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The Fourier–Laplace Generalized Convolutions and Applications to Integral Equations

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Abstract In this paper we introduce two generalized convolutions for the Fourier cosine, Fourier sine and Laplace integral transforms. Convolution properties and their applications to solving integral equations and systems of integral equations are considered.

Keywords Fourier sine transform · Fourier cosine transform · Laplace transform

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1 Introduction

Convolutions for integral transforms are studied in the early years of the 20th century, such as convolutions for the Fourier transform (see [2, 9, 13]), the Laplace transform (see [1, 2, 8, 13, 16–19]), the Mellin transform (see [8, 13]), the Hilbert transform (see [2, 3]), the Fourier cosine and sine transforms (see [5, 7, 13, 14]), and so on. These convolutions have many important applications in image processing, partial differential equations, integral equations, inverse heat problems (see [2–4, 8, 11–13, 15–18]).

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In 1998, in [6] the authors introduced the general method for defining a generalized convolution with a weight function γ for three arbitrary integral transforms K_1, K_2 and K_3, such that the following factorization identity holds:

$$K_1\big[f \overset{\gamma}{*} g\big](y) = \gamma(y)(K_2 f)(y)(K_3 g)(y).$$

This idea has opened up many new researches and new convolutions with interesting properties appearing in [7], but so far there is only one convolution for Laplace transform defined as follows (see [2, 19]):

$$\big(f \underset{L}{*} g\big)(x) = \int_0^x f(x-t)g(t)\,dt, \quad x > 0,$$

which satisfies the factorization identity

$$L\big(f \underset{L}{*} g\big)(y) = (Lf)(y)(Lg)(y).$$

Here L denotes the Laplace transform

$$(Lf)(y) = \int_0^\infty f(x)e^{-yx}\,dx, \quad y > 0.$$

In this paper, we introduce and study two new generalized convolutions with a weight function for the Fourier cosine-Laplace and Fourier sine-Laplace transforms. We also obtain some norm inequalities of these convolutions and algebraic properties of convolution operators on $L_1(\mathbb{R}_+)$ and $L_p^{\alpha,\beta}(\mathbb{R}_+)$. In the last section, we apply these convolutions to solve several classes of integral equations as well as systems of two integral equations.

2 Well-known Convolutions

The convolution of two functions f and g for the Fourier cosine transform is of the following form (see [13]):

$$\big(f \underset{F_c}{*} g\big)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[g(x+y) + g\big(|x-y|\big)\big]\,dy, \quad x > 0, \tag{1}$$

which satisfies the following factorization identity:

$$F_c\big(f \underset{F_c}{*} g\big)(y) = (F_c f)(y)(F_c g)(y) \quad \forall y > 0. \tag{2}$$

Here F_c is the Fourier cosine transform

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos xy\,dx, \quad y > 0.$$

The generalized convolution for the Fourier sine and Fourier cosine transforms of f and g is defined as follows (see [13]):

$$\big(f \underset{1}{*} g\big)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)\big[g\big(|x-u|\big) - g(x+u)\big]\,du, \quad x > 0, \tag{3}$$

which satisfies the following factorization identity:

$$F_s\big(f \underset{1}{*} g\big)(y) = (F_s f)(y)(F_c g)(y) \quad \forall y > 0. \tag{4}$$

Here F_s is the Fourier sine transform

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin xy \, dx, \quad y > 0.$$

The convolution of two functions f and g with a weight function for the Fourier sine transform is of the following form (see [5]):

$$\begin{aligned}\Big(f \overset{\gamma}{\underset{F_s}{*}} g\Big)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(y)\big[&\operatorname{sign}(x+y-1)g\big(|x+y-1|\big)\\ &- g(x+y+1) + \operatorname{sign}(x-y+1)g\big(|x-y+1|\big)\\ &- \operatorname{sign}(x-y-1)g\big(|x-y-1|\big)\big]\,dy, \quad x > 0,\end{aligned} \tag{5}$$

which satisfies the factorization equality

$$F_s\Big(f \overset{\gamma}{\underset{F_s}{*}} g\Big)(y) = \sin y (F_s f)(y)(F_s g)(y) \quad \forall y > 0. \tag{6}$$

