# The football pool problem for 5 matches

JOURNAL OF COMBINATORIAL THEORY 3, 315-325 (1967)

# The Football Pool Problem for 5 Matches

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*Communicated by N. G. deBruijn*


## ABSTRACT

We consider the set $\mathscr{R}$ of all 5-tuples $x = (x_1, x_2, x_3, x_4, x_5)$ with $x_i = 0, 1$, or 2 for $i = 1,..., 5$. The problem treated in this paper is determining the minimal $k$ for which a set $\mathscr{B}$ of 5-tuples exists such that for each $x$ in $\mathscr{R}$ there is an element in $\mathscr{B}$ that differs from $x$ in at most one coordinate.


## 1. INTRODUCTION

In this paper we shall consider the following problem. One wishes to forecast the outcome (win, lose, or draw) of 5 football matches. The question is what is the most efficient way of making a number of forecasts such that, no matter what the outcome of the matches, at least one of the forecasts will have 4 correct results. It has been shown (cf. [1], [3]) that, if the number of matches is not 5 but $n = \frac{1}{2}(3^k - 1)$ and if there must be a forecast with at least $n - 1$ correct results, then there is a solution with $3^{n-k}$ forecasts and this is the best possible. Hence for 4 matches there is a solution with 9 forecasts and trivially this gives a solution for the problem with 5 matches using 27 forecasts. In [2], O. Taussky and J. Todd asked the question whether this was the most efficient solution in the case of 5 matches. We shall show that there is no solution using less than 27 forecasts. This problem might be solvable with the aid of a computer but the program cannot be a simple search because that would take a tremendous amount of time. Generalizing our method to attempt to solve the problem for 6 matches does not look like a very pleasant task and we have done nothing in that direction.

## 2. DEFINITIONS AND NOTATION

We consider the set $\mathscr{R}$ of all 5-tuples from a 3-symbol alphabet. This set may be thought of as a 5-dimensional hypercube with sides of "length" 3.

The elements of $\mathscr{R}$ will be called *places*. There are $3^5$ places. Each place $P$ is determined by 5 *coordinates* $(x, y, z, u, v)$. For each coordinate we use the symbols 0, 1, 2. The only property we need is that the symbols are different. Without loss of generality we may permute these symbols. Furthermore we may interchange the roles of the coordinates $x, y, z, u, v$. We shall frequently make use of these possibilities. We choose three of the coordinates, generally $x$, $y$, and $z$, and then call these the *main coordinates* of the point and the other two the *local coordinates* (notation: $(x, y, z; u, v)$).

For two places $P$ and $Q$ we define the *distance* $\rho(P, Q)$ as the number of coordinates in which they differ. In the same way the *local distance* $r(P, Q)$ is defined as the number of local coordinates of $P$ not equal to the corresponding local coordinate of $Q$. We shall say that $P$ and $Q$ have different local coordinates if $r(P, Q) \neq 0$.

Every point $P$ for which $\rho(P, Q) \leqslant 1$ is said to be *covered* by $Q$. A set $\{Q_1, Q_2, ..., Q_k\}$ with the property that for every $P \in \mathscr{R}$ there is a $Q_j$ with $\rho(P, Q_j) \leqslant 1$ is called a *k-base of* $\mathscr{R}$ (or simply a *base*). An element of a $k$-base is called a *point*. In this terminology our problem now is to prove the following theorem:

(2.1) THEOREM. *The minimal value of $k$ for which there exists a $k$-base is* 27.

In the next sections we shall deduce a number of necessary conditions for the main coordinates of the points of a $k$-base if $k < 27$. In the last section it will follow that there is no $k$-base with $k < 27$.

First we introduce a few more notions and note some simple facts that will be used in the proof.

A set of nine places having the same main coordinates will be called a *block* and denoted by $(x, y, z; *, *)$.

A *row*, denoted $(x, y, *; *, *)$, e.g., is the union of 3 blocks with 2 main coordinates in common.

A *plane* is the union of 9 blocks with one main coordinate in common. We denote the plane consisting of the places with $x = 0$ as $(0, *, *; *, *)$ but sometimes we refer to this plane as "the plane $x = 0$."

The *rook domain* of a block is the set of 7 blocks having at least two main coordinates in common with the given block (notation: Rook $(x, y, z)$; we shall use the same symbol to denote the number of points in this set).

If a base is given, $N(x, y, z)$ denotes the number of points of that base in $(x, y, z; *, *)$. The numbers $N(x, y, *)$ and $N(x, *, *)$ are to be interpreted in the same obvious way. We refer to these numbers as the *order* of the block (row, plane).

