The Forces on a Body placed in a Curved or Converging Stream of Fluid.

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Introduction.

The force which a body experiences when placed in a converging stream of fluid has a certain practical interest in aeronautics because the flow in the centre of a parallel-walled wind tunnel is of this type. The convergence is due to the retardation of a layer of air close to the walls. This retarded layer increases in thickness as the air passes down the channel, thus causing a corresponding increase in the velocity in the central part of the channel. This increase in velocity is associated with a decrease in pressure in accordance with Bernouilli's equation, the pressure in a Pitot tube being very nearly constant down the channel at all points outside the retarded layer.

In measuring the resistance of models of airships it has been customary to correct the observed readings by subtracting what is called the "horizontal buoyancy," *i.e.*, the force which would act on the body if the air were a stationary fluid in which the existing pressure gradient down the channel was maintained by some external force like gravity. Expressed mathematically, if dp/dx is the pressure gradient, *i.e.*, the gradient of static pressure in the channel, and V the volume of the body, the "horizontal buoyancy" is $-\mathrm{V}\dfrac{dp}{dx}$. This correction to the measured resistance of an airship model is believed to be approximately correct from the point of view of wind tunnel practice, and the primary object of the present work was to find out how far it is justified from the point of view of hydrodynamical theory.

Previous Work.

An extremely simple treatment has already been given by Max M. Munk,* but this is limited to a very special class of body, namely, the body which can be represented as regards its external flow by a simple source and an equal sink placed at a finite distance apart. When the distance apart is large so that the flow due to the source at the position of the sink is small compared

* 'Some New Aerodynamical Relations,' Report No. 114, National Advisory Committee for Aeronautics. Washington, 1921.

with the general flow of the fluid, the body consists of two rounded ends connected by a middle body which is parallel to the stream lines of the converging or diverging stream, *i.e.*, the body is a long truncated cone, rounded off at the ends. For such a body the force acting in a converging stream of a perfect fluid is actually equal to the "horizontal buoyancy," $-\mathrm{V}\dfrac{dp}{dx}$.

In the other limiting case, when the source and sink are infinitely close together, Munk pointed out that the force is $-\frac{3}{2}\mathrm{V}\dfrac{dp}{dx}$, the body being in this case approximately a sphere; and he deduced that for bodies of intermediate length the factor by which the "horizontal buoyancy" must be multiplied in order to find the force acting on such a body when placed in a converging stream is intermediate between 1·0 and 1·5. Munk indicates that the force acting on bodies derived from any known distributions of sources and sinks could also be treated in the same way.

The forces acting on a small sphere in a stream circulating in a multiply-connected space has been studied by Lord Kelvin.* The limitation imposed by the supposition that the space is multiply-connected does not affect the result, however, because the force acting on the sphere must depend only on the direction, curvature and convergence of the stream in the neighbourhood of the sphere. Though he does not express it in this way, Kelvin's result may be reduced to the simple statement that the resultant force on the sphere is in the direction of the pressure gradient in the fluid, irrespective of the direction of flow, and it is equal to the pressure gradient multiplied by 1·5 times the volume of the sphere. This result may be regarded as a generalisation of Munk's result for a sphere in a straight converging flow, to the case when the flow is also curved.

Scope of the Present Work and Results.

In the present work these results are extended to bodies of any shape placed in a curved and converging or diverging stream, or even in a stream which converges in one place and diverges in a plane at right angles to it. In the case of a straight converging stream the actual value of the factor by which the "horizontal buoyancy" must be multiplied is calculated and shown to be equal to $1 + \alpha$, where $\alpha\rho\mathrm{V}$ is the virtual addition to the mass of the body which must be added to its own mass to account for its resistance to

* "On the Motion of Rigid Solids in a Liquid circulating irrotationally through perforations in them or a Fixed Solid," 'Phil. Mag.' (1873).

accelerated motion in the direction of the stream lines and ρV is the mass of fluid displayed by the body. This is true whether the convergence is symmetrical in all planes through the direction of the flow or not. It will be recognised at once that Munk's and Kelvin's expressions for a sphere are particular cases of this general result.

This result is extended to find the direction and magnitude of the force which acts on a small body of any shape placed in a curved or converging stream. It is found that Kelvin's result for a sphere (that the resultant force acts in the direction of the pressure gradient) is true for bodies of any shape provided that the body is placed so that one of its possible "directions of permanent translation" is parallel to the direction of the stream so that no couples act on it. When the body is placed in any other position, however, Kelvin's result for a sphere is no longer applicable. It is curious that Kelvin first published his result as a general one applicable to bodies of any shape, but on revising his paper later he recognised that his analysis applied only to the case of a sphere.

In general, a body placed in a straight uniform stream of fluid experiences couples about axes at right angles to the stream. The additional couples due to a small amount of convergence or curvature of the stream were calculated. These, however, appear of little interest because in a real fluid the observed couples usually differ very widely from those calculated. On the other hand, even in a real fluid there is a very large class of bodies which experience no couple about an axis parallel to the direction of the stream. Accordingly, the couples about the direction of the flow in a curved and converging stream were calculated and the results tested experimentally. The analysis shows that certain types of asymmetry in the stream react on certain types of asymmetry in the body and cause it to rotate about the stream direction into some definite orientation. Thus, if a body with the type of asymmetry possessed by a rod bent into a circular arc is suspended at its centre of gravity in a curved stream so that it can rotate about the direction of the stream which is parallel to the chord of the arc, it will take up a position so that the plane of the arc coincides with the plane containing the curved stream lines; but the direction of curvature of the arc is opposite to that of the stream lines. This was verified experimentally.

A body with the type of asymmetry possessed by a tetrahedron about the line joining the centres of opposite edges reacts with a stream in which the centre stream line is straight but the convergence is asymmetrical, being greatest in one plane through the direction of the stream and least in the plane at right angles to it. Such a stream was created by sucking air by means of a vacuum cleaner through a vertical channel two walls of which were parallel glass plates, the remaining two walls being made of bent metal sheet so that the stream first converged and then afterwards diverged. Bodies with the required type of symmetry were hung in this stream by fine silk threads, and it was found that in every case they set themselves in the positions indicated by theory. They rotated through 90° on being lowered through the point of maximum constriction from the converging to the diverging part of the channel. This also was predicted by theory.

Components of Force on a Body in a Curved and Converging Stream.

The kinetic energy of a system consisting of a fluid circulating in a cyclic space and containing a small sphere moving with velocity components $\dot{x}$, $\dot{y}$, $\dot{z}$ has been given by Lord Kelvin in the form

$$2T = (M + \tfrac{1}{2}\rho V)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + 2K, \tag{1}$$

where K is the energy of the cyclic motion when the sphere is held at rest, M is the mass of the sphere and V its volume. If x, y, z are the co-ordinates of the centre of the sphere, K may be regarded as a function of x, y, z.