The convolution of two functions f and g for the Fourier cosine and Fourier sine transform is of the following form (see [7]):

$$\big(f \underset{2}{*} g\big)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[g(x+y) + \operatorname{sign}(y-x)g\big(|y-x|\big)\big]\,dy, \quad x > 0, \tag{7}$$

which satisfies the following factorization identity:

$$F_c\big(f \underset{2}{*} g\big)(y) = (F_s f)(y)(F_s g)(y) \quad \forall y > 0.$$

In this paper we are interested in the weighted space $L_p^{\alpha,\beta}(\mathbb{R}_+) \equiv L_p(\mathbb{R}_+, x^\alpha e^{-\beta x} dx)$ with the norm defined as follows:

$$\|f(x)\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \left(\int_0^\infty |f(x)|^p x^\alpha e^{-\beta x}\, dx\right)^{1/p}, \quad 1 \le p < \infty.$$

3 The Fourier–Laplace Generalized Convolutions

Definition 1 The generalized convolutions with a weight function $\gamma(y) = e^{-\mu y}$, $\mu > 0$ of two functions f and g for the Fourier cosine-Laplace and Fourier sine-Laplace transforms are defined by

$$\begin{aligned}\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \bigg[&\frac{v+\mu}{(v+\mu)^2 + (x-u)^2}\\ &\pm \frac{v+\mu}{(v+\mu)^2 + (x+u)^2}\bigg] f(u)g(v)\,du\,dv,\end{aligned} \tag{8}$$

where $x > 0$.

Theorem 1 *For two arbitrary functions $f(x)$ and $g(x)$ in $L_1(\mathbb{R}_+)$, the generalized convolutions $(f * g)_{\{\frac{1}{2}\}}$ belong to $L_1(\mathbb{R}_+)$. Moreover, the following norm estimates and factorization identities hold*:

$$\begin{aligned}
&\big\|\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}\big\|_{L_1(\mathbb{R}_+)} \leq \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)},\\
&F_{\{\frac{c}{s}\}}\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(y) = e^{-\mu y}(F_{\{\frac{c}{s}\}}f)(y)(Lg)(y) \quad \forall y > 0.
\end{aligned} \tag{9}$$

Furthermore, the generalized convolutions $(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}}$ belong to $C_0(\mathbb{R}_+)$.

Proof We have

$$\begin{aligned}
&\int_0^\infty \left|\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right| dx\\
&\quad \leq \int_{-u}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}\,dt + \int_u^\infty \frac{v+\mu}{(v+\mu)^2+t^2}\,dt\\
&\quad = \int_{-\infty}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}\,dt = \pi.
\end{aligned} \tag{10}$$

From (8) and (10), we have

$$\int_0^\infty \big|\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x)\big|\,dx \leq \int_0^\infty \big|f(u)\big|\,du \int_0^\infty \big|g(v)\big|\,dv = \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Therefore

$$\big\|\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}\big\|_{L_1(\mathbb{R}_+)} \leq \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)} < \infty.$$

Thus

$$\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}} \in L_1(\mathbb{R}_+). \tag{11}$$

From (8) and by applying formula $\int_0^\infty e^{-\alpha x}\cos xy\,dx = \frac{\alpha}{\alpha^2+y^2}$ $(\alpha > 0)$ (see [2]), we obtain

$$\begin{aligned}
\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x) &= \frac{1}{\pi}\int_{\mathbb{R}_+^3} f(u)g(v)e^{-(v+\mu)y}\big[\cos(x-u)y \pm \cos(x+u)y\big]\,du\,dv\,dy\\
&= \frac{2}{\pi}\int_{\mathbb{R}_+^3} f(u)g(v)e^{-(v+\mu)y}\left\{\begin{matrix}\cos yx\cdot\cos yu\\ \sin yx\cdot\sin yu\end{matrix}\right\} du\,dv\,dy\\
&= \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty f(u)\left\{\begin{matrix}\cos yu\\ \sin yu\end{matrix}\right\} du \int_0^\infty g(v)e^{-vy}\,dv\right] e^{-\mu y}\left\{\begin{matrix}\cos xy\\ \sin xy\end{matrix}\right\} dy\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty (F_{\{\frac{c}{s}\}}f)(y)(Lg)(y)e^{-\mu y}\left\{\begin{matrix}\cos xy\\ \sin xy\end{matrix}\right\} dy.
\end{aligned} \tag{12}$$

