

The forcing connected detour number of a graph

A. P. Santhakumaran
Hindustan University, India

S. Athisayanathan
St. Xavier's College, India

Received : August 2013. Accepted : December 2013

Abstract

For two vertices u and v in a graph $G = (V, E)$, the detour distance $D(u, v)$ is the length of a longest u - v path in G . A u - v path of length $D(u, v)$ is called a u - v detour. A set $S \subseteq V$ is called a detour set of G if every vertex in G lies on a detour joining a pair of vertices of S . The detour number $dn(G)$ of G is the minimum order of its detour sets and any detour set of order $dn(G)$ is a detour basis of G . A set $S \subseteq V$ is called a connected detour set of G if S is detour set of G and the subgraph $G[S]$ induced by S is connected. The connected detour number $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a connected detour basis of G . A subset T of a connected detour basis S is called a forcing subset for S if S is the unique connected detour basis containing T . A forcing subset for S of minimum cardinality is a minimum forcing subset of S . The forcing connected detour number of S , denoted by $fcdn(S)$, is the cardinality of a minimum forcing subset for S . The forcing connected detour number of G , denoted by $fcdn(G)$, is $fcdn(G) = \min\{fcdn(S)\}$, where the minimum is taken over all connected detour bases S in G . The forcing connected detour numbers of certain standard graphs are obtained. It is shown that for each pair a, b of integers with $0 \leq a < b$ and $b \geq 3$, there is a connected graph G with $fcdn(G) = a$ and $cdn(G) = b$.

Key Words : *Detour, connected detour set, connected detour basis, connected detour number, forcing connected detour number.* **AMS**

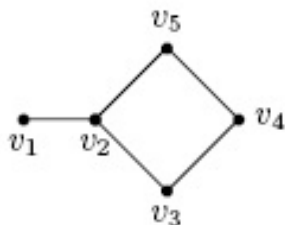
Subject Classification : *05C12.*

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1, 5]. For vertices u and v in a connected graph G , the detour distance $D(u, v)$ is the length of a longest u - v path in G . A u - v path of length $D(u, v)$ is called a u - v detour. It is known that the detour distance is a metric on the vertex set V . Detour distance and detour center of a graph were studied in [2, 4].

A vertex x is said to lie on a u - v detour P if x is a vertex of P including the vertices u and v . A set $S \subseteq V$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S . The detour number $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a detour basis of G . A vertex v that belongs to every detour basis of G is a detour vertex in G . If G has a unique detour basis S , then every vertex in S is a detour vertex in G . These concepts were studied in [3]. A set $S \subseteq V$ is called a connected detour set of G if S is a detour set of G and the subgraph $G[S]$ induced by S is connected. The connected detour number $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called a connected detour basis of G . A vertex v in G is a *connected detour vertex* if v belongs to every connected detour basis of G . If G has a unique connected detour basis S , then every vertex in S is a connected detour vertex of G . The connected detour number of a graph was introduced and studied in [6].

For the graph G given in Figure 1.1, the sets $S_1 = \{v_1, v_3\}$, $S_2 = \{v_1, v_5\}$ and $S_3 = \{v_1, v_4\}$ are the three detour bases of G so that $dn(G) = 2$. It is clear that no two element subset of V is a connected detour set of G . However the set $S_4 = \{v_1, v_2, v_3\}$ is a connected detour basis of G so that $cdn(G) = 3$. Also the set $S_5 = \{v_1, v_2, v_5\}$ is another connected detour basis of G . Thus there can be more than one connected detour basis for a graph G .

Figure 1.1: G

Graphs are often used to model network of real life problems and some definite part is always present in a minimum possible spanning set in a particular problem. For each connected detour basis S in a connected graph G , there is always some subset T of S that uniquely determines S as the connected detour basis containing T . Such subsets are called forcing subsets for S , and in this paper we briefly describe the properties satisfied by these sets in a graph.

The following theorem is used in the sequel.

Theorem 1.1. [6] All the end vertices and all the cut vertices of a connected graph G belong to every connected detour set of G .

Throughout this paper G denotes a connected graph with at least two vertices.

