The Formation of Black Holes in General Relativity

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In 1965 Penrose introduced the concept of a closed *trapped surface*. He defined a trapped surface to be a spacelike surface in spacetime, such that an infinitesimal virtual displacement of the surface along either family of future-directed null geodesic normals to the surface leads to a pointwise decrease of the area element. On the basis of this concept, Penrose proved an *incompleteness* theorem. In the light of subsequent work, namely the uniqueness theorem of the maximal development of given initial data by Choquet-Bruhat and Geroch, and the work of Rendall on the characteristic initial value problem, the incompleteness theorem of Penrose may be re-stated as follows:

Let us be given regular characteristic initial data on a complete null geodesic cone C_o of a point o. Let (M^*, g) be the maximal future development of the data on C_o. Suppose that M^* contains a closed trapped surface. Then (M^*, g) is future null geodesically incomplete.

An important remark here is that it is not *a priori* obvious that closed trapped surfaces are *evolutionary*. That is, it is not obvious whether closed trapped surfaces can form in evolution starting from initial conditions in which no such surfaces are present. What is more important, the physically interesting problem is the problem where the initial conditions are arbitrarily far from already containing closed trapped surfaces, and we are asked to follow the long time evolution and show that, under suitable circumstances, closed trapped surfaces eventually form. Only an analysis of the dynamics of gravitational collapse can achieve this aim.

John Wheeler, my teacher in physics, posed to me the following problem back in 1968: *to establish the formation of black holes in pure general relativity, by the focusing of incoming gravitational waves*. I have recently solved this problem. The solution is contained in a monograph which was published in the series "Monographs in Mathematics" of the European Mathematical Society (EMS Publishing House, January 2009).

I shall now state the simplest version of the theorem on the formation of closed trapped surfaces in pure general relativity which this monograph establishes. This is the limiting version, where we have an asymptotic characteristic initial value problem with initial data at past null infinity. Denoting by $\underline{u}$ the "advanced time", it is assumed that the initial data are trivial for $\underline{u} \leq 0$.

Let k, l be positive constants, $k > 1$, $l < 1$. Let us be given smooth asymptotic initial data at past null infinity which is trivial for advanced time $\underline{u} \leq 0$. Suppose that the incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$ is not less than $k/8\pi$. Then if δ is suitably small, the maximal development of the data contains a closed trapped surface S which is diffeomorphic to S^2 and has area

$$\text{Area}(S) \geq 4\pi l^2$$

We remark that by virtue of the scale invariance of the vacuum Einstein equations, the theorem holds with k, l, and δ, replaced by ak, al, and $a\delta$, respectively, for any positive constant a.

The above theorem is obtained through a theorem in which the initial data is given on a complete future null geodesic cone C_o. The generators of the cone are parametrized by an affine parameter s measured from the vertex o and defined so that the corresponding null geodesic vectorfield has projection T at o along a fixed unit future-directed timelike vector T at o. It is assumed that the initial data are trivial for $s \leq r_0$, for some $r_0 > 1$. The boundary of this trivial region is then a round sphere of radius r_0.

The advanced time $\underline{u}$ is then defined along C_o by

$$\underline{u} = s - r_0 \tag{1}$$

The formation of closed trapped surfaces theorem is similar in this case, the only difference being that the "incoming energy per unit solid angle in each direction in the advanced time interval $[0, \delta]$", a notion defined only at past null infinity, is replaced by the integral

$$\frac{r_0^2}{8\pi} \int_0^{\delta} e d\underline{u} \tag{2}$$

on the affine parameter segment $[r_0, r_0 + \delta]$ of each generator of C_o.

The function e is an invariant of the conformal intrinsic geometry of C_o, given by:

$$e = \frac{1}{2}|\hat{\chi}|^2_{\not{g}} \tag{3}$$

where $\not{g}$ is the induced metric on the sections of C_o corresponding to constant values of the affine parameter, and $\hat{\chi}$ is the *shear* of these sections, the trace-free part of their 2nd fundamental form relative to C_o.

The theorem for a cone C_o is established for any $r_0 > 1$ and the smallness condition on δ is independent of r_0. The domain of dependence, in the maximal development, of the trivial region in C_o is a domain in Minkowski spacetime bounded in the past by the trivial part of C_o and in the future by $\underline{C}_e$, the past null geodesic cone of a point e at arc length $2r_0$ along the timelike geodesic Γ_0 from o with tangent vector T at o. Considering then the corresponding complete timelike geodesic in Minkowski spacetime, fixing the origin on this geodesic to be the point e, the limiting form of the theorem is obtained by letting $r_0 \to \infty$, keeping the origin fixed, so that o tends to the infinite past along the timelike geodesic.

In the monograph, almost all the work goes into establishing an *existence theorem* for a development of the initial data which extends far enough into the future so that trapped spheres have eventually chance to form within this development. On the other hand, there is a wealth of information in this existence theorem, which gives us full knowledge of the geometry of spacetime when closed trapped surfaces begin to form.

The monograph relies on the methods which originated in my work with Klainerman on the stability of the Minkowski spacetime, in conjunction with a new method which I call the "short pulse method". The new method can, in principle, be extended to all systems of partial differential equations of hyperbolic type which are derivable from an action principle.

