The focusing Manakov system with nonzero boundary conditions

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Abstract. The initial value problem for the focusing Manakov system with nonzero boundary conditions at infinity is solved by developing an appropriate inverse scattering transform. The analyticity properties of the Jost eigenfunctions is investigated, and precise conditions on the potential that guarantee such analyticity are provided. The analyticity properties of the scattering coefficients is also established rigorously, and auxiliary eigenfunctions needed to complete the bases of analytic eigenfunctions are derived. The behavior of the eigenfunctions and scattering coefficients at the branch points is discussed, as are the symmetries of the analytic eigenfunctions and scattering coeffiecients. These symmetries are used to obtain a rigorous characterization of the discrete spectrum and to rigorously derive the symmetries of the associated norming constants. The asymptotic behavior of the Jost eigenfunctions is derived systematically. A general formulation of the inverse scattering problem as a Riemann-Hilbert problem is presented. Explicit relations among all reflection coefficients are given, and all entries of the scattering matrix are determined in the case of reflectionless solutions. New soliton solutions are explicitly constructed and discussed. These solutions, which have no analogue in the scalar case, are comprised of dark-bright soliton pairs as in the defocusing case. Finally, a consistent framework is formulated for obtaining solutions corresponding to any number of simple zeros of the analytic scattering coefficients, leading to any combination of bright and dark-bright soliton solutions.

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1. Introduction

Vector nonlinear Schrödinger (NLS) equations model the evolution of multi-component weakly nonlinear dispersive wave trains in many physical contexts [3, 20, 26, 29]. In some cases, these equations are completely integrable [2, 3, 6, 15, 18, 23], and the initial value problem can in principle be solved by the inverse scattering transform (IST).

This work is concerned with the Manakov system, i.e., the two-component vector nonlinear Schrödinger equation

$$i\mathbf{q}_t + \mathbf{q}_{xx} + 2\sigma(q_o^2 - \|\mathbf{q}\|^2)\mathbf{q} = \mathbf{0}, \tag{1.1}$$

with non-zero boundary conditions (NZBC) at infinity:

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm = \mathbf{q}_o \mathrm{e}^{i\theta_\pm}. \tag{1.2}$$

Hereafter: $\mathbf{q} = \mathbf{q}(x,t)$ and $\mathbf{q}_o$ are 2-component vectors, $\|\cdot\|$ is the standard Euclidean norm, $q_o = \|\mathbf{q}_o\|$, $\theta_\pm$ are real numbers, and subscripts x and t denote partial differentiation throughout.

The extra term q_o^2 in (1.1) was added so that the asymptotic values of the potential are independent of time.

The IST for the scalar NLS equation [i.e., the one-component reduction of (1.1)] was developed in [31] for the focusing case with zero boundary conditions (ZBC) (i.e., for $q_o = 0$) and in [32] for the defocusing case with NZBC (see also Refs. [1, 3, 4, 15]). The IST for (1.1) with ZBC was derived in [22] and generalized in [2]. On the other hand, the IST for the Manakov system (1.1) with NZBC remained an open problem for a long time, and even some questions for the scalar defocusing case were addressed only recently [10, 14]. A successful approach to the IST for the defocusing Manakov system was presented in [24] and rigorously revisited in [9]. The focusing case, however, remained completely open. In fact, even the IST for the scalar focusing NLS equation with NZBC remained a long-standing open problem until recently, when it was developed in [8] and used in [7] to study the behavior of solutions. Here we build on the work of [8] to develop the IST for the Manakov system (1.1) in the focusing case ($\sigma = -1$) with NZBC. We should note that, as in the defocusing case, the generalization of the IST from the scalar case to the vector case is highly nontrivial, which is a reflection of the added complexity of the corresponding solutions.

The outline of this work is the following: in Section 2 we formulate the direct problem (taking into account automatically the time evolution); in Section 3 we characterize the discrete spectrum; in Section 4 we formulate the inverse problem; and in Section 5 we derive the soliton solutions. Section 6 contains a final discussion. The proofs of all theorems, lemmas, and corollaries in the text are given in the Appendix. Throughout, asterisk denotes complex conjugation, and superscripts T and $\dagger$ denote, respectively, matrix transpose and matrix adjoint. We use $\mathbf{I}$ and $\mathbf{0}$ to denote the identity matrix and zero matrix of appropriate size, respectively. Also, we denote, respectively, with $\mathbf{A}_d$, $\mathbf{A}_o$, $\mathbf{A}_{bd}$ and $\mathbf{A}_{bo}$ the diagonal, off-diagonal, block diagonal, and block off-diagonal parts of a 3×3 matrix $\mathbf{A}$. In addition, we will use the shorthand notation

$$\hat{z} = -q_o^2/z. \tag{1.3}$$

2. Direct scattering

2.1. Lax pair, Riemann surface and uniformization

The focusing Manakov system [i.e., the 2-component VNLS equation (1.1) with $\sigma = -1$] is associated with the following Lax pair:

$$\phi_x = \mathbf{X}\phi, \qquad \phi_t = \mathbf{T}\phi, \tag{2.1}$$

where

$$\mathbf{X}(x,t,k) = -ik\mathbf{J} + \mathbf{Q}\,, \quad \mathbf{T}(x,t,k) = 2ik^2\mathbf{J} - i\mathbf{J}(\mathbf{Q}_x - \mathbf{Q}^2 - q_o^2) - 2k\mathbf{Q}\,, \tag{2.2a}$$

$$\mathbf{J} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \qquad \mathbf{Q}(x,t) = \begin{pmatrix} 0 & \mathbf{r}^T \\ \mathbf{q} & \mathbf{0} \end{pmatrix}, \tag{2.2b}$$

and $\mathbf{r} = -\mathbf{q}^*$. That is, (1.1) is the compatibility condition

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = \mathbf{0} \tag{2.3}$$

(also known as the zero-curvature condition [3, 23]) which ensures that $\phi_{xt} = \phi_{tx}$ (as is easily verified by direct calculation and noting that $\mathbf{JQ} = -\mathbf{QJ}$). As usual, the first half of (2.1) is referred to as the scattering problem. In the development of the IST, we take $\phi(x,t,k)$ as a

3×3 matrix. Moreover, we formulate the IST in a way that allows the reduction $q_o \to 0$ to be taken explicitly throughout.

As in the scalar case [8], in order to define the Jost eigenfunctions, one must first solve the asymptotic scattering problem as $x \to \pm\infty$, which is

$$\phi_x = \mathbf{X}_\pm \phi, \tag{2.4}$$

where $\mathbf{X}_\pm = -ik\mathbf{J} + \mathbf{Q}_\pm = \lim_{x\to\pm\infty} \mathbf{X}$. The eigenvalues of $\mathbf{X}_\pm$ are ik and $\pm i\lambda$, where

$$\lambda = (k^2 + q_o^2)^{1/2}. \tag{2.5}$$

As in the scalar case, $\lambda(k)$ is branched. To deal with this, we introduce the two-sheeted Riemann surface defined by (2.5). The branch points are the values of k for which $\lambda(k) = 0$, i.e., $k = \pm iq_o$. We take the branch cut on $i[-q_o, q_o]$, and we define $\lambda(k)$ as in [8]. Next, we introduce the uniformization variable by defining

$$z = k + \lambda. \tag{2.6}$$

The inverse transformation is

$$k = (z + \hat{z})/2, \qquad \lambda = (z - \hat{z})/2. \tag{2.7}$$

We can then express all k-dependence of eigenfunctions and scattering data in terms of z, thereby eliminating all square roots. Note that, formally, the uniformization variable has the same expression in terms of k and λ as in the defocusing case [15, 24], but the resulting map is quite different. Let C_o be the circle of radius q_o centered at the origin in the complex z-plane. The branch cuts on the two sheets of the Riemann surface are mapped onto C_o; The first sheet, $\mathbb{C}_{\mathrm{I}}$, is mapped onto the exterior of C_o; the second sheet, $\mathbb{C}_{\mathrm{II}}$, is mapped onto the interior of C_o. Moreover, $z(\infty_{\mathrm{I}}) = \infty$ (where ∞_{I} is the point at infinity in $\mathbb{C}_{\mathrm{I}}$), $z(\infty_{\mathrm{II}}) = 0$ (where ∞_{II} is the point at infinity in $\mathbb{C}_{\mathrm{II}}$), $z_{\mathrm{I}} z_{\mathrm{II}} = q_o^2$, $|k| \to \infty$ in $\mathbb{C}_{\mathrm{I}}$ corresponds to $z \to \infty$, and $|k| \to \infty$ in $\mathbb{C}_{\mathrm{II}}$ corresponds to $z \to 0$. Throughout this work, *subscripts* $\pm$ will denote normalization as $x \to -\infty$ or as $x \to \infty$, respectively, whereas *superscripts* $\pm$ will denote projections from the left and the right of the appropriate contour in the complex z-plane, respectively.

2.2. Jost solutions and scattering matrix

The continuous spectrum consists of all values of k (on either sheet) such that $\lambda(k) \in \mathbb{R}$. As in the scalar case, that is $k \in \mathbb{R} \cup i[-q_o, q_o]$ [8]. [This is in contrast with the defocusing case, where the continuous spectrum is the subset $(-\infty, -q_o] \cup [q_o, \infty)$ of the real k-axis [24, 32].] In the complex z-plane, the corresponding set is $\Sigma = \mathbb{R} \cup C_o$. For any 2-component vector $\mathbf{v} = (v_1, v_2)^T$, define

$$\mathbf{v}^\perp = (v_2, -v_1)^\dagger. \tag{2.8}$$

We may then write the eigenvalues and the corresponding eigenvector matrices of the asymptotic scattering problem (2.4) as

$$i\mathbf{\Lambda}(z) = \mathrm{diag}(-i\lambda, ik, i\lambda), \qquad \mathbf{E}_\pm(z) = \begin{pmatrix} 1 & 0 & iq_o/z \\ i\mathbf{q}_\pm/z & \mathbf{q}_\pm^\perp/q_o & \mathbf{q}_\pm/q_o \end{pmatrix}, \tag{2.9}$$

respectively, so that

$$\mathbf{X}_\pm \mathbf{E}_\pm = \mathbf{E}_\pm i\mathbf{\Lambda}. \tag{2.10}$$

It will be useful to note that

$$\det \mathbf{E}_\pm(z) = 1 + q_o^2/z^2 := \gamma(z), \tag{2.11a}$$

$$\mathbf{E}_\pm^{-1}(z) = \frac{1}{\gamma(z)} \begin{pmatrix} 1 & -i\mathbf{q}_\pm^\dagger/z \\ 0 & \gamma(z)(\mathbf{q}_\pm^\perp)^\dagger/q_o \\ -iq_o/z & \mathbf{q}_\pm^\dagger/q_o \end{pmatrix}. \tag{2.11b}$$

Let us now discuss the asymptotic time dependence. As $x \to \pm\infty$, we expect that the time evolution of the solutions of the Lax pair will be asymptotic to

$$\phi_t = \mathbf{T}_\pm \phi, \tag{2.12}$$

where $\mathbf{T}_\pm = 2ik^2\mathbf{J} + i\mathbf{J}\mathbf{Q}_\pm^2 + iq_o^2\mathbf{J} - 2k\mathbf{Q}_\pm$. The eigenvalues of $\mathbf{T}_\pm$ are $-i(k^2+\lambda^2)$ and $\pm 2ik\lambda$. Since the boundary conditions (BC) are constant, the zero-curvature condition (2.3) in the limit $x \to \pm\infty$ yields $[\mathbf{X}_\pm, \mathbf{T}_\pm] = \mathbf{0}$, so $\mathbf{X}_\pm$ and $\mathbf{T}_\pm$ admit common eigenvectors. In particular,

$$\mathbf{T}_\pm \mathbf{E}_\pm = -i\mathbf{E}_\pm \mathbf{\Omega}, \tag{2.13}$$

where $\mathbf{\Omega}(z) = \mathrm{diag}(-2k\lambda, k^2+\lambda^2, 2k\lambda)$. Then for all $z \in \Sigma$, we can define the Jost solutions $\phi_\pm(x,t,z)$ as the simultaneous solutions of both parts of the Lax pair satisfying the BC

$$\phi_\pm(x,t,z) = \mathbf{E}_\pm(z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)} + o(1), \qquad x \to \pm\infty, \tag{2.14}$$

where $\mathbf{\Theta}(x,t,z)$ is the 3×3 diagonal matrix

$$\mathbf{\Theta}(x,t,z) = \mathbf{\Lambda}(z)x - \mathbf{\Omega}(z)t = \mathrm{diag}(\theta_1(x,t,z), \theta_2(x,t,z), -\theta_1(x,t,z)), \tag{2.15}$$

and where, owing to (2.10) and (2.13),

$$\theta_2(x,t,z) = kx - (k^2+\lambda^2)t, \qquad \theta_1(x,t,z) = -\lambda x + 2k\lambda t. \tag{2.16}$$

The advantage of introducing simultaneous solutions of both parts of the Lax pair is that the scattering coefficients will be independent of time.

To make the above definitions rigorous, we factorize the asymptotic behavior of the potential and rewrite the first part of the Lax pair (2.1) as

$$(\phi_\pm)_x = \mathbf{X}_\pm \phi_\pm + \Delta\mathbf{Q}_\pm \phi_\pm, \tag{2.17}$$

where $\Delta\mathbf{Q}_\pm = \mathbf{Q} - \mathbf{Q}_\pm$. We remove the asymptotic exponential oscillations and introduce modified Jost eigenfunctions:

$$\mu_\pm(x,t,z) = \phi_\pm(x,t,z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)}, \tag{2.18}$$

so that

$$\lim_{x\to\pm\infty} \mu_\pm(x,t,z) = \mathbf{E}_\pm(z). \tag{2.19}$$

Introducing the integrating factor $\psi_\pm(x,t,z) = \mathrm{e}^{-i\mathbf{\Theta}(x,t,z)}\mathbf{E}_\pm^{-1}(z)\mu_\pm(x,t,z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)}$, we can then formally integrate the ODE for $\mu_\pm(x,t,z)$ obtain

$$\mu_-(x,t,z) = \mathbf{E}_- + \int_{-\infty}^{x} \mathbf{E}_-\mathrm{e}^{i(x-y)\mathbf{\Lambda}}\mathbf{E}_-^{-1}\Delta\mathbf{Q}_-\mu_-\mathrm{e}^{-i(x-y)\mathbf{\Lambda}}\mathrm{d}y, \tag{2.20a}$$

$$\mu_+(x,t,z) = \mathbf{E}_+ - \int_{x}^{\infty} \mathbf{E}_+\mathrm{e}^{i(x-y)\mathbf{\Lambda}}\mathbf{E}_+^{-1}\Delta\mathbf{Q}_+\mu_+\mathrm{e}^{-i(x-y)\mathbf{\Lambda}}\mathrm{d}y. \tag{2.20b}$$

One can now rigorously define the Jost eigenfunctions as the solutions of the integral equations (2.20). In fact, in Appendix A.1, we prove the following:

Theorem 2.1. *If $\mathbf{q}(\cdot,t)-\mathbf{q}_- \in L^1(-\infty,a)$ or, correspondingly, $\mathbf{q}(\cdot,t)-\mathbf{q}_+ \in L^1(a,\infty)$ for any constant $a\in\mathbb{R}$, the following columns of $\mu_-(x,t,z)$ or, correspondingly, $\mu_+(x,t,z)$ can be analytically extended onto the corresponding regions of the complex z-plane:*

$$\mu_{-,1}:\ z\in D_1,\qquad \mu_{-,2}:\ \operatorname{Im} z<0,\qquad \mu_{-,3}:\ z\in D_4, \tag{2.21a}$$

$$\mu_{+,1}:\ z\in D_2,\qquad \mu_{+,2}:\ \operatorname{Im} z>0,\qquad \mu_{+,3}:\ z\in D_3, \tag{2.21b}$$

where the domains of analyticity $D_1,\dots,D_4$ are

$$D_1=\{z:\operatorname{Im} z>0\wedge |z|>q_o\},\qquad D_2=\{z:\operatorname{Im} z<0\wedge |z|>q_o\}, \tag{2.22a}$$

$$D_3=\{z:\operatorname{Im} z<0\wedge |z|<q_o\},\qquad D_4=\{z:\operatorname{Im} z>0\wedge |z|<q_o\}. \tag{2.22b}$$

Note that $\overline{D_1}\cup\overline{D_2}\cup\overline{D_3}\cup\overline{D_4}=\mathbb{C}$.

Equation (2.18) implies that the same analyticity and boundedness properties also hold for the columns of $\phi_\pm(x,t,z)$. Note that four fundamental domains of analyticity are present for the focusing Manakov system with NZBC. This is in contrast to the defocusing Manakov system (where the eigenfunctions are analytic either in the upper-half plane or the lower-half plane [9, 24]) and to the scalar focusing NLS equation, where the fundamental domains are $D_1\cup D_3$ and $D_2\cup D_4$ [8]. (The difference from the scalar case can be traced to the presence of the additional eigenvalue ik in the 3×3 scattering problem.)

We now introduce the scattering matrix. If $\phi(x,t,z)$ solves (2.1), we have $\partial_x(\det\phi)=\operatorname{tr}\mathbf{X}\det\phi$ and $\partial_t(\det\phi)=\operatorname{tr}\mathbf{T}\det\phi$. Since $\operatorname{tr}\mathbf{X}=ik$ and $\operatorname{tr}\mathbf{T}=-i(k^2+\lambda^2)$, Abel's theorem yields

$$\frac{\partial}{\partial x}[\det(\phi_\pm(x,t,z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)})]=\frac{\partial}{\partial t}[\det(\phi_\pm(x,t,z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)})]=0. \tag{2.23}$$

Then (2.14) implies

$$\det\phi_\pm(x,t,z)=\gamma(z)\mathrm{e}^{i\theta_2(x,t,z)},\quad (x,t)\in\mathbb{R}^2,\quad z\in\Sigma\setminus\{\pm iq_o\}. \tag{2.24}$$

That is, $\phi_-(x,t,z)$ and $\phi_+(x,t,z)$ are two fundamental matrix solutions of the Lax pair, so there exists an invertible 3×3 matrix $\mathbf{A}(z)$ such that

$$\phi_-(x,t,z)=\phi_+(x,t,z)\mathbf{A}(z),\quad z\in\Sigma\setminus\{\pm iq_o\}. \tag{2.25}$$

As usual, $\mathbf{A}(z)=(a_{ij}(z))$ is referred to as the scattering matrix. Note that thanks to the explicit time dependence in the BCs (2.14) for the Jost eigenfunctions, $\mathbf{A}(z)$ is independent of time. Moreover, (2.24) and (2.25) imply

$$\det\mathbf{A}(z)=1,\quad z\in\Sigma\setminus\{\pm iq_o\}. \tag{2.26}$$

It is also convenient to introduce $\mathbf{B}(z):=\mathbf{A}^{-1}(z)=(b_{ij}(z))$. In the scalar case, the analyticity of the diagonal scattering coefficients follows trivially from their representations as Wronskians of analytic eigenfunctions. This approach, however, is not applicable to the vector case. Nonetheless, as in the defocusing case [9], this problem can be circumvented using an alternative integral representation for the eigenfunctions. Said representation is found in Appendix A.2. Combining the alternative integral representations with a Neumann series expansion yields the following result:

Lemma 2.2. *For all z in the interior of their corresponding domains of analyticity, the modified eigenfunctions $\mu_\pm(x,t,z)$ are bounded for all $x\in\mathbb{R}$.*

As in the defocusing case, this result will be important to the classification of the discrete spectrum (discussed in Section 3.1). Also, a straightforward combination of the scattering relation (2.25) and the alternative integral representation of the eigenfunctions yields the following result.

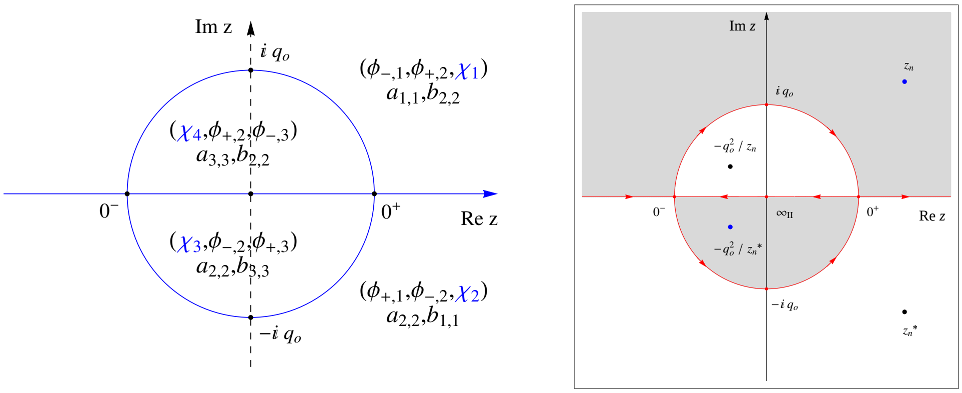

Figure 1. Left: The regions of analyticity of the Jost eigenfunctions and diagonal scattering coefficients in the complex z-plane (cf. section 2.2). Also indicated are the auxiliary eigenfunctions in each region (cf. section 2.3). Right: The symmetries of the discrete spectrum and the regions D^+ (gray) and D^- (white) and the orientation of Σ for the Riemann-Hilbert problem in section 4.

Proposition 2.3. *For $z \in \Sigma$, the Jost eigenfunctions exhibit the following asymptotic behavior as x tends to the opposite limit from the BC:*

$$\mu_+(x,t,z) = \mathbf{E}_-(z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)}\mathbf{B}(z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)} + o(1), \qquad x \to -\infty, \tag{2.27a}$$
$$\mu_-(x,t,z) = \mathbf{E}_+(z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)}\mathbf{A}(z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)} + o(1), \qquad x \to \infty. \tag{2.27b}$$

Finding explicit expressions for the limits of the modified eigenfunctions as x tends to the other infinity in the interior of the corresponding domains of analyticity would require the use of triangular decompositions of the scattering matrix (as in [25] for the defocusing case). Such expressions and their derivation are omitted for brevity.

In any case, using Lemma 2.2, in Appendix A.3, we obtain the analyticity properties of the scattering coefficients.

Theorem 2.4. *Under the same hypotheses as in Theorem 2.1, the following scattering coefficients can be analytically extended off of Σ in the following regions:*

$$a_{11}: \ z \in D_1, \qquad a_{22}: \ \mathrm{Im}\, z < 0, \qquad a_{33}: \ z \in D_4, \tag{2.28a}$$
$$b_{11}: \ z \in D_2, \qquad b_{22}: \ \mathrm{Im}\, z > 0, \qquad b_{33}: \ z \in D_3. \tag{2.28b}$$

Unlike the defocusing case [9, 24], all the diagonal entries of the scattering matrix are analytic in some part of the complex plane. The list of eigenfunctions and scattering coefficients that are analytic in each fundamental domain is shown in Fig. 1 (left). Note that the columns $\phi_{\pm,2}(x,t,z)$ are analytic in two domains, unlike the columns $\phi_{\pm,1}(x,t,z)$ and $\phi_{\pm,3}(x,t,z)$. Similarly, the scattering coefficients $a_{22}(z)$ and $b_{22}(z)$ are analytic in all of the lower-half plane and upper-half plane, respectively, unlike $a_{11}(z)$, $b_{11}(z)$, $a_{33}(z)$, and $b_{33}(z)$.

2.3. Adjoint problem and auxiliary eigenfunctions

Recall that, unlike in the defocusing case [24, 9], all of the columns of $\phi_\pm(x,t,z)$ are analytic in some portion of the complex z-plane. Nonetheless, a complete set of analytic eigenfunctions is needed to solve the inverse problem, and only two among the columns of $\phi_+(x,t,z)$ and $\phi_-(x,t,z)$ are analytic in any given domain. So one still needs to overcome a defect of analyticity.

