The Focusing Problem for the Eikonal Equation

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Dedicated to the Memory of Philippe Bénilan

ABSTRACT. We study the focusing problem for the eikonal equation

$$\partial_t u = |\nabla u|^2 ,$$

i.e., the initial value problem in which the support of the initial datum is outside some compact set in $\mathbf{R}^d$. The hole in the support will be filled in finite time and we are interested in the asymptotics of the hole as it closes. We show that in the radially symmetric case there are self-similar asymptotics, while in the absence of radial symmetry essentially any convex final shape is possible. However, for generic initial data the asymptotic shape will be either a vanishing triangle or the region between two parabolas moving in opposite directions (a closing eye). We compare these results with the known results for the porous medium pressure equation which approaches the eikonal equation in the limit as $m \to 1$.

1. Introduction

In this paper we compare the focusing problem for the eikonal equation

$$\partial_t u = |\nabla u|^2 \tag{EE}$$

with the corresponding problem for the porous medium (pressure) equation

$$\partial_t u = (m-1)u\Delta u + |\nabla u|^2 , \tag{PME}$$

in $\mathbb{R}^d \times \mathbb{R}^+$ for $d \geq 1$. Here ∇ denotes the gradient with respect to the spatial variables $(x_1, x_2, \cdots, x_d)$. The quantity $m > 1$ in (PME) is a constant.

Consider the initial value problem for either (EE) or (PME) with the initial datum $u(x,0) = u_0(x)$. We assume that u_0 is compactly supported and has a "regular free boundary", i.e.,

$$u_0(x) = \max(\widetilde{u}_0(x), 0)$$

for some $\widetilde{u}_0 \in C^\infty(\mathbf{R}^d)$, where 0 is a regular value of $\widetilde{u}_0$ and $\{x \mid \widetilde{u}_0(x) \geq 0\}$ is compact. The viscosity solution to the initial value problem for (EE) with initial datum $u(x,0) = u_0(x)$ is a continuous function of (x,t) (cf. [**17**]). The same is true for the generalized solution to the corresponding initial value problem for (PME) (cf. [8]). The zero set

$$Z_t = \{x \in \mathbf{R}^d : u(x,t) = 0\}$$

of either the viscosity or generalized solution has one unbounded component. Let K_t denote the union of all bounded components of Z_t. For the *focusing problem* we assume that K_0 is a compact simply connected set, i.e., that the support of u_0 "has a hole." The sets K_t form a nonincreasing and eventually strictly decreasing family of subsets of $\mathbf{R}^d$ which will become empty at some finite time T

$$T = \inf\{t : K_t = \varnothing\}$$

is called the *focusing time* and we say that the solution has focused (or filled the hole K_0) at time $t = T$.

The eikonal equation (EE) shows up as a formal limit of (PME) as $m \searrow 1$, and indeed, it has been shown in [**17**] that weak solutions to the initial value problem for (PME) converge uniformly as $m \searrow 1$ to viscosity solutions of the corresponding problem for (EE). The absence of diffusion makes the eikonal equation considerably simpler than (PME), and in fact there is even an explicit formula for the viscosity solution in terms of its initial data,

$$u(x,t) = \sup_{y\in\mathbb{R}^d}\left\{u_0(y) - \frac{|x-y|^2}{4t}\right\}. \tag{1}$$

This representation of the solution is called the *Lax-Hopf formula.* See [**17**] or [**15**] for a discussion. From this formula for the viscosity solution one immediately deduces the following representation of the free boundary. The free boundary is the graph in space-time of the *filling time*

$$T_*(x) = \inf_{u(x,t)>0} t = \inf_{u_0(y)>0} \frac{|x-y|^2}{4u_0(y)}. \tag{2}$$

The Lax-Hopf formula and the formula (2) for the filling time allow us to study focusing of solutions to (EE) in much more detail than can be done at present for (PME). In this note we compare the results on focusing for the eikonal equation with what is known for (PME).

In the non-generic case of radial symmetry, K_t is, of course, a d-dimensional ball. We show that in this case the viscosity solution u for (EE) behaves like a self-similar solution near focusing. More precisely, we show that

$$\lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} u(\varepsilon\xi, T+\varepsilon\tau) = c|\xi| + c^2\tau \tag{3}$$

uniformly for bounded $\xi \in \mathbb{R}^d$ and bounded $\tau \in \mathbb{R}$. This extends our results [**1**] for (PME) to the case $m = 1$. In [**1**] it was shown for any $1 < m < \infty$ that for a radially symmetric weak solution of (PME) which focusses at $t = T$ one has

$$\lim_{\varepsilon\searrow 0} \varepsilon^{\alpha-2} u(\varepsilon\xi, T-\varepsilon^{\alpha}\tau) = c^2 V_m\left(\frac{|\xi|}{c\tau^{1/\alpha}}\right),$$

in which V_m is the Aronson-Graveleau profile, $\alpha \in (1,2)$ is the corresponding exponent, and $0 < c < \infty$ is a positive constant depending on the initial data.

