof. As in the hypothesis of Proposition 4.9, let $|R| \leq \kappa$, where κ is some infinite cardinal, and let $\mathcal{G}$ be the set of all flat complexes $F \in \mathbf{Ch}(R)$ for which $|F| \leq \kappa$. Then $\mathcal{G}^\perp = dg\widetilde{\mathcal{C}}$:

($\supseteq$). This is clear since $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ is a cotorsion theory and $\mathcal{G} \subseteq \widetilde{\mathcal{F}}$.

($\subseteq$). Suppose we are given any chain complex C such that $\mathrm{Ext}^1(G, C) = 0$ for all $G \in \mathcal{G}$. Since any flat complex is a transfinite extension of complexes in $\mathcal{G}$, Lemma 4.5 tells us $\mathrm{Ext}^1(F, C) = 0$ for any flat complex F. Therefore $C \in dg\widetilde{\mathcal{C}}$. So $\mathcal{G}^\perp \subseteq dg\widetilde{\mathcal{C}}$. Since $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ is cogenerated by a set, it is complete. $\square$

Next we prove that $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ is complete. The method of proof is entirely analogous to the method we used above. We just need to derive the proper analogs to Lemmas 4.6 - 4.8. We will use the following well-known characterization of dg-flat complexes. The reader can find a proof of this in [GR99].

Proposition 4.11. *A chain complex F is dg-flat iff each F_n is flat and $F \otimes E$ is exact whenever E is an exact complex.*

With this in hand we start with the proper analog to the notion of *pure*.

Definition 4.12. A short exact sequence $0 \to X \to Y \to Z \to 0$ is called *dg-pure* if $0 \to E \overline{\otimes} X \to E \overline{\otimes} Y \to E \overline{\otimes} Z \to 0$ is exact whenever E is an exact complex.

Lemma 4.13. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Say $X \in \mathbf{Ch}(R)$ and we are given $x \in X$. Then there exists a dg-pure $P \subseteq X$ with $x \in P$ and $|P| \leq \kappa$.*

Proof. This is easy since any pure subcomplex is dg-pure. We just use Lemma 4.6. $\square$

Lemma 4.14. *Let F be a dg-flat complex. If $P \subseteq F$ is pure, then P and F/P are dg-flat.*

Proof. Let $M \in \mathbf{Ch}(R)$. Then

$$0 \to P \overline{\otimes} D^n(M) \to F \overline{\otimes} D^n(M) \to F/P \overline{\otimes} D^n(M) \to 0$$

is exact. By Proposition 4.2, we see that

$$0 \to \Sigma^n(P \otimes_R M) \to \Sigma^n(F \otimes_R M) \to \Sigma^n(F/P \otimes_R M) \to 0$$

is also exact. Therefore $P \subseteq F$ is pure in each degree.

By Proposition 4.11 it remains to show that for any exact complex E, $P \otimes E$ and $F/P \otimes E$ are exact. Notice however that $0 \to P \otimes E \to F \otimes E \to F/P \otimes E \to 0$ is exact (because it is exact in each degree). So by the fundamental lemma of homological algebra, showing $P \otimes E$ exact is equivalent to showing $F/P \otimes E$ exact. But it is not hard to see why $P \otimes E$ is exact: We know that $P \otimes E \to F \otimes E$ is injective, so if we let $z \in (P \otimes E)_n$ be a cycle, then we may view it as a cycle in $(F \otimes E)_n$. Since $F \otimes E$ is exact, $z \in B_n(F \otimes E)$. But by the very definition of $\overline{\otimes}$ and the fact that $P \overline{\otimes} E \to F \overline{\otimes} E$ is injective, z must be a boundary *in* $(P \otimes E)_n$. Hence $P \otimes E$ is exact. $\square$

Lemma 4.15. *If $0 \to X \to Y \to F \to 0$ is a short exact sequence of chain complexes and F is dg-flat, then the sequence is dg-pure.*

Proof. Let $\mathcal{S}\colon\ 0 \to X \to Y \to F \to 0$ be such a sequence and E an exact complex. We must show that $X \overline{\otimes} E \xrightarrow{f \overline{\otimes} 1} Y \overline{\otimes} E$ is injective. We know that $\mathcal{S}$ is pure in each degree since each F_n is flat. Therefore, for all pairs (m, n), $0 \to X_n \otimes E_m \to Y_n \otimes E_m \to F_n \otimes E_m \to 0$ is exact which implies that $0 \to X \otimes E \to Y \otimes E \to F \otimes E \to 0$ is exact.