As in [24], to circumvent this problem we consider the so-called "adjoint" Lax pair (following the terminology and the idea originally introduced for the three-wave interaction equations in [21]):

$$\tilde{\phi}_x = \tilde{\mathbf{X}}\tilde{\phi}, \quad \tilde{\phi}_t = \tilde{\mathbf{T}}\tilde{\phi}, \tag{2.29}$$

where $\tilde{\mathbf{X}} = ik\mathbf{J} + \mathbf{Q}^*$ and $\tilde{\mathbf{T}} = -2ik^2\mathbf{J} + i\mathbf{J}(\mathbf{Q}_x - \mathbf{Q}^2 - q_o^2) - 2k\mathbf{Q}$. Hereafter, tildes will denote that a quantity is defined for the adjoint problem (2.29) instead of the original one (2.1). Note that $\tilde{\mathbf{X}}(x,t,z) = \mathbf{X}^*(x,t,z^*)$ and $\tilde{\mathbf{T}}(x,t,z) = \mathbf{T}^*(x,t,z^*)$ for all $z \in \Sigma$. Denoting by "$\times$" the usual cross product, for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$ one has:

$$\begin{gathered}
(\mathbf{J}\mathbf{u}) \times \mathbf{v} + \mathbf{u} \times (\mathbf{J}\mathbf{v}) + \mathbf{u} \times \mathbf{v} + (\mathbf{J}\mathbf{u}) \times (\mathbf{J}\mathbf{v}) = \mathbf{0}\,, \\
\mathbf{J}(\mathbf{u} \times \mathbf{v}) = (\mathbf{J}\mathbf{u}) \times (\mathbf{J}\mathbf{v}), \\
\mathbf{Q}(\mathbf{u} \times \mathbf{v}) + (\mathbf{Q}^T\mathbf{u}) \times \mathbf{v} + \mathbf{u} \times (\mathbf{Q}^T\mathbf{v}) = \mathbf{0}\,, \\
\mathbf{J}\mathbf{Q}^2(\mathbf{u} \times \mathbf{v}) + (\mathbf{J}(\mathbf{Q}^T)^2\mathbf{u}) \times \mathbf{v} + \mathbf{u} \times (\mathbf{J}(\mathbf{Q}^T)^2\mathbf{v}) = \mathbf{0}\,.
\end{gathered}$$

Note also that in the focusing case, $\mathbf{Q}^T = -\mathbf{Q}^*$, implying $\mathbf{Q}^\dagger = -\mathbf{Q}$. Similarly to [21] and [24], using these identities it is straightforward to prove the following:

Proposition 2.5. *If $\tilde{\mathbf{v}}(x,t,z)$ and $\tilde{\mathbf{w}}(x,t,z)$ are two arbitrary solutions of the adjoint problem (2.29), then*

$$\mathbf{u}(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\mathbf{u}} \times \tilde{\mathbf{w}}](x,t,z) \tag{2.30}$$

is a solution of the Lax pair (2.1).

We use this result to construct four additional analytic eigenfunctions, one in each fundamental domain. We do so by constructing Jost eigenfunctions for the adjoint problem. The eigenvalues of $\tilde{\mathbf{X}}_\pm$ are $-ik$ and $\pm i\lambda$. Denoting the eigenvalue matrix as $-i\mathbf{\Lambda}(z) = \mathrm{diag}(i\lambda, -ik, -i\lambda)$, we can choose the eigenvector matrix as $\tilde{\mathbf{E}}_\pm(z) = \mathbf{E}_\pm^*(z^*)$. Note that $\det\tilde{\mathbf{E}}_\pm(z) = \gamma(z)$. As $x \to \pm\infty$, we expect that the solutions of the second equation in (2.29) will be asymptotic to those of $\tilde{\phi}_t = \tilde{\mathbf{T}}_\pm\tilde{\phi}$. The eigenvalues of $\tilde{\mathbf{T}}_\pm$ are $i(k^2 + \lambda^2)$ and $\pm 2ik\lambda$, and (2.13) imply $\tilde{\mathbf{T}}_\pm\tilde{\mathbf{E}}_\pm = \tilde{\mathbf{E}}_\pm i\mathbf{\Omega}$. As before, for all $z \in \Sigma$, we then define the Jost solutions of the adjoint problem as the simultaneous solutions $\tilde{\phi}_\pm(x,t,z)$ of (2.29) such that

$$\tilde{\phi}_\pm(x,t,z) = \tilde{\mathbf{E}}_\pm(z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)} + o(1), \qquad x \to \pm\infty. \tag{2.31}$$

Introducing modified adjoint eigenfunctions $\tilde{\mu}_\pm(x,t,z) = \tilde{\phi}_\pm(x,t,z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)}$ as before, one can show that the following columns of $\tilde{\mu}_\pm(x,t,z)$ can be extended into the complex plane:

$$\tilde{\mu}_{-,1}: \; z \in D_2, \qquad \tilde{\mu}_{-,2}: \; \mathrm{Im}\, z > 0, \qquad \tilde{\mu}_{-,3}: \; z \in D_3, \tag{2.32a}$$

$$\tilde{\mu}_{+,1}: \; z \in D_1, \qquad \tilde{\mu}_{+,2}: \; \mathrm{Im}\, z < 0, \qquad \tilde{\mu}_{+,3}: \; z \in D_4. \tag{2.32b}$$

Again, only two among the columns of $\tilde{\mu}_+(x,t,z)$ and $\tilde{\mu}_-(x,t,z)$ are analytic in the same region. And as before, $\tilde{\phi}_\pm(x,t,z)$ are both fundamental matrix solutions of the same problem, and therefore, we can introduce the adjoint scattering matrix as

$$\tilde{\phi}_-(x,t,z) = \tilde{\phi}_+(x,t,z)\tilde{\mathbf{A}}(z). \tag{2.33}$$

The same techniques used for the original scattering matrix show that for suitable potentials, the following coefficients can be analytically extended into the following regions:

$$\tilde{a}_{11}: z \in D_2, \qquad \tilde{a}_{22}: \; \mathrm{Im}\, z > 0, \qquad \tilde{a}_{33}: \; z \in D_3, \tag{2.34a}$$

$$\tilde{b}_{11}: \; z \in D_1, \qquad \tilde{b}_{22}: \; \mathrm{Im}\, z < 0, \qquad \tilde{b}_{33}: \; z \in D_4. \tag{2.34b}$$

where $\tilde{\mathbf{B}}(z) = \tilde{\mathbf{A}}^{-1}(z)$. In light of these results, we can define four new solutions of the original Lax pair (2.1):

$$\chi_1(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{+,1} \times \tilde{\phi}_{-,2}](x,t,z), \tag{2.35a}$$
$$\chi_2(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,1} \times \tilde{\phi}_{+,2}](x,t,z), \tag{2.35b}$$
$$\chi_3(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{+,2} \times \tilde{\phi}_{-,3}](x,t,z), \tag{2.35c}$$
$$\chi_4(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,2} \times \tilde{\phi}_{+,3}](x,t,z). \tag{2.35d}$$

We call $\chi_1(x,t,z),\dots,\chi_4(x,t,z)$ the *auxiliary* eigenfunctions. Note that here four different auxiliary eigenfunctions are needed, in contrast to the defocusing case [9, 24], where only two auxiliary eigenfunctions must be defined. This is because, in the focusing case, we have four different domains of analyticity, compared to just the upper-half plane and the lower-half plane in the defocusing case. Indeed, by construction, we have

Lemma 2.6. *For $j = 1,\dots,4$, the auxiliary eigenfunction $\chi_j(x,t,z)$ is analytic for $z \in D_j$.*

Note that a simple relation exists between the adjoint Jost eigenfunctions and the eigenfunctions of the original Lax pair (2.1):

Lemma 2.7. *For $z \in \Sigma$ and for all cyclic indices j, ℓ, and m,*

$$\phi_{\pm,j}(x,t,z) = \mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{\pm,\ell} \times \tilde{\phi}_{\pm,m}](x,t,z)/\gamma_j(z), \tag{2.36a}$$
$$\tilde{\phi}_{\pm,j}(x,t,z) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{\pm,\ell} \times \phi_{\pm,m}](x,t,z)/\gamma_j(z), \tag{2.36b}$$

where

$$\gamma_1(z) = 1, \qquad \gamma_2(z) = \gamma(z), \qquad \gamma_3(z) = 1. \tag{2.37}$$

This relation induces a relation between the corresponding scattering matrices:

Corollary 2.8. *The scattering matrices $\mathbf{A}(z)$ and $\tilde{\mathbf{A}}(z)$ are related by*

$$\tilde{\mathbf{A}}(z) = \mathbf{\Gamma}(z)(\mathbf{A}^{-1}(z))^T\mathbf{\Gamma}^{-1}(z), \tag{2.38}$$

where $\mathbf{\Gamma}(z) = \mathrm{diag}(1,\gamma(z),1)$.

Finally, using Lemma 2.7 and the adjoint scattering relation (2.33) in the definition (2.35) yields:

Corollary 2.9. *For all $z \in \Sigma$, the Jost eigenfunctions have the following decompositions:*

$$\phi_{-,1}(x,t,z) = \frac{1}{a_{22}(z)}\left[\chi_3(x,t,z) + a_{21}(z)\phi_{-,2}(x,t,z)\right] = \frac{1}{a_{33}(z)}\left[a_{31}(z)\phi_{-,3}(x,t,z) + \chi_4(x,t,z)\right], \tag{2.39a}$$

$$\phi_{-,3}(x,t,z) = \frac{1}{a_{22}(z)}\left[\chi_2(x,t,z) + a_{23}(z)\phi_{-,2}(x,t,z)\right] = \frac{1}{a_{11}(z)}\left[a_{13}(z)\phi_{-,1}(x,t,z) + \chi_1(x,t,z)\right], \tag{2.39b}$$

$$\phi_{+,1}(x,t,z) = \frac{1}{b_{22}(z)}\left[\chi_4(x,t,z) + b_{21}(z)\phi_{+,2}(x,t,z)\right] = \frac{1}{b_{33}(z)}\left[b_{31}(z)\phi_{+,3}(x,t,z) + \chi_3(x,t,z)\right], \tag{2.39c}$$

$$\phi_{+,3}(x,t,z) = \frac{1}{b_{22}(z)}\left[\chi_1(x,t,z) + b_{23}(z)\phi_{+,2}(x,t,z)\right] = \frac{1}{b_{11}(z)}\left[b_{13}(z)\phi_{+,1}(x,t,z) + \chi_2(x,t,z)\right]. \tag{2.39d}$$

All of these results are proved in Appendix A.4. In addition, similarly to the Jost eigenfunctions it will be useful to remove the exponential oscillations and define the modified auxiliary eigenfunctions as

$$m_j(x,t,z) = \chi_j(x,t,z)\mathrm{e}^{i\theta_1(x,t,z)}, \quad j = 1,2, \tag{2.40a}$$

$$m_j(x,t,z) = \chi_j(x,t,z)\mathrm{e}^{-i\theta_1(x,t,z)}, \quad j = 3,4. \tag{2.40b}$$

Then, using Lemma 2.2 and (2.35), we can characterize the asymptotic behavior of the modified auxiliary eigenfunctions as $x \to \pm\infty$:

Lemma 2.10. *For all z in the interior of their corresponding domains of analyticity, the modified auxiliary eigenfunctions $m_j(x,t,z)$ ($j = 1,\dots,4$) remain bounded for all $x \in \mathbb{R}$.*

Like Lemma 2.2, this result will be instrumental to characterizing the discrete spectrum (cf. Section 3.1).

2.4. Symmetries

For the Manakov system with ZBC, the only symmetry of the scattering problem is the mapping $k \mapsto k^*$. With NZBC, the symmetries are complicated by the presence of the Riemann surface, which requires one to keep track of each sheet. Correspondingly, one has two symmetries, one of which is the analogue of that with ZBC while the other involves a change of sheet. The symmetries with NZBC are also complicated by the fact that, after removing the asymptotic oscillations, the Jost solutions do not tend to the identity matrix. Recall that $\lambda_{\mathrm{II}}(k) = -\lambda_{\mathrm{I}}(k)$, $z = k+\lambda$, $\hat{z} = k-\lambda$, $\lambda = (z-\hat{z})/2$, and $k = (z+\hat{z})/2$.

2.4.1. First symmetry. Consider the transformation $z \mapsto z^*$ (mapping the upper-half plane into the lower-half plane and viceversa), implying $(k,\lambda) \mapsto (k^*,\lambda^*)$.

Proposition 2.11. *If ϕ is a non-singular solution of the Lax pair, so is* $\mathbf{w}(x,t,z) = (\phi^\dagger(x,t,z^*))^{-1}$.

Proposition 2.11 is proved in Appendix A.5. There, we also show that, as a consequence:

Lemma 2.12. *For all $z \in \Sigma$, the Jost eigenfunctions satisfy the symmetry*

$$(\phi_\pm^\dagger(x,t,z^*))^{-1}\mathbf{C}(z) = \phi_\pm(x,t,z), \tag{2.41}$$

where

$$\mathbf{C}(z) = \mathrm{diag}(\gamma(z), 1, \gamma(z)). \tag{2.42}$$

Note also that

$$(\phi_\pm^{-1}(x,t,z))^T = \frac{1}{\det\phi_\pm(x,t,z)}\big(\phi_{\pm,2}\times\phi_{\pm,3}, \phi_{\pm,3}\times\phi_{\pm,1}, \phi_{\pm,1}\times\phi_{\pm,2}\big)(x,t,z)\,.$$

Then, substituting (2.39) in (2.41) and using Schwarz reflection principle yields:

Lemma 2.13. *The Jost eigenfunctions obey the symmetry relations:*

$$\phi_{+,1}^*(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{b_{22}(z)}\left[\phi_{+,2}\times\chi_1\right](x,t,z), \tag{2.43a}$$

$$\phi_{-,1}^*(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{a_{22}(z)}\left[\phi_{-,2}\times\chi_2\right](x,t,z), \tag{2.43b}$$

$$\phi^*_{+,2}(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{\gamma(z)b_{11}(z)}\,[\chi_2\times\phi_{+,1}]\,(x,t,z) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{\gamma(z)b_{33}(z)}\,[\phi_{+,3}\times\chi_3]\,(x,t,z), \tag{2.43c}$$

$$\phi^*_{-,2}(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{\gamma(z)a_{11}(z)}\,[\chi_1\times\phi_{-,1}]\,(x,t,z) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{\gamma(z)a_{33}(z)}\,[\phi_{-,3}\times\chi_4]\,(x,t,z), \tag{2.43d}$$

$$\phi^*_{+,3}(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{b_{22}(z)}\,[\chi_4\times\phi_{+,2}]\,(x,t,z), \tag{2.43e}$$

$$\phi^*_{-,3}(x,t,z^*) = \frac{\mathrm{e}^{-i\theta_2(x,t,z)}}{a_{22}(z)}\,[\chi_3\times\phi_{-,2}]\,(x,t,z), \tag{2.43f}$$

where each equation involving $\chi_j(x,t,z)$ holds for $z\in D_j$, $j=1,\dots,4$.

Note that $\phi_{\pm,2}(x,t,z)$ have two different decompositions, one in each of their sub-domains of analyticity. Moreover, using (2.41) in the scattering relation (2.25), we conclude:

Lemma 2.14. *The scattering matrix and its inverse satisfy the symmetry relation:*

$$\mathbf{A}^\dagger(z^*) = \mathbf{C}(z)\mathbf{B}(z)\mathbf{C}^{-1}(z), \qquad z\in\Sigma. \tag{2.44}$$

Componentwise, for all $z\in\Sigma$ (2.44) yields

$$b_{11}(z) = a^*_{11}(z^*), \quad b_{12}(z) = \frac{1}{\gamma(z)}a^*_{21}(z^*), \quad b_{13}(z) = a^*_{31}(z^*), \tag{2.45a}$$

$$b_{21}(z) = \gamma(z)a^*_{12}(z^*), \quad b_{22}(z) = a^*_{22}(z^*), \quad b_{23}(z) = \gamma(z)a^*_{32}(z^*), \tag{2.45b}$$

$$b_{31}(z) = a^*_{13}(z^*), \quad b_{32}(z) = \frac{1}{\gamma(z)}a^*_{23}(z^*), \quad b_{33}(z) = a^*_{33}(z^*). \tag{2.45c}$$

The Schwarz reflection principle then allows us to conclude

$$b_{11}(z) = a^*_{11}(z^*), \qquad z\in D_2, \tag{2.46a}$$

$$b_{22}(z) = a^*_{22}(z^*), \qquad \operatorname{Im} z>0, \tag{2.46b}$$

$$b_{33}(z) = a^*_{33}(z^*), \qquad z\in D_3. \tag{2.46c}$$

We can also obtain symmetry relations for the auxiliary eigenfunctions:

Corollary 2.15. *The auxiliary eigenfunctions satisfy the following symmetry relations:*

$$\chi^*_1(x,t,z^*) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{+,1}\times\phi_{-,2}](x,t,z), \qquad z\in D_2, \tag{2.47a}$$

$$\chi^*_2(x,t,z^*) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{-,1}\times\phi_{+,2}](x,t,z), \qquad z\in D_1, \tag{2.47b}$$

$$\chi^*_3(x,t,z^*) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{+,2}\times\phi_{-,3}](x,t,z), \qquad z\in D_4, \tag{2.47c}$$

$$\chi^*_4(x,t,z^*) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{-,2}\times\phi_{+,3}](x,t,z), \qquad z\in D_3. \tag{2.47d}$$

In addition, the proof of Corollary 2.15 and (2.36) yield:

$$\phi^*_{\pm,j}(x,t,z^*) = \mathrm{e}^{-i\theta_2(x,t,z)}[\phi_{\pm,\ell}\times\phi_{\pm,m}](x,t,z)/\gamma_j(z), \tag{2.48}$$

where j, ℓ, and m are cyclic indices and $z\in\Sigma$.

2.4.2. Second symmetry. Consider the transformation $z\mapsto\hat{z}$ (mapping the exterior of the circle C_o of radius q_o centered at 0 into the interior, and viceversa), implying $(k,\lambda)\mapsto(k,-\lambda)$. This symmetry relates the values of the eigenfunctions on the two sheets when k is arbitrary but fixed (on either sheet). It is easy to show the following:

Proposition 2.16. *If $\phi(x,t,z)$ is a solution of the Lax pair, so is*

$$\mathbf{W}(x,t,z) = \phi(x,t,\hat{z}). \tag{2.49}$$

In appendix A.5 we then show that, as a consequence:

Lemma 2.17. *For all* $z \in \Sigma$*, the Jost eigenfunctions satisfy the symmetry*

$$\phi_\pm(x,t,z) = \phi_\pm(x,t,\hat{z})\mathbf{\Pi}(z),, \tag{2.50}$$

where

$$\mathbf{\Pi}(z) = \begin{pmatrix} 0 & 0 & iq_o/z \\ 0 & 1 & 0 \\ iq_o/z & 0 & 0 \end{pmatrix}. \tag{2.51}$$

As before, the analyticity properties of the eigenfunctions then allow us to extend all of the above relations:

$$\phi_{\pm,1}(x,t,z) = \frac{iq_o}{z}\phi_{\pm,3}(x,t,\hat{z}), \quad \operatorname{Im} z \lessgtr 0 \wedge |z| > q_o, \tag{2.52a}$$

$$\phi_{\pm,2}(x,t,z) = \phi_{\pm,2}(x,t,\hat{z}), \quad \operatorname{Im} z \gtrless 0, \tag{2.52b}$$

$$\phi_{\pm,3}(x,t,z) = \frac{iq_o}{z}\phi_{\pm,1}(x,t,\hat{z}), \quad \operatorname{Im} z \gtrless 0 \wedge |z| < q_o. \tag{2.52c}$$

Also, similarly as before, we can again use (2.25) to conclude

Lemma 2.18. *The scattering matrix satisfies the symmetry*

$$\mathbf{A}(\hat{z}) = \mathbf{\Pi}(z)\mathbf{A}(z)\mathbf{\Pi}^{-1}(z), \qquad z \in \Sigma. \tag{2.53}$$

Componentwise, we have

$$a_{11}(z) = a_{33}(\hat{z}), \quad a_{12}(z) = \frac{iq_o}{z}a_{32}(\hat{z}), \quad a_{13}(z) = a_{31}(\hat{z}), \tag{2.54a}$$

$$a_{21}(z) = -\frac{iz}{q_o}a_{23}(\hat{z}), \quad a_{22}(z) = a_{22}(\hat{z}), \quad a_{23}(z) = -\frac{iz}{q_o}a_{21}(\hat{z}), \tag{2.54b}$$

$$a_{31}(z) = a_{13}(\hat{z}), \quad a_{32}(z) = \frac{iq_o}{z}a_{12}(\hat{z}), \quad a_{33}(z) = a_{11}(\hat{z}). \tag{2.54c}$$

An identical set of relations obviously holds for the elements of $\mathbf{B}(z)$. The analyticity of the scattering matrix entries allows us to conclude

$$a_{11}(z) = a_{33}(\hat{z}), \quad z \in D_1, \qquad b_{11}(z) = b_{33}(\hat{z}), \quad z \in D_2, \tag{2.55a}$$

$$b_{22}(z) = b_{22}(\hat{z}), \quad \operatorname{Im} z \geq 0, \qquad a_{22}(z) = a_{22}(\hat{z}), \quad \operatorname{Im} z \leq 0. \tag{2.55b}$$

Finally, we combine (2.52) and (2.54) with (2.35) to conclude

Lemma 2.19. *The auxiliary eigenfunctions satisfy the symmetries*

$$\chi_1(x,t,z) = \frac{iq_o}{z}\chi_4(x,t,\hat{z}), \quad z \in D_1, \tag{2.56a}$$

$$\chi_2(x,t,z) = \frac{iq_o}{z}\chi_3(x,t,\hat{z}), \quad z \in D_2. \tag{2.56b}$$

2.4.3. Combined symmetry and reflection coefficients. Of course one can combine the above two symmetries to obtain relations between eigenfunctions and scattering coefficients evaluated at z and at $-q_o^2/z^*$. We omit these relations for brevity.

The following reflection coefficients will appear in the inverse problem:

$$\rho_1(z) = \frac{a_{21}(z)}{a_{11}(z)} = \gamma(z)\frac{b_{12}^*(z^*)}{b_{11}^*(z^*)}, \quad \rho_2(z) = \frac{a_{31}(z)}{a_{11}(z)} = \frac{b_{13}^*(z^*)}{b_{11}^*(z^*)}, \tag{2.57a}$$

$$\rho_3(z) = \frac{a_{32}(z)}{a_{22}(z)} = \frac{1}{\gamma(z)} \frac{b_{23}^*(z^*)}{b_{22}^*(z^*)}. \tag{2.57b}$$

The symmetries of the scattering matrix yield

$$\rho_1(\hat{z}) = \frac{iq_o}{z}\frac{a_{23}(z)}{a_{33}(z)} = \gamma(z)\frac{iq_o}{z}\frac{b_{32}^*(z^*)}{b_{33}^*(z^*)}, \quad \rho_2(\hat{z}) = \frac{a_{13}(z)}{a_{33}(z)} = \frac{b_{31}^*(z^*)}{b_{33}^*(z^*)}, \tag{2.58a}$$

$$\rho_3(\hat{z}) = -\frac{iz}{q_o}\frac{a_{12}(z)}{a_{22}(z)} = -\frac{iz}{q_o\gamma(z)}\frac{b_{21}^*(z^*)}{b_{22}^*(z^*)}. \tag{2.58b}$$

The definition of $\mathbf{B}(z)$ as $\mathbf{A}^{-1}(z)$ yields the following for $z \in \Sigma$:

$$a_{32}(z) = b_{12}(z)b_{31}(z) - b_{11}(z)b_{32}(z). \tag{2.59}$$

In terms of the reflection coefficents, we have

$$\rho_3(z) = \frac{b_{11}(z)b_{11}(\hat{z})}{a_{22}(z)\gamma(z)}\left[\rho_1^*(z^*)\rho_2^*(\hat{z}^*) - \frac{iz}{q_o}\rho_1^*(\hat{z}^*)\right], \qquad z \in \Sigma. \tag{2.60}$$

Thus, only two of the reflection coefficients are independent. Once the trace formulae for the analytic scattering coefficients has been obtained in section 4.3, we will show that one can combine all of the above symmetries to reconstruct the entire scattering matrix.

3. Discrete spectrum and asymptotic behavior

3.1. Discrete spectrum

As we show next, the discrete spectrum for the focusing Manakov system with NZBC is considerably richer than that of both the focusing case with ZBC and the defocusing case with NZBC.