From (12) and (11), we get the factorization identities (9). From (12) and Riemann–Lebesgue lemma, we obtain $(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}} \in C_0(\mathbb{R}_+)$. Theorem 1 is proved. □

Theorem 2 *Suppose that* $p > 1, r \geq 1, 0 < \beta \leq 1, f(x) \in L_p(\mathbb{R}_+), g(x) \in L_1(\mathbb{R}_+)$. *Then the generalized convolutions* $(f \overset{\gamma}{*} g)_{\{^1_2\}}$ *are well-defined, continuous and belong to* $L_r^{\alpha,\beta}(\mathbb{R}_+)$. *Moreover, we get the following estimates*:

$$\left\|\left(f \overset{\gamma}{*} g\right)_{\{^1_2\}}\right\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq C\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}, \tag{13}$$

where $C = (\frac{2}{\pi\mu})^{1/p}\beta^{-\frac{\alpha+1}{r}}\Gamma^{1/r}(\alpha+1)$ *and* $\Gamma(x)$ *is Gamma–Euler function*.

Furthermore, if $f(x) \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$ *then the generalized convolutions* $(f \overset{\gamma}{*} g)_{\{^1_2\}}$ *belong to* $C_0(\mathbb{R}_+)$, *and satisfy the factorization identity* (9).

Proof By applying Hölder's inequality for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and (10), we have

$$\begin{aligned}
&\left|\left(f \overset{\gamma}{*} g\right)_{\{^1_2\}}\right| \\
&\quad\leq \frac{1}{\pi}\left\{\int_{\mathbb{R}_+^2}\left|f(u)\right|^p\left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right]\left|g(v)\right| du\, dv\right\}^{1/p} \\
&\qquad\times\left\{\int_{\mathbb{R}_+^2}\left|g(v)\right|\left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right] du\, dv\right\}^{1/q} \\
&\quad\leq \frac{1}{\pi}\left[\int_{\mathbb{R}_+^2}\left|f(u)\right|^p\left|g(v)\right|\frac{2}{\mu}\, du\, dv\right]^{1/p}\left[\int_0^\infty\left|g(v)\right|\pi\, dv\right]^{1/q} \\
&\quad= \left(\frac{2}{\pi\mu}\right)^{1/p}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.
\end{aligned}$$

Thus, convolutions (8) exist and are continuous. Combining with formula (3.225.3) in [10, p. 115], we get

$$\int_0^\infty x^\alpha e^{-\beta x}\left|\left(f \overset{\gamma}{*} g\right)_{\{^1_2\}}(x)\right|^r dx \leq C^r\|f\|^r_{L_p(\mathbb{R}_+)}\|g\|^r_{L_1(\mathbb{R}_+)}.$$

Hence convolutions (8) are in $L_r^{\alpha,\beta}(\mathbb{R}_+)$ and identities (13) hold. From the hypothesis of Theorem 2, and by similar argument as in Theorem 1, we get the factorization identities (9). Combining with Riemann–Lebesgue lemma, we obtain $(f \overset{\gamma}{*} g)_{\{^1_2\}}(x) \in C_0(\mathbb{R}_+)$. Theorem 2 is proved. □