The simplest method for finding K is to use an artifice due, I think, to Dr. Horace Lamb. If M is made equal to the mass of the fluid displaced and if the sphere is made to move with the fluid in its neighbourhood, then the energy T is the same as what it would be if the sphere were absent, the space being occupied by fluid. This assumption implies that the sphere is so small that the changes in the velocity of the undisturbed stream in a distance equal to the diameter of the sphere is small compared with the velocity of the stream. Using this assumption, then

$$2K = \text{constant} - \tfrac{3}{2}\rho V(u^2 + v^2 + w^2), \tag{2}$$

where u, v, w are the components of velocity in the stream before the introduction of the sphere, at the point afterwards occupied by its centre.

To find the motion Kelvin used some general dynamical equations, proving that certain coefficients vanish in the case of a sphere. It is not necessary, however, to use these equations to find the force which acts on the body when held at rest in the circulating fluid. If the body be displaced slowly through a short distance whose components are δx, δy, δz, the change in K must be

$$-(X\delta x + Y\delta y + Z\delta z),$$

where X, Y, Z are the components of force acting on the body.

Hence

$$X = -\frac{\partial K}{\partial x} = \tfrac{3}{4}\rho V \frac{\partial}{\partial x}(u^2 + v^2 + w^2).$$

If p is the pressure in the undisturbed motion, *i.e.*, before the introduction of the sphere, $p + \frac{1}{2}\rho(u^2 + v^2 + w^2)$ is constant, since the motion is a steady one, so that

$$X = -\tfrac{3}{2}V\frac{\partial p}{\partial x},$$

and similarly

$$Y = -\tfrac{3}{2}V\frac{\partial p}{\partial y}, \quad Z = -\tfrac{3}{2}V\frac{\partial p}{\partial z}. \tag{3}$$

In this example the fluid is circulating in a cyclic region, but the force on the sphere must depend only on the convergence and curvature of the undisturbed stream lines in its vicinity; so that the formula (3) is general for any type of undisturbed flow provided that the changes in velocity in a length equal to the diameter of the sphere are small compared with the velocity of the stream.

To apply this method to bodies of any shape one may write T_0 for the energy in the fluid surrounding a body when it moves without rotation in a fluid at rest at infinity. If u, v, w are the components of its velocity*

$$2T_0/\rho = Au^2 + Bv^2 + Cw^2 + 2A'vw + 2B'wu + 2C'uv, \tag{4}$$

where A, B, C, A′, B′, C′ are six constants which depend only on the shape of the body. They are determined relative to some axes fixed in the body. Equation (2) then takes the form

$$2K/\rho = \text{constant} - V(u^2 + v^2 + w^2) - 2T_0/\rho. \tag{5}$$

As in the case of the sphere

$$X = -\frac{\partial K}{\partial x}, \quad Y = -\frac{\partial K}{\partial y}, \quad Z = -\frac{\partial K}{\partial z},$$

so that

$$\left.\begin{aligned}
X &= \tfrac{1}{2}\rho V \frac{\partial}{\partial x}(u^2 + v^2 + w^2) + \frac{\partial T_0}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial T_0}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial T_0}{\partial w}\frac{\partial w}{\partial x} \\
Y &= \tfrac{1}{2}\rho V \frac{\partial}{\partial y}(u^2 + v^2 + w^2) + \frac{\partial T_0}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial T_0}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial T_0}{\partial w}\frac{\partial w}{\partial y} \\
Z &= \tfrac{1}{2}\rho V \frac{\partial}{\partial z}(u^2 + v^2 + w^2) + \frac{\partial T_0}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial T_0}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial T_0}{\partial w}\frac{\partial w}{\partial z}
\end{aligned}\right\}. \tag{6}$$

* See Lamb's 'Hydrodynamics,' 4th edn., p. 155.

These expressions may be simplified by taking the axis of x parallel to the stream at the point considered. In that case $v = w = 0$, and

$$\frac{\partial T_0}{\partial u} = Au, \quad \frac{\partial T_0}{\partial v} = C'u, \quad \frac{\partial T_0}{\partial w} = B'u.$$

Also some of the partial differentials $u\frac{\partial u}{\partial x}$, $u\frac{\partial v}{\partial x}$, etc., are proportional to the components of the pressure gradient, thus with this choice of axes

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}, \quad \rho u \frac{\partial u}{\partial y} = \rho u \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}, \quad \rho u \frac{\partial u}{\partial z} = \rho u \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z},$$

so that

$$\left.\begin{aligned} X &= -(V + A)\frac{\partial p}{\partial x} - C'\frac{\partial p}{\partial y} - B'\frac{\partial p}{\partial z} \\ Y &= -(V + A)\frac{\partial p}{\partial y} + \rho C'u\frac{\partial v}{\partial y} + \rho B'u\frac{\partial w}{\partial y} \\ Z &= -(V + A)\frac{\partial p}{\partial z} + \rho C'u\frac{\partial v}{\partial z} + \rho B'u\frac{\partial w}{\partial z} \end{aligned}\right\}. \qquad (7)$$

From these equations it will be seen that in the particular case when $B' = C' = 0$, *i.e.*, when the body has one of its axes of permanent translation parallel to the direction of flow so that it would experience no couples in a uniform stream, the resultant force is in the direction of the pressure gradient and equal to $(V + A) \times$ (maximum pressure gradient).

An ellipsoid, for instance, placed with one of its principal axes in the direction of the stream would experience a force acting in the direction of the pressure gradient in the undisturbed stream. The magnitude of this force would differ according to which axis was parallel to the stream; thus, if the long axis were parallel to the stream the force would be smaller than if a shorter axis were in that direction. In general, however, when the direction of the stream is not parallel to one of the axes of permanent translation, the force acting on the body is not in the direction of the pressure gradient.

The particular case of a converging flow with a straight central stream line is of special interest because, as has already been pointed out, it is the condition which may be expected in the middle of a parallel-sided wind tunnel. In this case $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$, so that $X = -(V + A)\frac{\partial p}{\partial x}$ and no restriction is placed on the shape of the body. This is the generalised form of the expression obtained by Munk for a sphere and a certain class of elongated bodies of revolution. The "virtual mass" of a body in accelerated motion through a fluid is $A\rho$. This is sometimes expressed in the form $\alpha V\rho$, where $\alpha = A/V$ so that $X = -(1+\alpha) V \frac{\partial \rho}{\partial x}$. $(1+\alpha)$ is the factor by which the "horizontal buoyancy" should be multiplied in order to find the effect of convergence on the resistance of an airship model hung in a horizontal wind tunnel.

In order to estimate the value of $(1+\alpha)$ in the case of a body whose shape approximates to that of an airship, it seems useful to give the value of $(1+\alpha)$ for a prolate spheroid of eccentricity e moving parallel to its long axis.

In that case*

$$(1+\alpha)^{-1} = \frac{1}{e^2} - \frac{1-e^2}{2e^3} \log \frac{1+e}{1-e}. \tag{8}$$

It is customary in aerodynamical work to use the expression "fineness ratio" to indicate the ratio of the length to maximum diameter of an airship shape. The fineness ratio, β, for a spheroid is $(1-e^2)^{-\frac{1}{2}}$. From these formulæ the following Table I was calculated :—

Table I.