In the nonradial case we show that the eikonal equation admits many more self-similar focusing solutions than PME. In particular we show that *for any closed convex set $C \subset$*

$\mathbb{R}^d$ *containing the origin there is a self-similar viscosity solution* $S(x,t)$ (i.e. one which satisfies $S(\lambda x, \lambda t) = \lambda S(x,t)$ for all $\lambda > 0$) such that

$$\{x \in \mathbb{R}^d \mid S(x,t) = 0\} = (-t)C$$

for all $t < 0$. In other words, for the eikonal equation "a vanishing hole can have any convex shape."

This contrasts sharply with the porous medium equation. All self-similar focusing solutions for PME which have been constructed either analytically [**2**] or numerically [**3**] have some kind of discrete symmetry. Moreover, in [**2**] it is shown that self-similar focusing solutions satisfy a nonlinear elliptic free boundary problem which can be transformed to a nonlinear Fredholm equation. One therefore expects self-similar focusing solutions to PME to occur in discrete families, and not in infinite dimensional continua as is the case for the eikonal equation.

We next consider small perturbations of radially symmetric initial data. We show that, to leading order, ∂K_t propagates according to *Huygen's principle,* i.e., with constant normal velocity. More precisely, if $u^0(x,t) = U(|x|,t)$ is a radially symmetric solution which focusses at time $t = T$, then we observe that the viscosity solution u^ε with initial data $u^\varepsilon(x,0) = U(|x|) + \varepsilon g(x)$ satisfies

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} u^\varepsilon(\varepsilon\xi, T + \varepsilon\tau) = V(\xi, \tau) \tag{4}$$

where V is a viscosity solution of the Huygens-Hamilton-Jacobi equation

$$\partial_\tau V = c|\nabla V|. \tag{5}$$

The level sets of viscosity solutions to this equation are fronts which evolve by Huygens' principle, i.e. they propagate with constant normal velocity c. Moreover, we observe that *any solution of* (5) *can occur as the limit in* (4), provided $|\nabla V| \equiv c$.

This implies that radial focusing is unstable: a small perturbation will turn a radial focusing solution into one of the two generic focusing solutions described below. This instability also appears to be present in (PME). For (PME) radial holefilling is described by the Aronson-Graveleau profile, but if one perturbs a radial solution slightly, then numerical computations indicate that the perturbations will grow, and that the solution will generally not return to radial symmetry. However, for (PME) one expects a sufficiently *symmetric* peturbation to die out and disappear, at least if m is large enough. The instability of radial focusing for (EE) combined with Lions-Souganidis-Vázquez' convergence result [**17**] gives a heuristic reason for the occurrence of the infinite sequence of bifurcations found in [**2**]: for large values of m the diffusion term in (PME) is dominant, and the radial self-similar solution will only have a small number of unstable modes, but for $m = 1 + o(1)$ the solution to (PME) tries to follow the viscosity solution to (EE), where radial focussing has infinitely many unstable modes.

Finally, we consider generic solutions. For generic initial data, i.e. for an open and dense set of $\tilde{u}_0 \in C^\infty(\mathbb{R}^d)$, K_t shrinks to a point and, when $d = 2$, we can characterize the generic possibilities for the final form of ∂K_t. These are either

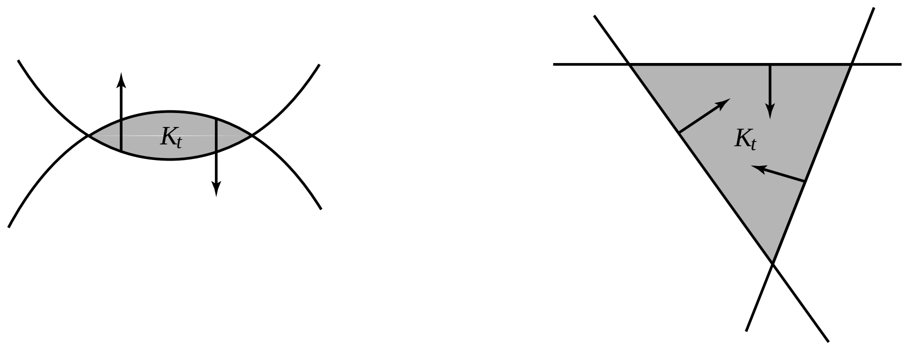

FIGURE 1. The closing eye and vanishing triangle solutions

A Closing Eye. K_t approximates the region between two parabolae

$$y = \pm\{c_\pm(T-t) - x^2\}.$$

The width of K_t is $O\left(\sqrt{T-t}\right)$ while its height is $O(T-t)$. The aspect ratio of K_t tends to infinity as $t \nearrow T$ (cf. Figure 1).