Theorem 3 *Let* $\alpha > -1, 0 < \beta \leq 1, p > 1, q > 1, r \geq 1$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $f(x) \in L_p(\mathbb{R}_+)$ *and* $g(x) \in L_q(\mathbb{R}_+, (1+x^2)^{q-1})$, *the convolutions* $(f \overset{\gamma}{*} g)_{\{^1_2\}}$ *are well-defined*, *continuous*, *bounded in* $L_r^{\alpha,\beta}(\mathbb{R}_+)$ *and*

$$\left\|\left(f \overset{\gamma}{*} g\right)_{\{^1_2\}}\right\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq C\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_q(\mathbb{R}_+,(1+x^2)^{q-1})}, \tag{14}$$

where $C = \mu^{-\frac{1}{p}}\pi^{-\frac{1}{q}}\beta^{-\frac{\alpha+1}{r}}\Gamma^{1/r}(\alpha+1)$. *Moreover, if* $f(x) \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$ *and* $g(x) \in L_1(\mathbb{R}_+) \cap L_q(\mathbb{R}_+, (1+x^2)^{q-1})$ *then convolutions* $(f \overset{\gamma}{*} g)_{\{^1_2\}}$ *belong to* $C_0(\mathbb{R}_+)$ *and satisfy factorization identities* (9).

Proof Applying Hölder's inequality for $p, q > 1$ and combining with (10), we have

$$\begin{aligned}
&\left|\left(f \overset{\gamma}{*} g\right)_{\{\frac{1}{2}\}}\right| \\
&\leq \frac{1}{\pi}\left\{\int_{\mathbb{R}_+^2} \left|f(u)\right|^p \left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right] \frac{1}{1+v^2}\, du\, dv\right\}^{1/p} \\
&\quad \times \left\{\int_{\mathbb{R}_+^2} \left|g(v)\right|^q \left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right]\right. \\
&\quad \times \left.\left(\frac{1}{1+v^2}\right)^{1-q} du\, dv\right\}^{1/q} \\
&\leq \frac{1}{\pi}\left[\int_0^\infty \left|f(u)\right|^p du \int_0^\infty \frac{2}{\mu}\frac{1}{1+v^2}\, dv\right]^{1/p} \left[\int_0^\infty \left|g(v)\right|^q \left(1+v^2\right)^{q-1} \pi\, dv\right]^{1/q} \\
&= \mu^{-\frac{1}{p}} \pi^{-\frac{1}{q}} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+,(1+x^2)^{q-1})}.
\end{aligned}$$

Therefore, the convolutions (8) are well-defined and continuous. From that and by applying formula (3.225.3) in [10, p. 115], we obtain

$$\int_0^\infty x^\alpha e^{-\beta x} \left|\left(f \overset{\gamma}{*} g\right)_{\{\frac{1}{2}\}}(x)\right|^r dx \leq C^r \|f\|^r_{L_p(\mathbb{R}_+)} \|g\|^r_{L_q(\mathbb{R}_+,(1+x^2)^{q-1})}.$$

It shows that the convolutions (8) are in $L_r^{\alpha,\beta}(\mathbb{R}_+)$, and estimates (14) hold. From hypothesis of Theorem 3, by similar argument as in Theorem 1, we get the factorization identities (9). Combining with the Riemann–Lebesgue lemma, we obtain $(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}}(x) \in C_0(\mathbb{R}_+)$. Theorem 3 is proved. □

Corollary 1 *Under the same hypothesis as in Theorem* 3, *the generalized convolutions* (8) *are well-defined*, *continuous*, *belong to* $L_p(\mathbb{R}_+)$, *and the following inequalities hold*:

$$\left\|\left(f \overset{\gamma}{*} g\right)_{\{\frac{1}{2}\}}\right\|_{L_p(\mathbb{R}_+)} \leq \left(\frac{\pi}{2}\right)^{1/p} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+,(1+x^2)^{q-1})}. \tag{15}$$

Furthermore, *in the case* $p = 2$, *we get the following Parseval identity*:

$$\int_0^\infty \left|\left(f \overset{\gamma}{*} g\right)_{\{\frac{1}{2}\}}(x)\right|^2 dx = \int_0^\infty \left|e^{-\mu y}\left(F_{\{\frac{c}{s}\}} f\right)(y)(Lg)(y)\right|^2 dy. \tag{16}$$