In order to characterize the discrete spectrum, it is convenient to introduce the following 3×3 matrices, each of which is analytic in one of the four fundamental domains:

$$\mathbf{\Phi}_1(x,t,z) = (\phi_{-,1}(x,t,z), \phi_{+,2}(x,t,z), \chi_1(x,t,z)), \qquad z \in D_1\,, \tag{3.1a}$$

$$\mathbf{\Phi}_2(x,t,z) = (\phi_{+,1}(x,t,z), \phi_{-,2}(x,t,z), \chi_2(x,t,z)), \qquad z \in D_2\,, \tag{3.1b}$$

$$\mathbf{\Phi}_3(x,t,z) = (\chi_3(x,t,z), \phi_{-,2}(x,t,z), \phi_{+,3}(x,t,z)), \qquad z \in D_3\,, \tag{3.1c}$$

$$\mathbf{\Phi}_4(x,t,z) = (\chi_4(x,t,z), \phi_{+,2}(x,t,z), \phi_{-,3}(x,t,z)), \qquad z \in D_4\,. \tag{3.1d}$$

Recalling (2.39), we obtain immediately

$$\mathrm{Wr}\,\mathbf{\Phi}_1(x,t,z) = a_{11}(z)b_{22}(z)\,\mathrm{e}^{i\theta_2(x,t,z)}, \quad z \in D_1, \tag{3.2a}$$

$$\mathrm{Wr}\,\mathbf{\Phi}_2(x,t,z) = a_{22}(z)b_{11}(z)\,\mathrm{e}^{i\theta_2(x,t,z)}, \quad z \in D_2, \tag{3.2b}$$

$$\mathrm{Wr}\,\mathbf{\Phi}_3(x,t,z) = a_{22}(z)b_{33}(z)\,\mathrm{e}^{i\theta_2(x,t,z)}, \quad z \in D_3, \tag{3.2c}$$

$$\mathrm{Wr}\,\mathbf{\Phi}_4(x,t,z) = a_{33}(z)b_{22}(z)\,\mathrm{e}^{i\theta_2(x,t,z)}, \quad z \in D_4. \tag{3.2d}$$

Thus, the columns of $\mathbf{\Phi}_1(x,t,z)$ become linearly dependent at the zeros of $a_{11}(z)$ and $b_{22}(z)$. Similarly, the columns of $\mathbf{\Phi}_2(x,t,z)$ are linearly dependent at the zeros of $a_{22}(z)$ and $b_{11}(z)$, etc. On the other hand, the symmetries of the scattering coefficients imply that these zeros are not independent of each other. Indeed, in Appendix A.6 we prove:

Lemma 3.1. *Let* $\mathrm{Im}\,z_o > 0$. *Then*

$$b_{22}(z_o) = 0 \iff a_{22}(z_o^*) = 0 \iff a_{22}(\hat{z}_o^*) = 0 \iff b_{22}(\hat{z}_o) = 0. \tag{3.3}$$

Lemma 3.2. *Let* $\operatorname{Im} z_o > 0$ *and* $|z_o| \geq q_o$. *Then*

$$a_{11}(z_o) = 0 \iff b_{11}(z_o^*) = 0 \iff b_{33}(\hat{z}_o^*) = 0 \iff a_{33}(\hat{z}_o) = 0. \tag{3.4}$$

Lemmas 3.2 and 3.1 imply that discrete eigenvalues appear in symmetric *quartets*: z_n, z_n^*, $-q_o^2/z_n$, $-q_o^2/z_n^*$. (This situation is similar to the scalar case with NZBC [8] and the defocusing Manakov system with NZBC [24].) It is therefore sufficient to study the zeros of $a_{11}(z)$ and $b_{22}(z)$ for $z \in D_1$. Clearly, there are three possible kinds of eigenvalue quartets corresponding to a given eigenvalue $z_o \in D_1$ (i.e., such that $\operatorname{Im} z_o > 0$ and $|z_o| > q_o$):

1. $a_{11}(z_o) = 0$ and $b_{22}(z_o) \neq 0$. We call this an *eigenvalue of the first kind*.
2. $a_{11}(z_o) \neq 0$ and $b_{22}(z_o) = 0$. We call this an *eigenvalue of the second kind*.
3. $a_{11}(z_o) = b_{22}(z_o) = 0$. We call this an *eigenvalue of the third kind*.

We next characterize each of these three types of eigenvalues. The following results will be instrumental to this end:

Lemma 3.3. *Suppose* $\operatorname{Im} z_o > 0$ *and* $|z_o| > q_o$. *Then the following statements are equivalent:*

(i) $\chi_1(x,t,z_o) = \mathbf{0}$,
(ii) $\chi_4(x,t,\hat{z}_o) = \mathbf{0}$,
(iii) There exists a constant b_o *such that* $\phi_{-,2}(x,t,z_o^*) = b_o \phi_{+,1}(x,t,z_o^*)$,
(iv) There exists a constant $\tilde{b}_o$ *such that* $\phi_{-,2}(x,t,\hat{z}_o^*) = \tilde{b}_o \phi_{+,3}(x,t,\hat{z}_o^*)$.

Lemma 3.4. *Suppose* $\operatorname{Im} z_o > 0$ *and* $|z_o| > q_o$. *Then the following statements are equivalent:*

(i) $\chi_2(x,t,z_o^*) = \mathbf{0}$,
(ii) $\chi_3(x,t,\hat{z}_o^*) = \mathbf{0}$,
(iii) There exists a constant $\hat{b}_o$ *such that* $\hat{b}_o \phi_{+,2}(x,t,z_o) = \phi_{-,1}(x,t,z_o)$,
(iv) There exists a constant $\check{b}_o$ *such that* $\check{b}_o \phi_{+,2}(x,t,\hat{z}_o) = \phi_{-,3}(x,t,\hat{z}_o)$.

Remark 3.5. *All the results up to this point are valid for zeros of* $a_{11}(z)$ *and/or* $b_{22}(z)$ *(in their appropriate domains of analyticity) of any order. For the remainder of this work we will only consider discrete eigenvalues that are simple zeros of* $a_{11}(z)$ *and/or* $b_{22}(z)$.

Using the results of this section and the assumption that the discrete eigenvalues are simple, in Appendix A.6 we prove the following:

Theorem 3.6. *Let* $z_o \in D_1$ *be a discrete eigenvalue of the scattering problem. That is,* $a_{11}(z_o)b_{22}(z_o) = 0$. *Then the following are true:*

(i) If z_o *is an eigenvalue of the first kind, there exist constants* c_o, $\hat{c}_o$, $\check{c}_o$, *and* $\overline{c}_o$ *such that*

$$\phi_{-,1}(x,t,z_o) = c_o \chi_1(x,t,z_o)/b_{22}(z_o), \quad \chi_2(x,t,z_o^*) = \hat{c}_o \phi_{+,1}(x,t,z_o^*),$$
$$\chi_3(x,t,\hat{z}_o^*) = \check{c}_o \phi_{+,3}(x,t,\hat{z}_o^*), \quad \phi_{-,3}(x,t,\hat{z}_o) = \overline{c}_o \chi_4(x,t,\hat{z}_o).$$

(ii) If z_o *is an eigenvalue of the second kind, there exist constants* d_o, $\hat{d}_o$, $\check{d}_o$, *and* $\overline{d}_o$ *such that*

$$\chi_1(x,t,z_o) = d_o \phi_{+,2}(x,t,z_o), \quad \phi_{-,2}(x,t,z_o^*) = \hat{d}_o \chi_2(x,t,z_o^*),$$
$$\phi_{-,2}(x,t,\hat{z}_o^*) = \check{d}_o \chi_3(x,t,\hat{z}_o^*), \quad \chi_4(x,t,\hat{z}_o) = \overline{d}_o \phi_{+,2}(x,t,\hat{z}_o).$$

(iii) If z_o is an eigenvalue of the third kind, then $\chi_1(x,t,z_o) = \chi_2(x,t,z_o^*) = \mathbf{0}$, and there exist constants f_o, $\hat{f}_o$, $\check{f}_o$, and $\overline{f}_o$ such that

$$\phi_{-,1}(x,t,z_o) = f_o\phi_{+,2}(x,t,z_o), \quad \phi_{-,2}(x,t,z_o^*) = \hat{f}_o\phi_{+,1}(x,t,z_o^*),$$
$$\phi_{-,2}(x,t,\hat{z}_o^*) = \check{f}_o\phi_{+,3}(x,t,\hat{z}_o^*), \quad \phi_{-,3}(x,t,\hat{z}_o) = \overline{f}_o\phi_{+,2}(x,t,\hat{z}_o).$$

Theorem 3.6 provides a full characterization of the discrete spectrum. In particular, taking into account the asymptotic behavior of the Jost eigenfunctions and auxiliary eigenfunctions as $x \to \pm\infty$ in Lemmas 2.2 and 2.10, it is straightforward to see that *a discrete eigenvalue of each kind corresponds to a bound state of the scattering problem* [i.e., an eigenfunction in $L^2(\mathbb{R})$]. This is in marked contrast to the defocusing case, where zeros of the analytic scattering coefficients off C_o do not lead to bound states [9, 24], and is a consequence of the fact that the scattering problem for the focusing case is not self-adjoint.

3.2. Symmetries of the norming constants

We first rewrite the results of Theorem 3.6 in terms of the modified eigenfunctions, which will be useful when deriving the residue conditions for the inverse problem. Let $\{w_n\}_{n=1}^{N_1}$ be the set of all eigenvalues of the first kind. Then

$$\mu_{-,1}(x,t,w_n) = c_n m_1(x,t,w_n)\mathrm{e}^{-2i\theta_1(x,t,w_n)}/b_{22}(w_n), \tag{3.5a}$$
$$m_2(x,t,w_n^*) = \hat{c}_n\mu_{+,1}(x,t,w_n^*)\mathrm{e}^{2i\theta_1(x,t,w_n^*)}, \tag{3.5b}$$
$$m_3(x,t,\hat{w}_n^*) = \check{c}_n\mu_{+,3}(x,t,\hat{w}_n^*)\mathrm{e}^{2i\theta_1(x,t,w_n^*)}, \tag{3.5c}$$
$$\mu_{-,3}(x,t,\hat{w}_n) = \overline{c}_n m_4(x,t,\hat{w}_n)\mathrm{e}^{-2i\theta_1(x,t,w_n)}. \tag{3.5d}$$

Let $\{z_n\}_{n=1}^{N_2}$ be the set of all eigenvalues of the second kind. Then

$$m_1(x,t,z_n) = d_n\mu_{+,2}(x,t,z_n)\mathrm{e}^{i(\theta_1+\theta_2)(x,t,z_n)}, \tag{3.6a}$$
$$\mu_{-,2}(x,t,z_n^*) = \hat{d}_n m_2(x,t,z_n^*)\mathrm{e}^{-i(\theta_1+\theta_2)(x,t,z_n^*)}, \tag{3.6b}$$
$$\mu_{-,2}(x,t,\hat{z}_n^*) = \check{d}_n m_3(x,t,\hat{z}_n^*)\mathrm{e}^{-i(\theta_1+\theta_2)(x,t,z_n^*)}, \tag{3.6c}$$
$$m_4(x,t,\hat{z}_n) = \overline{d}_n\mu_{+,2}(x,t,\hat{z}_n)\mathrm{e}^{i(\theta_1+\theta_2)(x,t,z_n)}. \tag{3.6d}$$

Let $\{\zeta_n\}_{n=1}^{N_3}$ be the set of all eigenvalues of the third kind. Then

$$\mu_{-,1}(x,t,\zeta_n) = f_n\mu_{+,2}(x,t,\zeta_n)\mathrm{e}^{-i(\theta_1-\theta_2)(x,t,\zeta_n)}, \tag{3.7a}$$
$$\mu_{-,2}(x,t,\zeta_n^*) = \hat{f}_n\mu_{+,1}(x,t,\zeta_n^*)\mathrm{e}^{i(\theta_1-\theta_2)(x,t,\zeta_n^*)}, \tag{3.7b}$$
$$\mu_{-,2}(x,t,\hat{\zeta}_n^*) = \check{f}_n\mu_{+,3}(x,t,\hat{\zeta}_n^*)\mathrm{e}^{i(\theta_1-\theta_2)(x,t,\zeta_n^*)}, \tag{3.7c}$$
$$\mu_{-,3}(x,t,\hat{\zeta}_n) = \overline{f}_n\mu_{+,2}(x,t,\hat{\zeta}_n)\mathrm{e}^{-i(\theta_1-\theta_2)(x,t,\zeta_n)}. \tag{3.7d}$$

Writing the norming constant relations in this manner will allow us to easily find the residue conditions of the Riemann-Hilbert problem, which will be introduced in Section 4.1. However, it will first be necessary to explore how the symmetries of the scattering matrix and eigenfunctions affect these norming constants. Said symmetries are combined to show:

Lemma 3.7. *The norming constants in Theorem 3.6 obey the following symmetry relations:*

$$\overline{c}_n = c_n/b_{22}(w_n), \quad \hat{c}_n = \check{c}_n = -c_n^*, \tag{3.8a}$$

$$\overline{d}_n = -\frac{iz_n}{q_o} d_n, \quad \hat{d}_n = -\frac{d_n^*}{\gamma(z_n^*) b_{11}(z_n^*)}, \quad \check{d}_n = -\frac{iq_o}{z_n^*} \frac{d_n^*}{\gamma(z_n^*) b_{11}(z_n^*)}, \tag{3.8b}$$

$$\overline{f}_n = -\frac{i\zeta_n}{q_o} f_n, \quad \hat{f}_n = -\frac{a'_{22}(\zeta_n^*)}{b'_{11}(\zeta_n^*)} \frac{f_n^*}{\gamma(\zeta_n^*)}, \quad \check{f}_n = -\frac{iq_o}{\zeta_n^*} \frac{a'_{22}(\zeta_n^*)}{b'_{11}(\zeta_n^*)} \frac{f_n^*}{\gamma(\zeta_n^*)}. \tag{3.8c}$$

3.3. Asymptotic behavior as $z \to \infty$ and $z \to 0$

To normalize the Riemann-Hilbert problem (RHP) (defined in section 4.1), it will be necessary to examine the asymptotic behavior of the eigenfunctions and scattering data as $k \to \infty$. In terms of the uniformization variable $z = k + \lambda$, this requires studying the behavior both as $z \to \infty$ and $z \to 0$. Consider the following formal expansion for $\mu_+(x,t,z)$:

$$\mu_+(x,t,z) = \sum_{n=0}^{\infty} \mu_n(x,t,z), \tag{3.9}$$

where

$$\mu_0(x,t,z) = \mathbf{E}_+(z), \tag{3.10a}$$

$$\mu_{n+1}(x,t,z) = -\int_x^\infty \mathbf{E}_+(z) \mathrm{e}^{i(x-y)\mathbf{\Lambda}(z)} \mathbf{E}_+^{-1}(z) \Delta\mathbf{Q}_+(y,t) \mu_n(y,t,z) \mathrm{e}^{-i(x-y)\mathbf{\Lambda}(z)} \mathrm{d}y. \tag{3.10b}$$

We will use (3.9) and (3.10) to characterize the asymptotic behavior of the eigenfunctions as $z \to \infty$ and $z \to 0$. Since doing so will require integration by parts, one must identify appropriate funcional classes for the potential which guarantee the validity of results. Denote by $W^{1,1}(a,b)$ the Sobolev space consisting of functions $f \in L^1(a,b)$ such that the first-order weak derivative of f is also in $L^1(a,b)$. In Appendix A.7, we prove the following:

Lemma 3.8. *Let $\mathbf{q}(\cdot,t) - \mathbf{q}_- \in W^{1,1}(-\infty,a)$ and $\mathbf{q}(\cdot,t) - \mathbf{q}_+ \in W^{1,1}(a,\infty)$ for any constant $a \in \mathbb{R}$. Then for all $m \geq 0$, (3.9) provides an asymptotic expansion for the columns of $\mu_+(x,t,z)$ as $z \to \infty$ in the appropriate region of the complex z-plane, with*

$$[\mu_{2m}]_{bd} = O(1/z^m), \qquad [\mu_{2m}]_{bo} = O(1/z^{m+1}), \tag{3.11a}$$

$$[\mu_{2m+1}]_{bd} = O(1/z^{m+1}), \qquad [\mu_{2m+1}]_{bo} = O(1/z^{m+1}). \tag{3.11b}$$

Lemma 3.9. *Let $\mathbf{q}(\cdot,t) - \mathbf{q}_- \in W^{1,1}(-\infty,a)$ and $\mathbf{q}(\cdot,t) - \mathbf{q}_+ \in W^{1,1}(a,\infty)$ for any constant $a \in \mathbb{R}$. Then for all $m \geq 0$, (3.9) provides an asymptotic expansion for the columns of $\mu_+(x,t,z)$ as $z \to 0$ in the appropriate region of the complex z-plane, with*

$$[\mu_{2m}]_{bd} = O(z^m), \qquad [\mu_{2m}]_{bo} = O(z^{m-1}), \tag{3.12a}$$

$$[\mu_{2m+1}]_{bd} = O(z^m), \qquad [\mu_{2m+1}]_{bo} = O(z^m). \tag{3.12b}$$

Then, evaluating explicitly the first few terms in (3.9), we obtain

Corollary 3.10. *As $z \to \infty$ in the appropriate regions of the z-plane,*

$$\mu_{\pm,1}(x,t,z) = \begin{pmatrix} 1 \\ (i/z)\mathbf{q}(x,t) \end{pmatrix} + O(1/z^2), \tag{3.13a}$$

$$\mu_{\pm,2}(x,t,z) = \begin{pmatrix} -(i/q_o z)\mathbf{q}^\dagger(x,t)\mathbf{q}_\pm^\perp \\ \mathbf{q}_\pm^\perp/q_o \end{pmatrix} + O(1/z^2), \tag{3.13b}$$

$$\mu_{\pm,3}(x,t,z) = \begin{pmatrix} (i/q_o z)\mathbf{q}^\dagger(x,t)\mathbf{q}_\pm \\ \mathbf{q}_\pm/q_o \end{pmatrix} + O(1/z^2). \tag{3.13c}$$

Similarly, as $z \to 0$ in the appropriate regions of the z-plane,

$$\mu_{\pm,1}(x,t,z) = \begin{pmatrix} \mathbf{q}^\dagger(x,t)\mathbf{q}_\pm/q_o^2 \\ (i/z)\mathbf{q}_\pm \end{pmatrix} + O(z), \tag{3.14a}$$

$$\mu_{\pm,2}(x,t,z) = \begin{pmatrix} 0 \\ \mathbf{q}_\pm^\perp/q_o \end{pmatrix} + O(z), \tag{3.14b}$$

$$\mu_{\pm,3}(x,t,z) = \begin{pmatrix} iq_o/z \\ \mathbf{q}(x,t)/q_o \end{pmatrix} + O(z). \tag{3.14c}$$

Next, we compute the asymptotic behavior of the auxiliary eigenfunctions $\chi_j(x,t,z)$, $j = 1,\dots,4$. It will be helpful to remove their exponential oscillations (as we did with the Jost eigenfunctions). Recall the definitions (2.40) of the modified auxiliary eigenfunctions. Combining (2.35) with (2.47) we then have:

Lemma 3.11. *As $z \to \infty$ in the appropriate regions of the z-plane,*

$$m_1(x,t,z) = \begin{pmatrix} (i/q_oz)\mathbf{q}^\dagger(x,t)\mathbf{q}_- \\ \mathbf{q}_-/q_o \end{pmatrix} + O(1/z^2), \quad m_2(x,t,z) = \begin{pmatrix} (i/q_oz)\mathbf{q}^\dagger(x,t)\mathbf{q}_+ \\ \mathbf{q}_+/q_o \end{pmatrix} + O(1/z^2),$$
$$m_3(x,t,z) = \begin{pmatrix} \mathbf{q}_-^\dagger\mathbf{q}_+/q_o^2 \\ (i/q_o^2z)\mathbf{q}_-^\dagger\mathbf{q}_+\mathbf{r}(x,t) \end{pmatrix} + O(1/z^2), \quad m_4(x,t,z) = \begin{pmatrix} \mathbf{q}_+^\dagger\mathbf{q}_-/q_o^2 \\ (i/q_o^2z)\mathbf{q}_+^\dagger\mathbf{q}_-\mathbf{r}(x,t) \end{pmatrix} + O(1/z^2).$$

Similarly, as $z \to 0$ in the appropriate regions of the z-plane,

$$m_1(x,t,z) = \begin{pmatrix} (i/q_oz)\mathbf{q}_+^\dagger\mathbf{q}_- \\ 0 \end{pmatrix} + O(1), \qquad m_2(x,t,z) = \begin{pmatrix} (i/q_oz)\mathbf{q}_-^\dagger\mathbf{q}_+ \\ 0 \end{pmatrix} + O(1),$$
$$m_3(x,t,z) = \begin{pmatrix} 0 \\ (i/z)\mathbf{q}_+ \end{pmatrix} + O(1), \qquad m_4(x,t,z) = \begin{pmatrix} 0 \\ (i/z)\mathbf{q}_- \end{pmatrix} + O(1).$$

Next, we find the asymptotic behavior of the scattering matrix entries. Combining the results in Corollary 3.10 with the scattering relation (2.25) and the symmetry (2.41) yields

Corollary 3.12. *As $z \to \infty$ in the appropriate regions of the z-plane,*

$$a_{11}(z) = 1 + O(1/z), \quad a_{22}(z) = \mathbf{q}_-^\dagger\mathbf{q}_+/q_o^2 + O(1/z), \quad a_{33}(z) = \mathbf{q}_+^\dagger\mathbf{q}_-/q_o^2 + O(1/z),$$
$$b_{11}(z) = 1 + O(1/z), \quad b_{22}(z) = \mathbf{q}_+^\dagger\mathbf{q}_-/q_o^2 + O(1/z), \quad b_{33}(z) = \mathbf{q}_-^\dagger\mathbf{q}_+/q_o^2 + O(1/z).$$

Similarly, as $z \to 0$ in the appropriate regions of the z-plane,

$$a_{11}(z) = \mathbf{q}_+^\dagger\mathbf{q}_-/q_o^2 + O(z), \quad a_{22}(z) = \mathbf{q}_-^\dagger\mathbf{q}_+/q_o^2 + O(z), \quad a_{33}(z) = 1 + O(z),$$
$$b_{11}(z) = \mathbf{q}_-^\dagger\mathbf{q}_+/q_o^2 + O(z), \quad b_{22}(z) = \mathbf{q}_+^\dagger\mathbf{q}_-/q_o^2 + O(z), \quad b_{33}(z) = 1 + O(z).$$

Finally, we find the asymptotic behavior of the off-diagonal scattering matrix entries. Again, combining Corollary 3.10 with the scattering relation (2.25) and the symmetry (2.41) yields

Corollary 3.13. *As $z \to \infty$ on the real z-axis,*

$$[\mathbf{A}^{\pm 1}(z)]_o = \frac{1}{q_o^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{r}_\mp^\dagger\mathbf{r}_\pm^\perp \\ 0 & \mathbf{q}_\pm^\dagger\mathbf{q}_\mp^\perp & 0 \end{pmatrix} + O(1/z).$$

Similarly, as $z \to 0$ on the real z-axis,

$$[\mathbf{A}^{\pm 1}(z)]_o = \frac{iq_o}{z}\begin{pmatrix} 0 & 0 & 0 \\ \mathbf{r}_\mp^\dagger \mathbf{r}_\pm^\perp & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(1).$$

Note that not all off-diagonal entries vanish as $z \to \infty$. (The same happens in the defocusing case [9].) This, however, does not complicate the inverse problem since the appropriate combinations of reflection coefficients will still vanish as $z \to \infty$.