Now suppose $\overline{x} \in (X \overline{\otimes} E)_n$ and $(f \overline{\otimes} 1)(x) = 0$. This means

$$(f \otimes 1)(x) \in B_n(Y \otimes E).$$

But $f \otimes 1$ is a chain map and $X \otimes E \to Y \otimes E$ is injective, so we must have $x \in Z_n(X \otimes E)$. Now $F \otimes E$ is exact since F is dg-flat and it follows from the fundamental lemma of homological algebra that $f \otimes 1$ is an H_*-isomorphism. Since the isomorphism induced by $f \otimes 1$ is exactly the definition of the map $f \overline{\otimes} 1$, we see that $x \in B_n(X \otimes E)$. I.e. $\overline{x} = 0$. So $X \overline{\otimes} E \xrightarrow{f \overline{\otimes} 1} Y \overline{\otimes} E$ is injective. $\square$

Lemma 4.16. *A direct limit of dg-pure sequences is dg-pure. In particular, a direct union of dg-pure subcomplexes is dg-pure.*

Proof. Let $0 \to P_i \to X_i$ be dg-pure $(i \in I)$. Then for any exact complex E, $0 \to P_i \overline{\otimes} E \to X_i \overline{\otimes} E$ is exact. So $0 \to \mathrm{colim}_{i \in I}(P_i \overline{\otimes} E) \to \mathrm{colim}_{i \in I}(X_i \overline{\otimes} E)$ is exact. By Rozas' adjointness Proposition 4.2, we have $0 \to (\mathrm{colim}_{i \in I} P_i) \overline{\otimes} E \to (\mathrm{colim}_{i \in I} X_i) \overline{\otimes} E$ is exact, so that $\mathrm{colim}_{i \in I} P_i$ is dg-pure. $\square$

Proposition 4.17. *Let $|R| \leq \kappa$, where κ is some infinite cardinal. Let $\mathcal{G}$ be the set of all dg-flat complexes $F \in \mathbf{Ch}(R)$ for which $|F| \leq \kappa$ (take one representative for each isomorphism class). Then any dg-flat complex $F \in \mathbf{Ch}(R)$ is (isomorphic to) a transfinite extension of $\mathcal{G}$.*

Proof. This follows exactly as the proof of Proposition 4.9. Just replace the word "flat" by "dg-flat" and the word "pure" by "dg-pure" and quote the analogous Lemmas 4.13 - 4.16. $\square$

The following corollary follows as well by referring to Lemma 4.5.

Corollary 4.18. *Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair on* **Rmod**. *Then the induced cotorsion pair $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$ on $\mathbf{Ch}(R)$ is complete.*

We finish this section by observing that each chain complex X has a (dg-)flat cover and a (dg-)injective envelope. This problem was recently solved by the group of authors in [Ald01] for the case of flat covers and dg-cotorsion envelopes.

Recall that if $\mathcal{A}$ is a class in an abelian category $\mathcal{C}$ and $C \in \mathcal{C}$, then an $\mathcal{A}$-precover of C is a morphism $\phi\colon\ A \to C$ with $A \in \mathcal{A}$ such that given any other morphism $\phi'\colon\ A' \to C$ with $A' \in \mathcal{A}$, there exists a map $\psi\colon\ A' \to A$ such that $\phi' = \phi\psi$. An $\mathcal{A}$-precover ϕ is called an $\mathcal{A}$-cover if whenever ψ satisfies $\phi = \phi\psi$ we must have ψ as an automorphism. $\mathcal{A}$-pre-envelopes and $\mathcal{A}$-envelopes are defined dually. Note that if $(\mathcal{A}, \mathcal{B})$ is a cotorsion theory with enough projectives and injectives, then we have that every object has an $\mathcal{A}$-precover and a $\mathcal{B}$-pre-envelope.