The short pulse method is a method which allows us to establish an existence theorem for a development of the initial data which is large enough so that interesting things have chance to occur within this development, if a nonlinear system is involved. One may ask at this point: what does it mean for a length to be small in the context of the vacuum Einstein equations? For, the equations are scale invariant. Here *small* means *by comparison to the area radius of the trapped sphere to be formed*.

With initial data on a complete future null geodesic cone C_o, as previously explained, which are trivial for $s \leq r_0$, we consider the restriction of the initial data to $s \leq r_0 + \delta$. In terms of the advanced time $\underline{u}$, we restrict attention to the interval $[0, \delta]$, the data being trivial for $\underline{u} \leq 0$. The retarded time u is set equal to $u_0 = -r_0$ at o and therefore on C_o. Also, $u - u_0$ is defined along Γ_0 to be one half the arc length from o. This determines u everywhere. The level sets of $\underline{u}$, denoted $\underline{C}_{\underline{u}}$, are incoming null hypersurfaces, while the level sets of u, denoted by C_u, are outgoing null hypersurfaces, future null geodesic cones with vertices on Γ_0. In particular, the initial future null geodesic cone C_o is denotes C_{u_0}.

We denote by $S_{\underline{u},u}$ the spacelike surfaces of intersection $S_{\underline{u},u} = \underline{C}_{\underline{u}} \bigcap C_u$. These may be thought of as the "wave fronts".

The development whose existence we want to establish is that bounded in the future by the spacelike hypersurface H_{-1} where $\underline{u} + u = -1$ and by the incoming null hypersurface $\underline{C}_\delta$. We denote this development M_{-1}.

The first step is the analysis of the equations along the initial hypersurface C_{u_0}. This is particularly clear and simple because of the fact that C_{u_0} is a null hypersurface, so we are dealing with the characteristic initial value problem.

There is a way of formulating the problem in terms of free data which are not subject to any constraints. The full set of data which includes all the curvature components and their transversal derivatives, up to any given order, along C_{u_0}, is then determined by integrating ordinary differential equations along the generators of C_{u_0}. We show that the free data may be described as a 2-covariant symmetric positive definite tensor density m, of weight -1 and unit determinant, on S^2, depending on $\underline{u}$.

This is of the form:

$$m = \exp \psi \tag{4}$$

where ψ is a 2-dimensional symmetric trace-free matrix valued "function" on S^2, depending on $\underline{u} \in [0, \delta]$, and transforming under change of charts on S^2 in such a way so as to make m a 2-covariant tensor density of weight -1. The transformation rule is particularly simple if stereographic charts on S^2 are used.

Then there is a function O defined on the intersection of the domains of the north and south polar stereographic charts on S^2, with values in the 2-dimensional symmetric orthogonal matrices of determinant -1 such that in going from the north polar chart to the south polar chart or vise-versa, $\psi \mapsto \tilde{O}\psi O$ and $m \mapsto \tilde{O} m O$.

The crucial ansatz of the short pulse method is the following. We consider an arbitrary smooth 2-dimensional symmetric trace-free matrix valued "function" ψ_0 on S^2, depending on $s \in [0, 1]$, which extends smoothly by 0 to $s \leq 0$, and we set:

$$\psi(\underline{u}, \vartheta) = \frac{\delta^{1/2}}{|u_0|} \psi_0 \left(\frac{\underline{u}}{\delta}, \vartheta \right), \quad (\underline{u}, \vartheta) \in [0, \delta] \times S^2 \tag{5}$$

A question that immediately comes up when one ponders this ansatz, is why is the "amplitude" of the pulse proportional to the square root of the "length" of the pulse? (the factor $|u_0|^{-1}$ is the standard decay factor in 3 spatial dimensions, the square root of the area of the wave fronts). Where does this relationship come from? Obviously, there is no such relationship in a linear theory.

The answer is that it comes from our desire to form trapped surfaces in the development M_{-1}. If a problem involving the focusing of incoming waves in a different context was the problem under study, for example the formation of electromagnetic shocks by the focusing of incoming electromagnetic waves in a nonlinear medium, the relationship between length and amplitude would be dictated by the desire to form such shocks within our development.

The analysis of the equations along C_{u_0} then gives a certain hierachy with respect to δ for the geometric quantities associated to the foliation of C_{u_0} by the surfaces $S_{\underline{u},u_0}$, in particular for the spacetime curvature componets relative to this foliation. That is, the size of different components has a different power low dependence on δ. I call this the "short pulse hierarchy". And this hierarchy is *nonlinear*. For, if only the linearized form of the equations was considered, a different hierarchy would be obtained.

The short pulse hierarchy is the key to the existence theorem as well as to the trapped surface formation theorem. The main step is of course to establish that the short pulse hierarchy is preserved in evolution. That is, that the geometric quantities associated to the foliation of the development M_{-1} by the surfaces $S_{\underline{u},u}$, in particular the spacetime curvature componets relative to this foliation, satisfy the same hierachy.

This is established through a bootstrap argument which at the same time establishes the existence of the solution in the domain M_{-1}. The proof is by refining the methods which originated in the work on the stability of the Minkowski spacetime in the light of the short pulse hierarchy. It is thus by combining the new idea of the short pulse method with the ideas involved in the stabililty of Minkowski that the present results are reached.