3.4. Behavior at the branch points

We now discuss the behavior of the Jost eigenfunctions and the scattering matrix at the branch points $k = \pm iq_o$. The complication there is due to the fact that $\lambda(\pm iq_o) = 0$, and therefore, at $z = \pm iq_o$, the two exponentials $\mathrm{e}^{\pm i\lambda x}$ reduce to unity. Correspondingly, at $z = \pm iq_o$, the matrices $\mathbf{E}_\pm(z)$ are degenerate. Nonetheless, the term $\mathbf{E}_\pm(z)\mathrm{e}^{i(x-y)\mathbf{\Lambda}(z)}\mathbf{E}_\pm^{-1}(z)$ appearing in the integral equations for the Jost eigenfunctions remains finite as $z \to \pm iq_o$:

$$\lim_{z\to\pm iq_o} \mathbf{E}_\pm(z)\mathrm{e}^{i\xi\mathbf{\Lambda}(z)}\mathbf{E}_\pm^{-1}(z) = \begin{pmatrix} 1 \pm q_o\xi & \xi\mathbf{q}_\pm^\dagger \\ \xi\mathbf{q}_\pm & \frac{1}{q_o^2}\mathbf{U}_\pm(\xi) \end{pmatrix},$$

where $\xi = x - y$ and $\mathbf{U}_\pm(\xi) = (1 \mp q_o\xi)\mathbf{q}_\pm\mathbf{q}_\pm^\dagger + \mathrm{e}^{\mp q_o\xi}\mathbf{q}_\pm^\perp(\mathbf{q}_\pm^\perp)^\dagger$. Thus, if $(1+|x|)(\mathbf{q}(x,t) - \mathbf{q}_\pm) \in L^1(\mathbb{R}^\pm)$, the integrals in (2.20) are also convergent at $z = \pm iq_o$, and the Jost solutions admit a well-defined limit at the branch points. (This is identical to what happens in the scalar and defocusing cases [8, 9, 14].) Nonetheless, $\det\phi_\pm(x,t,\pm iq_o) = 0$ for all $(x,t) \in \mathbb{R}^2$. Thus, the columns of $\phi_\pm(x,t,iq_o)$ [as well as those of $\phi_\pm(x,t,-iq_o)$] are linearly dependent. Comparing the asymptotic behavior of the columns of $\phi_\pm(x,t,\pm iq_o)$ as $x \to \pm\infty$, we obtain

$$\phi_{\pm,1}(x,t,iq_o) = \phi_{\pm,3}(x,t,iq_o), \qquad \phi_{\pm,1}(x,t,-iq_o) = -\phi_{\pm,3}(x,t,-iq_o). \tag{3.15}$$

Next, we characterize the limiting behavior of the scattering matrix near the branch points. It is easy to combine the identity (2.24) with the scattering relation (2.25) to express all entries of the scattering matrix $\mathbf{A}(z)$ as Wronskians:

$$a_{j\ell}(z) = \frac{z^2}{z^2+q_o^2} W_{j\ell}(x,t,z)\mathrm{e}^{-i\theta_2(x,t,z)},$$

where

$$W_{j\ell}(x,t,z) = \mathrm{Wr}(\phi_{-,\ell}(x,t,z), \phi_{+,j+1}(x,t,z), \phi_{+,j+2}(x,t,z)),$$

and $j+1$ and $j+2$ are calculated modulo 3. We then have the following Laurent series expansions about $z = \pm iq_o$:

$$a_{ij}(z) = \frac{a_{ij,\pm}}{z \mp iq_o} + a_{ij,\pm}^{(o)} + O(z \mp iq_o), \quad z \in \Sigma \setminus \{\pm iq_o\}, \tag{3.16}$$

where, for example,

$$a_{11,\pm} = \pm\frac{iq_o}{2} W_{11}(x,t,\pm iq_o)\mathrm{e}^{\pm q_o(x\mp iq_o t)},$$

$$a_{11,\pm}^{(o)} = \left[\pm\frac{iq_o}{2}\frac{\mathrm{d}}{\mathrm{d}z}W_{11}(x,t,z)|_{z=\pm iq_o} + W_{11}(x,t,\pm iq_o)\right]\mathrm{e}^{\pm q_o(x\mp iq_o t)}.$$

Note that in (3.16), the subscript "+" is used to indicate quantities associated with the Laurent series expansion of $a_{ij}(z)$ about $z = iq_o$, while the subscript "−" corresponds to quantities associated with the Laurent series expansion of $a_{ij}(z)$ about $z = -iq_o$. This is in contrast to the rest of this work, where such subscripts denote normalization as $x \to \pm\infty$. Summarizing, the asymptotic expansion of $\mathbf{A}(z)$ in a neighborhood of the branch point is

$$\mathbf{A}(z) = \frac{1}{z \mp iq_o}\mathbf{A}_\pm + \mathbf{A}_\pm^{(o)} + O(z \mp iq_o),$$

where $\mathbf{A}_\pm^{(o)} = (a_{ij,\pm}^{(o)})$,

$$\mathbf{A}_\pm = a_{11,\pm}\begin{pmatrix} 1 & 0 & \pm 1 \\ 0 & 0 & 0 \\ \mp 1 & 0 & -1 \end{pmatrix} + a_{12,\pm}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & \mp 1 & 0 \end{pmatrix},$$

and $a_{12,\pm} = \pm(iq_o/2)W_{12}(x,t,\pm iq_o)\mathrm{e}^{\pm q_o(x \mp iq_o t)}$. Note that the second row of $\mathbf{A}_\pm$ is identically zero because $a_{2j,\pm} = \pm(iq_o/2)W_{j2}(x,t,\pm iq_o)\mathrm{e}^{\pm q_o(x \mp iq_o t)}$, which is zero by virtue of (3.15).

4. Inverse problem

4.1. Riemann-Hilbert problem

As usual, the inverse scattering problem is formulated in terms of an appropriate RHP. To this end, we need suitable jump conditions that express eigenfunctions meromorphic in D_1 in terms of eigenfunctions that are meromorphic in D_2 (and similarly for the other regions). The desired eigenfunctions are the columns of $\mathbf{\Phi}_j(x,t,z)$ ($j = 1,\ldots,4$) in (3.1), and, as in the defocusing case, the jump conditions are provided by the scattering relation (2.25). Using these relations, in Appendix A.8 we then prove:

Lemma 4.1. *Define the piecewise meromorphic function $\mathbf{M}(x,t,z)$ as $\mathbf{M}(x,t,z) = \mathbf{M}_j(x,t,z)$ for $z \in D_j$ ($j = 1,\ldots,4$), where*

$$\mathbf{M}_1(x,t,z) = \mathbf{\Phi}_1\mathrm{e}^{-i\mathbf{\Theta}}[\mathrm{diag}(a_{11},1,b_{22})]^{-1} = \left(\frac{\mu_{-,1}}{a_{11}}, \mu_{+,2}, \frac{m_1}{b_{22}}\right), \quad z \in D_1\,, \qquad (4.1a)$$

$$\mathbf{M}_2(x,t,z) = \mathbf{\Phi}_2\mathrm{e}^{-i\mathbf{\Theta}}[\mathrm{diag}(1,a_{22},b_{11})]^{-1} = \left(\mu_{+,1}, \frac{\mu_{-,2}}{a_{22}}, \frac{m_2}{b_{11}}\right), \quad z \in D_2\,, \qquad (4.1b)$$

$$\mathbf{M}_3(x,t,z) = \mathbf{\Phi}_3\mathrm{e}^{-i\mathbf{\Theta}}[\mathrm{diag}(b_{33},a_{22},1)]^{-1} = \left(\frac{m_3}{b_{33}}, \frac{\mu_{-,2}}{a_{22}}, \mu_{+,3}\right), \quad z \in D_3\,, \qquad (4.1c)$$

$$\mathbf{M}_4(x,t,z) = \mathbf{\Phi}_4\mathrm{e}^{-i\mathbf{\Theta}}[\mathrm{diag}(b_{22},1,a_{33})]^{-1} = \left(\frac{m_4}{b_{22}}, \mu_{+,2}, \frac{\mu_{-,3}}{a_{33}}\right), \quad z \in D_4\,. \qquad (4.1d)$$

Then $\mathbf{M}_j(x,t,z)$ satisfy the jump conditions

$$\mathbf{M}^+(x,t,z) = \mathbf{M}^-(x,t,z)[\mathbf{I} - \mathrm{e}^{i\mathbf{\Theta}(x,t,z)}\mathbf{L}(z)\mathrm{e}^{-i\mathbf{\Theta}(x,t,z)}], \quad z \in \Sigma, \qquad (4.2)$$

where $\mathbf{M} = \mathbf{M}^+$ for $z \in D^+ = D_1 \cup D_3$ and $\mathbf{M} = \mathbf{M}^-$ for $z \in D^- = D_2 \cup D_4$ (namely, $\mathbf{M}^+ = \mathbf{M}_1$ for $z \in D_1$, $\mathbf{M}^+ = \mathbf{M}_3$ for $z \in D_3$, $\mathbf{M}^- = \mathbf{M}_2$ for $z \in D_2$, and $\mathbf{M}^- = \mathbf{M}_4$ for $z \in D_4$) and where the superscripts $\pm$ denote, respectively, projections from the left and the right of the appropriate contour in the complex z-plane. Here $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, where Σ_j is the boundary of

$\overline{D}_j \cap \overline{D}_{j+1 \mod 4}$ *(oriented so that D^+ is always to its left), and the matrix $\mathbf{L}(x,t,z)$ is given on each portion of the contour as*

$$\mathbf{L}(z) = \begin{pmatrix} \frac{iq_o}{z}\rho_1\hat{\rho}_3 - R_2 - \rho_1 R_2\rho_3 & \frac{iq_o}{z}\hat{\rho}_3 + R_2\rho_3 & -R_2 - \frac{iq_o}{z}\gamma\hat{\rho}_3 R_3 - \gamma R_2\rho_3 R_3 \\ -\rho_1 & 0 & \gamma R_3 \\ -\rho_1 - \rho_2 & \rho_3 & -\gamma\rho_3 R_3 \end{pmatrix}, \quad z \in \Sigma_1,$$

$$\mathbf{L}(z) = \begin{pmatrix} \hat{R}_2 R_2 & 0 & -R_2 \\ 0 & 0 & 0 \\ \hat{R}_2 & 0 & 0 \end{pmatrix}, \quad z \in \Sigma_2,$$

$$\mathbf{L}(z) = \begin{pmatrix} -\hat{\rho}_1\hat{R}_2 & \hat{\rho}_2\rho_3 - \frac{iq_o}{z}\hat{\rho}_3 & \hat{\rho}_2 \\ \frac{iz}{q_o}\hat{\rho}_1\hat{R}_2 - \frac{iz}{q_o}\gamma\hat{R}_3 - \gamma\hat{\rho}_2\hat{R}_2 R_3 & \gamma\left[\hat{\rho}_3\hat{R}_3 + \hat{\rho}_2\rho_3 R_3\right] & \gamma\hat{\rho}_2 R_3 \\ \hat{R}_2 & -\rho_3 & 0 \end{pmatrix}, \quad z \in \Sigma_3,$$

$$\mathbf{L}(z) = \begin{pmatrix} \rho_2\hat{\rho}_2 & 0 & \hat{\rho}_2 \\ \frac{iq_o\gamma}{z}\hat{R}_3[1-\rho_2\hat{\rho}_2] - \rho_1 + \frac{iz}{q_o}\hat{\rho}_1\rho_2 & 0 & \gamma[\rho_3 - \frac{iq_o}{z}\hat{\rho}_2\hat{R}_3] - \frac{iz}{q_o}\hat{\rho}_1 \\ -\rho_2 & 0 & 0 \end{pmatrix}, \quad z \in \Sigma_4,$$

where $\rho_j = \rho_j(z)$ and $R_j = \rho_j^*(z^*)$ for $j = 1,\dots,4$, and where the circumflex accent denotes evaluation at $\hat{z} = -q_o^2/z$.

The various sections of the contour are illustrated in Fig. 1 (right). In order for the above RHP to admit a unique solution, one must also specify a suitable normalization condition. In this case, this condition is provided by the leading-order asymptotic behavior of $\mathbf{M}^\pm$ as $z \to \infty$ and the pole contribution at 0 to help regularize the RHP (4.2). More precisely, using the results from Section 3.3 together with the definitions in (4.1), we have that

$$\mathbf{M}(x,t,z) = \mathbf{M}_\infty + O(1/z), \qquad z \to \infty, \tag{4.3a}$$

$$\mathbf{M}(x,t,z) = (i/z)\mathbf{M}_0 + O(1), \qquad z \to 0, \tag{4.3b}$$

where

$$\mathbf{M}_\infty = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{0} & \mathbf{q}_+^\perp/q_o & \mathbf{q}_+/q_o \end{pmatrix}, \quad \mathbf{M}_0 = \begin{pmatrix} 0 & 0 & q_o \\ \mathbf{q}_+ & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{4.4}$$

Note that each limit is expressed in terms of the asymptotic behavior of the potential as $x \to \infty$ (instead of $x \to -\infty$). This is because the definition of $\mathbf{M}(x,t,z)$ in (4.1) breaks the symmetry between the limits $x \to \infty$ and $x \to -\infty$. In addition, note that $\mathbf{M}_\infty + (i/z)\mathbf{M}_0 = \mathbf{E}_+(z)$. This is analogous to what happens in the scalar case.

In addition to the asymptotics in (4.3), to fully specify the RHP (4.2) one must also specify residue conditions. This is done using the characterization of the discrete spectrum obtained in section 3.1, where we also assumed that all discrete eigenvalues are simple. As a result of this assumption, the poles of the Riemann-Hilbert problem at the discrete eigenvalues are all simple. For brevity, we denote by $\mathbf{M}^\pm_{-1,w}(x,t)$ the residue of $\mathbf{M}^\pm$ at $z = w$. Also, we introduce the notation $\mathbf{M}^\pm = (m_1^\pm, m_2^\pm, m_3^\pm)$. In what follows, we must be careful to remember the piecewise definitions of $\mathbf{M}^\pm$ from Lemma 4.1. Then in Appendix A.8 we prove:

Lemma 4.2. *The meromorphic matrices defined in Lemma 4.1 satisfy the following residue conditions:*

$$\mathbf{M}^+_{-1,w_n}(x,t) = C_n\left(m_3^+(w_n), \mathbf{0}, \mathbf{0}\right), \quad \mathbf{M}^-_{-1,w_n^*}(x,t) = \hat{C}_n\left(\mathbf{0}, \mathbf{0}, m_1^-(w_n^*)\right), \tag{4.5a}$$

$$\mathbf{M}^+_{-1,\hat{w}_n^*}(x,t) = -\frac{iw_n^*}{q_o}\check{C}_n\left(m_1^-(w_n^*), \mathbf{0}, \mathbf{0}\right), \; \mathbf{M}^-_{-1,\hat{w}_n}(x,t) = -\frac{iw_n}{q_o}\bar{C}_n\left(\mathbf{0}, \mathbf{0}, m_3^+(w_n)\right), \tag{4.5b}$$

$$\mathbf{M}^{+}_{-1,z_n}(x,t) = D_n\left(\mathbf{0},\mathbf{0},m_2^+(z_n)\right), \quad \mathbf{M}^{-}_{-1,z_n^*}(x,t) = \hat{D}_n\left(\mathbf{0},m_3^-(z_n^*),\mathbf{0}\right), \tag{4.5c}$$

$$\mathbf{M}^{+}_{-1,\hat{z}_n^*}(x,t) = -\frac{iz_n^*}{q_o}\check{D}_n\left(\mathbf{0},m_3^-(z_n^*),\mathbf{0}\right), \quad \mathbf{M}^{-}_{-1,\hat{z}_n}(x,t) = \bar{D}_n\left(m_2^+(z_n),\mathbf{0},\mathbf{0}\right), \tag{4.5d}$$

$$\mathbf{M}^{+}_{-1,\zeta_n}(x,t) = F_n\left(m_2^+(\zeta_n),\mathbf{0},\mathbf{0}\right), \quad \mathbf{M}^{-}_{-1,\zeta_n^*}(x,t) = \hat{F}_n\left(\mathbf{0},m_1^-(\zeta_n^*),\mathbf{0}\right), \tag{4.5e}$$

$$\mathbf{M}^{+}_{-1,\hat{\zeta}_n^*}(x,t) = -\frac{i\zeta_n^*}{q_o}\check{F}_n\left(\mathbf{0},m_1^-(\zeta_n^*),\mathbf{0}\right), \quad \mathbf{M}^{+}_{-1,\hat{\zeta}_n}(x,t) = \bar{F}_n\left(\mathbf{0},\mathbf{0},m_2^+(\zeta_n)\right), \tag{4.5f}$$

with norming constants

$$C_n(x,t) = \frac{c_n}{a'_{11}(w_n)}\mathrm{e}^{-2i\theta_1(w_n)}, \quad \hat{C}_n(x,t) = \frac{\hat{c}_n}{b'_{11}(w_n^*)}\mathrm{e}^{2i\theta_1(w_n^*)}, \tag{4.6a}$$

$$\check{C}_n(x,t) = \frac{\check{c}_n}{b'_{33}(\hat{w}_n^*)}\mathrm{e}^{2i\theta_1(w_n^*)}, \quad \overline{C}_n(x,t) = \frac{\overline{c}_n b_{22}(w_n)}{a'_{33}(\hat{w}_n)}\mathrm{e}^{-2i\theta_1(w_n)}, \tag{4.6b}$$

$$D_n(x,t) = \frac{d_n}{b'_{22}(z_n)}\mathrm{e}^{i(\theta_1+\theta_2)(z_n)}, \quad \hat{D}_n(x,t) = \frac{\hat{d}_n b_{11}(z_n^*)}{a'_{22}(z_n^*)}\mathrm{e}^{-i(\theta_1+\theta_2)(z_n^*)}, \tag{4.6c}$$

$$\check{D}_n(x,t) = \frac{\check{d}_n b_{11}(z_n^*)}{a'_{22}(\hat{z}_n^*)}\mathrm{e}^{-i(\theta_1+\theta_2)(z_n^*)}, \quad \overline{D}_n(x,t) = \frac{\overline{d}_n}{b'_{22}(\hat{z}_n)}\mathrm{e}^{i(\theta_1+\theta_2)(z_n)}, \tag{4.6d}$$

$$F_n(x,t) = \frac{f_n}{a'_{11}(\zeta_n)}\mathrm{e}^{-i(\theta_1-\theta_2)(\zeta_n)}, \quad \hat{F}_n(x,t) = \frac{\hat{f}_n}{a'_{22}(\zeta_n^*)}\mathrm{e}^{i(\theta_1-\theta_2)(\zeta_n^*)}, \tag{4.6e}$$

$$\check{F}_n(x,t) = \frac{\check{f}_n}{a'_{22}(\hat{\zeta}_n^*)}\mathrm{e}^{i(\theta_1-\theta_2)(\zeta_n^*)}, \quad \bar{F}_n(x,t) = \frac{\bar{f}_n}{a'_{33}(\hat{\zeta}_n)}\mathrm{e}^{-i(\theta_1-\theta_2)(\zeta_n)}, \tag{4.6f}$$

where the (x,t)-dependence was omitted from the right-hand sides of all equations for simplicity and where $n = 1,\dots,N_1$ for equations involving w_n, $n = 1,\dots,N_2$ for equations involving z_n, and $n = 1,\dots,N_3$ for equations involving ζ_n.

It is important to realize that the norming constants in (4.6) are not all independent. More precisely, the symmetries of the norming constants combined with the symmetries of the scattering matrix yield:

Lemma 4.3. *The norming constants in Theorem 4.4 obey the following symmetry relations:*

$$\hat{C}_n(x,t) = -C_n^*(x,t), \quad \check{C}_n(x,t) = -\frac{q_o^2}{(w_n^*)^2}C_n^*(x,t),$$
$$\overline{C}_n(x,t) = \frac{q_o^2}{w_n^2}C_n(x,t), \quad \hat{D}_n(x,t) = -\frac{D_n^*(x,t)}{\gamma(z_n^*)},$$
$$\check{D}_n(x,t) = -\frac{iq_o^3}{(z_n^*)^3}\frac{D_n^*(x,t)}{\gamma(z_n^*)}, \quad \overline{D}_n(x,t) = -\frac{iq_o}{z_n}D_n(x,t),$$
$$\bar{F}_n(x,t) = -\frac{iq_o}{\zeta_n}F_n(x,t), \quad \check{F}_n(x,t) = -\frac{iq_o^3}{(\zeta_n^*)^3}\frac{F_n^*(x,t)}{\gamma(\zeta_n^*)}, \quad \hat{F}_n(x,t) = -\frac{F_n^*(x,t)}{\gamma(\zeta_n^*)}.$$

4.2. Solution of the Riemann-Hilbert problem

The RHP defined in the previous section can be formally solved by converting it into a mixed system of algebraic-integral equations by subtracting the asymptotic behavior at infinity, by regularizing (i.e., subtracting any pole contributions from the discrete spectrum), and then applying Cauchy projectors. In this way, in Appendix A.8 we prove:

Theorem 4.4. *The solution of the RHP defined by (4.3) and Lemmas 4.1 and 4.2 is given by the system of matrix algebraic-integral equations*

$$\mathbf{M}(x,t,z) = \mathbf{E}_+(z) + \sum_{n=1}^{N} \left(\frac{\mathbf{M}^+_{-1,\upsilon_n}}{z-\upsilon_n} + \frac{\mathbf{M}^-_{-1,\upsilon_n^*}}{z-\upsilon_n^*} + \frac{\mathbf{M}^+_{-1,\hat{\upsilon}_n^*}}{z-\hat{\upsilon}_n^*} + \frac{\mathbf{M}^-_{-1,\hat{\upsilon}_n}}{z-\hat{\upsilon}_n} \right) - \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{M}^-(\zeta)}{\zeta - z} \overline{\mathbf{L}}(\zeta) \mathrm{d}\zeta, \quad (4.7)$$

where $\{\upsilon_n\}_{n=1}^N$ *denotes the set of all discrete eigenvalues,* $\overline{\mathbf{L}} = \mathrm{e}^{i\mathbf{\Theta}}\mathbf{L}\mathrm{e}^{-i\mathbf{\Theta}}$*, and* $\mathbf{M}(x,t,z) = \mathbf{M}^\pm(x,t,z)$ *for* $z \in D^\pm$*. Moreover, the eigenfunctions in the residue conditions (4.5) are given by*

$$m_2^+(x,t,z) = \begin{pmatrix} 0 \\ \frac{1}{q_o}\mathbf{q}_+^\perp \end{pmatrix} + \sum_{n=1}^{N_2} \left[\frac{\hat{D}_n}{z-z_n^*} - \frac{iz_n^*}{q_o} \frac{\check{D}_n}{z-\hat{z}_n^*} \right] m_3^-(z_n^*) + \sum_{n=1}^{N_3} \left[\frac{\hat{F}_n}{z-\zeta_n^*} - \frac{i\zeta_n^*}{q_o} \frac{\check{F}_n}{z-\hat{\zeta}_n^*} \right] m_1^-(\zeta_n^*)$$
$$- \frac{1}{2\pi i} \int_\Sigma \frac{(\mathbf{M}^-\overline{\mathbf{L}})_2(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z = z_{i'}, \zeta_{\ell'}, \qquad (4.8a)$$

$$m_3^-(x,t,z) = \begin{pmatrix} \frac{iq_o}{z} \\ \frac{1}{q_o}\mathbf{q}_+ \end{pmatrix} + \sum_{n=1}^{N_1} \left[\frac{\hat{C}_n m_1^-(w_n^*)}{z-w_n^*} - \frac{iw_n}{q_o} \frac{\overline{C}_n m_3^+(w_n)}{z-\hat{w}_n} \right] + \sum_{n=1}^{N_2} \frac{D_n m_2^+(z_n)}{z-z_n}$$
$$+ \sum_{n=1}^{N_3} \frac{\bar{F}_n m_2^+(\zeta_n)}{z-\hat{\zeta}_n} - \frac{1}{2\pi i} \int_\Sigma \frac{(\mathbf{M}^-\overline{\mathbf{L}})_3(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z = z_{i'}, \qquad (4.8b)$$

$$m_1^-(x,t,z) = \begin{pmatrix} 1 \\ \frac{i}{z}\mathbf{q}_+ \end{pmatrix} + \sum_{n=1}^{N_1} \left[\frac{C_n m_3^+(w_n)}{z-w_n} - \frac{iw_n^*}{q_o} \frac{\check{C}_n m_1^-(w_n^*)}{z-\hat{w}_n^*} \right] + \sum_{n=1}^{N_2} \frac{\overline{D}_n m_2^+(z_n)}{z-\hat{z}_n} + \sum_{n=1}^{N_3} \frac{F_n m_2^+(\zeta_n)}{z-\zeta_n}$$
$$- \frac{1}{2\pi i} \int_\Sigma \frac{(\mathbf{M}^-\overline{\mathbf{L}})_1(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z = \zeta_{\ell'}^*, w_{j'}^*, \qquad (4.8c)$$

$$m_3^+(x,t,z) = \begin{pmatrix} \frac{iq_o}{z} \\ \frac{1}{q_o}\mathbf{q}_+ \end{pmatrix} + \sum_{n=1}^{N_1} \left[\frac{\hat{C}_n m_1^-(w_n^*)}{z-w_n^*} - \frac{iw_n}{q_o} \frac{\overline{C}_n m_3^+(w_n)}{z-\hat{w}_n} \right] + \sum_{n=1}^{N_2} \frac{D_n m_2^+(z_n)}{z-z_n}$$
$$+ \sum_{n=1}^{N_3} \frac{\bar{F}_n m_2^+(\zeta_n)}{z-\hat{\zeta}_n} - \frac{1}{2\pi i} \int_\Sigma \frac{(\mathbf{M}^-\overline{\mathbf{L}})_3(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z = w_{j'}, \qquad (4.8d)$$

where $i' = 1,\ldots,N_1$*,* $j' = 1,\ldots,N_2$*, and* $\ell' = 1,\ldots,N_3$ *and where for brevity the* (x,t)*-dependence was omitted in the right-hand side of (4.7)–(4.8).*

A question that can be considered at this point is that of identifying conditions on the scattering data that guarantee the existence and uniqueness of solutions of the above system of equations in Theorem 4.4. These questions can be addressed using similar techniques as in the defocusing case, which was discussed in detail in [9] (even though the vanishing lemmas for the two cases are different). The upshot is that, notwithstanding the larger size of the RHP and the fact that the residue conditions are more involved, the issue of existence and uniqueness of solutions for the focusing and defocusing vector cases is essentially the same as that of the corresponding scalar cases [4, 5, 33]. We omit the details for brevity.