Corollary 4.19. *Every chain complex X has a flat cover, a dg-flat cover, a cotorsion envelope, and a dg-cotorsion envelope.*

Proof. We refer to [Xu96], pp. 30–37. Since the class of flat and dg-flat complexes are closed under direct limits, Xu's Theorem 2.2.6 and 2.2.12 give us the result. Although the proofs given are for R-modules, they clearly hold for complexes, too. □

5. An alternate definition of Ext

In this section we assume that the reader has some familiarity with Quillen's notion of a model category introduced in [Qui67]. This is a category $\mathcal{M}$ in which we can do homotopy theory. We refer the reader to [DS95] for a readable introduction to model categories and to [Hov99] for a more in-depth presentation. We will show that we have a "flat" model structure on $\mathbf{Ch}(R)$ and that this gives us an alternate description of Ext. We also show that this model structure is monoidal. For R-modules M and N, $\operatorname{Ext}^n_R(M,N)$ is normally defined by taking a projective resolution $P_\bullet$ of M and taking the homology of $\operatorname{Hom}_R(P_\bullet, N)$. Alternatively, it is often defined by taking an injective coresolution $I_\bullet$ of N and taking the homology of $\operatorname{Hom}_R(M, I_\bullet)$. We will show that we can also compute $\operatorname{Ext}^n_R(M,N)$ by taking a "flat" resolution $F_\bullet$ of M and a "cotorsion" coresolution $C_\bullet$ of N and taking the homology of the "enriched" complex $Hom(F_\bullet, C_\bullet)$. Essentially this works because in each situation there is a cotorsion pair $(\mathcal{A}, \mathcal{B})$ of R-modules which induces two complete cotorsion pairs on $\mathbf{Ch}(R)$ that are compatible in the sense of Definition 3.7.

The next corollary follows from Mark Hovey's Theorem 1.2 of [Hov00] which relates cotorsion pairs to model structures on abelian categories.

Corollary 5.1. *There is a monoidal model category structure on $\mathbf{Ch}(R)$, where the weak equivalences are the H_*-isomorphisms, the (trivial) cofibrations are the injections with (exact) dg-flat cokernels, and the (trivial) fibrations are the surjections with (exact) dg-cotorsion kernels. In particular $dg\widetilde{\mathcal{F}}$ is the class of cofibrant objects and $dg\widetilde{\mathcal{C}}$ is the class of fibrant objects.*

Proof. As we have seen in section 4, both of the induced cotorsion pairs $(dg\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}} \cap \mathcal{E})$ and $(dg\widetilde{\mathcal{F}} \cap \mathcal{E}, dg\widetilde{\mathcal{C}})$ are complete. To get the model structure use the converse of Hovey's Theorem 1.2 (taking $\mathcal{P}$ to be the class of all short exact sequences in the theorem) along with his Definition 4.1. To see that the model structure is monoidal (with respect to the usual tensor product $\otimes$) we will now prove the hypotheses of Hovey's Theorem 6.2.

First we observe that Hovey's notion of a $\mathcal{P}$-pure short exact sequence in this case just means a short exact sequence of complexes that is pure in each degree. According to the theorem we now must check:

(1) Every cofibration is a pure injection in each degree.
(2) If X and Y are dg-flat, then $X \otimes Y$ is dg-flat.
(3) If X is dg-flat and Y is flat, then $X \otimes Y$ is flat.
(4) $S(R)$ is dg-flat

(1) is obvious since a cofibration is an injection with dg-flat cokernel. Also (4) is obvious since the complex is bounded and R is flat. Now for (2), since for any pair of integers i, j we know that $X_i \otimes Y_j$ is flat, it follows that $\bigoplus(X_i \otimes Y_j) = (X \otimes Y)_n$ is flat. Also, for any exact complex E, $Y \otimes E$ is exact. So $X \otimes (Y \otimes E) = (X \otimes Y) \otimes E$ is exact. For (3), say X is dg-flat and Y is flat. Then Y is also dg-flat, so by (2) $X \otimes Y$ is dg-flat. But Y is also exact, so $X \otimes Y$ must be exact, too. Therefore, $X \otimes Y$ is flat. $\square$