We remark that:

| | | | | | | |
|---|---|---|---|---|---|---|
| (2.2) | every point covers | 5 | places | in | its | block, |
| (2.3) | ” ” ” | 7 | ” | ” | ” | row, |
| (2.4) | ” ” ” | 9 | ” | ” | ” | plane, |
| (2.5) | ” ” ” | 11 | ” | ” | | $\mathscr{R}$. |

If we consider a subset of a base $\mathscr{B}$, then it is possible that certain places in $\mathscr{R}$ are covered by more than one point of this subset. Hence (2.2) to (2.5) will give us inequalities for the total number of places covered by the points of this subset. The following property will be used several times:

(2.6) If $\mathscr{B}$ is a base with $N(x, y, z) = 0$, then $\text{Rook}(x, y, z) \geqslant 9$ and, if $N(x, y, z) = 1$, then Rook $(x, y, z) \geqslant 5$. (This follows immediately from (2.2) and (2.3).) If in either of these cases equality holds, the block is said to be *critical* (no place in the block is covered more than once).

## 3. Planes

We now consider the order of the planes in the case of a $k$-base with $k < 27$.

(3.1) Lemma. *If $\mathscr{B}$ is a k-base for which there is a plane of order* $\leqslant 6$ *then* $k \geqslant 33$.

Proof: Assume $N(0, *, *) \leqslant 6$. Each of the 6 points of $\mathscr{B}$ in $(0, *, *; *, *)$ covers 9 places in that plane. Hence at least 27 places of this plane are covered by points not in the plane. As each point not in $(0, *, *; *, *)$ covers one place in that plane we have $k \geqslant 6 + 27 = 33$.

(3.2) Lemma. *If $\mathscr{B}$ is a k-base for which there is a plane of order* 7 *then* $k \geqslant 27$.

Proof: By (3.1) we may assume that all planes have order $\geqslant 7$. Let $N(0, *, *)$ be 7. We now distinguish two cases, namely that there are exactly two empty blocks in the plane and that there is a block of order $\geqslant 2$ in the plane.

In the first case there is a row, say $(0, 0, *; *, *)$ containing 3 points of $\mathscr{B}$. There are at least 4 points of $\mathscr{B}$ in $(*, 0, *; *, *)$ outside of $(0, 0\ *; *, *)$. Let $S$ denote the set of places in $(0, *, *; *, *)$ but not in $(*, 0, *; *, *)$.

The 4 points of $\mathscr{B}$ in $S$ each cover 8 places of $S$. The 3 points in (0, 0, *; *, *) each cover 2 places of $S$. Hence at least 16 points of $\mathscr{B}$ are not in $S$ and not in (*, 0, *; *, *). Hence $k \geqslant 7 + 4 + 16 = 27$.

In the second case there is a block with at least 2 points. These 2 points cover at most 8 places in this block. Hence the 7 points in (0, *, *; *, *) cover at most $8 + 2 \cdot 4 + 5 \cdot 9 = 61$ places in this plane, which implies $k \geqslant 7 + 20 = 27$.

A consequence of (3.1) and (3.2) is

(3.3) LEMMA. *If there is a k-base with* $k < 27$ *then all the planes have order* 8, 9 *or* 10.

REMARK. By (3.3) we have $k \geqslant 24$.

## 4. ROWS

In the same way as in Section 3 we now derive inequalities for the order of a row in a $k$-base with $k < 27$.

(4.1) LEMMA. *If* $\mathscr{B}$ *is a k-base for which there is a row of order* $\leqslant 1$ *then* $k \geqslant 27$.

PROOF: Let $R = (0, 0, *; *, *)$ be a row with

$$N(0, 0, 0) = N(0, 0, 1) = 0, \qquad N(0, 0, 2) \leqslant 1.$$

As the blocks (0, 0, 0; *, *), (0, 0, 1; *, *), and (0, 0, 2; *, *) are covered, we have by (2.6):

$$N(0, 1, 0) + N(0, 2, 0) + N(1, 0, 0) + N(2, 0, 0) \geqslant 8,$$

$$N(0, 1, 1) + N(0, 2, 1) + N(1, 0, 1) + N(2, 0, 1) \geqslant 8,$$

$$N(0, 1, 2) + N(0, 2, 2) + N(1, 0, 2) + N(2, 0, 2) + N(0, 0, 2) \geqslant 5.$$

This implies that at least one of the planes containing $R$ has order $> 10$, which by (3.3) cannot happen if $k < 27$.