4.3. Reconstruction formula, trace formulae and asymptotic phase difference

We can now reconstruct the potential in terms of the norming constants and scattering coefficients by examining the solution (4.7) of the regularized RHP. Specifically, the first of Eq. (3.13a) gives the potential in terms of the Jost eigenfunction $\mu_{+,1}(x,t,z)$ (as seen in (A.31)), while the first column of (4.7) with $\mathrm{Im}\, z < 0$ yields an expression for $\mu_{+,1}(x,t,z)$ in terms of the scattering data. We combine this information to find:

Theorem 4.5. *Let* $\mathbf{M}(x,t,z)$ *be the solution of the Riemann-Hilbert problem in Theorem 4.4. The corresponding solution* $\mathbf{q}(x,t) = (q_1(x,t), q_2(x,t))^T$ *of the focusing Manakov system with NZBC (1.2) is reconstructed as*

$$q_k(x,t) = q_{+,k} - i\sum_{j=1}^{N_1}\Big[C_j m^+_{(k+1)3}(w_j) + \check{C}_j m^+_{(k+1)3}(\hat{w}_j^*)\Big] - i\sum_{j=1}^{N_2}\overline{D}_j m^+_{(k+1)2}(\hat{z}_j)$$
$$- i\sum_{j=1}^{N_3} F_j m^+_{(k+1)2}(\zeta_j) - \frac{1}{2\pi}\sum_{j=1}^{4}\int_{\Sigma_j}(\mathbf{M}^-\overline{\mathbf{L}}_j)_{(k+1)1}(\zeta)\mathrm{d}\zeta, \qquad k = 1,2, \quad (4.9)$$

where again the (x,t)*-dependence on the right hand side was omitted for brevity.*

The last task in the inverse problem is the derivation of the trace formulae, namely the reconstruction of the analytic scattering coefficients in terms of the scattering data. This is accomplished by formulating another, appropriate Riemann-Hilbert problem, similar to the one used to find the trace formulae for the defocusing Manakov system [9, 24]. Using this approach, in Appendix A.8 we prove:

Lemma 4.6. *The analytic scattering coefficients are given explicitly by*

$$a_{11}(z) = \exp\left(\frac{1}{2\pi i}\int_{\Sigma}\frac{J(\zeta)}{\zeta - z}\mathrm{d}\zeta\right)\prod_{n=1}^{N_1}\frac{z-w_n}{z-w_n^*}\frac{z-\hat{w}_n^*}{z-\hat{w}_n}\prod_{n=1}^{N_2}\frac{z-\hat{z}_n}{z-\hat{z}_n^*}\prod_{n=1}^{N_3}\frac{z-\zeta_n}{z-\zeta_n^*}, \qquad (4.10a)$$

$$b_{22}(z) = \exp\left(-i\Delta\theta - \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{J_o(\zeta)}{\zeta - z}\mathrm{d}\zeta\right)\prod_{n=1}^{N_2}\frac{z-z_n}{z-z_n^*}\frac{z-\hat{z}_n}{z-\hat{z}_n^*}\prod_{n=1}^{N_3}\frac{z-\zeta_n}{z-\zeta_n^*}\frac{z-\hat{\zeta}_n}{z-\hat{\zeta}_n^*}, \qquad (4.10b)$$

where the reflection coefficients $\rho_j(z)$ $(j = 1,2,3)$ *are as defined in (2.57) and the jump conditions* $J_o(z)$ *and* $J(z)$ *are given in (A.40), (A.41), and (A.42).*

Trace formulae for the remaining analytic scattering coefficients follow trivially from the symmetries of the scattering matrix [i.e., the symmetries (2.46) and (2.55)]. It is important to note that in the reflectionless case, the integrals in (4.10a) and (4.10b) vanish. Note that the trace formula for $a_{11}(z)$ here includes a contribution from the eigenvalues of type 2, even though $a_{22}(z_n) \neq 0$ for all eigenvalues z_n of type 2. This is in contrast to the defocusing Manakov system, where the trace formulae were much simpler, and is a result of the existence of four fundamental domains of analyticity instead of two [9, 24].

Next, letting $z \to 0$ in (4.10a) and comparing with the asymptotics in Corollary 3.12 yields an expression for the asymptotic phase difference $\Delta\theta = \theta_+ - \theta_-$ for the BC (1.2) of the potential:

Corollary 4.7. *The asymptotic phase difference* $\Delta\theta = \theta_+ - \theta_-$ *is given by*

$$\Delta\theta = \frac{1}{2\pi}\int_{\Sigma}\frac{J(\zeta)}{\zeta}\mathrm{d}\zeta - 4\sum_{n=1}^{N_1}\arg w_n + 2\sum_{n=1}^{N_2}\arg z_n - 2\sum_{n=1}^{N_3}\arg\zeta_n. \qquad (4.11)$$

5. Reflectionless potentials and exact soliton solutions

We now look at potentials $\mathbf{q}(x,t)$ for which there is no jump across the continuum spectrum. In this case, the reflection coefficients (2.57) vanish identically, implying that $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are diagonal matrices and that the inverse problem reduces to an algebraic system (namely, equations (4.8a)-(4.8d) without the integrals) whose solution yields the soliton solutions of the integrable nonlinear equation. We will again make use of the assumption made in section 3.1 that every discrete eigenvalue is simple.

Theorem 5.1. *In the reflectionless case, the solution (4.9) of the focusing Manakov system with NZBC may be written*

$$\mathbf{q}(x,t) = \frac{1}{\det \mathbf{G}} \begin{pmatrix} \det \mathbf{G}_1^{\mathrm{aug}} \\ \det \mathbf{G}_2^{\mathrm{aug}} \end{pmatrix}, \qquad k = 1,2,$$

where $\mathbf{G} = \mathbf{I} - \mathbf{F}$, *the augmented matrix* $\mathbf{G}^{\mathrm{aug}}$ *is*

$$\mathbf{G}_k^{\mathrm{aug}} = \begin{pmatrix} q_{+,k} & \mathbf{y}^T \\ \mathbf{b}_k & \mathbf{G} \end{pmatrix},$$

the vectors $\mathbf{b}_k$ *and* $\mathbf{y}$ *are*

$$\mathbf{b}_k = (b_{k1}, \ldots, b_{k(2N_1+N_2+N_3)})^T, \quad \mathbf{y} = (y_1, \ldots, y_{2N_1+N_2+N_3})^T,$$

and the entries F_{ij}, b_{kj}, *and* y_j *are given by (A.45)–(A.48) in Appendix A.10.*

In addition, the trace formulae have simpler expressions in the reflectionless case. Specifically, as mentioned before, the integrals in (4.10a) and (4.10b) vanish identically in the reflectionless case, and we obtain:

$$a_{11}(z) = \prod_{n=1}^{N_1} \frac{z - w_n}{z - w_n^*} \frac{z - \hat{w}_n^*}{z - \hat{w}_n} \prod_{n=1}^{N_2} \frac{z - \hat{z}_n}{z - \hat{z}_n^*} \prod_{n=1}^{N_3} \frac{z - \zeta_n}{z - \zeta_n^*}, \tag{5.1a}$$

$$b_{22}(z) = \mathrm{e}^{-i\Delta\theta} \prod_{n=1}^{N_2} \frac{z - z_n}{z - z_n^*} \frac{z - \hat{z}_n}{z - \hat{z}_n^*} \prod_{n=1}^{N_3} \frac{z - \zeta_n}{z - \zeta_n^*} \frac{z - \hat{\zeta}_n}{z - \hat{\zeta}_n^*}, \tag{5.1b}$$

where, as before, N_1, N_2, N_3 denote respectively the number of discrete eigenvalues of type I, type II and type III. The framework is now in place for the construction of explicit soliton solutions. We construct explicit solutions for each of the three different types of eigenvalues and examine their various properties. In doing so, we will be able to clearly see the similarities and differences among the three types of eigenvalues.

5.1. Type I solitons

Here, we assume $N_1 = 1$ and $N_2 = N_3 = 0$. We may also assume without any loss of generality that $\mathbf{q}_+ = (1,0)^T$. In addition, suppose

$$c_1 = \mathrm{e}^{\xi + i\phi}, \quad \xi, \phi \in \mathbb{R}, \qquad w_1 = iZ\mathrm{e}^{i\alpha}, \quad Z > 1, \quad \alpha \in (-\pi/2, \pi/2). \tag{5.2}$$

We then use Theorem 5.1 to find the following explicit solution:

$$\mathbf{q}(x,t) = \frac{\cosh[U + 2i\alpha] + \frac{1}{2}A[c_{+2}(Z^2 \sin(s+2\alpha) - \sin s) - ic_{-2}(Z^2 \cos(s+2\alpha) - \cos s)]}{\cosh U + A[Z^2 \sin(s+2\alpha) - \sin s]} \mathbf{q}_+,$$

where

$$U(x,t) = c_- x \cos\alpha - c_{+2} t \sin(2\alpha) + c_o' + \xi,$$
$$s(x,t) = c_+ x \sin\alpha + c_{-2} t \cos(2\alpha) + \phi,$$

and where

$$c_\pm = Z \pm 1/Z, \quad c_{+2} = Z^2 + 1/Z^2, \quad c_{-2} = Z^2 - 1/Z^2 = c_+ c_-, \quad A = 1/(c_+' c_-'),$$
$$c_o' = \log(c_+'/c_-'), \qquad c_+' = |1 - Z^2 \mathrm{e}^{-2i\alpha}|, \qquad c_-' = (Z + 1/Z)/(2\cos\alpha).$$

An example of such a solution is shown in Fig. 2. The properties of these solutions are discussed in section 5.4.

5.2. Type II solitons

Here, we assume $N_2 = 1$ and $N_1 = N_3 = 0$. In addition, suppose

$$d_1 = \mathrm{e}^{\xi + i\phi}, \quad \xi, \phi \in \mathbb{R}, \qquad z_1 = Z\mathrm{e}^{i\alpha}, \quad Z > q_o, \quad \alpha \in (0, \pi).$$

We then use Theorem 5.1 to find the following explicit solution:

$$\mathbf{q}(x,t) = [\cos\alpha + i\sin\alpha\tanh V(x,t)]\,\mathrm{e}^{-i\alpha}\mathbf{q}_+ - V_o \operatorname{sech} V(x,t)\,\mathrm{e}^{-\frac{iq_o^2}{Z}x\cos\alpha - \frac{iq_o^4}{Z^2}t\cos(2\alpha)}\mathbf{q}_+^\perp,$$

where

$$V(x,t) = \frac{q_o^2}{Z}x\sin\alpha + \frac{q_o^4}{Z^2}t\sin(2\alpha) - \frac{1}{2}\log\gamma(Z) - \xi, \qquad V_o = i\sqrt{\gamma(Z)}\,\sin\alpha\,\mathrm{e}^{i\alpha + i\phi}.$$

It is straightforward to see both that the dark part of the solution achieves its minimum and the bright part of the solution achieves its maximum when $V(x,t) = 0$. These values are, respectively,

$$q_{\text{dark,min}} = q_o|\cos\alpha|, \qquad q_{\text{bright,max}} = q_o\sqrt{\gamma(Z)}\,|\sin\alpha|.$$

An example of such a solution is shown in Fig. 3. As with solutions of type I, these solutions are discussed in section 5.4.

5.3. Type III solitons

Here, we assume $N_1 = N_2 = 0$ and $N_3 = 1$. In addition, suppose

$$f_1 = \mathrm{e}^{\xi + i\phi}, \quad \xi, \phi \in \mathbb{R}, \qquad \zeta_1 = Z\mathrm{e}^{i\alpha}, \quad Z > q_o, \quad \alpha \in (0, \pi). \tag{5.3}$$

We then use Theorem 5.1 to find the following explicit solution:

$$\mathbf{q}(x,t) = [\cos\alpha - i\sin\alpha\tanh W(x,t)]\,\mathrm{e}^{i\alpha}\mathbf{q}_+ - W_o \operatorname{sech} W(x,t)\,\mathrm{e}^{iZx\cos\alpha - iZ^2t\cos(2\alpha)}\mathbf{q}_+^\perp,$$

where

$$W(x,t) = Z\sin\alpha(x - 2Zt\cos\alpha) + \log\sqrt{\gamma(Z)} - \xi, \qquad W_o = -(Z/q_o)\sqrt{\gamma(Z)}\,\sin\alpha\,\mathrm{e}^{i\phi}.$$

It is straightforward to see both that the dark part of the solution achieves its minimum and the bright part of the solution achieves its maximum along the straight line $W(x,t) = 0$. These values are, respectively,

$$q_{\text{dark,min}} = q_o|\cos\alpha|, \qquad q_{\text{bright,max}} = Z\sqrt{\gamma(Z)}\,|\sin\alpha|.$$

An example of such a solution is shown in Fig. 4.

5.4. Discussion

Solutions of type I are the trivial vectorization of the bright soliton solutions of the scalar NLS equation with NZBC [8]. Solutions of type II and type III have a bright component in addition to the expected dark component (which is required to connect the asymptotic boundary values), as seen in Fig. 3 and Fig. 4. These solutions are the analogue of the dark-bright soliton solutions of the defocusing Manakov system with NZBC [9]. Note, however,

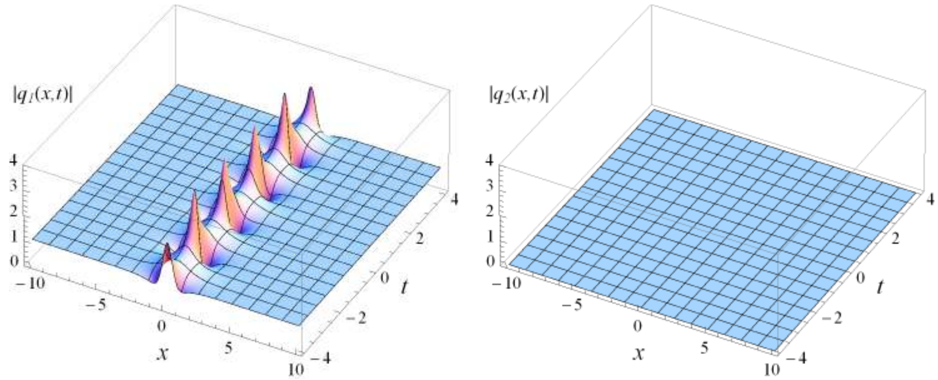

Figure 2. One bright soliton solution of the focusing Manakov system obtained by taking $N_1 = 1$, $N_2 = N_3 = 0$, $\mathbf{q}_+ = (1,0)^T$, $w_1 = 2e^{i\pi/2}$.

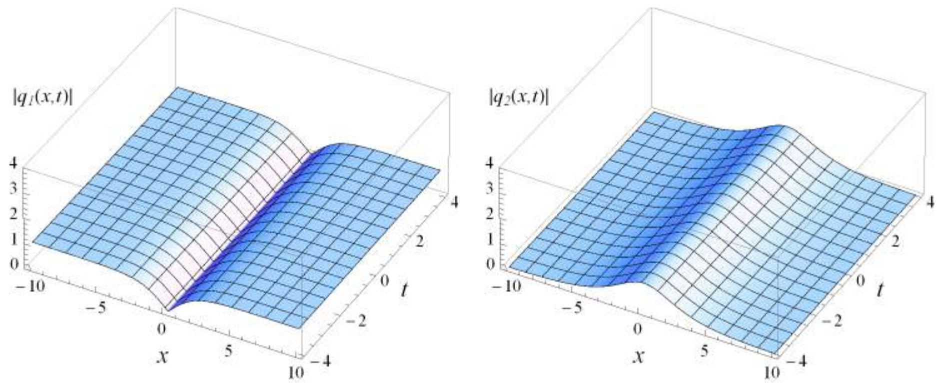

Figure 3. One dark-bright soliton solution of the focusing Manakov system obtained by taking $N_2 = 1$, $N_1 = N_3 = 0$, $\mathbf{q}_+ = (1,0)^T$, $z_1 = 2e^{i\pi/2}$.

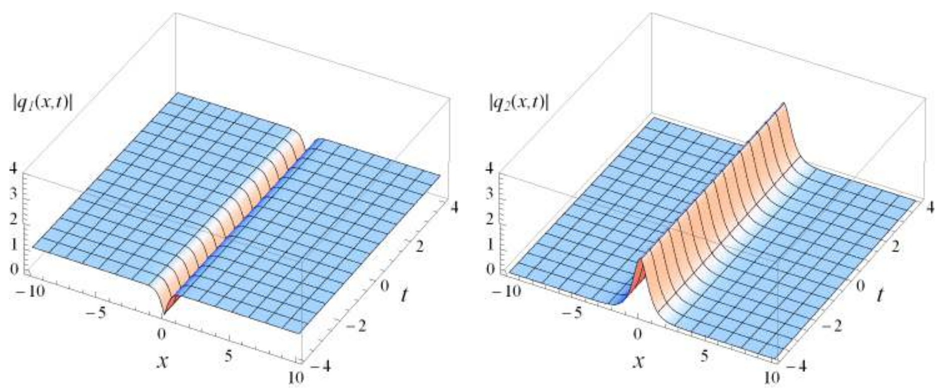

Figure 4. One dark-bright soliton solution of the focusing Manakov system obtained by taking $N_3 = 1$, $N_1 = N_2 = 0$, $\mathbf{q}_+ = (1,0)^T$, $\zeta_1 = 2e^{i\pi/2}$.

that while in the defocusing case only one kind of dark-bright soliton exists, here two types of dark-bright solutions are possible. Note that the two types are distinct by the fact that, while the minimum of the dark component is the same in both cases, the maximum of the bright

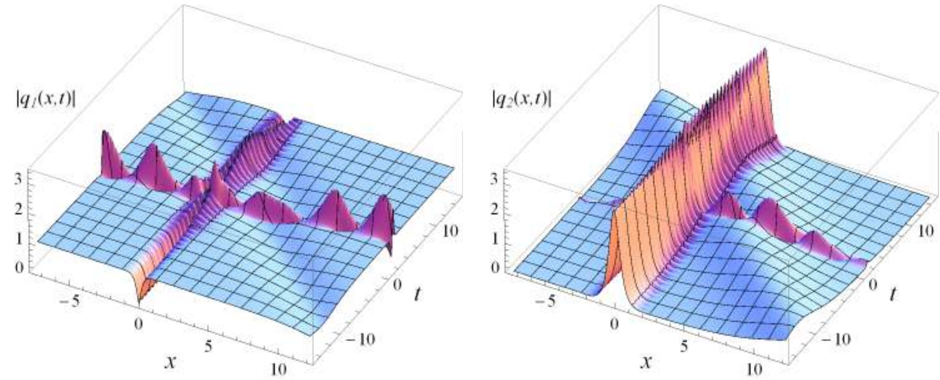

Figure 5. A one bright, two dark-bright soliton solution of the focusing Manakov system obtained by taking $N_1 = N_2 = N_3 = 1$, $\mathbf{q}_+ = (1,0)^T$, $w_1 = 2.5e^{i\pi/4}$, $z_1 = 1.1e^{i\pi/4}$, $\zeta_1 = 3e^{i\pi/2}$.

component takes on different values. Namely, for solutions of type II one has

$$|q_1|_{\min} = q_o\,|\cos\alpha|\,, \qquad |q_2|_{\max} = q_o\,\sqrt{\gamma(Z)}\,|\sin\alpha|\,, \tag{5.4}$$

while for solutions of type III one has

$$|q_1|_{\min} = q_o\,|\cos\alpha|\,, \qquad |q_2|_{\max} = Z\,\sqrt{\gamma(Z)}\,|\sin\alpha|\,. \tag{5.5}$$

On the other hand, it is straightforward to see that solutions of type III can be obtained formally by taking the analytic continuation of solutions of type II when the eigenvalue $Z e^{i\alpha}$ is taken inside the circle of radius q_o (upon proper redefinition of the norming constants). While such a situation is not strictly allowed due to the restrictions on the analyticity properties of the eigenfunctions and scattering coefficients, such an extension is allowed by the final result for the soliton solution. Of course, the algebraic system discussed earlier allows one to also easily construct multi-soliton solutions containing a combination of any number of the three types of solitons.

An example of such a solution is shown in Fig. 5, which describes the interaction between two dark-bright solitons (whose dark and bright profiles are in the form of a traveling wave solution) and a bright soliton (whose dark and bright profiles are in the form of a breather-type solution). Note how the bright soliton experiences a polarization shift as an effect of the interaction, resulting in a redistribution of energy between the two components, similarly to what happens in the focusing case with zero BC [22].

6. Conclusions

We have developed the IST for the focusing Manakov system with NZBC, and we have shown that the problem is significantly more complex than its defocusing counterpart. In particular, we have seen that the discrete spectrum yields three types of discrete eigenvalues, each corresponding to a different type of soliton solutions. We expect the results of this paper to be useful in characterizing recent experiments in nonlinear optics [11, 16, 27] and Bose-Einstein condensation [19, 30].

It should be noted that the soliton solutions presented in this work might be unstable to long-wavelength perturbations due to the modulational instability of the constant background.

On the other hand, some recent studies suggest that some of the soliton solutions themselves may be the vehicle for the instability [17]. If that scenario is correct, the same phenomenon might also play out in the vector case. In any case, the results of this work provide a framework that can be used to study the stability of the soliton solutions, which is still an open question even in the scalar case.

More in general, a characterization of the nonlinear stage of modulational instability is still by and large an open problem even in the scalar case. In the case of the scalar NLS equation with periodic BC, the modulational instability can be attributed to the existence of homoclinic solutions [1, 28]. On the other hand, even in the case of periodic BC there is no characterization of what is the long-time behavior of the solutions to the best of our knowledge. Also, there is no reason to expect that the instability mechanisms in the case of periodic BC and on the whole line will be the same. (For example, while a threshold for instability exists in the case of periodic BC, no such threshold exists on the whole line.) Finally, we emphasize that, since the linearization ceases to be valid as soon as the perturbations have become comparable to the background, the IST is the only tool with which one can hope to characterize the nonlinear stage of the modulational instability. Indeed, in [7] we were able to precisely identify the mechanism for instability within the context of the IST for the focusing NLS equation with NZBC, using the framework that we had developed in [9]. It is hoped that those results will enable researchers to answer the above questions about soliton stability and nonlinear stage of modulational instability, and that the results of this work will provide the tools to do the same in the vector case.

From a theoretical point of view, the results in this paper open the door for studying several open problems: (i) An investigation of the case where the analytic scalar coefficients have double zeros. Soliton solutions corresponding to double poles in the RHP are known to exist in the scalar case with both ZBC and NZBC; (ii) A study of the long-time asymptotics using the Deift-Zhou method [12, 13]; (iii) The development of an appropriate perturbation theory; (iv) The extension of the present approach to the N-component case. Also, a non-trivial technical issue that was omitted for simplicity is a explicit and detailed proof of existence of the solutions of the RHP. We should remark that the initial-value problem for the defocusing N-component coupled nonlinear Schrödinger equation was recently studied in [25] using the approach of [5], where a rigorous construction was given of the fundamental analytic eigenfunctions. We believe that a similar formulation to that of [25] and the approach to the symmetries presented in this work will allow one to construct non-trivial explicit multi-component solutions. We plan to study some of these problems in the near future.