A Vanishing Triangle. K_t is approximately the region enclosed by three straight lines which propagate with constant normal velocity, and which pass through a common point at $t = T$ (cf. Figure 1)

Numerics and asymptotic analysis strongly suggest that there are solutions to (PME) close to the closing eye ([**3, 11**]) for any $m > 1$, even though there is as yet no rigorous theory.

The situation with the vanishing triangle is different: numerics again suggest that there exists a self-similar solution whose free boundary approximates a triangle when $m-1$ is small enough, but this triangle *must be an equilateral triangle.* Thus it seems that the second kind of generic hole filling for (EE) is highly nongeneric for (PME). Again, rigorous proofs for the statements about (PME) are lacking.

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2. Radial Solutions

If the initial function $u_0(x)$ is radially symmetric, i.e. $u_0(x) = U_0(r)$, $r = |x|$, then the viscosity solution $u(t,x)$ will be radially symmetric for all $t > 0$. Indeed, the Lax-Hopf formula reduces to

$$U(t,r) = \sup_{\rho > 0}\left\{U_0(\rho) - \frac{(r-\rho)^2}{4t}\right\}. \tag{6}$$

This equation is obtained from (1) by maximizing over all y with $|y| = \rho$. So we see that the radial solution is exactly the same as the viscosity solution in one space dimension.

This makes it easy to understand generic radial hole filling. Consider an initial function $U_0(r)$ as in Figure 2.

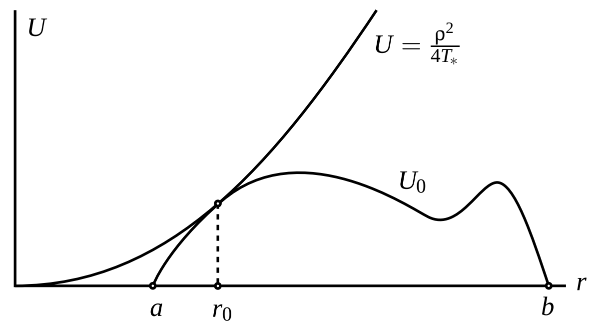

FIGURE 2. Computing the Radial Hole Filling Time

The unique hole in the center will fill up in time

$$T_* = \sup_{\rho>0, U_0(\rho)>0} \frac{\rho^2}{4U_0(\rho)}.$$

Let us assume that the function $\rho \mapsto \rho^2/4U_0(\rho)$ has a unique maximum at $\rho = r_0$ as will be the case for a generic smooth initial U_0. Then, at the hole filling time T_* the solution near $r = 0$ will be a smooth function of r, and if we allow $r < 0$ in (6), then it extends to a smooth function in a full neighborhood in $\mathbb{R}^2$ of $(r = 0, t = T_*)$. At $r = 0$, $t = T_*$ the method of characteristics tells us that $U_r = U'(r_0)$; the eikonal equation enforces $U_t(0, T_*) = (U_r)^2 = U'(r_0)^2$. Thus, locally we have

$$U(r,t) = U'(r_0)r + U'(r_0)^2(t - T_*) + O\left(r^2 + (t - T_*)^2\right).$$

Hence

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} U(\varepsilon R, T_* + \varepsilon\tau) = U'(r_0)R + U'(r_0)^2\tau.$$

As we noted in the Introduction, this is analogous to the radial hole-filling asymptotics for the porous medium equation. Here we see that that radial hole filling is described by a self-similar solution

$$U_c(r,t) = c(r + ct)_+,$$

and that these self-similar solutions occur in a one parameter family. For the porous medium equation the self-similar solutions are the Aronson-Graveleau solutions, for the eikonal equation they are simply plane waves converging upon the origin.

3. Nonradial Self-Similar Solutions

For any vector $\xi \in \mathbb{R}^d$ the plane wave solution

$$W_\xi(x,t) = \left(\xi \cdot x + |\xi|^2 t\right)_+$$

is a self-similar solution which reaches the origin at $t = 0$.

LEMMA 1. *For any closed subset $F \subset \mathbb{R}^d$ the function*

$$S_F(x,t) = \sup_{\xi \in F} W_\xi(x,t) \tag{7}$$

is a self-similar solution which reaches the origin at time $t = 0$.