(4.2) LEMMA. *If* $\mathscr{B}$ *is a k-base with* $k < 27$ *then a row of order* 2 *cannot be the intersection of* 2 *planes of order* 8.

PROOF: This immediately follows from (2.2), (2.3), (2.4) because there are 27 places in a row.

REMARK: $k = 24$ is impossible (by (4.1) and (4.2)).

(4.3) LEMMA. *If $\mathscr{B}$ is a $k$-base for which there is a row in which the blocks have order* 0, 0, 2, *respectively, then* $k \geqslant 27$.

PROOF: Assume $N(0, 0, 0) = N(0, 0, 1) = 0$, $N(0, 0, 2) = 2$. Then Rook(0, 0, 0) has at least 7 points in the plane $z = 0$, Rook(0, 0, 1) has at least 7 points in the plane $z = 1$, and Rook(0, 0, 2) has at least 3 points in the plane $z = 2$ (2 points in a block cover at most 8 places of that block). Now we write $N(*, *, 2) = 3 + a$, $N(*, *, 0) + N(*, *, 1) = 14 + b$, and define $S$ to be the union of the blocks outside the three planes $x = 0$, $y = 0$, $z = 2$. The number of places of $S$ covered by $\mathscr{B}$ is at most $2a + 28 + 8b$. Hence

$$2a + 28 + 8b \geqslant 72$$

and, as a $\geqslant 5$, this implies $a + b \geqslant 10$. Then we have

$$k \geqslant 3 + 7 + 7 + 10 = 27.$$

## 5. A PLANE OF ORDER 8

We are especially interested in planes of order 8, because if there is a $k$-base $\mathscr{B}$ with $k < 27$ then there must be, for every main coordinate, a plane of order 8.

Configurations which can be transformed into each other by permuting the symbols 0, 1, and 2 are considered equivalent. For convenience we introduce the matrices $C = (c_{ij})$ and $C^* = (c_{ij}^*)$ $(i, j = 0, 1, 2)$ as follows: Let

$$N(*, *, 0) = 8.$$

Then

$$c_{ij} = N(i, j, 0); \qquad c_{ij}^* = N(i, j, 1) + N(i, j, 2).$$

We now give a list of all possible matrices $C$ and for each the minimal entries for $C^*$ which follow from (2.6), (4.1), and (4.3). We use

$$k \geqslant \Sigma(c_{ij} + c_{ij}^*) = 8 + \Sigma c_{ij}^*.$$

| $C$ | $C^*$ | Remarks |
|---|---|---|
| $\begin{pmatrix}4&0&0\\0&1&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}0&3&3\\3&2&2\\3&2&2\end{pmatrix}$ | $k \geqslant 27$ |
| $\begin{pmatrix}3&1&0\\1&0&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}0&1&3\\1&5&2\\3&2&2\end{pmatrix}$ | $k \geqslant 27$ |
| $\begin{pmatrix}3&1&0\\0&1&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}0&1&3\\4&1&2\\4&1&2\end{pmatrix}$ | $N(*, 0, *) \geqslant 11$, which contradicts (3.3) |
| $\begin{pmatrix}2&2&0\\1&0&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}1&1&3\\1&4&2\\4&1&2\end{pmatrix}$ | $k \geqslant 27$ |
| $\begin{pmatrix}2&1&1\\1&0&1\\1&1&0\end{pmatrix}$ | $\begin{pmatrix}1&1&1\\1&5&2\\1&2&5\end{pmatrix}$ | $k \geqslant 27$ |
| $\begin{pmatrix}2&1&1\\1&0&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}1&1&1\\1&5&1\\4&2&1\end{pmatrix}$ | By (3.2) we need 2 more points (to assure that $N(*, 2, *) \geqslant 8$). Hence $k \geqslant 27$. |

Therefore a row of order 4 is not possible.