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Appendix

A.1. Analyticity of the eigenfunctions

Proof of Theorem 2.1. We start by rewriting the first of the integral equations (2.20) that define the Jost eigenfunctions:

$$\mu_-(x,t,z) = \mathbf{E}_-(z)\Big[\mathbf{I} + \int_{-\infty}^{x} \mathrm{e}^{i(x-y)\mathbf{\Lambda}(z)}\mathbf{E}_-^{-1}(z)\Delta\mathbf{Q}_-(y,t)\mu_-(y,t,z)\mathrm{e}^{-i(x-y)\mathbf{\Lambda}(z)}\,\mathrm{d}y\Big]. \tag{A.1}$$

The limits of integration imply that $x-y$ is always positive for μ_- (and always negative for μ_+). Also, note that the matrix products on the RHS of (A.1) operate column-wise. In particular, letting $W(x,z) = \mathbf{E}_-^{-1}\mu_-$, for the first column w of W, one has

$$w(x,t,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \mathbf{G}(x-y,z)\Delta\mathbf{Q}_-(y,t)\mathbf{E}_-(z)\, w(y,t,z)\,\mathrm{d}y\,, \tag{A.2}$$

where

$$\mathbf{G}(\xi,z) = \operatorname{diag}\big(1, \mathrm{e}^{i(k(z)+\lambda(z))\xi}, \mathrm{e}^{2i\lambda(z)\xi}\big)\,\mathbf{E}_-^{-1}(z)\,. \tag{A.3}$$

Now, we introduce a Neumann series representation for w:

$$w(x,z) = \sum_{n=0}^{\infty} w^{(n)}\,, \tag{A.4a}$$

with

$$w^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad w^{(n+1)}(x,t,z) = \int_{-\infty}^{x} \mathbf{C}(x,y,t,z)\, w^{(n)}(y,t,z)\,\mathrm{d}y\,, \tag{A.4b}$$

and where $\mathbf{C}(x,y,t,z) = \mathbf{G}(x-y,z)\Delta\mathbf{Q}(y,t)\mathbf{E}_-(z)$. Introducing the L^1 vector norm $\|w\| = |w_1|+|w_2|+|w_3|$ and the corresponding subordinate matrix norm $\|C\|$, we then have

$$\|w^{(n+1)}(x,t,z)\| \le \int_{-\infty}^{x} \|\mathbf{C}(x,y,t,z)\|\,\|w^{(n)}(y,t,z)\|\,\mathrm{d}y\,. \tag{A.5}$$

Note that $\|\mathbf{E}_\pm\| \le 1 + q_o/|z|$ and $\|\mathbf{E}_\pm^{-1}\| \le (1+q_o/|z|)/|\gamma(z)|$. The properties of the matrix norm imply

$$\begin{aligned} \|\mathbf{C}(x,y,t,z)\| &\le \|\operatorname{diag}(1, \mathrm{e}^{i(k(z)+\lambda(z))(x-y)}, \mathrm{e}^{2i\lambda(z)(x-y)})\|\,\|\mathbf{E}_-(z)\|\,\|\Delta\mathbf{Q}(y,t)\|\,\|\mathbf{E}_-^{-1}(z)\| \\ &\le c(z)\,(1 + \mathrm{e}^{-(k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z))(x-y)} + \mathrm{e}^{-2\lambda_{\mathrm{im}}(z)(x-y)})\,\|\mathbf{q}(y,t)-\mathbf{q}_-\|\,, \end{aligned} \tag{A.6}$$

where $\lambda_{\mathrm{im}}(z) = \operatorname{Im}\lambda(z)$, $k_{\mathrm{im}}(z) = \operatorname{Im}k(z)$, and $c(z) = (1+q_o/|z|)^2/|\gamma(z)|$. Now, recall that $\operatorname{Im}\lambda(z) > 0$ for z in D_1. On the other hand, $c(z) \to \infty$ as $z \to \pm iq_o$. Thus, given $\varepsilon > 0$, we restrict our attention to the domain $(D_1)_\varepsilon = D_1 \setminus (B_\varepsilon(iq_o) \cup B_\varepsilon(-iq_o))$, where $B_\varepsilon(z_o) = \{z \in \mathbb{C} : |z - z_o| < \varepsilon q_o\}$. It is straightforward to show that $c_\varepsilon = \max_{z\in(D_1)_\varepsilon} c(z) = 2 + 2/\varepsilon$. Next, we prove that for all $z \in (D_1)_\varepsilon$ and for all $n \in \mathbb{N}$,

$$\|w^{(n)}(x,t,z)\| \le \frac{M^n(x,t)}{n!}\,, \tag{A.7a}$$

where

$$M(x,t) = 3c_\varepsilon \int_{-\infty}^{x} \|\mathbf{q}(y,t)-\mathbf{q}_-\|\,\mathrm{d}y\,. \tag{A.7b}$$

We will prove the result by induction, following [2]. The claim is trivially true for $n = 0$. Also, note that for all $z \in \overline{D_1}$ and for all $y \le x$, one has $1 + \mathrm{e}^{-(k_{\mathrm{im}}(z)+\lambda_{\mathrm{im}}(z))(x-y)} + \mathrm{e}^{-2\lambda_{\mathrm{im}}(x-y)} \le 3$. Then, if (A.7$a$) holds for $n = j$, (A.5) implies

$$\|w^{(j+1)}(x,t,z)\| \le \frac{3c_\varepsilon}{j!} \int_{-\infty}^{x} \|\mathbf{q}(y,t)-\mathbf{q}_-\|\, M^j(y,t)\,\mathrm{d}y = \frac{1}{j!(j+1)} M^{j+1}(x,t)\,, \tag{A.8}$$

proving the induction step (namely, that the validity of (A.7a) for $n = j$ implies its validity for $n = j+1$). Thus, if $\mathbf{q}(x,t) - \mathbf{q}_- \in L^1(-\infty, a]$ for all finite $a \in \mathbb{R}$ and for all $\varepsilon > 0$, then the Neumann series converges absolutely and uniformly with respect to $x \in (-\infty, a)$ and to $z \in (D_1)_\varepsilon$. Similar results hold for $\mu_+(x,t,z)$. Since a uniformly convergent series of analytic functions converges to an analytic function, $\mu_{-,1}(x,t,z)$ is analytic for $z \in D_1$. The rest of the theorem is proved similarly. Note that since $\mathbf{q}_+ \neq \mathbf{q}_-$ in general, $\mathbf{q}(\cdot,t) - \mathbf{q}_- \notin L_1(\mathbb{R})$,and therefore, one cannot take $a = \infty$. This problem can be resolved using an approach similar to [25] or alternatively by deriving a different set of integral equations for the Jost eigenfunctions, as discussed in the following section. Note also that, as in the scalar case, additional conditions need to be imposed on the potential to establish convergence of the Neumann series at the branch points [14]. □

A.2. Alternative integral representation for the Jost eigenfunctions

In order to derive the analyticity properties of the scattering coefficients, we found it necessary to introduce an alternative integral representation for the Jost eigenfunctions. While the resulting equations are slightly more complicated than the standard integral equations (2.20), this representation has the advantage of allowing one to prove explicitly that $\mu_\pm(x,t,z)$ remain bounded for all $x \in \mathbb{R}$ in their regions of analyticity.

We follow a similar approach to that used in Ref. [14] for the defocusing scalar case. Since the scattering matrix is time-independent, it is sufficient to do the calculations at $t = 0$. With this understanding, we omit the time dependence from the potential and the eigenfunctions throughout this subsection.

We first note that the scattering problem (2.17) is equivalent to the following problem:

$$\phi_x = \bar{\mathbf{X}}(x,z)\,\phi + (\mathbf{Q}(x) - \mathbf{Q}_f(x))\,\phi, \tag{A.9}$$

where

$$\bar{\mathbf{X}}(x,z) = H(x)\mathbf{X}_+(z) + H(-x)\mathbf{X}_-(z), \qquad \mathbf{Q}_f(x) = H(x)\mathbf{Q}_+ + H(-x)\mathbf{Q}_-, \tag{A.10}$$

and $H(x)$ denotes the Heaviside function (namely, $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ otherwise). The advantage of using (A.9) instead of (2.17) is that the "forcing" term $\mathbf{Q} - \mathbf{Q}_f$ vanishes both as $x \to -\infty$ and as $x \to \infty$, which leads to integral equations that are better behaved. [Correspondingly, the factorized problem (A.9) is now the same for both ϕ_- and ϕ_+.] For $z \in \Sigma$, we introduce *fundamental eigenfunctions* $\bar{\phi}_\pm(x,z)$ as square matrix solutions of (A.9) satisfying

$$\bar{\phi}_\pm(x,z) = \mathrm{e}^{x\mathbf{X}_\pm(z)}[\mathbf{I} + o(1)], \quad x \to \pm\infty. \tag{A.11}$$

By solving (A.9) in a similar way as (2.20), we obtain

$$\bar{\phi}_-(x,z) = \mathbf{G}_f(x,0,z) + \int_{-\infty}^{x} \mathbf{G}_f(x,y,z)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\bar{\phi}_-(y,z)\,\mathrm{d}y, \tag{A.12a}$$

$$\bar{\phi}_+(x,z) = \mathbf{G}_f(x,0,z) - \int_{x}^{\infty} \mathbf{G}_f(x,y,z)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\bar{\phi}_+(y,z)\,\mathrm{d}y, \tag{A.12b}$$

where $\mathbf{G}_f(x,y,z)$ is the special solution of the homogeneous problem, $\mathbf{G}_x(x,y,z) = \bar{\mathbf{X}}(x,z)\mathbf{G}(x,y,z)$, satisfying the "initial conditions" $\mathbf{G}(x,x,z) = \mathbf{I}$. Namely:

$$\mathbf{G}_f(x,y,z) = \begin{cases} \mathrm{e}^{(x-y)\mathbf{X}_+(z)}, & x,y \geq 0, \\ \mathrm{e}^{(x-y)\mathbf{X}_-(z)}, & x,y \leq 0, \\ \mathrm{e}^{x\mathbf{X}_+(z)}\mathrm{e}^{-y\mathbf{X}_-(z)}, & x,-y \geq 0, \\ \mathrm{e}^{x\mathbf{X}_-(z)}\mathrm{e}^{-y\mathbf{X}_+(z)}, & x,-y \leq 0. \end{cases} \tag{A.13}$$

Using (A.12), we conclude

$$\bar{\phi}_\pm(x,z) = \mathbf{G}_f(x,0,z)\,[\mathbf{A}_\mp(z) + o(1)]\,, \quad x \to \mp\infty, \quad z \in \mathbb{R}, \tag{A.14}$$

where

$$\mathbf{A}_\mp(z) = \mathbf{I} \mp \int_{-\infty}^{\infty} \mathbf{G}_f(0,y,z)\Big[\mathbf{Q}(y) - \mathbf{Q}_f(y)\Big]\bar{\phi}_\pm(y,z)\,\mathrm{d}y. \tag{A.15}$$

Since $\mathrm{e}^{x\mathbf{X}_\pm(z)}$ are bounded for $x \in \mathbb{R}$ when $z \in \Sigma$, the assumption that $\mathbf{Q}(x) - \mathbf{Q}_f(x) \in L^1(\mathbb{R})$ and an application of Gronwall's inequality imply $\bar{\phi}_\pm(x,z)$ are bounded as $x \to \mp\infty$. In addition, comparing (A.13) with the solutions of the asymptotic scattering problem (2.4) yields $\bar{\phi}_\pm(x,z)\mathbf{E}_\pm(z) = \phi_\pm(x,z)$, so (A.12) imply

$$\phi_-(x,z) = \mathbf{G}_f(x,0,z)\mathbf{E}_-(z) + \int_{-\infty}^{x} \mathbf{G}_f(x,y,z)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\phi_-(y,z)\,\mathrm{d}y, \tag{A.16a}$$

$$\phi_+(x,z) = \mathbf{G}_f(x,0,z)\mathbf{E}_+(z) - \int_{x}^{\infty} \mathbf{G}_f(x,y,z)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\phi_+(y,z)\,\mathrm{d}y. \tag{A.16b}$$

Note that (A.16a) coincides with (2.20a) for all $x \le 0$ and (A.16b) coincides with (2.20b) for all $x \ge 0$. Additionally, $\mathbf{q}(x) - \mathbf{q}_\pm \in L^1(\mathbb{R}^\pm)$ implies $\mathbf{Q}(x) - \mathbf{Q}_f(x) \in L^1(\mathbb{R})$, so we can use this information and (A.16) to prove Theorem 2.1 as well as to establish that $\mu_\pm(x,z) = \phi_\pm(x,z)\mathrm{e}^{-ix\mathbf{\Lambda}(z)}$ remain bounded as $x \to \mp\infty$. This result will be instrumental in proving the analyticity of the entries of the scattering matrix (see Theorem 2.4 and the following section).

A.3. Analyticity of the scattering matrix entries

Note first that, for all $z \in \mathbb{C}$,

$$z \in D_1 \Leftrightarrow \mathrm{Im}(k+\lambda), \mathrm{Im}\,\lambda > 0, \quad z \in D_2 \Leftrightarrow \mathrm{Im}(k+\lambda), \mathrm{Im}\,\lambda < 0,$$
$$z \in D_3 \Leftrightarrow \mathrm{Im}(k-\lambda) < 0, \mathrm{Im}\,\lambda > 0, \quad z \in D_4 \Leftrightarrow \mathrm{Im}(k-\lambda) > 0, \mathrm{Im}\,\lambda < 0.$$

The above results can be trivially obtained after noting that $2\,\mathrm{Im}\,\lambda = (1 - q_o^2/|z|^2)\,\mathrm{Im}\,z$, $\mathrm{Im}(k+\lambda) = \mathrm{Im}\,z$, and $\mathrm{Im}(k-\lambda) = (q_o^2/|z|^2)\,\mathrm{Im}\,z$.

Proof of Theorem 2.4. We compare the asymptotics as $x \to \infty$ of $\phi_-(x,z)$ from (A.14) with those of $\phi_+(x,z)\mathbf{A}(z)$ from (2.14) to obtain

$$\mathbf{A}(z) = \mathbf{E}_+^{-1}(z)\mathbf{A}_+(z)\mathbf{E}_-(z). \tag{A.17}$$

The expression in (A.17) simplifies to the following integral representation for the scattering matrix:

$$\begin{aligned}\mathbf{A}(z) = &\int_0^{\infty} \mathrm{e}^{-iy\mathbf{\Lambda}(z)}\mathbf{E}_+^{-1}(z)[\mathbf{Q}(y) - \mathbf{Q}_+]\phi_-(y,z)\,\mathrm{d}y \\ &+ \mathbf{E}_+^{-1}(z)\mathbf{E}_-(z)\Big[\mathbf{I} + \int_{-\infty}^{0} \mathrm{e}^{-iy\mathbf{\Lambda}(z)}\mathbf{E}_-^{-1}(z)[\mathbf{Q}(y) - \mathbf{Q}_-]\phi_-(y,z)\,\mathrm{d}y\Big].\end{aligned} \tag{A.18}$$

A similar expression can be found for $\mathbf{B}(z)$. We can now examine the individual entries of (A.18). In particular, the 1,1 entry of (A.18) yields an integral representation for $a_{11}(z)$, and the corresponding two integrands from (A.18) are, respectively,

$$\frac{\mathrm{e}^{i\lambda y}}{\gamma(z)}\Big[-\frac{i}{z}\mathbf{q}_+^\dagger\Delta\mathbf{q}\,\phi_{-,11} + \Delta r_1\phi_{-,21} + \Delta r_2\phi_{-,31}\Big], \tag{A.19a}$$

$$\sum_{j=1}^{3}\Big[c_{11}T_{1j}+c_{12}T_{2j}(\mathrm{e}^{-i(k+\lambda)y}+c_{13}T_{3j}\mathrm{e}^{-2i\lambda y}\Big]\phi_{-,j1}\mathrm{e}^{i\lambda y}, \tag{A.19b}$$

where $\Delta\mathbf{q}(x) = \mathbf{q}(x)-\mathbf{q}_f(x)$ (similarly for $\Delta\mathbf{r}(x)$) and

$$\mathbf{E}_+^{-1}(z)\mathbf{E}_-(z) = (c_{ij}(z)), \qquad \mathbf{E}_-^{-1}(z)[\mathbf{Q}(y)-\mathbf{Q}_-] = (T_{ij}(y,z)).$$

Recall that $\phi_{-,1}(y,z)\mathrm{e}^{i\lambda(z)y}$ is analytic for $z \in D_1$ and bounded over $y \in \mathbb{R}$, so each term in (A.19a) is analytic for $z \in D_1$ and bounded when $y > 0$. Thus, the first integral in the representation (A.18) for $a_{11}(z)$ defines an analytic function for all $z \in D_1$. Further, recalling that $\operatorname{Im}\lambda$ and $\operatorname{Im}(k+\lambda)$ have the same sign when $z \in D_1$, we conclude that each term in (A.19b) is analytic for $z \in D_1$ and bounded when $y < 0$, so the second integral also defines an analytic function for all $z \in D_1$. Thus, the integral representation (A.18) for $a_{11}(z)$ can be analytically extended off the real z-axis onto D_1. The remainder of Theorem 2.4 is proved similarly. □

A.4. Adjoint problem

Proof of Lemma 2.7. We verify (2.36a) with $j = 3$. The rest of Lemma 2.7 is proved similarly. Equations (2.14) and (2.30) yield

$$\mathbf{v}_\pm(x,t,z) = \mathrm{e}^{-i\theta_1(x,t,z)}\mathbf{E}_{\pm,3}(z)+o(1), \qquad x \to \pm\infty.$$

However, $\mathbf{v}_\pm$ must be a linear combination of the columns of $\phi_\pm$, so there exist scalar functions $a_\pm(z)$, $b_\pm(z)$, and $c_\pm(z)$ such that $\mathbf{v}_\pm(x,t,z) = a_\pm(z)\phi_{\pm,1}(x,t,z)+b_\pm(z)\phi_{\pm,2}(x,t,z)+c_\pm(z)\phi_{\pm,3}(x,t,z)$. Comparing the asymptotics as $x \to \pm\infty$ in (2.14) with those of $\mathbf{v}_\pm$ yields $a_\pm(z) = b_\pm(z) = 0$ and $c_\pm(z) = 1$. □

Proof of Corollary 2.8. We suppress the x, t, and z dependence for simplicity. Combining (2.36) with (2.25) yields $\tilde{\phi}_{+,1} = (b_{22}b_{33}-b_{32}b_{23})\tilde{\phi}_{-,1}+\gamma(b_{32}b_{13}-b_{12}b_{33})\tilde{\phi}_{-,2}+(b_{22}b_{13}-b_{12}b_{23})\tilde{\phi}_{-,3}$. Combining this with (2.33) yields

$$\tilde{b}_{11} = b_{22}b_{33}-b_{32}b_{23}, \quad \tilde{b}_{21} = \gamma(b_{32}b_{13}-b_{12}b_{33}), \quad \tilde{b}_{31} = b_{12}b_{23}-b_{22}b_{13}.$$

Using a similar process, we find that

$$\begin{aligned}\tilde{b}_{12} = \frac{1}{\gamma}(b_{23}b_{31}-b_{33}b_{21}), \quad \tilde{b}_{22} = b_{33}b_{11}-b_{13}b_{31}, \quad \tilde{b}_{32} = \frac{1}{\gamma}(b_{13}b_{21}-b_{23}b_{11}),\\ \tilde{b}_{13} = b_{21}b_{32}-b_{31}b_{22}, \quad \tilde{b}_{23} = \gamma(b_{31}b_{12}-b_{11}b_{32}), \quad \tilde{b}_{33} = b_{11}b_{22}-b_{21}b_{12}.\end{aligned}$$

Next, note that

$$(\mathbf{A}(z))^T = \begin{pmatrix} b_{22}b_{33}-b_{23}b_{32} & b_{23}b_{31}-b_{21}b_{33} & b_{21}b_{32}-b_{22}b_{31} \\ b_{13}b_{32}-b_{12}b_{33} & b_{11}b_{33}-b_{13}b_{31} & b_{12}b_{31}-b_{11}b_{32} \\ b_{12}b_{23}-b_{13}b_{22} & b_{13}b_{21}-b_{11}b_{23} & b_{11}b_{22}-b_{12}b_{21} \end{pmatrix}.$$

Combining all this information, we finally obtain (2.38). □

Proof of Corollary 2.9. Substituting (2.33) into (2.35) yields the following for $z \in \Sigma$:

$$\gamma\chi_2 = \tilde{b}_{22}\mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,1}\times\tilde{\phi}_{-,2}]+\tilde{b}_{32}\mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,1}\times\tilde{\phi}_{-,3}], \tag{A.20a}$$

$$\gamma\chi_3 = \tilde{b}_{12}\mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,1}\times\tilde{\phi}_{-,3}]+\tilde{b}_{22}\mathrm{e}^{i\theta_2(x,t,z)}[\tilde{\phi}_{-,2}\times\tilde{\phi}_{-,3}]. \tag{A.20b}$$

Applying (2.36) to (A.20) yields the following for $z \in \Sigma$:

$$\gamma(z)\chi_2(x,t,z) = \tilde{b}_{22}(z)\phi_{-,3}(x,t,z) - \tilde{b}_{32}(z)\gamma(z)\phi_{-,2}(x,t,z), \tag{A.21a}$$

$$\gamma(z)\chi_3(x,t,z) = -\tilde{b}_{12}(z)\gamma(z)\phi_{-,2}(x,t,z) + \tilde{b}_{22}(z)\phi_{-,1}(x,t,z). \tag{A.21b}$$

We apply (2.38) to (A.21) to obtain the first of (2.39a) and the first of (2.39b). The rest of (2.39) is obtained similarly. □

A.5. Symmetries

Proof of Proposition 2.11. Let $\phi(x,t,z)$ be a non-singular solution of the Lax pair. Then, $\phi_x^\dagger = \phi^\dagger \mathbf{X}^\dagger$ and $\phi_t^\dagger = \phi^\dagger \mathbf{T}^\dagger$. But since $\mathbf{Q}^\dagger = -\mathbf{Q}$, we have

$$\mathbf{w}_x = -(\phi^\dagger(z^*))^{-1}\phi_x^\dagger(z^*)(\phi^\dagger(z^*))^{-1} = -(\phi^\dagger(z^*))^{-1}\phi^\dagger(z^*)[ik\mathbf{J} - \mathbf{Q}](\phi^\dagger(z^*))^{-1} = \mathbf{X}\,\mathbf{w},$$
$$\mathbf{w}_t = -(\phi^\dagger(z^*))^{-1}\phi_t^\dagger(z^*)(\phi^\dagger(z^*))^{-1} = -[-2ik^2\mathbf{J} + i\mathbf{J}(\mathbf{Q}_x - \mathbf{Q}^2 - q_o^2) + 2k\mathbf{Q}]\mathbf{w} = \mathbf{T}\,\mathbf{w},$$

where the (x,t)-dependence was omitted for brevity. Thus, $\mathbf{w}(x,t,z)$ is a solution of the Lax pair. □

Proof of Lemma 2.12. Define

$$\mathbf{w}_\pm(x,t,z) = (\phi_\pm^\dagger(x,t,z^*))^{-1}, \quad z \in \Sigma. \tag{A.22}$$

Also, note that for all $z \in \mathbb{C}$,

$$(\mathrm{e}^{i\mathbf{\Theta}(x,t,z^*)})^\dagger = \mathrm{e}^{-i\mathbf{\Theta}(x,t,z)}.$$

As before, we restrict our attention to $z \in \Sigma$. The BC (2.14) imply

$$\mathbf{w}_\pm(x,t,z) = (\mathbf{E}_\pm^\dagger(z^*))^{-1}\mathrm{e}^{i\mathbf{\Theta}(x,t,z)} + o(1), \quad x \to \pm\infty. \tag{A.23}$$

Since both $\mathbf{w}_\pm(x,t,z)$ and $\phi_\pm(x,t,z)$ are fundamental matrix solutions of the Lax pair (2.1), there must exist an invertible 3×3 matrix $\mathbf{C}(z)$ such that (2.41) holds. Comparing the asymptotics from (A.23) to those from (2.14), we then obtain (2.42). □

Proof of Corollary 2.15. Taking into account the boundary conditions (2.14) and the corresponding boundary conditions for the adjoint problem, we obtain $\phi_\pm^*(x,t,z^*) = \tilde{\phi}_\pm(x,t,z)$ for $z \in \Sigma$. Thus, by the Schwarz reflection principle,

$$\phi_{\pm,1}^*(x,t,z^*) = \tilde{\phi}_{\pm,1}(x,t,z), \qquad \operatorname{Im}(z) \gtrless 0 \wedge |z| > q_o, \tag{A.24a}$$

$$\phi_{\pm,2}^*(x,t,z^*) = \tilde{\phi}_{\pm,2}(x,t,z), \qquad \operatorname{Im}(z) \lessgtr 0, \tag{A.24b}$$

$$\phi_{\pm,3}^*(x,t,z^*) = \tilde{\phi}_{\pm,3}(x,t,z), \qquad \operatorname{Im}(z) \gtrless 0 \wedge |z| < q_o. \tag{A.24c}$$

We can then combine (2.35) with (A.24) to obtain (2.47). □

Proof of Lemma 2.17. For $z \in \Sigma$, define $\mathbf{W}_\pm(x,t,z) = \phi_\pm(x,t,\hat{z})$. Since $\mathbf{W}_\pm$ and $\phi_\pm$ both solve the Lax pair (2.1), there must exist an invertible 3×3 matrix $\mathbf{\Pi}(z)$ such that (2.50) holds. Comparing the asymptotics of (2.50) with the asymptotics from (2.14) yields

$$\mathbf{E}_\pm(\hat{z})\mathrm{e}^{i\mathbf{K}\mathbf{\Theta}(x,t,z)}\mathbf{\Pi}(z) = \mathbf{E}_\pm(z)\mathrm{e}^{i\mathbf{\Theta}(x,t,z)}, \qquad z \in \Sigma, \tag{A.25}$$

where $\mathbf{K} = \operatorname{diag}(-1,1,-1)$. From this, we obtain (2.51). □

A.6. Discrete eigenvalues and symmetries of the norming constants

Proof of Lemma 3.1. The desired results follow from the symmetries (2.44) and (2.53). □

The proof of Lemma 3.2 is similar to the proof of Lemma 3.1 and is omitted.