Let M and N be R-modules. Recall the usual definition of $\mathrm{Ext}^n_R(M, N)$. We let $(P_\bullet, \epsilon)$ be a projective resolution of M, so that $\cdots P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$ is exact with each P_n a projective module. Then

$$\mathrm{Ext}^n(M, N) = H_{-n}(\mathrm{Hom}(P_\bullet, N)). \tag{1}$$

It is easy to see that this is the same as $\mathbf{Ch}(R)(P_\bullet, S^n(N))/\sim$. So we get the equation

$$\mathrm{Ext}^n(M, N) = \mathbf{Ch}(R)(P_\bullet, S^n(N))/\sim, \tag{2}$$

where $\sim$ is chain homotopy.

Definition 5.2. Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of modules and let M be a module. Then $(A_\bullet, \epsilon) = \cdots \to A_1 \to A_0 \xrightarrow{\epsilon} M \to 0$ is called an $(\mathcal{A}, \mathcal{B})$-*resolution* of M if the sequence is exact with each $A_n \in \mathcal{A}$ and each cycle module in $\mathcal{B}$. Dually, $(B_\bullet, \eta) = 0 \to M \xrightarrow{\eta} B_0 \to B_1 \to \cdots$ is called an $(\mathcal{A}, \mathcal{B})$-*coresolution* of M if the sequence is exact with each $B_n \in \mathcal{B}$ and each cycle module in $\mathcal{A}$.

Clearly if $(\mathcal{A}, \mathcal{B})$ is complete, then $(\mathcal{A}, \mathcal{B})$-resolutions and coresolutions exist for all modules. In particular, $(\mathcal{F}, \mathcal{C})$ resolutions and coresolutions exist. We will simply call them *flat resolutions* and *cotorsion coresolutions*, respectively. The next lemma just relates this language to Hovey's model structure induced by $(\mathcal{F}, \mathcal{C})$.

Lemma 5.3. *Let M be a module and $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair for which both induced cotorsion pairs on $\mathbf{Ch}(R)$ are complete.*

(a) *If $(A_\bullet, \epsilon)$ is an $(\mathcal{A}, \mathcal{B})$-resolution of M, then $A_\bullet$ is a cofibrant replacement of $S(M)$ in the model structure induced by $(dg\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}} \cap \mathcal{E})$ and $(dg\widetilde{\mathcal{A}} \cap \mathcal{E}, dg\widetilde{\mathcal{B}})$.*

(b) *If $(B_\bullet, \eta)$ is an $(\mathcal{A}, \mathcal{B})$-coresolution of M, then $B_\bullet$ is a fibrant replacement of $S(M)$ in the model structure induced by $(dg\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}} \cap \mathcal{E})$ and $(dg\widetilde{\mathcal{A}} \cap \mathcal{E}, dg\widetilde{\mathcal{B}})$.*

Proof. We will just prove (a); part (b) is dual. Since $A_\bullet$ is bounded below and each entry belongs to $\mathcal{A}$, it is a dg-$\mathcal{A}$ complex. I.e. $\mathcal{A}_\bullet$ is cofibrant. Furthermore, the map $\bar{\epsilon}\colon A_\bullet \to S(M)$ defined by ϵ in degree 0 is clearly a surjective H_*-isomorphism with $\ker \bar{\epsilon}$ a $\mathcal{B}$ complex (trivially fibrant). $\square$

Now we apply the power of model categories. With any model structure on $\mathbf{Ch}(R)$ in which the weak equivalences are the H_*-isomorphisms we have

$$\mathbf{HoCh(R)}(S(M), S^n(N)) = \mathbf{Ch}(R)(Q, R)/\sim,$$

where Q is a cofibrant replacement of $S(M)$ and R is a fibrant replacement of $S^n(N)$. First consider the usual "projective" model structure on $\mathbf{Ch}(R)$ (induced by the usual projective cotorsion pair of R-modules using Theorem 3.12 and Theorem 1.2 of [Hov00]). By letting $P_\bullet$ be a projective resolution of M and using