The "hole" in support of the self-similar solutions constructed here is the set

$$\begin{aligned} K_F(t) &= \{x \in \mathbb{R}^d \mid S_F(x,t) = 0\} \\ &= \{x \in \mathbb{R}^d \mid \forall_{\xi \in F} \xi \cdot x + |\xi|^2 t \le 0\} \\ &= \{x \in \mathbb{R}^d \mid \forall_{\xi \in F} \frac{\xi}{|\xi|} \cdot x + |\xi| t \le 0\} \\ &= \bigcap_{\xi \in F} \left\{ x \mid \frac{\xi}{|\xi|} \cdot x \le -|\xi| t \right\}. \end{aligned} \tag{8}$$

Thus we see that for $t < 0$ the hole $K_F(t)$ is a nonempty convex set whose support function $p : S^{d-1} \to \mathbb{R}$ is given by

$$p(\omega) = \inf\{p > 0 \mid -tp\omega \in F\}$$

Furthermore, *any compact convex set $C \subset \mathbb{R}^d$ containing the origin can be represented as $C = K_F(-1)$, with $K_F(-1)$ defined as in* (8) *for some suitably chosen $F \subset \mathbb{R}^d$.* Therefore there are *uncountably many* self similar solutions, and that self-similarity of a solution to the eikonal equation does not guarantee any kind of symmetry. This is in contrast with the porous medium equation. For PME not all self-similar solutions are known either rigrously or computationally. However all self-similar solutions whose existence has been proven analytically or by numerical computation have some kind of symmetry. This poses the following

Open problem: *which of the self-similar solutions S_F of the eikonal equation arise as limits of self-similar solutions of PME as $m \searrow 1$?*

PROOF OF LEMMA 1. We apply the Lax-Hopf formula to find the viscosity solution v at time $s > 0$ with initial datum $S_F(\cdot, t)$:

$$\begin{aligned} v(s,x) &= \sup_y \left\{ S_F(t,y) - \frac{|x-y|^2}{4s} \right\} \\ &= \sup_y \left\{ \sup_{\xi \in F} W_\xi(y,t) - \frac{|x-y|^2}{4s} \right\} \\ &= \sup_{\xi \in F} \left\{ \sup_y W_\xi(y,t) - \frac{|x-y|^2}{4s} \right\} \\ &= \sup_{\xi \in F} W_\xi(x, t+s) \\ &= S_F(x, t+s) \end{aligned}$$

where we have used the fact that W_ξ itself is a smooth solution to (EE) and hence also a viscosity solution. □

4. Perturbation of Radial Solutions

In this section we observe that radial focusing is unstable.

Let $U_1 : \mathbb{R}^d \to \mathbb{R}$ be any given function, and consider the initial data

$$u^\varepsilon(x) = U_0(r) + \varepsilon U_1(r\theta),$$

where $x = r\theta$, $r > 0$ and $|\theta| = 1$, i.e. $\theta \in S^{d-1}$ is a unit vector.

The viscosity solution to the initial value problem for (EE) with initial datum u_0^ε is

$$u^\varepsilon(x,t) = \sup_{y \in \mathbb{R}^d} F(x,t;y,\varepsilon),$$

where

$$F(x,t;y,\varepsilon) = U_0(|y|) + \varepsilon U_1(y) - \frac{|x-y|^2}{4t}.$$

Set $x = \varepsilon\xi$, $t = T_* + \varepsilon\tau$, and $y = \rho\omega$ for $\omega \in S^{d-1}$, and maximize first over $\rho > 0$ keeping ω fixed. For $\varepsilon = 0$ the function to be maximized $\rho \mapsto F(0, T_*; \rho\omega, 0)$ has a unique nondegenerate maximum at $\rho = r_0$. since this maximum describes focusing we have

$$F(0, T_*; r_0\omega, 0) = u^\varepsilon(0, T_*) = 0. \tag{9}$$

and

$$\omega \cdot F_y(0, T_*; r_0\omega, 0) = \frac{\partial}{\partial\rho} F(0, T_*; \rho\omega, 0)\Big|_{\rho = r_0} = 0. \tag{10}$$

By the Implicit Function Theorem $\rho \mapsto F(x,t;\rho\omega,\varepsilon)$ will have a nondegenerate maximum at $\rho = r_\varepsilon(x,t)$ where $r_\varepsilon(x,t)$ depends smoothly on ε, x and t. We now compute

$$\frac{1}{\varepsilon} u^\varepsilon(x,t) = \max_{|\omega|=1} \varepsilon^{-1} F(x,t; r_\varepsilon(x,t)\omega, \varepsilon)$$

by expanding F in a Taylor series. At the maximizing ω we have $F_y = 0$, so we get

$$\begin{aligned} &F(x,t,;r_\varepsilon(x,t)\omega,\varepsilon) \\ &\quad = F(\varepsilon\xi, T_* + \varepsilon\tau; (r_0 + O(\varepsilon))\omega, \varepsilon) \\ &\quad = \varepsilon\{\xi \cdot F_x(0,T_*;r_0\omega,0) + \tau F_t(0,T_*;r_0\omega,0) + F_\varepsilon(0,T_*;r_0\omega,0)\} + O(\varepsilon^2) \end{aligned}$$