2. The Forcing Connected Detour Number

Definition 2.1. Let G be a connected graph and S a connected detour basis of G . A subset $T \subseteq S$ is called a forcing subset for S if S is the unique connected detour basis containing T . A forcing subset for S of minimum cardinality is a minimum forcing subset of S . The forcing connected detour number of S , denoted by $fc dn(S)$, is the cardinality of a minimum forcing subset for S . The forcing connected detour number of G , denoted by $fc dn(G)$, is $fc dn(G) = \min\{fc dn(S)\}$, where the minimum is taken over all connected detour bases S in G .

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{u, s, w, t, v\}$ is the unique connected detour basis of G so that $fcdn(G) = 0$ and for the graph G given in Figure 1.1, $S_2 = \{v_1, v_2, v_3\}$ and $S_3 = \{v_1, v_2, v_5\}$ are the only connected detour bases of G so that $fcdn(G) = 1$.

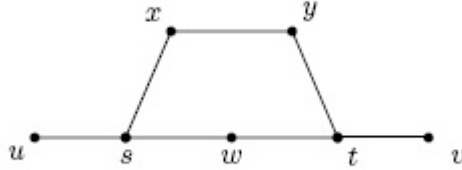


Figure 2.1: G

The next theorem follows immediately from the definitions of connected detour number and forcing connected detour number of a connected graph G .

Theorem 2.3. For every connected graph G , $0 \leq fcdn(G) \leq cdn(G)$.

Remark 2.4. The lower bound in Theorem 2.3 is sharp. For the graph G given in Figure 2.2, $fcdn(G) = 0$. Also, all the inequalities in Theorem 2.3 can be strict. For the graph G given in Figure 1.1, $cdn(G) = 3$ and $fcdn(G) = 1$. Thus $0 < fcdn(G) < cdn(G)$.

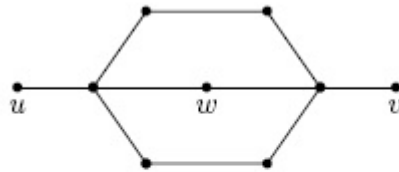


Figure 2.2: G

The following theorem is an easy consequence of the definition of forcing connected detour number of a graph.

Theorem 2.5. *Let G be a connected graph. Then*

- a) $fcdn(G) = 0$ if and only if G has a unique connected detour basis,
- b) $fcdn(G) = 1$ if and only if G has at least two connected detour bases, one of which is a unique connected detour basis containing one of its elements, and
- c) $fcdn(G) = cdn(G)$ if and only if no connected detour basis of G is the unique connected detour basis containing any of its proper subsets.

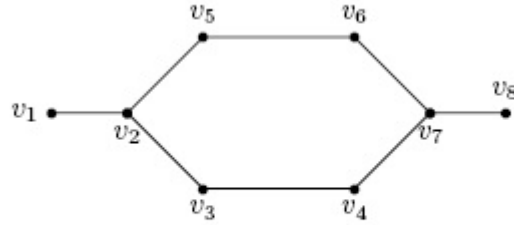
Theorem 2.6. *Let G be a connected graph and W be the set of all connected detour vertices of G . Then $fcdn(G) \leq cdn(G) - |W|$.*

Proof. *Let S be a connected detour basis S of G . Then $cdn(G) = |S|$, $W \subseteq S$ and S is the unique connected detour basis containing $S - W$. Thus $fcdn(S) \leq |S - W| = |S| - |W| = cdn(G) - |W|$ and the result follows. \square*

Corollary 2.7. *If G is a connected graph with k end-vertices and l cut-vertices, then $fcdn(G) \leq cdn(G) - k - l$.*

Proof. *This follows from Theorems 1.1 and 2.6. \square*

Remark 2.8. *The bound in Theorem 2.6 is sharp. For the graph G given in Figure 1.1, $cdn(G) = 3$, $|W| = 2$ and $fcdn(G) = 1$ as in Remark 2.4. Also, the inequality in Theorem 2.6 can be strict. For the graph G of Figure 2.3, the sets $S_1 = \{v_1, v_2, v_3, v_4, v_7, v_8\}$, and $S_2 = \{v_1, v_2, v_5, v_6, v_7, v_8\}$ are the two connected detour bases of G and $W = \{v_1, v_2, v_7, v_8\}$ so that $cdn(G) = 6$, $|W| = 4$ and $fcdn(G) = 1$. Thus $fcdn(G) < cdn(G) - |W|$. Moreover, the bound in Corollary 2.7 is also sharp. For the graph G given in Figure 1.1, $cdn(G) = 3$, $k = 1$, $l = 1$ and $fcdn(G) = 1$. Also, the inequality in Corollary 2.7 can be strict. For the graph G of Figure 2.3, $cdn(G) = 6$, $k = 2$, $l = 2$ and $fcdn(G) = 1$. Thus $fcdn(G) < cdn(G) - k - l$.*

Figure 2.3: G

In the following theorems we proceed to find the forcing numbers of certain graphs G .