Proof of Lemma 3.3. [(i) ⇔ (ii)] The symmetry (2.56) gives the desired results.

[(i) ⇔ (iii)] Follows directly from (2.47*a*).

[(iii) ⇔ (iv)] Assume that there exists a constant b_o such that $\phi_{-,2}(x,t,z_o^*) = b_o\phi_{+,1}(x,t,z_o^*)$. Applying the symmetry (2.52) and taking $\tilde{b}_o = iq_ob_o/z_o^*$ yields the desired result. The converse is proved similarly. □

The proof of Lemma 3.4 is similar to the proof of Lemma 3.3 and is therefore omitted.

Proof of Theorem 3.6. (i) If $\chi_1(x,t,z_o) = \mathbf{0}$, then (2.43*a*) implies $\phi_{+,1}(z_o^*) = \mathbf{0}$. This is a contradiction, so $\chi_1(x,t,z_o) \neq \mathbf{0}$. Note that the left hand side of (2.43*d*) (an analytic function) will have a pole at $z = z_o$ unless $[\chi_1 \times \phi_{-,1}](x,t,z_o) = \mathbf{0}$. This is equivalent to the existence of the desired constant c_o. The presence of the factor of $b_{22}(z_o)$ in the denominator is for convenience in the formulation of a Riemann-Hilbert problem in later sections. The other results are proved similarly.

(ii) If $\chi_1(x,t,z_o) = \mathbf{0}$, then (2.43*d*) implies $\phi_{-,2}(x,t,z_o^*) = \mathbf{0}$. This is a contradiction, so $\chi_1(x,t,z_o) \neq \mathbf{0}$. Note that the left hand side of (2.43*a*) (an analytic function) will have a pole at $z = z_o$ unless $[\phi_{+,2} \times \chi_1](x,t,z_o) = \mathbf{0}$. This is equivalent to the existence of the desired constant d_o. The other results are proved similarly.

(iii) Suppose $\chi_1(x,t,z_o) \neq \mathbf{0}$. Since $\det \mathbf{\Phi}_1(x,t,z_o) = 0$, there exist constants g_1 and g_2 such that $\chi_1(x,t,z_o) = g_1\phi_{-,1}(x,t,z_o) + g_2\phi_{+,2}(x,t,z_o)$. However, (2.43*a*) will have a pole unless $g_1 = 0$. We use the same argument with (2.43*d*) to conclude $g_2 = 0$. Thus, $\chi_1(x,t,z_o) = \mathbf{0}$. The proof that $\chi_2(x,t,z_o^*) = \mathbf{0}$ is similar. The existence of the desired norming constants then follows trivially from Lemmas 3.3 and 3.4. □

Proof of Lemma 3.7. The symmetries (2.52) and (2.56) yield $\hat{d}_n = -(iz_n^*/q_o)\check{d}_n$ and (3.8*b*). Then, combining (2.43) with (2.35) and comparing the result with (3.6*a*) yields the rest of (3.8*b*). The rest of Lemma 3.7 is proved similarly. □

A.7. Asymptotics as $z \to \infty$ and $z \to 0$

In this section we show how to evaluate the asymptotic behavior of the eigenfunctions. Throughout this section, we will use the shorthand notation

$$\mathrm{e}^{i\hat{\mathbf{\Lambda}}}(M) = \mathrm{e}^{i\mathbf{\Lambda}}M\,\mathrm{e}^{-i\mathbf{\Lambda}} = \begin{pmatrix} m_{11} & \mathrm{e}^{-i(k+\lambda)}m_{12} & \mathrm{e}^{-2i\lambda}m_{13} \\ \mathrm{e}^{i(k+\lambda)}m_{21} & m_{22} & \mathrm{e}^{i(k-\lambda)}m_{23} \\ \mathrm{e}^{2i\lambda}m_{31} & \mathrm{e}^{-i(k-\lambda)}m_{32} & m_{33} \end{pmatrix},$$

where $\mathbf{M}$ is any 3×3 matrix. In order to prove Lemmas 3.8 and 3.9, it will be convenient to decompose (3.10*b*) into block-diagonal and block-off-diagonal terms. First, note that for any 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$,

$$[\mathbf{AB}]_{bd} = \mathbf{A}_{bd}\mathbf{B}_{bd} + \mathbf{A}_{bo}\mathbf{B}_{bo}, \qquad [\mathbf{AB}]_{bo} = \mathbf{A}_{bd}\mathbf{B}_{bo} + \mathbf{A}_{bo}\mathbf{B}_{bd}. \tag{A.26a}$$

$$[\mathbf{A}_{bd}\mathbf{B}_{bd}]_d = [\mathbf{A}]_d[\mathbf{B}]_d + [\mathbf{A}_{bd}]_o[\mathbf{B}_{bd}]_o, \tag{A.26b}$$

$$[\mathbf{A}_{bd}\mathbf{B}_{bd}]_o = [\mathbf{A}]_d[\mathbf{B}_{bd}]_o + [\mathbf{A}_{bd}]_o[\mathbf{B}]_d. \tag{A.26c}$$

We denote the integrand of (3.10*b*) as

$$\mathbf{M}_+(x,y,t,z) = \mathbf{E}_+(z)\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}(z)}\Big(\mathbf{E}_+^{-1}(z)\Delta\mathbf{Q}_+(y,t)\mu_n(y,t,z)\Big).$$

In the following calculations we suppress some x, t, and z dependence for brevity when doing so introduces no confusion. Since $\mathrm{e}^{i(x-y)\mathbf{\Lambda}(z)}$ is a diagonal matrix, and since $\Delta\mathbf{Q}_+$ is a block off-diagonal matrix,

$$\begin{aligned}[\mathbf{M}_+]_{bd} = {} & [\mathbf{E}_+]_{bd}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^{-1}]_{bo}\Delta\mathbf{Q}_+(y,t)[\mu_n(y,t,z)]_{bd} + [\mathbf{E}_+^{-1}]_{bd}\Delta\mathbf{Q}_+(y,t)[\mu_n(y,t,z)]_{bo}\Big)\\ & + [\mathbf{E}_+]_{bo}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^{-1}]_{bd}\Delta\mathbf{Q}_+(y,t)[\mu_n(y,t,z)]_{bd} + [\mathbf{E}_+^{-1}]_{bo}\Delta\mathbf{Q}_+(y,t)[\mu_n(y,t,z)]_{bo}\Big).\end{aligned}$$

Equation (2.11) implies

$$[\mathbf{E}_\pm^{-1}]_{bd} = \frac{1}{\gamma(z)}\mathbf{\Gamma}(z)[\mathbf{E}_\pm^\dagger]_{bd}, \qquad [\mathbf{E}_\pm^{-1}]_{bo} = \frac{1}{\gamma(z)}[\mathbf{E}_\pm^\dagger]_{bo},$$

with $\mathbf{\Gamma}(z) = \mathrm{diag}(1,\gamma(z),1)$ as before. We then obtain

$$\begin{aligned}[\mathbf{M}_+]_{bd} = {} & \frac{[\mathbf{E}_+]_{bd}}{\gamma}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bd} + \mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}\Delta\mathbf{Q}_+[\mu_n]_{bo}\Big)\\ & + \frac{[\mathbf{E}_+]_{bo}}{\gamma}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}^\dagger\Delta\mathbf{Q}_+[\mu_n]_{bd} + [\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bo}\Big).\end{aligned}$$

Similarly,

$$\begin{aligned}[\mathbf{M}_+]_{bo} = {} & \frac{[\mathbf{E}_+]_{bd}}{\gamma}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bo} + \mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}\Delta\mathbf{Q}_+[\mu_n]_{bd}\Big)\\ & + \frac{[\mathbf{E}_+]_{bo}}{\gamma}\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}\Delta\mathbf{Q}_+[\mu_n]_{bo} + [\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bd}\Big).\end{aligned}$$

We now use these relations to decompose the integral in (3.10*b*). Namely,

$$\begin{aligned}-\gamma[\mu_{n+1}]_{bd} = {} & [\mathbf{E}_+]_{bd}\int_x^\infty\Big[[\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_d + \mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[[\mu_n]_{bd}]_o\Big)\Big]\mathrm{d}y\\ & + [\mathbf{E}_+]_{bd}\mathbf{\Gamma}\int_x^\infty\Big[[\mathbf{E}_+^\dagger]_d\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_d + [[\mathbf{E}_+^\dagger]_{bd}]_o\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_o\Big]\mathrm{d}y\\ & + [\mathbf{E}_+]_{bd}\int_x^\infty\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[\mathbf{E}_+^\dagger]_d\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_o + \mathbf{\Gamma}[[\mathbf{E}_+^\dagger]_{bd}]_o\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_d\Big)\mathrm{d}y\\ & + [\mathbf{E}_+]_{bo}\int_x^\infty\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}\Delta\mathbf{Q}_+[\mu_n]_{bd} + [\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bo}\Big)\mathrm{d}y, \qquad \text{(A.27}a\text{)}\end{aligned}$$

$$\begin{aligned}-\gamma[\mu_{n+1}]_{bo} = {} & [\mathbf{E}_+]_{bd}\int_x^\infty\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big([\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+[\mu_n]_{bo} + \mathbf{\Gamma}[\mathbf{E}_+^\dagger]_{bd}\Delta\mathbf{Q}_+[\mu_n]_{bd}\Big)\mathrm{d}y\\ & + [\mathbf{E}_+]_{bo}\int_x^\infty\Big[\mathbf{\Gamma}[[\mathbf{E}_+^\dagger]_{bd}]_o\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_o + \mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[\mathbf{E}_+^\dagger]_d\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_o\Big)\Big]\mathrm{d}y\\ & + [\mathbf{E}_+]_{bo}\int_x^\infty\Big[\mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}}\Big(\mathbf{\Gamma}[[\mathbf{E}_+^\dagger]_{bd}]_o\big[\Delta\mathbf{Q}_+[\mu_n]_{bo}\big]_d\Big) + [[\mathbf{E}_+^\dagger]_{bo}\Delta\mathbf{Q}_+]_d[\mu_n]_d\Big]\mathrm{d}y\end{aligned}$$

$$+ [\mathbf{E}_+]_{bo} \int_x^\infty \Big[[[\mathbf{E}_+^\dagger]_{bo} \Delta \mathbf{Q}_+]_o [[\mu_n]_{bd}]_o + \mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}} \Big([[\mathbf{E}_+^\dagger]_{bo} \Delta \mathbf{Q}_+]_o [\mu_n]_d \Big) \Big] \mathrm{d}y$$

$$+ [\mathbf{E}_+]_{bo} \int_x^\infty \mathrm{e}^{i(x-y)\hat{\mathbf{\Lambda}}} \Big([[\mathbf{E}_+^\dagger]_{bo} \Delta \mathbf{Q}_+]_d [[\mu_n]_{bd}]_o \Big) \mathrm{d}y. \quad \text{(A.27}b\text{)}$$

Equations (A.27a) and (A.27b) will allow us to use induction to prove Lemmas 3.8 and 3.9.

Proof of Lemma 3.8. The claims in (3.11a) are trivially true for μ_0. Suppose the claims in (3.11) are true for some $n \geq 0$. We then use integration by parts and the facts that $k = z/2 + O(1/z)$ and $\lambda = z/2 + O(1/z)$ as $z \to \infty$ to see that the terms on the right hand side of (A.27a) are $O([\mu_n]_{bd}/z)$, $O([\mu_n]_{bd}/z^2)$, $O([\mu_n]_{bo})$, $O([\mu_n]_{bo})$, $O([\mu_n]_{bo}/z)$, $O([\mu_n]_{bo}/z)$, $O([\mu_n]_{bd}/z^2)$, and $O([\mu_n]_{bo}/z^3)$, respectively, as $z \to \infty$.

When $n = 2m$ for some $m \in \mathbb{N}$, the first, third, and fourth terms on the right hand side of (A.27a) are $O(1/z^{m+1})$, the second, fifth, sixth, and seventh terms are $O(1/z^{m+2})$, and the eighth term is $O(1/z^{m+4})$ (all as $z \to \infty$). Then $[\mu_{n+1}]_{bd} = O(1/z^{m+1})$, as $z \to \infty$.

When $n = 2m + 1$ for some $m \in \mathbb{N}$, the third and fourth terms on the right hand side of (A.27a) are $O(1/z^{m+1})$, the first, fifth, and sixth terms are $O(1/z^{m+2})$, the second and seventh terms are $O(1/z^{m+3})$, and the eighth term is $O(1/z^{m+4})$ (all as $z \to \infty$). Then $[\mu_{n+1}]_{bd} = O(1/z^{m+1})$ as $z \to \infty$.

Similar results hold for the terms in (A.27b) using the same analysis. Also, the same results hold for $\mu_-(x, t, z)$ when it is expanded as a series similar to (3.9). □

Proof of Lemma 3.9. The claims in (3.12a) are trivially true for μ_0. Suppose the claims in (3.12) are true for some $n \geq 0$. We then use integration by parts and the facts that $k = O(1/z)$ and $\lambda = O(1/z)$ as $z \to 0$ to see that the eight terms on the right hand side of (A.27a) are, respectively, $O(z[\mu_n]_{bd})$, $O(z^2[\mu_n]_{bd})$, $O(z^2[\mu_n]_{bo})$, $O(z^2[\mu_n]_{bo})$, $O(z^3[\mu_n]_{bo})$, $O(z^3[\mu_n]_{bo})$, $O(z^2[\mu_n]_{bd})$, and $O(z[\mu_n]_{bo})$ as $z \to 0$.

When $n = 2m$ for some $m \in \mathbb{N}$, the eighth term on the right hand side of (A.27a) is $O(z^m)$, the first, third, and fourth terms are $O(z^{m+1})$, and the rest are $O(z^{m+2})$. Then $[\mu_{n+1}]_{bd} = O(z^m)$ as $z \to 0$.

When $n = 2m + 1$ for some $m \in \mathbb{N}$, the first and eighth terms on the right hand side of (A.27a) are $O(z^{m+1})$, the second, third, fourth, and seventh terms are $O(z^{m+2})$, and the rest are $O(z^{m+3})$. Then $[\mu_{n+1}]_{bd} = O(z^{m+1})$ as $z \to 0$.

Similar results hold for the terms in (A.27b) using the same analysis. Also, the same results hold for $\mu_-(x, t, z)$ when it is expanded as a series similar to (3.9). □

A.8. Riemann-Hilbert problem

Proof of Lemma 4.1. Combining (2.25) with the second of (2.39d) yields

$$\phi_{+,2}(x,t,z) = -\left[\frac{a_{32}(z)}{a_{22}(z)} \frac{b_{13}(z)}{b_{11}(z)} + \frac{a_{12}(z)}{a_{22}(z)} \right] \phi_{+,1}(x,t,z) + \frac{\phi_{-,2}(x,t,z)}{a_{22}(z)} - \frac{a_{32}(z)}{a_{22}(z)} \frac{\chi_2(x,t,z)}{b_{11}(z)}. \quad \text{(A.28)}$$

Equation (A.28) expresses $\phi_{+,2}(x, t, z)$ in terms of eigenfunctions meromorphic in D_2. Examining (2.25) once again as well as equating the two expressions in (2.39d) and solving for χ_1/b_{22} yields

$$\frac{\phi_{-,1}(x,t,z)}{a_{11}(z)} = \phi_{+,1}(x,t,z) + \frac{a_{21}}{a_{11}(z)} \phi_{+,2}(x,t,z) + \frac{a_{31}(z)}{a_{11}(z)} \phi_{+,3}(x,t,z), \quad \text{(A.29}a\text{)}$$

$$\frac{\chi_1(x,t,z)}{b_{22}(z)} = \frac{b_{13}(z)}{b_{11}(z)}\phi_{+,1}(x,t,z) - \frac{b_{23}(z)}{b_{22}(z)}\phi_{+,2}(x,t,z) + \frac{\chi_2(x,t,z)}{b_{11}(z)}. \tag{A.29b}$$

Combining (A.29a) with the second of (2.39d) and then (A.28) expresses $\phi_{-,1}(x,t,z)/a_{11}(z)$ in terms of eigenfunctions meromorphic in D_2. The same is done for $\chi_1(x,t,z)/b_{22}(z)$ by combining (A.29b) with (A.28). The columns of $\mathbf{L}_1(z)$ are then obtained by applying (2.57). The rest of the jump matrices are obtained similarly. □

Proof of Lemma 4.2. The residue conditions (4.6) are trivial results of equations (3.5), (3.6), and (3.7). □

Proof of Lemma 4.3. We use the following symmetries (obtained from (2.44) and (2.53)):

$$b'_{22}(z_o) = (a'_{22}(z_o))^*\Big|_{z=z_o^*}, \quad b'_{22}(z_o) = \frac{q_o^2}{z_o^2}(b'_{22}(z))\Big|_{z=\hat{z}_o},$$

$$a'_{22}(z_o^*) = \frac{q_o^2}{(z_o^*)^2}(a'_{22}(z))\Big|_{z=\hat{z}_o^*},$$

$$a'_{11}(z_o) = (b'_{11}(z))^*\Big|_{z=z_o^*}, \quad a'_{33}(\hat{z}_o) = (b'_{33}(z))^*\Big|_{z=\hat{z}_o^*},$$

$$a'_{11}(z_o) = \frac{q_o^2}{z_o^2}(a'_{33}(z))\Big|_{z=\hat{z}_o}, \quad b'_{11}(z_o^*) = \frac{q_o^2}{(z_o^*)^2}(b'_{33}(z))\Big|_{z=\hat{z}_o^*},$$

along with the symmetries in Lemma 3.7 to obtain the desired results. □

Proof of Theorem 4.4. Define the following Cauchy projectors:

$$P^{\pm}(f)(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(\zeta)}{\zeta-(z\pm i0)}\mathrm{d}\zeta, \quad \bar{P}^{\pm}(f)(z) = \frac{1}{2\pi i}\int_{\Sigma} \frac{f(\zeta)}{\zeta-(z\pm i0)}\mathrm{d}\zeta, \tag{A.30}$$

where the orientation of Σ is given in Fig. 1 (right). (Note that this is the same as in the scalar case [8].) To solve (4.2), we subtract from both sides of (4.2) the asymptotic behavior (4.3) as well as the residue contributions from the poles. Namely, we subtract

$$\mathbf{M}_\infty + (i/z)\mathbf{M}_0 + \sum_{n=1}^{N}\left(\frac{\mathbf{M}^+_{-1,v_n}}{z-v_n} + \frac{\mathbf{M}^-_{-1,v_n^*}}{z-v_n^*} + \frac{\mathbf{M}^+_{-1,\hat{v}_n^*}}{z-\hat{v}_n^*} + \frac{\mathbf{M}^-_{-1,\hat{v}_n}}{z-\hat{v}_n}\right).$$

Note that the left hand side of the resulting regularized RHP is analytic in D^+ and is $O(1/z)$ as $z\to\infty$ there. Also, the right hand side is analytic in D^- and is $O(1/z)$ as $z\to\infty$ there. Applying the projector $\bar{P}^{\pm}$ from (A.30) to the regularized RHP and using Plemelj's formulae yields (4.7). In order to determine the solution $\mathbf{M}(x,t,z)$ completely, we need to compute the eigenfunctions $m_1(x,t,w_n)$, $\mu_{+,1}(x,t,w_n^*)$, etc. We take $\mathbf{M}=\mathbf{M}_1$ in (4.7), evaluate its second column at $z=z_{i'}$ or $z=\zeta_{\ell'}$, and apply the symmetries of the eigenfunctions to obtain (4.8a). Next, we take $\mathbf{M}=\mathbf{M}_2$ and evaluate its third column at $z_{i'}^*$ to obtain (4.8b). Thirdly, we take $\mathbf{M}=\mathbf{M}_2$ and evaluate its first column at $z=\zeta_{\ell'}^*$ or $z=w_{j'}^*$ to obtain (4.8c). Finally, we take $\mathbf{M}=\mathbf{M}_1$ and evaluate its third column at $w_{j'}$ to obtain (4.8d). This mixed algebraic-integral system of equations is closed, so we have determined the solution $\mathbf{M}(x,t,z)$ of the RHP (4.2) given in (4.7). □

Proof of Theorem 4.5. The asymptotics in (3.13a) imply

$$q_k(x,t) = -i\lim_{z\to\infty}\left(z\mu_{+,(k+1)1}(x,t,z)\right), \qquad k=1,2. \tag{A.31}$$

Comparing the 2,1 and 3,1 elements in the limit as $z\to\infty$ of (4.8c) with the corresponding elements found in (3.13a) yields (4.9). □

A.9. Trace formulae

Proof of Lemma 4.6. We first derive (4.10b). A cofactor expansion of $\mathbf{A}(z)$ along its second column, combined with the definition (2.57) of the reflection coefficients, yields

$$\log b_{22}(z) - \log(1/a_{22}(z)) = -\log[1+\gamma(z)\rho_3(\hat{z})\rho_3^*(\hat{z}^*)+\gamma(z)\rho_3(z)\rho_3^*(z^*)], \quad z\in\Sigma. \qquad \text{(A.32)}$$

Since $b_{22}(z)$ and $a_{22}(z)$ are analytic in the upper- and lower-half plane, respectively, (A.32) is a jump condition that defines a scalar, additive Riemann-Hilbert problem. To remove the pole singularities coming from the zeros of $b_{22}(z)$ and $a_{22}(z)$, we can define

$$\beta^+(z) = b_{22}(z)\mathrm{e}^{i\Delta\theta}\prod_{n=1}^{N_2}\frac{z-z_n^*}{z-z_n}\frac{z-\hat{z}_n^*}{z-\hat{z}_n}\prod_{n=1}^{N_3}\frac{z-\zeta_n^*}{z-\zeta_n}\frac{z-\hat{\zeta}_n^*}{z-\hat{\zeta}_n}, \quad z\in\mathbb{C}^+\,, \qquad \text{(A.33}a\text{)}$$

$$\beta^-(z) = (1/a_{22}(z))\mathrm{e}^{i\Delta\theta}\prod_{n=1}^{N_2}\frac{z-z_n^*}{z-z_n}\frac{z-\hat{z}_n^*}{z-\hat{z}_n}\prod_{n=1}^{N_3}\frac{z-\zeta_n^*}{z-\zeta_n}\frac{z-\hat{\zeta}_n^*}{z-\hat{\zeta}_n}, \quad z\in\mathbb{C}^-\,. \qquad \text{(A.33}b\text{)}$$

Now $\beta^\pm(z)$ are analytic in $\mathbb{C}^\pm$, respectively, $\beta^\pm(z)$ have no zeros (or poles) in $\mathbb{C}^\pm$, respectively, and each approaches 1 as $z\to\infty$ in the appropriate region of the complex plane. Applying the projectors $P^\pm$ defined in (A.30) and using Plemelj's formulae we obtain $\log\beta(z) = P(\log[\beta^+\beta^-])$ for all $z\in\mathbb{C}\setminus\Sigma$. Taking into account the explicit form of the jump condition in (A.32) and taking exponentials then yields (4.10b).