Computing the relevant derivatives of F and using (9) we get finally,

$$\frac{1}{\varepsilon}u^\varepsilon(x,t) = \max_{|\omega|=1}\left\{U_1(r_0\omega) + \frac{r_0}{2T_*}\xi\cdot\omega\right\} + \frac{r_0^2}{4T_*^2}\tau + O(\varepsilon).$$

so that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}u^\varepsilon(x,t) = cP(\xi) + c^2\tau, \tag{11}$$

in which

$$P(\xi) = \max_{|\omega|=1}\frac{1}{c}U_1(r_0\omega) + \xi\cdot\omega$$

and

$$c = \frac{r_0}{2T_*}.$$

At time $\tau < 0$ the zeroset of the limiting solution in (11) is given by

$$K_\tau = \bigcap_{|\omega|=1}\{\xi \mid \xi\cdot\omega \le -c\tau - \frac{1}{c}U_1(r_0\omega)\}.$$

For sufficently large $\tau < 0$ we see that K_τ is a convex set with smooth boundary, whose support function is $p(\tau,\omega) = -c\tau - \frac{1}{c}U_1(r_0\omega)$. This support function decreases with constant rate $\partial_\tau p = -c$, so ∂K_τ shrinks with constant speed c. At some moment ∂K_τ will develop a singularity, and after that K_τ still shrinks with constant velocity in the sense of viscosity solutions.

Note that the support function $-c - c^{-1}U_1(r_0\omega)$ of ∂K_{-1} can be arbitrary, so that *the limiting shape of the shrinking hole can be* ***any*** *convex front moving by Huygens' principle.*

5. Generic Solutions and the Method of Characteristics

We briefly recall the method of characteristics. In this approach to the solution of (EE) one allows the solution $u(x,t)$ to become multiply valued and considers the graph of this solution together with its space-time gradient, i.e.

$$\Lambda_u = \{(x,t,u(x,t),\nabla u(x,t),u_t(x,t)) \mid (x,t) \in \Omega_t \times [0,\infty)\}$$

where $\Omega_t \subset \mathbb{R}^d$ is the domain of the possibly multiply valued function $x \mapsto u(x,t)$. In the language of geometric optics, one calls Λ_u a *front*.

This graph Λ_u is a subset of the 1-jet space

$$J^1(\mathbb{R}^{d+1}) = \{(x,t,u,p,\dot{u}) \mid x \in \mathbb{R}^d, t \in \mathbb{R}, u \in \mathbb{R}, p \in \mathbb{R}^d, \dot{u} \in \mathbb{R}\}$$

of $\mathbb{R}^{d+1}$. Points in $J^1(\mathbb{R}^{d+1})$ are called *1-jets*, or also *contact elements.*

On the space of contact elements one has the contact form

$$\theta = du - p_1dx_1 - \cdots - p_ddx_d - \dot{u}dt.$$

Λ_u is a $d+1$ dimensional submanifold of $J^1(\mathbb{R}^{d+1})$, and it is even an integral manifold for θ.

The initial datum we prescribe is a smooth function $u_0 : \Omega_0 \to \mathbb{R}$, on some bounded domain $\Omega_0 \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega_0$, such that $\nabla u_0 \neq 0$ on $\partial\Omega_0$, while $u_0(x) > 0$ on Ω_0. The front Λ_u is determined by the initial data through the characteristic flow: namely,

(i) Λ_u contains the initial surface

$$\Lambda_{u_0} = \left\{(x, 0, u_0(x), \nabla u_0(x), \dot{u}_0(x)) \mid x \in \mathbb{R}^d\right\},$$

where $\dot{u}_0(x) = |\nabla u_0(x)|^2$ is the initial velocity, and

(ii) Λ_u is invariant under the characteristic flow. For (EE) this flow is given by

$$\begin{cases} \dfrac{dx_k}{d\tau} = 2p_k, \quad \dfrac{dt}{d\tau} = 1, \quad \dfrac{dp_k}{d\tau} = 0, \\ \dfrac{du}{d\tau} = -\{p_1^2 + \cdots + p_d^2\}, \quad \dfrac{d\dot{u}}{d\tau} = 0 \end{cases} \tag{12}$$

(τ is the time variable for the flow.)