β.	$1+\alpha$.	β.	$1+\alpha$.
1·00	1·50	3·64	1·093
1·34	1·35	5·08	1·057
1·81	1·24	7·12	1·035
2·5	1·16		

It will be seen that for this series of shapes the factor by which the "horizontal buoyancy" must be multiplied decreases from 1·5 for the sphere ($\beta = 1 \cdot 00$) to 1·057 for a body five times as long as its maximum diameter.

In a recent paper† I have given a simple method by which the "virtual mass" of any airship shape derived from a system of sources and sinks may be found. Suppose that a system of sources and sinks $m_1 U$, $m_2 U$, ... $m_r U$, ..., placed at points (x_1, y_1, z_1), (x_2, y_2, z_2) ..., in a stream of fluid flowing with uniform velocity $-U$ parallel to the axis of x has been found to represent the

* This expression was first calculated by integrating the pressure over a spheroid due to the flow from a source placed on the axis. It was only after finishing the calculation that it was found that (8) gives also the expression for the "virtual mass."

† "The Energy of a Body moving in an Infinite Fluid, with an Application to Airships," 'Roy. Soc. Proc.,' *supra*, p. 13.

flow round a body of volume V. The "virtual mass" of this body in accelerated motion was shown to be*

$$\rho A = \rho \Sigma m_r x_r - \rho V. \tag{9}$$

When placed in a converging stream this body experiences a force

$$X = -\rho (A + V) \frac{\partial p}{\partial x}.$$

Hence from (9)

$$X = -(\Sigma m_r x_r) \frac{\partial p}{\partial x}. \tag{9A}$$

This is equivalent to Munk's formula. It can only be applied when the requisite distribution of sources and sinks is known.

Verification of Equation (7) *in a Simple Case.*

It is of interest to notice why the terms in C′ and B′ must come into the expressions for the forces. Consider, for example, the case of a body, O, placed between two concentric circular cylinders (see fig. 1) with liquid circulating between them. Let u be the velocity of the fluid at the radius r where the body is situated.

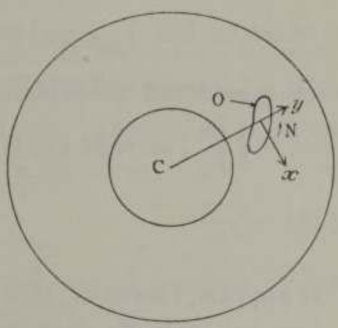

Fig. 1.

Now consider a displacement in which the body rotates through an angle $\delta\theta$ about the axis C of the cylinders. Since there is no change in the circulation round the inner cylinder, such a displacement does not alter the energy of the system. If there is a couple N acting on the body owing to its being placed so that the direction of the stream is not one of its axes of permanent translation, then there must also be a force X in the direction of motion so that the work done during the displacement is zero. Hence

$$Xr\,\delta\theta - N\,\delta\theta = 0 \quad \text{or} \quad X = N/r. \tag{10}$$

Now the couple on a body moving parallel to the axis of x, which may be taken in the direction of flow as shown in fig. 1, is†

$$N/\rho = v \frac{\partial T_0}{\partial u} - u \frac{\partial T_0}{\partial v} = -C' u^2. \tag{11}$$

* *Loc. cit.*, equation (21). There given as $\rho A = \rho \int_{\kappa_1}^{\kappa_2} mx\,dx - \rho V$, which is the form of (9) above, suitable for a continuous distribution of sources along a line.

† Lamb, 'Hydrodynamics,' 4th edn., p. 159.

Hence from (10) and (11)

$$X = -\rho C' u^2/r.$$

But

$$\frac{\partial p}{\partial r} = \rho u^2/r.$$

Hence

$$X = -C'\frac{\partial p}{\partial r} = -C'\frac{\partial p}{\partial y}. \tag{12}$$

Comparing (12) with (7) it will be seen that for the particular case when

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0,$$

the first of formulæ (7) is verified by this very simple argument.

As regards the practical application of equations (7) it will be seen that convergence in a stream of fluid produces an effect on lift as well as on drag. Thus, for a wind tunnel for which $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$, the convergence produces an effect on lift, Y, equal to $+\rho C' u \frac{\partial v}{\partial y}$. In the case where the convergence is the same in all axial planes, as in a wind tunnel of square or circular section, $\frac{\partial v}{\partial y} = -\frac{1}{2}\frac{\partial u}{\partial x}$, so that the effect on lift is

$$+\tfrac{1}{2}C'\frac{\partial p}{\partial x}. \tag{13}$$

It appears, therefore, that the effect of convergence on lift may be of the same order of magnitude as that on drag.

Determination of Forces and Couples by Integration of Pressures over the Surface of the Body.

The formulæ (7) were not originally obtained by the simple process described in the first part of this paper. They were first derived by laborious integration of pressures over the surface of the body, a process made possible by Green's theorem, which enabled the integrations to be transformed into integrations over a spherical surface.

For finding the couples acting on the body, equation (5) giving K in terms of u, v, w is of no value. It is true that couples about axes perpendicular to the direction of the stream could be derived from (5), but these would only be the couples which would act on the body in a uniform stream. This can be illustrated by reversing the argument in the example given in the preceding section. To determine the changes in these couples owing to convergence, or the couple acting about the direction of the stream, it would be necessary to determine K to a higher order of approximation than can be done by Dr. Lamb's artifice. It is clear, for instance, that rotation of the body about the direction of the stream makes no change in (5), for T_0 is unaltered by such a rotation, so that as far as (5) goes one might conclude that the couple is zero.

To find the couples, therefore, I found it necessary to revert to my original method and to expand both the disturbed and the undisturbed stream in a series of spherical harmonics.

Representation of the Undisturbed Stream.

The velocity potential of a uniform stream of fluid is represented by a spherical harmonic function of the first degree. In general three terms are necessary to determine the three components of velocity, but we shall simplify the formulæ by taking the direction of the stream at the origin as the line $\theta = 0$ when we use spherical polar co-ordinates, or the axis of x when we use Cartesian co-ordinates.

Convergence or curvature in the stream lines near the origin may be represented, to the degree of approximation required, by spherical harmonics of the second degree. The general expression for the velocity potential of a curved and converging stream may, therefore, be written in the form

$$\phi_0 = r\mathrm{S}_1 + r^2\mathrm{S}_2, \tag{14}$$

where S_1 and S_2 are surface harmonics of the first and second degrees. When the velocity at the origin is parallel to $\theta = 0$ and equal to U,

$$\mathrm{S}_1 = -\mathrm{U}\cos\theta, \tag{15}$$

the most general form of S_2 is

$$\mathrm{S}_2 = \mathrm{G}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) + (\mathrm{H}\cos\omega + \mathrm{J}\sin\omega)\cos\theta\sin\theta + (\mathrm{E}\cos 2\omega + \mathrm{F}\sin 2\omega)\sin^2\theta, \tag{16}$$

when ω is the angular co-ordinate of an axial plane so that

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta\cos\omega \\ z &= r\sin\theta\sin\omega \end{aligned}\right\}. \tag{17}$$

If G is positive the flow is diverging; if G is negative it is converging. If $\mathrm{H} = \mathrm{J} = 0$ the stream line through the origin is straight. If $\mathrm{H} = \mathrm{J} = \mathrm{E} = \mathrm{F} = 0$ the flow is symmetrical about the axis of x. If

$H = J = F = 0$ the stream is symmetrical about the planes $y = 0$, $z = 0$, and if, in addition, $G = 2E$ the motion is two-dimensional, the stream lines being parallel to the plane $y = 0$.