Proof By applying Hölder's inequality and (10), we have

$$\begin{aligned}
&\int_0^\infty \left|\left(f \overset{\gamma}{*} g\right)_{\{\frac{1}{2}\}}(x)\right|^p dx \\
&\leq \frac{1}{\pi^p} \int_0^\infty \left\{\left[\int_{\mathbb{R}_+^2} \frac{1}{1+v^2} \left|f(u)\right|^p \left|\frac{v+\mu}{(v+\mu)^2+(x-u)^2}\right.\right.\right. \\
&\quad \left.\left.\pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right|^p du\, dv\right]^{1/p} \left[\int_{\mathbb{R}_+^2} \left(\frac{1}{1+v^2}\right)^{1-q} \left|g(v)\right|^q du\, dv\right]^{1/q} \left.\right\}^p dx
\end{aligned}$$

$$\leq \frac{1}{\pi^p}\left[\int_{\mathbb{R}_+^2} \frac{1}{1+v^2}\left|f(u)\right|^p \pi^p \, du\, dv\right]\left[\int_0^\infty \left(1+v^2\right)^{q-1}\left|g(v)\right|^q dv\right]^{p/q}$$

$$= \left[\int_0^\infty \frac{1}{1+v^2} dv \int_0^\infty \left|f(u)\right|^p du\right]\left[\int_0^\infty \left(1+v^2\right)^{q-1}\left|g(v)\right|^q dv\right]^{p/q}$$

$$= \frac{\pi}{2}\|f\|^p_{L_p(\mathbb{R}_+)}\|g\|^p_{L_q(\mathbb{R}_+,(1+x^2)^{q-1})}.$$

Therefore, the convolutions $(f \overset{\gamma}{*} g)_{\{^1_2\}}(x)$ are continuous in $L_p(\mathbb{R}_+)$ and (15) hold. On the other hand, we get the following Parseval equalities in $L_2(\mathbb{R}_+)$:

$$\|F_{\{^c_s\}} f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

Combining with factorization identities (9), we get the Fourier-type Parseval identity (16). □

Corollary 2

(a) *Let* $f(x) \in L_2(\mathbb{R}_+)$, $g(x) \in L_1(\mathbb{R}_+)$. *Then the generalized convolutions* (8) *are well-defined in* $L_r^{\alpha,\beta}(\mathbb{R}_+)$ $(r \geq 1, \beta \geq 0, \alpha > -1)$, *and the following estimates hold*:

$$\left\|(f \overset{\gamma}{*} g)_{\{^1_2\}}\right\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi\mu}}\beta^{-\frac{\alpha+1}{r}}\Gamma^{1/r}(\alpha+1)\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \tag{17}$$

(b) *If* $f(x), g(x) \in L_1(\mathbb{R}_+)$ *then convolutions* (8) *are well-defined in* $L_r^{\alpha,\beta}(\mathbb{R}_+)$ $(r \geq 1,$ $\beta \geq 0, \alpha > -1)$ *and the following estimates hold*:

$$\left\|(f \overset{\gamma}{*} g)_{\{^1_2\}}\right\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \frac{2}{\pi\mu}\beta^{-\frac{\alpha+1}{r}}\Gamma^{1/r}(\alpha+1)\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \tag{18}$$

Proof

(a) By applying Schwarz's inequality and (10), we have

$$\left|(f \overset{\gamma}{*} g)_{\{^1_2\}}(x)\right| \leq \frac{1}{\pi}\left[\int_0^\infty \pi\left|g(v)\right|dv\right]^{1/2}\left[\int_{\mathbb{R}_+^2}\left|f(u)\right|^2\left|g(v)\right|\frac{2}{\mu}\,du\,dv\right]^{1/2}$$

$$= \sqrt{\frac{2}{\pi\mu}}\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Combining with formula (3.225.3) in [10, p.115], we get

$$\left\|(f \overset{\gamma}{*} g)_{\{^1_2\}}\right\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi\mu}}\beta^{-\frac{\alpha+1}{r}}\Gamma^{1/r}(\alpha+1)\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Thus, (17) is proved.