Theorem 2.9. Let G be the complete graph K_p ($p \geq 2$) or the cycle C_p or the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$). Then a set S of vertices is a connected detour basis if and only if S consists of any two adjacent vertices of G . Furthermore, $cdn(G) = 2$ for each of these graphs.

Proof. If G is the complete graph K_p ($p \geq 2$) or the cycle C_p , then it is clear that any set of two adjacent vertices is a connected detour basis of G . Let G be the complete bipartite graph $K_{m,n}$ ($2 \leq m \leq n$). Let X and Y be the bipartite sets of $K_{m,n}$ ($2 \leq m \leq n$) with $X = \{x_1, x_2, \dots, x_m\}$. Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m - 1$. Let $y \in Y - \{v\}$. Then the vertex y lies on a u - v detour $P : u = x_1, y, x_2, y_1, x_3, y_2, \dots, x_{m-1}, y_{m-2}, x_m, v$, where $y_1, y_2, \dots, y_{m-2} \in Y - \{v, y\}$. Thus the set $\{u, v\}$ is a connected detour basis of $K_{m,n}$.

Now, let S be a connected detour basis of G . Let S' be any set consisting of two adjacent vertices of G . Then as in the first part of this theorem S' is a connected detour basis of G . Hence $|S| = |S'| = 2$ and it follows that the two vertices of S are adjacent. The converse is obvious. \square

Theorem 2.10. a) If G is the complete graph K_p ($p \geq 3$) or the cycle C_p or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$), then $cdn(G) = fcdn(G) = 2$.

b) If G is a tree of order $p \geq 2$, then $cdn(G) = p$ and $fcdn(G) = 0$.

Proof. a) By Theorem 2.9, a set S of vertices is a connected detour basis of G if and only if S consists of two adjacent vertices of G . For each vertex v in G there are at least two vertices adjacent with v . Thus the vertex v belongs to more than one connected detour basis of G . Hence it follows that no set consisting of a single vertex is a forcing subset for any connected detour basis of G . Thus $fcdn(G) = 2$. Also, by Theorem 2.9, $cdn(G) = 2$ and the result follows.

b) By Theorem 1.1, $cdn(G) = p$. The set of all vertices of a tree is the unique connected detour basis so that $fcdn(G) = 0$ by Theorem 2.5(a). \square

Theorem 2.11. Let G be a connected graph with cut-vertices and S a connected detour set of G . Then for any cut-vertex v of G , every component of $G - v$ contains an element of S .

Proof. Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S . Let $u \in V(C)$. Since S is a connected detour set of G , there exist vertices $x, y \in S$ such that the vertex u lies on some x - y detour $P : x = u_0, u_1, \dots, u, \dots, u_t = y$ in G . Let P_1 be the x - u subpath of P and P_2 be the u - y subpath of P . Since v is a cut-vertex of G both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Thus every component of $G - v$ contains an element of S . \square

Theorem 2.12. Let $G = (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \dots + n_r + k = p - 1$. Then $cdn(G) = r + k + 1$.

Proof. Let u_1, u_2, \dots, u_k be the end-vertices of G . Let S be any connected detour set of G . Then by Theorem 1.1, $v \in S$ and $u_i \in S$ ($1 \leq i \leq k$). Also by Theorem 2.11, S contains a vertex from each component K_{n_i} ($1 \leq i \leq r$). Now, choose exactly one vertex v_i from each K_{n_i} such that $v_i \in S$. Then $|S| \geq r + k + 1$. Let $T = \{v, v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_k\}$. Since every vertex of G lies on a detour joining a pair of vertices of T , it follows that T is a detour basis of G . Also, since $G[T]$ is connected, $cdn(G) = r + k + 1$. \square

Now, in view of Theorem 2.3, we have the following realization result.