If $G = E = F = 0$ the stream lines are curved but do not converge in any axial plane and the direction of maximum pressure gradient is at right angles to the stream lines. If, in addition, $J = 0$ the stream lines lie in the plane $z = 0$, and if H is then positive the centre of curvature of the central stream line is in the direction $\theta = \frac{1}{2}\pi$, $\omega = \pi$, *i.e.*, it lies on the negative side of the axis of y.

If $H = 0$ and J is positive the central stream line lies in the plane $y = 0$ and its centre of curvature lies in the negative side of the axis of z.

Representation of the Disturbed Flow.

The disturbed flow may be represented by the velocity potential

$$\phi = \phi_0 + \phi_1 = r^2 S_2 + r S_1 + r^{-2} s_1 + r^{-3} s_2 + \dots + r^{-m-1} s_m + \dots, \tag{18}$$

where s_m is a surface spherical harmonic of degree m.

Forces found by Integration of Pressures.

Since the motion is steady $p/\rho + \frac{1}{2}(u^2 + v^2 + w^2)$* is constant. If the components of the resultant force on the body are X, Y, Z,

$$\frac{X}{\rho} = \frac{1}{2}\iint_i l\,(u^2 + v^2 + w^2)\,ds, \qquad \frac{Y}{\rho} = \frac{1}{2}\iint_i m\,(u^2 + v^2 + w^2)\,ds,$$

$$\frac{Z}{\rho} = \frac{1}{2}\iint_i n\,(u^2 + v^2 + w^2)\,ds,$$

the suffix i shows that the integration extends over the surface of the body and l, m, n are the direction cosines of the outward drawn normal to the surface of the body.

These expressions may be transformed by Green's theorem in the forms

$$\left.\begin{aligned}
\frac{X}{\rho} &= \frac{1}{2}\iint_0 (u^2 + v^2 + w^2)\,l\,ds - \iint_0 (lu + mv + nw)\,u\,ds \\
\frac{Y}{\rho} &= \frac{1}{2}\iint_0 (u^2 + v^2 + w^2)\,m\,ds - \iint_0 (lu + mv + nw)\,v\,ds \\
\frac{Z}{\rho} &= \frac{1}{2}\iint_0 (u^2 + v^2 + w^2)\,n\,ds - \iint_0 (lu + mv + nw)\,w\,ds
\end{aligned}\right\}, \tag{19}$$

* Note that u, v, w are used here and in succeeding pages as the components of velocity at any point in the fluid. In the first part of the paper up to equation (13) they represent the components of velocity in the fluid before the introduction of the solid.

the integrations extending over any surface 0 which completely surrounds the body. Taking 0 as a sphere of radius r it is found that

$$\frac{X}{\rho} = \tfrac{1}{2}\iint_0 \cos\theta \left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\phi}{\partial\omega}\right)^2\right\} ds$$

$$- \iint_0 \left\{\left(\frac{\partial\phi}{\partial r}\cos\theta - \frac{\partial\phi}{r\partial\theta}\sin\theta\right)\frac{\partial\phi}{\partial r}\right\} ds, \qquad (20)$$

with similar expressions for Y/ρ and Z/ρ.

Now the value of X/ρ is independent of r, and ds is $r^2 \sin\theta\, d\omega\, d\theta$, so that it is only necessary to pick out terms in the integrands of (20) which contain r^{-2} as a factor. All other terms must vanish when integrated over the sphere of radius r. It is hardly worth while to complete the whole operation because we should ultimately only find the expressions given in equations (7), but it seems worth while to verify, say, the first term in the expression for X/ρ.

Substituting from (14) and (18) in (20) and picking out only terms in r^{-2} in the integrands, it is found that

$$\frac{X}{\rho} = \iint_0 \left\{4S_2 s_1 \cos\theta - 2\sin\theta\left(S_2\frac{\partial s_1}{\partial\theta} - s_1\frac{\partial S_2}{\partial\theta}\right) + \frac{\partial S_2}{\partial\theta}\frac{\partial s_1}{\partial\theta}\cos\theta\right.$$

$$\left. + \cos\theta\,\mathrm{cosec}^2\,\theta\,\frac{\partial S_2}{\partial\omega}\frac{\partial s_1}{\partial\omega}\right\} r^{-2}\, ds. \qquad (21)$$

Taking the case when $H = J = 0$ so that the centre line of the undisturbed flow is straight, and substituting for s_1, its most general value

$$a_1\cos\theta + a_2\sin\theta\cos\omega + a_3\sin\theta\sin\omega,$$

and for S_2 from (16), it is found that the only terms in (21) which do not vanish are those containing a_1G thus:

$$\iint_0 4S_2\, s_1\cos\theta\, r^{-2}\, ds = \tfrac{32}{15}\pi a_1 G,$$

$$\iint_0 2\sin\theta\left(S_2\frac{\partial s_1}{\partial\theta} - s_1\frac{\partial S_2}{\partial\theta}\right) r^{-2}\, ds = \tfrac{64}{15}\pi a_1 G,$$

$$\iint_0 \frac{\partial S_2}{\partial\theta}\frac{\partial s_1}{\partial\theta}\cos\theta\, r^{-2}\, ds = \tfrac{24}{15}\pi a_1 G,$$

$$\iint_0 \cos\theta\,\mathrm{cosec}^2\,\theta\,\frac{\partial S_2}{\partial\omega}\frac{\partial s_1}{\partial\omega}\, r^{-2}\, ds = 0,$$

so that from (21)

$$X = \frac{32 + 64 + 24}{15}\pi a_1 S\rho = 8\pi a_1 G\rho. \qquad (22)$$

Remembering that $H = J = 0$ in (16), it can be shown that

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0,$$

so that the value of X from (7) is

$$X = -(V + A)\frac{\partial p}{\partial x}. \tag{23}$$

In comparing the two expressions for X given in (22) and (23) we must first calculate the value of $\partial p/\partial x$ at the origin in the undisturbed flow. The velocity along the central stream line before the introduction of the body is

$$u = -\left[\frac{\partial \phi_0}{\partial r}\right]_{\theta=0} = [-S_1 - 2rS_2]_{\theta=0},$$

so that the rate of change in pressure along the axis of x is

$$\frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}(\tfrac{1}{2}\rho u^2) = -\rho\left[u\frac{\partial u}{\partial x}\right]_{v=0} = -\rho U[-2S_2]_{\theta=0} = 2\rho GU. \tag{24}$$

It remains to find the relationship between a_1 and A. a_1 is the coefficient of $\cos\theta/r^2$ or x/r^3 in the expansion of the velocity potential of the disturbed motion in spherical harmonics. We can regard the disturbed motion as consisting of two parts, one due to the term rS_1 in the undisturbed motion and the other to the term r^2S_2. Let the coefficients of x/r^3 in these two parts be a_1' and a_1'' respectively, so that

$$a_1 = a_1' + a_1''. \tag{25}$$

So far we have made no assumptions as to the relative magnitudes of the terms r_1S_1 and r^2S_2 in ϕ_0. If now we make the assumption made in obtaining (7) that the changes in velocity of the undisturbed stream in a distance comparable with the linear dimensions of the body are small, then a_1'' is small compared with a_1'. Now a_1' is the coefficient of x/r^3 in the expansion of the disturbed motion due to a uniform stream of velocity U flowing past the body. This is identical with the coefficient of x/r^3 in the expansion of the velocity potential in spherical harmonics of the flow produced by moving the body with velocity $-$ U in an infinite fluid at rest.