Lemma 5.3 we get

$$\mathbf{HoCh}(\mathbf{R})(S(M), S^n(N)) = \mathbf{Ch}(R)(P_\bullet, S^n(N))/\sim \tag{3}$$

because $P_\bullet$ is a cofibrant replacement of $S(M)$ and $S^n(N)$ is already fibrant. Our new "flat" model structure gives us a new description of $\mathbf{HoCh}(\mathbf{R})(S(M), S^n(N))$ since we have changed the fibrations and cofibrations. Now letting $F_\bullet$ be a flat resolution of M and $C_\bullet$ be a cotorsion resolution of N, we see that

$$\mathbf{HoCh}(\mathbf{R})(S(M), S^n(N)) = \mathbf{Ch}(R)(F_\bullet, \Sigma^n C_\bullet)/\sim . \tag{4}$$

Putting equations (2), (3), and (4) together we get

$$\operatorname{Ext}^n(M, N) = \mathbf{Ch}(R)(F_\bullet, \Sigma^n C_\bullet)/\sim .$$

Using Lemma 2.1 we see that this just expresses $\operatorname{Ext}^n(M, N)$ as the homology of the enriched hom-complex:

$$\operatorname{Ext}^n(M, N) = H_{-n} Hom(F_\bullet, C_\bullet).$$

Compare this to the original definition (1) above.

References

[Ald01] S. Tempest Aldrich, Edgar E. Enochs, Luis Oyonarte, and J.R. García-Rozas, *Covers and envelopes in Grothendieck categories: flat covers of complexes with applications*, Journal of Algebra 243, 2001, pp. 615-630. MR **2002i:**18010

[DS95] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (Amsterdam), North-Holland, Amsterdam, 1995, pp. 73-126. MR **96h:**55014

[BBE00] L. Bican, R. El Bashir, and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. 33, 2001, pp. 385–390. MR **2002e:**16002

[ET99] Paul C. Eklof and Jan Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. 33, 2001, pp. 41–51. MR **2001i:**16015

[EEGO] E. Enochs, S. Estrada, J.R. García-Rozas, and L. Oyonarte, *Flat covers of quasi-coherent sheaves*, preprint, 2000.

[EJ01] E. Enochs and O. Jenda, *Relative homological algebra*, De Gruyter Expositions in Mathematics no. 30, Walter De Gruyter, New York, 2001. MR **2001h:**16013

[EO01] E. Enochs and L. Oyonarte, *Flat covers and cotorsion envelopes of sheaves*, Proceedings of the American Mathematical Society vol. 130, no. 5, 2001, pp. 1285-1292. MR **2003d:**18023

[EGR97] E. Enochs and J.R. García-Rozas, *Tensor products of chain complexes*, Math J. Okayama Univ. 39, 1997, pp. 19-42. MR **2001b:**16006

[GR99] J. R. García-Rozas, *Covers and envelopes in the category of complexes of modules*, Research Notes in Mathematics no. 407, Chapman & Hall/CRC, Boca Raton, FL, 1999. MR **2001i:**16009

[Gri99] Pierre Antoine Grillet, *Algebra*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1999. MR **2000g:**20001

[Hov00] Mark Hovey, *Cotorsion theories, model category structures, and representation theory*, preprint, 2000.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs vol. 63, American Mathematical Society, 1999. MR **99h:**55031

[Joy84] A. Joyal, Letter to A. Grothendieck, 1984.

[Mac71] Saunders Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematatics vol. 5, Springer-Verlag, New York, second edition, 1998. MR **2001j:**18001

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics no. 43, Springer-Verlag, 1967. MR **36:**6480

[Sal79] L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. 23, 1979. MR **81j:**20078

[Spa66] Edwin H. Spanier, *Algebraic Topology*, McGraw-Hill series in higher mathematics, McGraw-Hill, New York, 1966. MR **35:**1007

[Wis91] Robert Wisbauer, *Foundations of module and ring theory*, Algebra, Logic and Applications series vol. 3, Gordon and Breach Science Publishers,1991. MR **92i:**16001

[Xu96] Jinzhong Xu, *Flat covers of modules*, Lecture Notes in Mathematics no. 1634, Springer-Verlag, Berlin, 1996. MR **98b:**16003

Department of Mathematics, 4000 University Drive, Penn State–McKeesport, McKeesport, Pennsylvania 15132-7698

E-mail address: jrg21@psu.edu