These equations are easily integrated, and they lead to the following parametrization of Λ_u

$$\begin{cases} X(t,y) = y - 2t\nabla u_0(y), \\ U(t,y) = u_0(y) - t|\nabla u_0(y)|^2, \\ P(t,y) = \nabla u_0(y) \end{cases} \tag{13}$$

LEMMA 2. *The map from $\Omega_0 \times [0,\infty)$ to $J^1(\mathbb{R}^{d+1})$ given by*

$$(y,t) \mapsto (X(t,y), t, U(t,y), P(t,y), \dot{U}(t,y))$$

with $\dot{U}(y) = |\nabla u_0(y)|^2$ is a Legendre embedding.

PROOF. The map $(X,t,U,P) \mapsto (t,y) = (t, X + 2tP)$ is a differentiable inverse for our given map, which shows that it is an embedding.

To verify that it is a Legendre embedding one must show that $dU - P \cdot dX - \dot{U}dt = 0$. This can be done by a short direct computation, but it is also guaranteed by the method of characteristics. □

Thus Λ_u is in particular a smooth submanifold of $J^1(\mathbb{R}^{d+1})$.

If one only looks at the graph of the solution u, rather than of $(u, \nabla u, u_t)$, then one finds the following set

$$\Gamma = \{(X(t,y), t, U(t,y)) \mid y \in \Omega_0, t \geq 0\}$$

This is a submanifold of $\mathbb{R}^d \times [0,\infty) \times \mathbb{R}$ which generally has singularities. The map

$$(t,y) \mapsto (X(t,y), t, U(t,y))$$

is called a *Legendre mapping* (see **[7]**.)

If one extends the initial data u_0 by setting $u_0(x) = 0$ for $x \notin \Omega_0$, then the Lax-Hopf formula (1) provides us with a viscosity solution $u^*(x,t)$. One can recover u^* from Γ by considering the upper envelope of Γ,

$$u^*(x,t) = \max_{(x,t,u)\in\Gamma} u = \begin{cases} \max_{X(t,y)=x} U(t,y) & \text{if } \exists_{y\in\Omega_0} X(t,y) = x \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

This follows from the Lax-Hopf formula (1): if $y \in \mathbb{R}^d$ maximizes $u_0(y) - |x-y|^2/4t$, then, since the viscosity solution is nonnegative one has $u_0(y) \geq |x-y|^2/4t$. Thus either $u_0(y) > 0$ and hence $y \in \Omega_0$, or $u_0(y) = 0$, $x = y$ and $u^*(x,t) = 0$. If $y \in \Omega_0$, then the fact that $u_0(y) - |x-y|^2/4t$ has a maximum at y implies $x = y - 2t\nabla u_0(y) = X(t,y)$ and $u(x,t) = U(t,y)$.

The singularities of Legendre mappings in general have been studied by the Arnol'd school [**7**]. The singularities which arise in generic solutions to (EE) and more general Hamilton-Jacobi equations, have been studied by I.A.Bogaevski [**12, 13, 14**]. He presents a classification the possible singularities by their topological type, at least in dimensions $d = 2$ and 3 (the case $d = 1$ is classical.) In higher dimensions the classification seems to be very complicated, and even unknown.

Before going on to the free boundary of a solution, we briefly recall some of the singularities found by Bogaevski.

For $d = 1$ a generic choice of initial data u_0 (i.e. for u_0 in an open and dense subset in the class of C^∞ functions) the viscosity solution u^* will only have the following singularities:

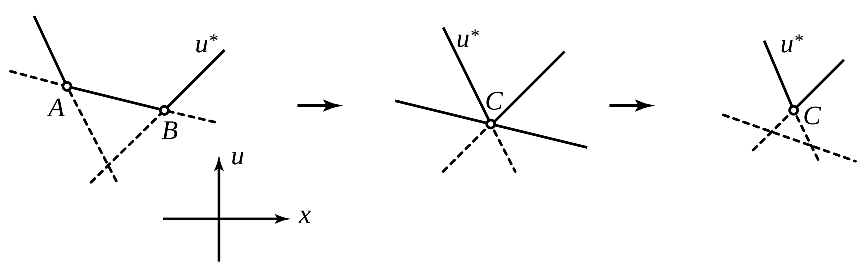

FIGURE 3. Two corners A and B merge into one new corner C $(d = 1)$

Except at a finite number of times the viscosity solution u^* has only simple corners, which come from transverse self-intersections of the Legendre map defined in (14). At isolated moments in time one such corner can be created in a so-called swallow tail singularity (see figure 4), or two corners can meet and combine to form one new corner (figure 3.)