The connection between this coefficient and the expression for the energy of a body moving without rotation in an infinite fluid has been discussed by the author in a previous paper.* As a particular case of the formulæ in that paper the coefficient of x/r^3 in the expansion of the velocity potential due to a velocity $-$ U parallel to the axis of x is

$$-U(A + V)/4\pi, \tag{26}$$

* *Loc. cit.*, p. 13.

where A and V have the same meaning as in (7). If we neglect a_1'' compared with a_1', we can therefore write

$$4\pi a_1 = -(\mathrm{A} + \mathrm{V})\,\mathrm{U}. \tag{27}$$

Taking the value of a_1 from (27) and of G from (24) and substituting them in (22), it will be found that

$$\mathrm{X} = 8\pi a_1 \mathrm{G}\rho = (4\pi a_1)(2\mathrm{G}\rho) = -2(\mathrm{A}+\mathrm{V})\,\mathrm{UG}\rho = -(\mathrm{A}+\mathrm{V})\frac{\partial p}{\partial x},$$

which is identical with the expression (23) derived from the first part of the paper.

In the same way the rest of formulæ (7) can be found by direct integration of the pressures over the surface of the body.

Couples found by Integration of Pressures over the Surface of the Body.

If $\mathrm{L}\rho$, $\mathrm{M}\rho$, $\mathrm{N}\rho$ are the component couples,

$$\mathrm{L} = \frac{1}{\rho}\iint_i p\,(mz - ny)\,ds = \tfrac{1}{2}\iint_i (ny - mz)(u^2 + v^2 + w^2)\,ds. \tag{28}$$

First the integral will be transformed into an integral over an outer surface 0 completely surrounding the body, thus:

$$\tfrac{1}{2}\iint_i ny\,(u^2+v^2+w^2)\,ds = \tfrac{1}{2}\iint_0 ny\,(u^2+v^2+w^2)\,ds$$
$$-\tfrac{1}{2}\iiint y\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dx\,dy\,dz, \tag{29}$$

the volume integral extending between the body and the outer surface 0. Remembering that $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ and at the surface of the body $lu + mv + nw = 0$,

$$-\tfrac{1}{2}\iiint y\frac{\partial}{\partial z}(u^2+v^2+w^2)\,dx\,dy\,dz$$
$$= -\iiint y\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)dx\,dy\,dz$$
$$= -\iint_0 yw\,(lu + mv + nw)\,ds$$
$$+\iiint w\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(yv) + \frac{\partial}{\partial z}(yw)\,dx\,dy\,dz$$
$$= -\iint_0 yw\,(lu + mv + nw)\,ds + \iiint wv\,dx\,dy\,dz.$$

The integral $-\frac{1}{2}\iint_i mz\,(u^2+v^2+w^2)\,ds$ can be treated in the same way, and by adding the two results it is found that

$$L = \tfrac{1}{2}\iint_0 (ny-mz)(u^2+v^2+w^2)\,ds - \iiint_0 (yw-zv)\,(lu+mv+nw)\,ds. \tag{30}$$

If now the outer surface be taken on a sphere of radius r, $ny-mz=0$ so that

Similarly,

$$\left.\begin{aligned} L &= \iint_0 (yw-zv)\frac{\partial\phi}{\partial r}\,ds.\\ M &= \iint_0 (zu-xw)\frac{\partial\phi}{\partial r}\,ds.\\ N &= \iint_0 (xv-yu)\frac{\partial\phi}{\partial r}\,ds.\end{aligned}\right\}. \tag{31}$$

Expressing u, v, w in terms of the velocity potential

$$\left.\begin{aligned} yw-zv &= -\frac{\partial\phi}{\partial\omega}\\ zu-xw &= \frac{\partial\phi}{\partial\theta}\sin\omega + \frac{\partial\phi}{\partial\omega}\cot\theta\cos\omega\\ xv-yu &= -\frac{\partial\phi}{\partial\theta}\cos\omega + \frac{\partial\phi}{\partial\omega}\cot\theta\sin\omega\end{aligned}\right\}, \tag{32}$$

and substituting these expressions in (31)

$$\left.\begin{aligned} L &= -\iint_0 \frac{\partial\phi}{\partial\omega}\frac{\partial\phi}{\partial r}\,ds\\ M &= \iint_0 \left(\frac{\partial\phi}{\partial\theta}\sin\omega + \frac{\partial\phi}{\partial\omega}\cot\theta\cos\omega\right)\frac{\partial\phi}{\partial r}\,ds\\ N &= \iint_0 \left(\frac{\partial\phi}{\partial\theta}\cos\omega + \frac{\partial\phi}{\partial\omega}\cot\theta\sin\omega\right)\frac{\partial\phi}{\partial r}\,ds\end{aligned}\right\}. \tag{33}$$

The remainder of this paper will be devoted to the discussion of L.

Since L has a definite value independent of the radius of the sphere over which the integration is taken, it is only necessary to pick out terms in $\frac{\partial\phi}{\partial\omega}\frac{\partial\phi}{\partial r}$

which contain the factor r^{-2}. Differentiating the expression (18) for ϕ with respect to r and ω, multiplying and picking out terms in r^{-2},

$$-\iint_0 \frac{\partial\phi}{\partial\omega}\frac{\partial\phi}{\partial r}\,ds = \iint_0\left(2s_1\frac{\partial S_1}{\partial\omega} - S_1\frac{\partial s_1}{\partial\omega}\right)r^{-2}\,ds + \iint_0\left(3s_2\frac{\partial S_2}{\partial\omega} - 2S_2\frac{\partial s_2}{\partial\omega}\right)r^{-2}\,ds, \quad (34)$$

and since S_1 does not contain ω when the axis of x is parallel to the stream direction at the origin, the first of these integrals vanishes so that

$$L = \iint_0\left(3s_2\frac{\partial S_2}{\partial\omega} - 2S_2\frac{\partial s_2}{\partial\omega}\right)r^{-2}\,ds. \quad (35)$$

Now S_2 is

$$G\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) + (H\cos\omega + J\sin\omega)\cos\theta\sin\theta + (E\cos 2\omega + F\sin 2\omega)\sin^2\theta \quad (36)$$

and s_2 may be expressed in its most general form

$$s_2 = g\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) + (h\cos\omega + j\sin\omega)\cos\theta\sin\theta + (e\cos 2\omega + f\sin 2\omega)\sin^2\theta. \quad (37)$$

It will be seen that only four kinds of terms can occur in (35), namely, those containing Hj, hJ, Ef, eF; all others vanish when integrated over the surface of the sphere.