Theorem 2.13. For each pair a, b of integers with $0 \leq a < b$ and $b \geq 3$, there is a connected graph G with $fcdn(G) = a$ and $cdn(G) = b$.

Proof. **Case 1:** $a = 0$. For each $b \geq 3$, let G be a tree with b vertices. Then $fdn(G) = 0$ and $cdn(G) = b$ by Theorem 2.10(b).

Case 2: $a \geq 1$. For each integer i with $1 \leq i \leq a$, let F_i be a copy of the complete graph K_2 , where $V(F_i) = \{u_i, v_i\}$ and let $H = K_{1, b-a-1}$ be the star whose vertex set is $W = \{z_1, z_2, \dots, z_{b-a-1}, v\}$. Then the graph G is obtained by joining the central vertex v of H to the vertices of F_1, F_2, \dots, F_a . The graph G is connected and is shown in Figure 2.4.

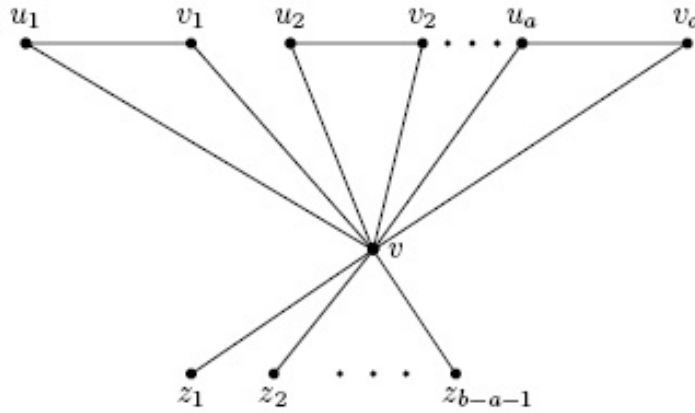


Figure 2.4: G

By Theorem 2.12, $cdn(G) = b$. Now, we show that $fdn(G) = a$. It is clear that W is the set all connected detour vertices of G . Hence it follows from Theorem 2.6 that $fdn(G) \leq cdn(G) - |W| = b - (b - a) = a$. Now, since $cdn(G) = b$, it follows from Theorem 2.11 that any connected detour basis of G is of the form $S = W \cup \{x_1, x_2, \dots, x_a\}$, where $x_i \in \{u_i, v_i\}$ ($1 \leq i \leq a$). Let T be a subset of S with $|T| < a$. Then there is a vertex x_j ($1 \leq j \leq a$) such that $x_j \notin T$. Let y_j be a vertex of F_j distinct from x_j . Then $S' = (S - \{x_j\}) \cup \{y_j\}$ is also a connected detour basis such that it contains T . Thus S is not the unique connected detour basis containing

T and so T is not a forcing set of S . Since this is true for all connected detour bases of G , it follows that $fcdn(G) \geq a$ and so $fcdn(G) = a$. \square

References

- [1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Reading MA, (1990).
- [2] G. Chartrand, H. Escudro and P. Zhang, Detour distance in graphs, *J. Combin. Math. Combin. Comput.*, 53, pp. 75-94, (2005).
- [3] G. Chartrand, L. Johns, and P. Zhang, Detour Number of a Graph, *Util. Math.* **64**, pp. 97–113, (2003).
- [4] G. Chartrand and P. Zhang, *Distance in Graphs—Taking the Long View*, *AKCE J. Graphs.1. No.1*, pp. 1–13, (2004).
- [5] G. Chartrand and P. Zang, *Introduction to Graph Theory*, Tata McGraw-Hill, (2006).
- [6] A. P. Santhakumaran and S. Athisayanathan, *The connected detour number of a graph*, *J. Combin. Math. Combin. Comput.*, 69, pp. 205–218, (2009).

A. P. Santhakumaran

Department of Mathematics
Hindustan University
Hindustan Institute of Technology and Science
Padur, Chennai-603 103, India
e-mail : apskumar1953@yahoo.co.in

and

S. Athisayanathan

Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai - 627 002,
India
e-mail: athisayanathan@yahoo.co.in