The more complicated singularities which can only occur at isolated instances in time (swallowtail, and merging corners) have been called *perestroikas* by the Russian school since they describe how one constellation of simpler singularities can transform into another.

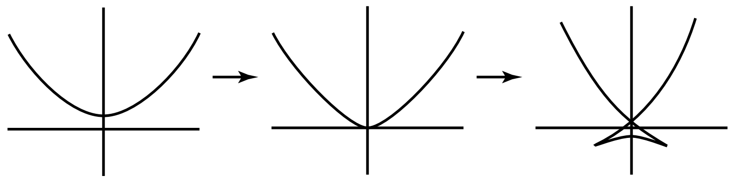

FIGURE 4. Birth of a corner through a swallowtail perestroika $(d = 1)$

By combining the curves in figure 4 into one surface one obtains the set Γ (figure 5.) The graph of the viscosity solution is obtained by removing the curved tetrahedron $ABCD$ which is "at the bottom" of the surface Γ.

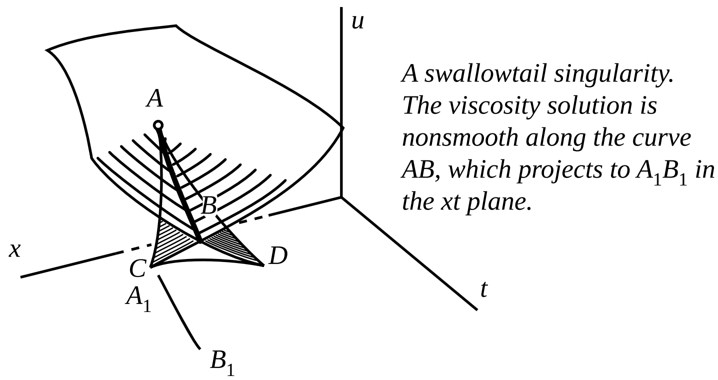

FIGURE 5. A Typical Swallowtail Singularity

6. Generic Hole Filling

We now consider the free boundary of a solution. The free boundary of the viscosity solution is the graph of the filling time $T_*(x)$ defined in the introduction (see (2).) The filling time is, in general, not a smooth function. To analyze its singularities we use the method of characteristics which shows us that the free boundary is a subset of a Legendre map, and we find that (for generic initial data $u_0 \in C^\infty(\overline{\Omega}_0)$) its singularities are those of a generic Legendre map. The generic singularities of the filling time are then those obtained by Bogaevski for generic minimum functions [**12**].

The free boundary is given by setting $U(t, y) = 0$. This leads to an equation $(u_0(y) - t|\nabla u_0(y)|^2 = 0)$ for the time t it takes the contact element $(y,\ 0,\ u_0(y),\ \nabla u_0(y),\ \dot{u}_0(y))$ to reach the level $\{u = 0\}$ under the characteristic flow. Let

$$\Theta(y) = u_0(y)/|\nabla u_0(y)|^2$$

be this time. When the contact element reaches $\{u = 0\}$, it finds itself at $x = X(y)$, where

$$X(y) = y - \frac{2u_0(y)}{|\nabla u_0(y)|^2}\nabla u_0(y).$$

The map $y \mapsto (X(y), \Theta(y))$ then parametrizes the extended free boundary (extended, because it will also include those points where Γ hits the set $u = 0$.) In general Θ is not single valued, and the map $y \mapsto (X(y), \Theta(y))$ will have singularities. One can again analyze these by considering the graph of Θ and its gradient. This graph is contained in the 1-jet space $J^1(\mathbb{R}^d)$, it is parametrized by

$$\Theta(y) = \frac{u_0(y)}{|\nabla u_0(y)|^2}, \qquad N(y) = -\frac{\nabla u_0(y)}{|\nabla u_0(y)|^2}.$$

LEMMA 3. *Let $\Omega_0^* = \{y \in \Omega_0 \mid \nabla u_0(y) \neq 0\}$. The map Φ from Ω_0^* into $J^1(\mathbb{R}^d)$ given by*

$$\Phi : y \mapsto (X(y), \Theta(y), N(y))$$

is a smooth, proper Legendre embedding.

PROOF. A smooth inverse is obtained by direct computation,

$$\nabla u_0 = -\frac{N}{|N|^2} \Rightarrow u_0 = \Theta|\nabla u_0|^2 = \Theta|N|^{-2} \Rightarrow y = X + \frac{2u_0}{|\nabla u_0|^2}\nabla u_0 = X + \frac{2\Theta}{|N|^2}N.$$

So the map is a smooth embedding.

As y approaches the zeroset of $\nabla u_0(y)$ the quantity $\Theta(y)$ becomes unbounded, so the map Φ is proper.