To find the coefficient of Hj take $J = E = F = 0$, then

$$L = Hj\int_0^\pi\int_0^{2\pi}(-3\sin^2\theta\cos^2\theta\sin^2\omega - 2\sin^2\theta\cos^2\theta\cos^2\omega)\sin\theta\,d\omega\,d\theta = -\tfrac{4}{3}\pi Hj. \quad (38)$$

Similarly the term in Jh is $+\tfrac{4}{3}\pi Jh$.

To find the coefficient of Ef put $H = J = F = 0$.

Then

$$L = Ef\int(-6\sin^4\theta\sin^2 2\omega - 4\sin^4\theta\cos^2 2\omega)\,r^{-2}\,ds = -\tfrac{32}{3}\pi Ef.$$

Hence this complete expression for L is

$$L = \tfrac{4}{3}\pi(Jh - Hj) + \tfrac{32}{3}\pi(Fe - Ef). \quad (39)$$

Each of these terms represents the reaction between some type of asymmetry in the body and a corresponding type of asymmetry in the flow. If the changes in velocity of the undisturbed stream in a distance comparable with the linear dimensions of the body are small compared with the velocity of the undisturbed stream, we can use the same argument to find h, j, e and f that we previously used to find the coefficient a_1 which occurs in finding the force component X in (22). Thus h, j, e and f are the coefficients of four of the terms in the harmonic of the second degree in the expansion of the velocity potential due to the motion of the body with velocity $-$ U parallel to the axis of x in an infinite fluid.

The first two harmonics in the velocity potential due to a movement of the body with velocity $+$ U are therefore

$$\phi' = r^{-2}(-a_1 \cos\theta - a_2 \sin\theta\cos\omega - a_3\sin\theta\sin\omega) \\ - r^{-3}\{g(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) + (h\cos\omega + j\sin\omega)\sin\theta\cos\theta \\ + (e\cos 2\omega + f\sin 2\omega)\sin^2\theta\}. \quad (40)$$

Though it is not possible to express the coefficients in the harmonics of the second degree in terms of the coefficients occurring in the energy equation, as it is in the case of the harmonics of the first degree; yet it is possible to see what kind of asymmetry in the body will give rise to positive values of the coefficients h, j, e and f.

Consider, for instance, what kind of body might be expected to give rise to a flow for which $j = e = f = 0$ but h is not zero. The simplest such body is a sphere with its centre not at the origin.

The velocity potential due to a sphere of radius a with its centre at small distance η along the axis of y and moving with velocity U parallel to the axis of x is

$$\phi' = \tfrac{1}{2}\mathrm{U}\frac{a^3 x}{\{x^2 + (y-\eta)^2\}^{3/2}} = \tfrac{1}{2}\mathrm{U}a^3\left\{\frac{x}{r^3} + \frac{3\eta xy}{r^5}\right\}.$$

The harmonic of the second degree is therefore

$$\tfrac{3}{2}\mathrm{U}a^3\eta r^{-3}\cos\theta\sin\theta\cos\omega,$$

so that comparing this with (40) it will be seen that

$$g = j = c = f = 0 \quad \text{and} \quad -h = \tfrac{3}{2}\eta\mathrm{U}a^3.$$

Hence from (39)

$$\mathrm{L} = \tfrac{4}{3}\pi h\mathrm{J} = -2\pi\eta\mathrm{U}a^3\mathrm{J}. \quad (41)$$

Since with positive values of J and η, L is negative, the couple tends to turn the sphere about the axis of x as though its centre were attracted towards the centre of curvature of the central stream line, which for positive values of J is in the negative part of the axis of z.

This result can be verified because in the case of a sphere the resultant force necessarily acts through its centre, and from (7) is

$$\mathrm{Z} = -2\pi a^3\frac{\partial p}{\partial z}.$$

It can be shown that $\frac{\partial p}{\partial z} = \mathrm{JU}$, so that

$$\mathrm{Z} = -2\pi a^3 \mathrm{JU} \quad \text{and} \quad \mathrm{L} = \mathrm{Z}\eta = -2\pi a^3 \mathrm{JU}\eta,$$

which agrees with (41).

Experimental Verification of Expression for Couples on Bodies in Curved Streams.

In the case of a sphere capable of rotation about an axis which does not pass through its centre there is a resultant force when it is placed in a curved stream. This lateral force makes it difficult to use such a body for experimental demonstration; accordingly a body was devised which when suspended under gravity might be expected to give rise to a pronounced value for the coefficient h of (40) without necessarily giving rise to lateral force. Such a body is shown in fig. 2. It was an elongated body of revolution the centre line of which was afterwards bent into an arc of a circle so that it looked like a small bologna sausage.

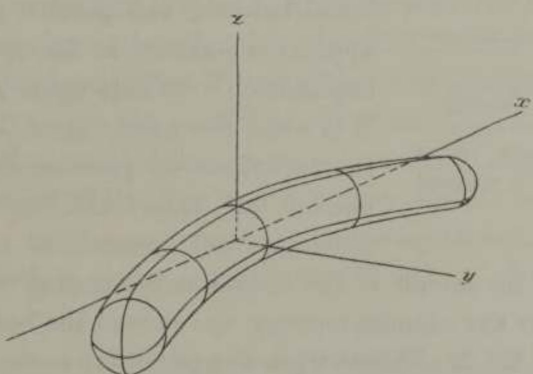

FIG. 2.

If such a body be set in the position shown in fig. 2 so that the curvature of the centre line lies in the plane $z = 0$ and the centre of curvature lies in the positive part of the axis of y, then the effect of the curvature of the centre line would be to increase the velocity of the flow at points in the negative side of the axis of y and to decrease it at points on the positive side. This is exactly opposite to the effect of moving the centre of a sphere from the origin to a distance η out along the positive side of the axis of y. It will be seen, therefore, that if the body shown in fig. 2 is moved parallel to the axis of x in the positive direction, the coefficient of the harmonic $r^{-3} \cos \theta \sin \theta \cos \omega$

in the corresponding expansion of ϕ' will be negative while that of $r^{-3}\cos\theta\sin\theta\sin\omega$ will be zero.

Hence from (40) h is positive and $j = 0$.

Taking the case of the undisturbed flow for which $H = E = F = 0$ and J is positive, *i.e.*, curved flow in which the stream lines are parallel to the plane $y = 0$ and the centre of curvature in the negative part of the axis of z, it will be seen from (39) that L is positive so that a couple acts on the body tending to turn it towards the position where the centre of curvature of the middle line of the body is in the positive side of the axis of z. It appears, therefore, that a body of this shape should set itself so that the curvature of its centre line is in the same plane as the curvature of the stream lines but in the opposite direction. It has two positions of equilibrium, one stable and the other unstable.