| $C$ | $C^*$ | Remarks |
|---|---|---|
| $\begin{pmatrix}3&0&0\\0&2&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}0&3&4\\3&1&1\\4&1&2\end{pmatrix}$ | $k \geqslant 27$ |
| $\begin{pmatrix}2&1&0\\1&1&1\\0&1&1\end{pmatrix}$ | $\begin{pmatrix}1&1&4\\1&1&1\\4&1&2\end{pmatrix}$ | By (3.2), $N(1, *, *) \geqslant 8$ and $N(*, 1, *) \geqslant 8$. If $k < 27$, only 2 more points may be added, i.e., $N(1, 1, *) = 4$ in a plane of order 8. This has just been excluded. |
| $\begin{pmatrix}2&1&0\\1&0&1\\0&1&2\end{pmatrix}$ $\begin{pmatrix}2&1&0\\1&0&2\\0&1&1\end{pmatrix}$ | $\Bigg\}$ | type B |
| $\begin{pmatrix}1&1&1\\1&1&1\\1&1&0\end{pmatrix}$ | | type A |

The last 3 possibilities cannot be excluded at this point of the proof. A plane of order 8 with 1 empty block (the last type in the list) will be called of type A; a plane of order 8 with 3 empty blocks is of type B. We remark that a search for all possible planes of order 8 on an $ELX8$-computer took 1 minute and of course gave all permutations of the planes of type A and B.

## 6. Three Orthogonal Planes of Order 8

From now on we shall consider a $k$-base $\mathscr{B}$ and we shall assume that $k < 27$. Hence in each of the 3 main directions there is a plane of order 8. We assume that $N(0, *, *) = N(*, 0, *) = N(*, *, 0) = 8$. A consequence of (4.2) and Section 5 is that these 3 planes are all of type A or all of type B. We shall refer to these 2 possibilities as configuration A and configuration B.

First consider configuration A. The planes $x = 0$ and $z = 0$ have an empty block. Suppose there are in $y = 1$ and $y = 2$, i.e., not both in the same plane $y = y_0$. Then by (2.6) and (4.1) these planes have rows of order $\geqslant 2, \geqslant 5, 3$, respectively, i.e., both of these planes have order $\geqslant 10$, which is impossible. We have proved:

(6.1) Lemma. *In configuration* A *the empty blocks of the planes* $x = 0, y = 0, z = 0$ *belong to one rook domain.*

By permuting the symbols 0, 1, 2 we may assume that this is Rook (2, 2, 2). Configuration A can be represented by Figure 1, in which each

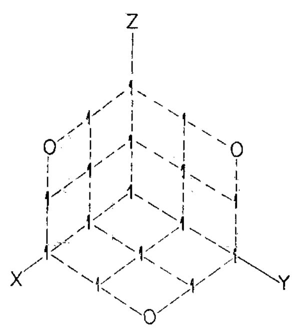


Figure 1

number denotes the order of a block in one of the planes $x = 0$, $y = 0$, $z = 0$.

We now consider configuration B.

(6.2) LEMMA. *In configuration* B *we may assume* $N(0, 0, 0) = 0$.

PROOF: If $N(0, 0, 0)$ were 1, then the rook domain of an empty block on an axis would contain only 7 points, which contradicts (2.6). Now assume $N(0, 0, 0) = 2$. After permuting 0, 1, 2 we may assume that the configuration is represented by Figure 2.

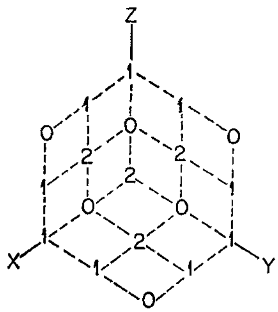


FIGURE 2

As $(0, 0, 2; *, *)$, $(0, 2, 0; *, *)$, $(2, 0, 0; *, *)$ are critical, we see that the points in these blocks have local distance 2 to each of the points in $(0, 0, 0; *, *)$. This is only possible if two of them, e.g., the points in $(0, 2, 0; *, *)$ and in $(0, 0, 2; *, *)$, have the same local coordinates. This implies $N(*, 2, 2) = 6$, hence $N(*, 2, *) = 10$ and then $N(*, 1, *) = 8$. Permutation of $y = 0$ and $y = 1$ proves the lemma.

The result of lemma (6.2) is that we know that the blocks on the main axis outside the origin contain one or two points. We assume $N(0, 0, 2) = N(0, 2, 0) = N(2, 0, 0) = 1$. Now we see that there are 2 possible cases for each of the 3 planes of order 8, namely:

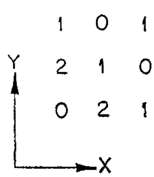


FIGURE 3

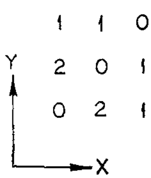


FIGURE 4

Then, using the fact that $(0, 0, 0; *, *)$, $(2, 0, 0; *, *)$, and $(0, 2, 0; *, *)$ are critical, we see that the 6 points in $(0, *, 0; *, *)$ and $(*, 0, 0; *, *)$ have different local coordinates. If the local coordinates of the other 2 points in $(*, *, 0; *, *)$ coincide with 2 of these 6 then it follows that

$$N(1, 1, *) + N(1, 2, *) + N(2, 1, *) + N(2, 2, *) \geqslant 14$$

which contradicts $k < 27$. Hence the plane $z = 0$ contains at least 7 points with different local coordinates.