Next we derive (4.10a). Using appropriate cofactor expansions, similarly as above, and the definition (2.57) of the reflection coefficients yields:

$$\log a_{11}(z) - \log(1/b_{11}(z)) = -\log\left[1+\frac{1}{\gamma(z)}\rho_1(z)\rho_1^*(z^*)+\rho_2(z)\rho_2^*(z^*)\right], \quad z\in\Sigma, \qquad \text{(A.34}a\text{)}$$

$$\log b_{33}(z) - \log(1/a_{33}(z)) = -\log\left[1+\frac{1}{\gamma(z)}\rho_1(\hat{z})\rho_1^*(\hat{z}^*)+\rho_2(\hat{z})\rho_2^*(\hat{z}^*)\right], \quad z\in\Sigma. \qquad \text{(A.34}b\text{)}$$

Now, however, the situation is complicated by the fact that $a_{11}(z)$, $a_{33}(z)$, $b_{11}(z)$ and $b_{33}(z)$ are each only analytic in one of the fundamental domains $D_1,\dots,D_4$. In order to formulate a Riemann-Hilbert problem, one needs a sectionally analytic function over the whole complex plane. Moreover, since we have four fundamental domains of analyticity, we need additional jump conditions. Recalling that $\mathbf{A}(z)\mathbf{B}(z)=\mathbf{I}$, we have $a_{22}(z) = b_{11}(z)b_{33}(z)-b_{13}(z)b_{31}(z)$ and $b_{22}(z) = a_{11}(z)a_{33}(z)-a_{13}(z)a_{31}(z)$ for all $z\in\Sigma$. Using again the definitions (2.57) of the reflection coefficients we then obtain

$$\log b_{33}(z) - \log(1/b_{11}(z)) = \log a_{22}(z) - \log[1-\rho_2^*(z^*)\rho_2^*(\hat{z}^*)], \quad z\in\Sigma, \qquad \text{(A.35}a\text{)}$$

$$\log a_{11}(z) - \log(1/a_{33}(z)) = \log b_{22}(z) - \log[1-\rho_2(z)\rho_2(\hat{z})], \quad z\in\Sigma. \qquad \text{(A.35}b\text{)}$$

We therefore define

$$\bar{\beta}^+(z) = \begin{cases}\beta_1(z), & z\in D_1,\\ \beta_3(z), & z\in D_3,\end{cases} \qquad \bar{\beta}^-(z) = \begin{cases}\beta_2(z), & z\in D_2,\\ \beta_4(z), & z\in D_4,\end{cases} \qquad \text{(A.36)}$$

where

$$\beta_1(z) = \frac{a_{11}(z)}{p_1(z)}\prod_{n=1}^{N_1}\frac{z-w_n^*}{z-w_n}\frac{z-\hat{w}_n}{z-\hat{w}_n^*}\prod_{n=1}^{N_2}\frac{z-z_n}{z-z_n^*}\frac{z-\hat{z}_n^*}{z-\hat{z}_n}, \quad z\in D_1, \qquad \text{(A.37}a\text{)}$$

$$\beta_2(z) = \frac{p_1^*(z^*)(z)}{b_{11}(z)}\prod_{n=1}^{N_1}\frac{z-w_n^*}{z-w_n}\frac{z-\hat{w}_n}{z-\hat{w}_n^*}\prod_{n=1}^{N_2}\frac{z-z_n}{z-z_n^*}\frac{z-\hat{z}_n^*}{z-\hat{z}_n}, \quad z\in D_2, \qquad \text{(A.37}b\text{)}$$

$$\beta_3(z) = \frac{b_{33}(z)}{p_2^*(z^*)} \prod_{n=1}^{N_1} \frac{z-w_n^*}{z-w_n} \frac{z-\hat{w}_n}{z-\hat{w}_n^*} \prod_{n=1}^{N_2} \frac{z-z_n}{z-z_n^*} \frac{z-\hat{z}_n^*}{z-\hat{z}_n}, \quad z \in D_3, \tag{A.37c}$$

$$\beta_4(z) = \frac{p_2(z)}{a_{33}(z)} \prod_{n=1}^{N_1} \frac{z-w_n^*}{z-w_n} \frac{z-\hat{w}_n}{z-\hat{w}_n^*} \prod_{n=1}^{N_2} \frac{z-z_n}{z-z_n^*} \frac{z-\hat{z}_n^*}{z-\hat{z}_n}, \quad z \in D_4, \tag{A.37d}$$

$$p_1(z) = \prod_{n=1}^{N_2} \frac{z-z_n}{z-z_n^*} \prod_{n=1}^{N_3} \frac{z-\zeta_n}{z-\zeta_n^*}, \qquad p_2(z) = \prod_{n=1}^{N_2} \frac{z-\hat{z}_n}{z-\hat{z}_n^*} \prod_{n=1}^{N_3} \frac{z-\hat{\zeta}_n}{z-\hat{\zeta}_n^*}. \tag{A.37e}$$

Note that $\bar{\beta}^{\pm}(z)$ are analytic in $D^{\pm}$, respectively, have no zeros (or poles) there, and each approaches 1 as $z \to \infty$ in the appropriate region of the complex plane. Equations (A.34) and (A.35) can be written in terms of $\beta_1(z),\ldots,\beta_4(z)$. Specifically, (A.34) yields

$$\log\beta_1(z) - \log\beta_2(z) = -\log\left[1 + \frac{1}{\gamma(z)}\rho_1(z)\rho_1^*(z^*) + \rho_2(z)\rho_2^*(z^*)\right], \quad z \in \Sigma, \tag{A.38a}$$

$$\log\beta_3(z) - \log\beta_4(z) = -\log\left[1 + \frac{1}{\gamma(z)}\rho_1(\hat{z})\rho_1^*(\hat{z}^*) + \rho_2(\hat{z})\rho_2^*(\hat{z}^*)\right], \quad z \in \Sigma. \tag{A.38b}$$

Moreover, using (A.37) and (4.10b) to simplify (A.35) yields

$$\log\beta_3(z) - \log\beta_2(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_o^*(\zeta)}{\zeta - z}\mathrm{d}\zeta - \log[1 - \rho_2^*(z^*)\rho_2^*(\hat{z}^*)], \qquad z \in \Sigma, \tag{A.39a}$$

$$\log\beta_1(z) - \log\beta_4(z) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_o(\zeta)}{\zeta - z}\mathrm{d}\zeta - \log[1 - \rho_2(z)\rho_2(\hat{z})], \qquad z \in \Sigma, \tag{A.39b}$$

where

$$J_o(z) = \log[1 + \gamma(z)\rho_3(\hat{z})\rho_3^*(\hat{z}^*) + \gamma(z)\rho_3(z)\rho_3^*(z^*)]. \tag{A.40}$$

Together, (A.38) and (A.39) are the jump conditions for the Riemann-Hilbert problem for the sectionally analytic function $\bar{\beta}(z)$ defined in (A.36), with jump conditions $J_1(z),\ldots,J_4(z)$, where

$$J_1(z) = -\log\left[1 + \frac{1}{\gamma(z)}\rho_1(z)\rho_1^*(z^*) + \rho_2(z)\rho_2^*(z^*)\right], \tag{A.41a}$$

$$J_2(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_o^*(\zeta)}{\zeta - z}\mathrm{d}\zeta - \log[1 - \rho_2^*(z^*)\rho_2^*(\hat{z}^*)], \tag{A.41b}$$

$$J_3(z) = -\log\left[1 + \frac{1}{\gamma(z)}\rho_1(\hat{z})\rho_1^*(\hat{z}^*) + \rho_2(\hat{z})\rho_2^*(\hat{z}^*)\right], \tag{A.41c}$$

$$J_4(z) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{J_o(\zeta)}{\zeta - z}\mathrm{d}\zeta - \log[1 - \rho_2(z)\rho_2(\hat{z})], \tag{A.41d}$$

and with $J_o(z)$ defined as in (A.40). More precisely, we have

$$\log\bar{\beta}^+(z) - \log\bar{\beta}^-(z) = J_j(z), \qquad z \in \Sigma_j \tag{A.42}$$

for $j = 1,\ldots,4$, and where the Σ_j are as defined in Lemma 4.1.

Applying the projectors $\bar{P}^{\pm}$ defined in (A.30), using Plemelj's formulae and taking into account the jump conditions (A.41) then yields

$$\log\bar{\beta}(z) = \frac{1}{2\pi i}\int_{\Sigma} \frac{J(\zeta)}{\zeta - z}\mathrm{d}\zeta, \quad z \in \mathbb{C}\setminus\Sigma. \tag{A.43}$$

Taking the exponential of both sides of (A.43) with $z \in D_1$ yields

$$\frac{a_{11}(z)}{p_1(z)} = \exp\left(\frac{1}{2\pi i}\int_\Sigma \frac{J(\zeta)}{\zeta - z}\mathrm{d}\zeta\right)\prod_{n=1}^{N_1}\frac{z-w_n}{z-w_n^*}\frac{z-\hat{w}_n^*}{z-\hat{w}_n}\prod_{n=1}^{N_2}\frac{z-z_n^*}{z-z_n}\frac{z-\hat{z}_n}{z-\hat{z}_n^*}. \tag{A.44}$$

Simplifying this expression for $a_{11}(z)$ yields (4.10a). □

Proof of Corollary 4.7. It is straightforward to see that taking the limit as $z \to 0$ in the trace formula (4.10a) for $a_{11}(z)$ and comparing with the asymptotics in Corollary 3.12 yields the desired result. □

A.10. Reflectionless solutions

The coefficients b_{kj} and y_j appearing in Theorem 5.1 are defined as follows:

$$b_{kj}(x,t) = \begin{cases} q_{+,k}/q_o, & j = 1,\dots,N_1,\\ iq_{+,k}/w^*_{j-N_1}, & j = N_1+1,\dots,2N_1,\\ (-1)^{k+1}\frac{q^*_{+,\bar{k}}}{q_o} + \frac{q_{+,k}}{q_o}\sum_{n=1}^{N_2} G_n^{(1)}(z_{j-2N_1}) & \\ \quad +iq_{+,k}\sum_{n=1}^{N_3} G_n^{(2)}(z_{j-2N_1})/\zeta_n^*, & j = 2N_1+1,\dots,2N_1+N_2,\\ (-1)^{k+1}\frac{q^*_{+,\bar{k}}}{q_o} + \frac{q_{+,k}}{q_o}\sum_{n=1}^{N_2} G_n^{(1)}(\zeta_{j-2N_1-N_2}) & \\ \quad +iq_{+,k}\sum_{n=1}^{N_3} G_n^{(2)}(\zeta_{j-2N_1-N_2})/\zeta_n^*, & j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3, \end{cases} \tag{A.45}$$

$$y_j(x,t) = \begin{cases} iC_j, & j = 1,\dots,N_1,\\ -(iw^*_{j-N_1}/q_o)\check{C}_{j-N_1}, & j = N_1+1,\dots,2N_1,\\ i\bar{D}_{j-2N_1}, & j = 2N_1+1,\dots,2N_1+N_2,\\ iF_{j-2N_1-N_2}, & j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3, \end{cases} \tag{A.46}$$

where $k = 1,2$ and $\bar{k} = 3-k$. For simplicity, we define

$$G_n^{(1)}(x,t,z) = \frac{\hat{D}_n(x,t)}{z-z_n^*} - \frac{iz_n^*}{q_o}\frac{\check{D}_n(x,t)}{z-\hat{z}_n^*}, \qquad G_n^{(2)}(x,t,z) = \frac{\hat{F}_n(x,t)}{z-\zeta_n^*} - \frac{i\zeta_n^*}{q_o}\frac{\check{F}_n(x,t)}{z-\hat{\zeta}_n^*}, \tag{A.47a}$$

$$G_n^{(3)}(x,t,z) = \frac{\hat{C}_n(x,t)}{z-w_n^*}, \quad G_n^{(4)}(x,t,z) = -\frac{iw_n}{q_o}\frac{\bar{C}_n(x,t)}{z-\hat{w}_n}, \quad G_n^{(5)}(x,t,z) = \frac{D_n(x,t)}{z-z_n}, \tag{A.47b}$$

$$G_n^{(6)}(x,t,z) = \frac{\bar{F}_n(x,t)}{z-\hat{\zeta}_n}, \quad G_n^{(7)}(x,t,z) = \frac{C_n(x,t)}{z-w_n}, \quad G_n^{(8)}(x,t,z) = -\frac{w_n^*}{q_o}\frac{\check{C}_n(x,t)}{z-\hat{w}_n^*}, \tag{A.47c}$$

$$G_n^{(9)}(x,t,z) = \frac{\bar{D}_n(x,t)}{z-\hat{z}_n}, \qquad G_n^{(10)}(x,t,z) = \frac{F_n(x,t)}{z-\zeta_n}. \tag{A.47d}$$

Using (A.47), the coefficients F_{ij} are defined as follows. For $i,j = 1,\dots,N_1$,

$$F_{ij}(x,t) = G_j^{(4)}(w_i). \tag{A.48a}$$

For $i = 1,\dots,N_1$ and $j = N_1+1,\dots,2N_1$,

$$F_{ij}(x,t) = G_{j-N_1}^{(3)}(w_i). \tag{A.48b}$$

For $i = 1,\dots,N_1$ and $j = 2N_1+1,\dots,2N_1+N_2$,
$$F_{ij}(x,t) = G^{(5)}_{j-2N_1}(w_i)\,. \tag{A.48c}$$
For $i = 1,\dots,N_1$ and $j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$,
$$F_{ij}(x,t) = G^{(6)}_{j-2N_1-N_2}(w_i)\,. \tag{A.48d}$$
For $i = N_1+1,\dots,2N_1$ and $j = 1,\dots,N_1$,
$$F_{ij}(x,t) = G^{(7)}_{j}(w^*_{i-N_1})\,. \tag{A.48e}$$
For $i, j = N_1+1,\dots,2N_1$,
$$F_{ij}(x,t) = G^{(8)}_{j-N_1}(w^*_{i-N_1})\,. \tag{A.48f}$$
For $i = N_1+1,\dots,2N_1$ and $j = 2N_1+1,\dots,2N_1+N_2$,
$$F_{ij}(x,t) = G^{(9)}_{j-2N_1}(w^*_{i-N_1})\,. \tag{A.48g}$$
For $i = N_1+1,\dots,2N_1$ and $j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$,
$$F_{ij}(x,t) = G^{(10)}_{j-2N_1-N_2}(w^*_{i-N_1})\,. \tag{A.48h}$$
For $i = 2N_1+1,\dots,2N_1+N_2$ and $j = 1,\dots,N_1$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(z_{i-2N_1})G^{(4)}_j(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(z_{i-2N_1})G^{(4)}_j(z^*_n)\,. \tag{A.48i}$$
For $i = 2N_1+1,\dots,2N_1+N_2$ and $j = N_1+1,\dots,2N_1$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(z_{i-2N_1})G^{(3)}_{j-N_1}(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(z_{i-2N_1})G^{(8)}_{j-N_1}(\zeta^*_n)\,. \tag{A.48j}$$
For $i, j = 2N_1+1,\dots,2N_1+N_2$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(z_{i-2N_1})G^{(5)}_{j-2N_1}(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(z_{i-2N_1})G^{(9)}_{j-2N_1}(\zeta^*_n)\,. \tag{A.48k}$$
For $i = 2N_1+1,\dots,2N_1+N_2$ and $j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_3} [G^{(1)}_n(z_{i-2N_1})G^{(6)}_{j-2N-1-N_2}(z^*_n) + G^{(2)}_n(z_{i-2N_1})G^{(10)}_{j-2N_1-N_2}(\zeta^*_n)]\,. \tag{A.48l}$$
For $i = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$ and $j = 1,\dots,N_1$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(\zeta_{i-2N_1-N_2})G^{(4)}_j(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(\zeta_{i-2N_1-N_2})G^{(4)}_j(z^*_n)\,. \tag{A.48m}$$
For $i = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$ and $j = N_1+1,\dots,2N_1$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(\zeta_{i-2N_1-N_2})G^{(3)}_{j-N_1}(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(\zeta_{i-2N_1-N_2})G^{(8)}_{j-N_1}(\zeta^*_n)\,. \tag{A.48n}$$
For $i = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$ and $j = 2N_1+1,\dots,2N_1+N_2$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_2} G^{(1)}_n(\zeta_{i-2N_1-N_2})G^{(5)}_{j-2N_1}(z^*_n) + \sum_{n=1}^{N_3} G^{(2)}_n(\zeta_{i-2N_1-N_2})G^{(9)}_{j-2N_1}(\zeta^*_n)\,. \tag{A.48o}$$
Finally, for $i, j = 2N_1+N_2+1,\dots,2N_1+N_2+N_3$,
$$F_{ij}(x,t) = \sum_{n=1}^{N_3} [G^{(1)}_n(\zeta_{i-2N_1-N_2})G^{(6)}_{j-2N-1-N_2}(z^*_n) + G^{(2)}_n(\zeta_{i-2N_1-N_2})G^{(10)}_{j-2N_1-N_2}(\zeta^*_n)]\,. \tag{A.48p}$$
As usual, the (x,t)-dependence was omitted from the right hand side of the above equations for brevity.

Proof of Theorem 5.1. We consider the equations for the eigenfunctions in Theorem 4.4 in the reflectionless case. Evaluating the second and third entries of these equations at the appropriate eigenvalues, we obtain the following algebraic system of equations for $k = 1,2$ and $\bar{k} = 3-k$:

$$m^+_{(k+1)2}(z_{i'}) = (-1)^{k+1}\frac{q^*_{+,\bar{k}}}{q_o} + \sum_{n=1}^{N_2}\left[\frac{\hat{D}_n}{z_{i'}-z^*_n} - \frac{iz^*_n}{q_o}\frac{\check{D}_n}{z_{i'}-\hat{z}^*_n}\right]m^-_{(k+1)3}(z^*_n)$$
$$+\sum_{n=1}^{N_3}\left[\frac{\hat{F}_n}{z_{i'}-\zeta^*_n} - \frac{i\zeta^*_n}{q_o}\frac{\check{F}_n}{z_{i'}-\hat{\zeta}^*_n}\right]m^-_{(k+1)1}(\zeta^*_n), \qquad i' = 1,\dots,N_2, \tag{A.49a}$$

$$m^+_{(k+1)2}(\zeta_{\ell'}) = (-1)^{k+1}\frac{q^*_{+,\bar{k}}}{q_o} + \sum_{n=1}^{N_2}\left[\frac{\hat{D}_n}{\zeta_{\ell'}-z^*_n} - \frac{iz^*_n}{q_o}\frac{\check{D}_n}{\zeta_{\ell'}-\hat{z}^*_n}\right]m^-_{(k+1)3}(z^*_n)$$
$$+\sum_{n=1}^{N_3}\left[\frac{\hat{F}_n}{\zeta_{\ell'}-\zeta^*_n} - \frac{i\zeta^*_n}{q_o}\frac{\check{F}_n}{\zeta_{\ell'}-\hat{\zeta}^*_n}\right]m^-_{(k+1)1}(\zeta^*_n), \qquad \ell' = 1,\dots,N_3, \tag{A.49b}$$

$$m^-_{(k+1)3}(z^*_{i'}) = \frac{q_{+,k}}{q_o} + \sum_{n=1}^{N_1}\left[\frac{\hat{C}_n m^-_{(k+1)1}(w^*_n)}{z^*_{i'}-w^*_n} - \frac{iw_n}{q_o}\frac{\overline{C}_n m^+_{(k+1)3}(w_n)}{z^*_{i'}-\hat{w}_n}\right]$$
$$+\sum_{n=1}^{N_2}\frac{D_n m^+_{(k+1)2}(z_n)}{z^*_{i'}-z_n} + \sum_{n=1}^{N_3}\frac{\bar{F}_n m^+_{(k+1)2}(\zeta_n)}{z^*_{i'}-\hat{\zeta}_n}, \qquad i' = 1,\dots,N_2, \tag{A.49c}$$

$$m^-_{(k+1)1}(w^*_{j'}) = \frac{iq_{+,k}}{w^*_{j'}} + \sum_{n=1}^{N_1}\left[\frac{C_n m^+_{(k+1)3}(w_n)}{w^*_{j'}-w_n} - \frac{iw^*_n}{q_o}\frac{\check{C}_n m^-_{(k+1)1}(w^*_n)}{w^*_{j'}-\hat{w}^*_n}\right] + \sum_{n=1}^{N_2}\frac{\overline{D}_n m^+_{(k+1)2}(z_n)}{w^*_{j'}-\hat{z}_n}$$
$$+\sum_{n=1}^{N_3}\frac{F_n m^+_{(k+1)2}(\zeta_n)}{w^*_{j'}-\zeta_n}, \qquad j' = 1,\dots,N_1, \tag{A.49d}$$

$$m^-_{(k+1)1}(\zeta^*_{\ell'}) = \frac{iq_{+,k}}{\zeta^*_{\ell'}} + \sum_{n=1}^{N_1}\left[\frac{C_n m^+_{(k+1)3}(w_n)}{\zeta^*_{\ell'}-w_n} - \frac{iw^*_n}{q_o}\frac{\check{C}_n m^-_{(k+1)1}(w^*_n)}{\zeta^*_{\ell'}-\hat{w}^*_n}\right] + \sum_{n=1}^{N_2}\frac{\overline{D}_n m^+_{(k+1)2}(z_n)}{\zeta^*_{\ell'}-\hat{z}_n}$$
$$+\sum_{n=1}^{N_3}\frac{F_n m^+_{(k+1)2}(\zeta_n)}{\zeta^*_{\ell'}-\zeta_n}, \qquad \ell' = 1,\dots,N_3, \tag{A.49e}$$

$$m^+_{(k+1)3}(w_{j'}) = \frac{q_{+,k}}{q_o} + \sum_{n=1}^{N_1}\left[\frac{\hat{C}_n m^-_{(k+1)1}(w^*_n)}{w_{j'}-w^*_n} - \frac{iw_n}{q_o}\frac{\overline{C}_n m^+_{(k+1)3}(w_n)}{w_{j'}-\hat{w}_n}\right] + \sum_{n=1}^{N_2}\frac{D_n m^+_{(k+1)2}(z_n)}{w_{j'}-z_n}$$
$$+\sum_{n=1}^{N_3}\frac{\bar{F}_n m^+_{(k+1)2}(\zeta_n)}{w_{j'}-\hat{\zeta}_n}, \qquad j' = 1,\dots,N_1, \tag{A.49f}$$

where, as before, the (x,t)-dependence was omitted for simplicity. Next, we substitute (A.49c) and (A.49e) into both (A.49a) and (A.49b) and combine the result with (A.47) to obtain the following for $z = z_{i'}$ and $z = \zeta_{\ell'}$:

$$m^+_{(k+1)2}(z) = (-1)^{k+1}\frac{q^*_{+,\bar{k}}}{q_o} + \frac{q_{+,k}}{q_o}\sum_{n=1}^{N_2}G^{(1)}_n(z) + iq_{+,k}\sum_{n=1}^{N_3}\frac{G^{(2)}_n(z)}{\zeta^*_n}$$
$$+\sum_{n=1}^{N_2}\sum_{n'=1}^{N_1}G^{(1)}_n(z)[G^{(3)}_{n'}(z^*_n)m^-_{(k+1)1}(w^*_{n'}) + G^{(4)}_{n'}(z^*_n)m^+_{(k+1)3}(w_{n'})]$$
$$+\sum_{n=1}^{N_2}\sum_{n'=1}^{N_2}G^{(1)}_n(z)G^{(5)}_{n'}(z^*_n)m^+_{(k+1)2}(z_{n'})$$
$$+\sum_{n=1}^{N_2}\sum_{n'=1}^{N_3}G^{(1)}_n(z)G^{(6)}_{n'}(z^*_n)m^+_{(k+1)2}(\zeta_{n'})$$

$$+\sum_{n=1}^{N_3}\sum_{n'=1}^{N_1} G_n^{(2)}(z)[G_{n'}^{(7)}(\zeta_n^*)m_{(k+1)3}^{+}(w_{n'})+G_{n'}^{(8)}(\zeta_n^*)m_{(k+1)1}^{-}(w_{n'}^*)]$$
$$+\sum_{n=1}^{N_3}\sum_{n'=1}^{N_2} G_n^{(2)}(z)G_{n'}^{(9)}(\zeta_n^*)m_{(k+1)2}^{+}(z_{n'})+\sum_{n=1}^{N_3}\sum_{n'=1}^{N_3} G_n^{(2)}(z)G_{n'}^{(10)}(\zeta_n^*)m_{(k+1)2}^{+}(\zeta_{n'}). \quad \text{(A.50)}$$

Together, equations (A.49*d*), (A.49*f*), and (A.50) comprise closed systems of linear equations. We now rewrite this system so as to solve it using Cramer's rule. First, we define $\mathbf{x}_k = (x_{k1},\dots,x_{k(2N_1+N_2+N_3)})^T$ for $k=1,2$, where

$$x_{kj} = \begin{cases} m_{(k+1)3}^{+}(w_j), & j=1,\dots,N_1,\\ m_{(k+1)1}^{-}(w_{j-N_1}^*), & j=N_1+1,\dots,2N_1,\\ m_{(k+1)2}^{+}(z_{j-2N_1}), & j=2N_1+1,\dots,2N_1+N_2,\\ m_{(k+1)2}^{+}(\zeta_{j-2N_1-N_2}), & j=2N_1+N_2+1,\dots,2N_1+N_2+N_3. \end{cases}$$

Then we may rewrite the above closed systems of equations as $(\mathbf{I}-\mathbf{F})\mathbf{x}_k = \mathbf{b}_k$, where the remaining quantities are as defined in the theorem. Using Cramer's rule, it is easy to see that the components of the solutions of the closed systems are

$$x_{kj} = \frac{\det \mathbf{G}_{kj}^{\text{aug}}}{\det \mathbf{G}}, \qquad j=1,\dots,2N_1+N_2+N_3, \quad k=1,2,$$

where $G_{kj}^{\text{aug}} = (\mathbf{G}_1,\dots,\mathbf{G}_{j-1},\mathbf{b}_k,\mathbf{G}_{j+1},\dots,\mathbf{G}_{2N+N_2+N_3})$. Substituting this into the reconstruction formula (4.9) and using the definition (A.46) of the y_j yields the desired results. □

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