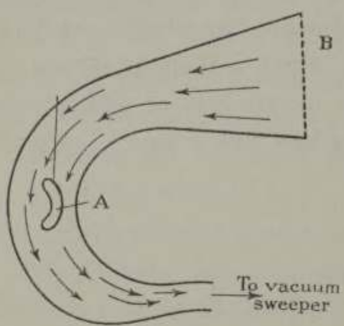

Fig. 3. — Experiment on a Curved Body in a Curved Stream of Air.

A body of this form was hung by a fine silk thread in the curved stream created by a vacuum sweeper sucking air through a curved channel formed by two bent pieces of sheet metal between two parallel glass plates. The apparatus is shown in fig. 3, which needs little explanation. In that figure A is the body and B is some fine gauze fitted into the mouth of the apparatus to prevent disturbances in the outside air from affecting the flow. The apparatus was mounted so that the air in the neighbourhood of the middle of the body was descending vertically.

Result.—Directly the vacuum sweeper was started the body swung into the position shown in fig. 3. This is what was predicted mathematically.

Experimental Verification of Formulæ for Converging and Diverging Flow.

Next suppose the central stream line of the undisturbed flow is straight so that $H = J = 0$, and let us consider the expression $L = \frac{32}{3}\pi\,(Fe - Ef)$.

The quantities E and F represent differences in the amount of convergence or divergence of stream lines in different axial planes through the central stream line. We have already pointed out, for instance, that if $H = J = F = 0$ and $G = 2E$ the stream lines are all parallel to the plane $y = 0$. A convenient method for obtaining a stream having a pronounced value of E is therefore to make a channel two of whose walls are parallel (in the actual apparatus glass plates) and the remaining two are made from sheet metal bent into the form shown in fig. 4. If the axes are chosen so that the plane $y = 0$ is parallel to the glass sheets, and if the axis of x is the central line of the apparatus, the

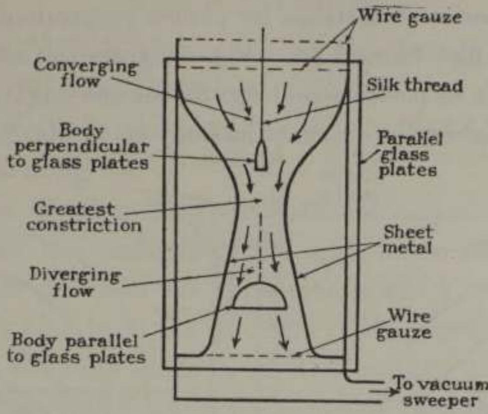

FIG. 4.—Experiment on a Flattened Body in Converging and Diverging Flow.

positive direction being downwards, then $\mathrm{E} = \frac{1}{2}\mathrm{G}$ is negative in the converging upper part of the apparatus and positive in the lower diverging part.

Referring to the expression (39) for L, it will be seen that E occurs associated with f. For an experimental verification, therefore, we must find what shape a body must be made in order that it may give rise to a pronounced value for f.

Referring to (40), $-f$ is the coefficient of $r^{-3}\sin^2\theta\sin 2\omega$ in ϕ'. We shall first find what kind of small alteration must be made to a sphere in order that the velocity potential of the flow round it may contain a term of the type $r^{-3}\sin^2\theta\sin 2\omega$. Let $r = a + b\chi(\theta, \omega)$ be the equation to the body, b being small compared with a, and χ a function of θ and ω. This will produce a velocity potential

$$\phi' = \tfrac{1}{2}\mathrm{U}a^3r^{-2}\cos\theta - fr^{-3}\sin 2\omega\sin^2\theta,$$

provided $l\mathrm{U} = -\dfrac{\partial\phi'}{\partial r}$, where l is the direction cosine of the normal to the surface of the body.

Since b is small compared with a

$$l = \cos\theta + \frac{b}{a}\sin\theta\frac{\partial}{\partial\theta}\chi(\theta, \omega),$$

so that

$$b\frac{\partial}{\partial\theta}\chi(\theta, \omega) = -3f\mathrm{U}^{-1}a^{-3}\sin 2\omega\sin\theta,$$

or

$$b\chi(\theta, \omega) = 3f\mathrm{U}^{-1}a^{-3}\sin 2\omega\cos\theta = b'\sin 2\omega\cos\theta. \tag{42}$$

In order to form a picture of the shape of the body whose equation is

$$r = a + b' \sin 2\omega \cos \theta, \tag{43}$$

one can imagine a series of sections by planes perpendicular to the axis of x. These are shown in figs. 5A and 5B. Fig. 5A gives the sections of the half of the body for which x is positive and fig. 5B for the negative half. It will be seen that the essential feature is that the body is circular in the central section

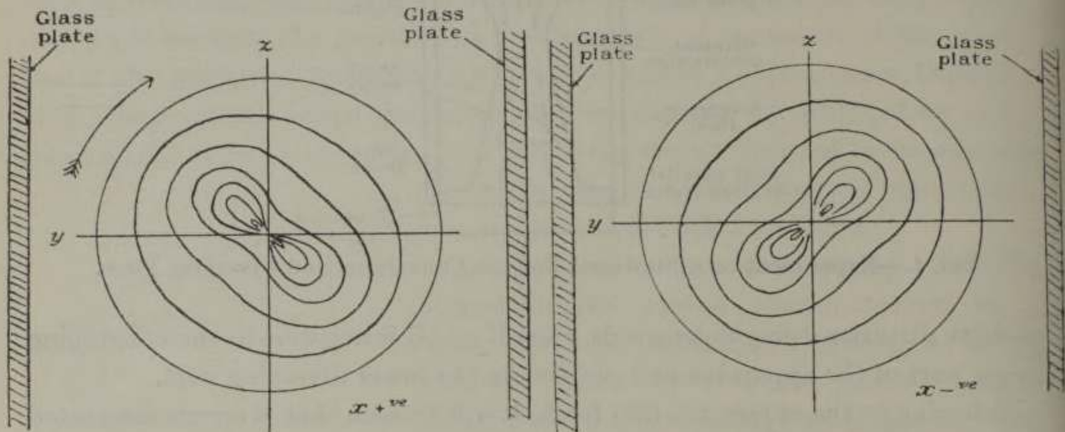

FIG. 5A. FIG. 5B.

Sections of the Body $r = a + b' \cos\theta \sin 2\omega$ by Plane Perpendicular to the Axis, *i.e.*, $\theta = 0$.