Switching to $z$, $u$, $v$ as main coordinates we see that the plane $z = 0$ has order 8 and at most 2 empty blocks, i.e., it is of type A. We have proved:

(6.3) LEMMA. *If $\mathscr{B}$ is a k-base with $k < 27$ then it is possible to choose main coordinates in such a way that we have configuration* A.

## 7. CONFIGURATION A

We now consider the situation of Figure 1, and turn to the blocks for which the order has not been determined. We define the matrices $C$ and $D$ as follows:

$$c_{ij} = N(i, j, 2) \qquad (i, j = 1, 2),$$
$$d_{ij} = N(i, j, 1) \qquad (i, j = 1, 2).$$

As $k < 27$ we have

$$\sum_{i,j=1}^{2} (d_{ij} + c_{ij}) \leqslant 10 \tag{$*$}$$

If $c_{11} = 0$ then by (2.6) we have

$$c_{12} + c_{21} + d_{11} \geqslant 6,$$
$$c_{22} + d_{22} \geqslant 5,$$

which contradicts (*). By symmetry we therefore have $c_{11} \geqslant 1$, $d_{12} \geqslant 1$, $d_{21} \geqslant 1$. If $d_{11} = 0$ then by (2.6)

$$c_{11} + d_{12} + d_{21} \geqslant 6,$$
$$c_{22} + d_{22} \geqslant 5,$$

which again contradicts (*). Hence $d_{11} \geqslant 1$.

If $c_{22} < 5$ then by (2.6) and the previous arguments

$$c_{22} + c_{12} + c_{21} + d_{22} \geqslant 7,$$
$$c_{11} + d_{12} + d_{21} + d_{11} \geqslant 4,$$

which again contradicts (*).

This implies that the only possible matrices $C$, $D$ are

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

or any pair differing from this by 1 in one entry.

We now permute 0, 1, 2 in such a way that the block of order 5 (or 6) is (0, 0, 0; *, *). Configuration A is then represented by Figure 5, which

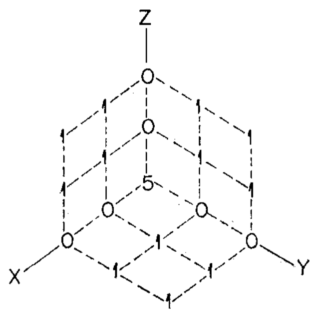


FIGURE 5

should be interpreted as follows. The order of the blocks is as in Figure 5, and the order of each of the blocks not in the figure is 1, but it is possible that one of the blocks (0, 0, 0; *, *), (0, 0, 1; *, *), (1, 0, 1; *, *), (1, 1, 1; *, *) is 1 more than in the figure (we can restrict the increase of 1 to these 4 blocks by symmetry). For each of these possibilities the blocks (2, 1, 0; *, *), (2, 2, 0; *, *), (1, 2, 0; *, *), (2, 0, 2; *, *), and (0, 2, 2; *, *) are critical.

We shall now prove that configuration A is impossible, completing the proof of Theorem (2.1). We now also need the local coordinates. We permute these in such a way that the points in (*, *, 0; *, *) outside (0, 0, 0; *, *) are

$$(1, 1, 0; 0, 0), (1, 2, 0; 1, 1), (2, 1, 0; 1, 2), (2, 2, 0; 2, 0).$$

(It is clear that this is essentially the only possibility because 3 of the 4 blocks considered are critical.)

Now it follows that the other 2 points in (2, 2, *; *, *) must have local coordinates (0,1) and (0,2); the other 2 points in (1, 2, *; *, *) have local coordinates (0,2) and (2,2), and finally the other 2 points in (2, 1, *; *, *) have local coordinates (0,1) and (2,1). Now, keeping in mind that (2, 0, 2; *, *) and (0, 2, 2; *, *) are critical, there are a number of ways in which the local coordinates just mentioned can be divided over the blocks under consideration. No matter how this is done, at most 8 points in (2, 2, 1; *, *) are covered. Hence configuration A is impossible.

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