To verify that Φ is a Legendre embedding one must show $d\Theta - N \cdot dX = 0$. This can be done by an straightforward computation in which one verifies that $\partial\Theta/\partial y_j = N \cdot (\partial X/\partial y_j)$ for $j = 1, \ldots, d$. Alternatively one can deduce this from the fact that Φ parametrizes the Legendre submanifold of $J^1(\mathbb{R}^d)$ obtained by slicing Λ_u with $\{u = 0\}$. $\square$

The following (easy) observation is the key to the analysis of generic singularities of the free boundary.

LEMMA 4. *Let $u_0 \in C^\infty(\bar{\Omega}_0)$ be given, and let Φ_0 be its corresponding generalized free boundary. Then any sufficiently small perturbation of Φ_0 with in the class of C^∞ Legendre immersions can be achieved by an appropriately chosen small perturbation of the initial data $u_0 \in C^\infty(\bar{\Omega}_0)$.*

We use the characteristic flow to solve the inverse problem in which one determines the initial data from a given free boundary. This is of course only possible for the generalized solution in the sense of Legendre submanifolds: For viscosity solutions the free boundary does not determine the initial data uniquely.

PROOF. Let $\tilde{\Lambda} \subset J^1(\mathbb{R}^d)$ be any Legendre submanifold (such as the extended free boundary, i.e. the image of Φ). We embed this Legendre submanifold in $J^1(\mathbb{R}^{d+1})$ by mapping the contact element $(X, \Theta, N) \in \tilde{\Lambda} \subset J^1(\mathbb{R}^d)$ to the contact element

$$(x, t, u, p, \dot{u}) = \left(X, \Theta, 0, -\frac{N}{|N|^2}, \frac{1}{|N|^2}\right) \in J^1(\mathbb{R}^{d+1}).$$

The union of all characteristics passing throught the points in $J^1(\mathbb{R}^{d+1})$ thus obtained gives us a Legendre submanifold of $J^1(\mathbb{R}^{d+1})$ which satisfies $\dot{u} = |p|^2$. Slicing this large Legendre submanifold with the section $\{t = 0\}$ then gives a Legendre submanifold $\Sigma \subset$

$J^1(\mathbb{R}^d)$. If this last Legendre submanifold Σ is the graph of some smooth function u_1 and its gradient ∇u_1, then u_1 is the initial function whose free boundary is $\tilde{\Lambda}$.

In our situation $\tilde{\Lambda}$ is a small perturbation of the free boundary of the solution with initial function u_0. In this case the initial Legendre submanifold $\Sigma \subset J^1(\mathbb{R}^d)$ corresponding to the perturbed free boundary $\tilde{\Lambda}$ will be a small perturbation of the graph of $(u_0, \nabla u_0)$. Hence Σ will also be a graph of some function $x \mapsto (u_1(x), p(x))$. The Legendre condition forces $p(x) = \nabla u_1(x)$. □

From this lemma we see that the filling time is just a generic minimum function, i.e. generic choices of u_0 lead to functions with the same kind of singularities as functions of the type

$$T(x) = \min_{y \in \Omega_0} F(x, y)$$

in which $F : \Omega_0 \times \Omega_0 \to \mathbb{R}$ is a generically chosen smooth function of two variables.

Close scrutiny of Bogaevski's list of possible singularities of minimum functions reveals that there are only two possible ways in which a hole can be filled when $d = 2$, namely, the vanishing triangle and the closing eye.

The "vanishing triangle" is locally described to leading order by

$$u(x,t) = \max\{0, w_1(x,t), w_2(x,t), w_3(x,t)\}$$

where the w_j are three plane waves converging upon the focusing point at time $t = T$. It is thus an asymptotically self-similar solution. In the language of singularity theory this singularity should be labelled "A_1^3" (there are three values of $y \in \mathbb{R}^2$ which minimize the quantity $Q^x(y) = |x-y|^2/4u_0(y)$ in the definition of the filling time, and these three minima each are nondegenerate critical points of Q^x.)

The "closing eye" is locally described by

$$u(x,t) = \max\{0, q_+(x,t), q_-(x,t)\}.$$

where near $(x,y,t) = (0,0,0)$ one has

$$q_\pm(x,t) = \pm\left\{y - c_\pm x^2 + (1 + 4c_\pm^2 x^2)tx^2 + \cdots\right\}.$$

In the language of singularity theory this singularity should be labelled "A_1^2" (there are two values of $y \in \mathbb{R}^2$ which minimize the quantity $Q^x(y) = |x-y|^2/4u_0(y)$, and these two minima each are nondegenerate critical points of Q^x.)

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