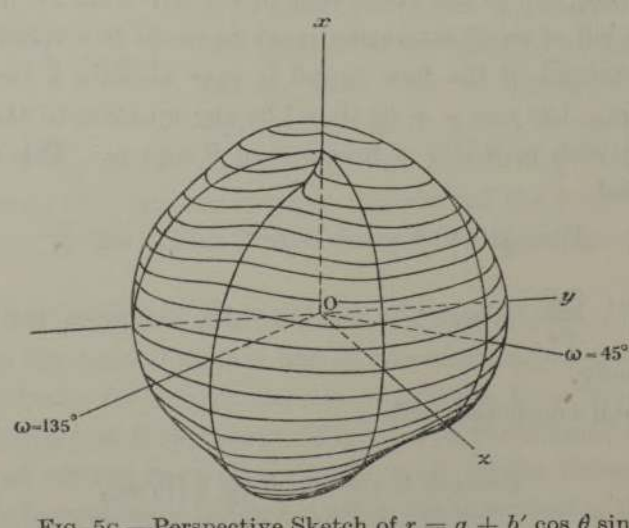

FIG. 5C.—Perspective Sketch of $r = a + b' \cos\theta \sin 2\omega$.

and that it has a flattened nose and tail, but the direction of flattening at the nose is at right angles to the direction at the tail and each is at 45° to the axes of y and z. The asymmetry is, in fact, that possessed by a tetrahedron about the line joining the mid-points of opposite edges. If a body of this shape is placed in a converging stream of fluid it gives rise to a value of f equal to $\frac{1}{3}b'Ua^3$, so that

$$L = -\tfrac{32}{9}\pi E b' U a^3. \tag{44}$$

In the converging part of the apparatus shown in fig. 4 E is negative, so that L is positive. The reaction between the fluid and the body is therefore such that the body tends to turn in the direction of increasing ω, namely, that shown by the arrow in fig. 5A.

Since the wind is blowing downwards in the apparatus shown in fig. 4, figs. 5A and 5B show the contours of the body when seen from *below*. The positions of the glass plates in the apparatus of fig. 4 are also shown in figs. 5A and 5B: it will be seen from the direction of the arrow in fig. 5A that in the converging channel the body tends to set itself so that the flattened lower end is parallel to the glass plates while the flattened upper end is perpendicular to them. This is a position of stable equilibrium. Another stable position can evidently be found by rotating the body through 180° round the wind direction, and these are two intermediate unstable positions of equilibrium at right angles to them.

A perspective sketch of the body $r = a + b' \cos\theta \sin 2\omega$ is shown in fig. 5C.

If the body is lowered through the point of maximum constriction of flow into the diverging part of the channel in the apparatus shown in fig. 4, E changes sign. The stable positions of equilibrium become unstable and unstable positions become stable. The body should therefore rotate through a right angle as it passes from the converging to the diverging part of the channel.

These predictions were completely verified. The body used was not, in fact, the body whose equation is $r = a + b' \sin 2\omega \cos\theta$, because the flow round bodies which are nearly spherical is known to be quite unlike the flow contemplated in hydrodynamical theory. In order that the actual flow may resemble at all closely the theoretical irrotational flow, the body must be smooth and elongated. Accordingly a body was made in which the essential feature of the symmetry about the wind direction, or axis of x, of the body represented by (43) was preserved, but the length was six times the maximum diameter. This body was made by flattening the two ends of a circular cylinder into two knife edges at right angles to one another, as shown in fig. 6. It was then rounded off carefully so that there were no sharp edges to spoil the flow except the knife edges at the nose and tail. It was suspended by a silk thread at S and hung in the wind tunnel shown

Fig. 6.

in fig. 4, so that it would be raised or lowered through the central part of the channel.

When a draught was created by applying a vacuum sweeper to the lower end of the channel, the body set itself so that the upper knife edge was perpendicular to the glass plates when it was in the upper converging part; but it turned through a right angle when lowered through the point of maximum constriction into the diverging part of the channel.

It appears, therefore, that the predictions of mathematical theory as to the effect of two types of asymmetry in the flow on corresponding types of asymmetry in the body are completely verified by experiment.

It is well known that the flow of fluid past a body differs considerably from that contemplated by irrotational theory even in the case of elongated bodies, but the flow at the forward end is far more like the theoretical flow than that behind the mid-ship section. In order to separate the effects of asymmetry of the flow and the body at the forward end from those at the after end, the body shown in the perspective sketch (fig. 7) was made. One end was cut to a knife edge like the body previously described, while the other end was turned to a point in a symmetrical ogival shape like the nose of an airship. There were no sharp edges except at the nose and tail of the body.

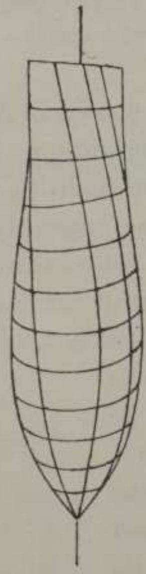

FIG. 7. — Body with circular cross-sections in lower half, finishing in a point ϵ, and elliptical cross-sections in upper half with knife-edge on top.

When hung in the converging channel with its point downwards and knife edge upwards, this body set itself with its knife edge perpendicular to the glass plates. When hung in the converging channel with the point on top, the knife edge at the bottom set itself parallel to the glass plates. It appears, therefore, that the agreement between theory and experiment extends to the tail end of the body when the body is of the "easy" shape shown in fig. 7.

A body cut into the shape of half a magnifying lens and hung from the middle point of its curved edge set itself perpendicular to the glass plates in the converging part and parallel to them in the diverging part. The two positions of this body are shown in fig. 4.

A body shaped like a whole magnifying lens or disc sets itself in the same position as that assumed by the half disc. If the flow were accurately the irrotational flow contemplated by theory, the stream should exert no directive effect on a body with this kind of symmetry. Failure of the stream lines to close in at the after end of the body in manner indicated by irrotational theory would, however, weaken the negative directive effect of the rear portion compared with the positive effect of the front portion. It seems clear that this is the reason for the observed orientation of a lenticular-shaped body hung from a point in its curved edge. An oblate spheroid with one of its maximum diameters along the wind direction behaves in the same way. The experiments here described are very easy to carry out with a domestic vacuum cleaner, but in making a body like that shown in fig. 7 great care has to be exercised to keep it symmetrical and to hang it symmetrically. In my experiments I controlled it with a magnet, inserting a small magnet in the body perpendicular to the axis. When the flow was established I removed the control.

In conclusion I should like to express my thanks to Mr. W. S. Farren for preparing the three perspective sketches of figs. 2, 5c and 7, and to Sir Ernest Rutherford for facilities for making the experiments.

On the Decomposition of Ammonia by High-Speed Electrons.

By Prof. J. C. McLennan, F.R.S., and Gilbert Greenwood, M.Sc., University of Toronto.

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Introduction.

During recent years a knowledge of the ammonia equilibrium has attained a great technical importance. The fact that ammonia decomposes under the action of an electric spark or on passing through a red-hot tube was known as long ago as in the time of Priestly. It was, at this time, regarded as a complete decomposition. The first indication of a certain quantity of the ammonia remaining undecomposed, and thus of the balanced nature of the action, was obtained by Deville* in 1805. Measurements on the variation of the equilibrium with temperature, at atmospheric pressure, were made by Haber and Oordt.† They found that at 1020° C. the equilibrium mixture contained

* 'C. R.,' vol. 60, p. 317 (1865).

† 'Z. f. anorg Chem.,' vol. 44, p. 341 (1905).