(b) By applying Schwarz's inequality, we have

$$\left|\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x)\right| \le \frac{1}{\pi}\left[\int_{\mathbb{R}_+^2} \big|f(u)\big|\big|g(v)\big|\frac{2}{\mu}\,du\,dv\right]^{1/2}\left[\int_{\mathbb{R}_+^2} \big|f(u)\big|\big|g(v)\big|\frac{2}{\mu}\,du\,dv\right]^{1/2}$$
$$= \frac{2}{\pi\mu}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Combining with formula (3.225.3) in [10, p.115], we get (18). □

Theorem 4 (Titchmarch's Type Theorem) *Given two continuous functions $g \in L_1(\mathbb{R}_+)$, $f \in L_1(\mathbb{R}_+, e^{\gamma x})$, $\gamma > 0$. If $(f \overset{\gamma}{*} g)_1(x) = 0\ \forall x > 0$ then either $f(x) = 0\ \forall x > 0$ or $g(x) = 0$ $\forall x > 0$.*

Proof We have

$$\left|\frac{d^n}{dy^n}\big(\cos yx f(x)\big)\right| = \left|f(x)x^n \cos\left(yx + n\frac{\pi}{2}\right)\right|$$
$$\le \left|e^{-\gamma x}x^n\right|\left|e^{\gamma x} f(x)\right| \le \frac{n!}{\gamma^n}\left|e^{\gamma x} f(x)\right|. \tag{19}$$

Here we used the following estimate:

$$0 \le e^{-\gamma x}x^n = e^{-\gamma x}\frac{(\gamma x)^n}{n!}\frac{n!}{\gamma^n} \le e^{-\gamma x}e^{\gamma x}\frac{n!}{\gamma^n} = \frac{n!}{\gamma^n},$$

and $f \in L_1(\mathbb{R}_+, e^{\gamma x})$. Combining with (19) we get $\frac{d^n}{dy^n}(\cos yx f(x)) \in L_1(\mathbb{R}_+)$.

Since $L_1(\mathbb{R}_+, e^{\gamma x}) \subset L_1(\mathbb{R}_+)$, $(F_c f)(y)$ are analytic in $\mathbb{R}_+$. On the other hand, we find that $(Lg)(y)$ is analytic in $\mathbb{R}_+$. By using the factorization properties (9) for $(f \overset{\gamma}{*} g)_1(x) = 0$ we have $(F_c f)(y)(Lg)(y) = 0\ \forall y > 0$. It implies that either $f(x) = 0\ \forall x > 0$ or $g(x) = 0$ $\forall x > 0$. Theorem 4 is proved. □

Corollary 3 *Under the same hypothesis as in Theorem 4, if $(f \overset{\gamma}{*} g)_2(x) = 0\ \forall x > 0$ then either $f(x) = 0\ \forall x > 0$ or $g(x) = 0\ \forall x > 0$.*

Proposition 1 *Let $f(x)$ and $g(x)$ be two functions in $L_1(\mathbb{R}_+)$. Then*

$$\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(v)\left(f(u) \underset{\{\substack{F_c\\1}\}}{*} \frac{v+\mu}{(v+\mu)^2 + u^2}\right)(x)\,dv.$$

Here, the convolutions $(\cdot \underset{F_c}{*} \cdot)$, $(\cdot \underset{1}{*} \cdot)$ are defined by (1), (3), respectively.

Proof From (8), (1) and (3), we have

$$\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}}(x)$$
$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty\left[\frac{v+\mu}{(v+\mu)^2 + (x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2 + (x+u)^2}\right]f(u)g(v)\,du\,dv$$

$$= \frac{1}{\pi}\int_0^\infty g(v)\left\{\int_0^\infty f(u)\left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right]du\right\}dv$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty g(v)\left(f(u) \underset{\left\{\substack{F_c\\1}\right\}}{*} \frac{v+\mu}{(v+\mu)^2+u^2}\right)(x)\,dv. \qquad \square$$

Proposition 2 *Let $f(x), g(x)$ and $h(x)$ be functions in $L_1(\mathbb{R}_+)$. Then convolutions (8) are not commutative and associative but satisfy the following equalities*:

(a) $f \overset{\gamma}{\underset{F_s}{*}} (g \overset{\gamma}{*} h)_2 = \left(\left(f \overset{\gamma}{\underset{F_s}{*}} g\right) \overset{\gamma}{*} h\right)_2$,

(b) $f \underset{F_c}{*} (g \overset{\gamma}{*} h)_1 = \left(\left(f \underset{F_c}{*} g\right) \overset{\gamma}{*} h\right)_1$,

(c) $f \underset{1}{*} (g \overset{\gamma}{*} h)_1 = \left(\left(f \underset{1}{*} g\right) \overset{\gamma}{*} h\right)_2$,

(d) $f \underset{2}{*} (g \overset{\gamma}{*} h)_2 = \left(\left(f \underset{2}{*} g\right) \overset{\gamma}{*} h\right)_1$.

Here the convolutions $\left(\cdot \overset{\gamma}{\underset{F_s}{*}} \cdot\right), (\cdot \underset{F_c}{*} \cdot), (\cdot \underset{1}{*} \cdot)$ *and* $(\cdot \underset{2}{*} \cdot)$ *are defined by* (5), (1), (3) *and* (7), *respectively.*

Proof From (6) and (9), we have

$$\begin{aligned} F_s\left(f \overset{\gamma}{\underset{F_s}{*}} \left(g \overset{\gamma}{*} h\right)_2\right)(y) &= \sin y (F_s f)(y) F_s\left(g \overset{\gamma}{*} h\right)_2(y) \\ &= e^{-\mu y}\sin y (F_s f)(y)(F_s g)(y)(Lh)(y) \\ &= e^{-\mu y} F_s\left(f \overset{\gamma}{\underset{F_s}{*}} g\right)(y)(Lh)(y) = F_s\left(\left(f \overset{\gamma}{\underset{F_s}{*}} g\right) \overset{\gamma}{*} h\right)_2(y). \end{aligned}$$

Hence $f \overset{\gamma}{\underset{F_s}{*}} (g \overset{\gamma}{*} h)_2 = \left(\left(f \overset{\gamma}{\underset{F_s}{*}} g\right) \overset{\gamma}{*} h\right)_2$.

The proofs of (b), (c), and (d) are similar. $\square$

4 Integral Equations and Systems of Integral Equations

In this section we introduce several classes of integral equations and systems of two integral equations related to convolutions (8) which can be solved in a closed form.

(a) Consider integral equations of the first kind

$$\int_0^\infty \theta_{\left\{\substack{1\\2}\right\}}(x,u) f(u)\,du = g(x), \quad x > 0, \tag{20}$$

where

$$\theta_{\left\{\substack{1\\2}\right\}}(x,u) = \frac{1}{\pi}\int_0^\infty \varphi(v)\left[\frac{v+\mu}{(v+\mu)^2+(x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+u)^2}\right]dv, \quad \mu > 0.$$

Put $H(\mathbb{R}_+) = \{h \in L_1(\mathbb{R}_+), h = (F_{\left\{\substack{c\\s}\right\}} f)(y)\}$. We consider the restriction mapping $F_{\left\{\substack{c\\s}\right\}} : H(\mathbb{R}_+) \to L_1(\mathbb{R}_+)$.

Theorem 5 *Let $g(x), \varphi(x) \in L_1(\mathbb{R}_+)$ and suppose that $g_1(x), g_2(x)$ be such that $g(x) = (g_1 \overset{\gamma}{*} g_2)_{\{^1_2\}}(x)$. Then the necessary and sufficient condition to ensure that the equations* (20) *have solutions in $L_1(\mathbb{R}_+)$ is that $\frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)} \in H(\mathbb{R}_+)$. Moreover, the solutions are given in the following closed form*:

$$f(x) = \int_0^\infty \frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)} \begin{Bmatrix} \cos xy \\ \sin xy \end{Bmatrix} dy. \tag{21}$$

Proof Necessity. By the hypothesis, equations (20) has solutions in $L_1(\mathbb{R}_+)$ given by (21). Since $g(x) \in L_1(\mathbb{R}_+)$ therefore $(f \overset{\gamma}{*} \varphi)_{\{^1_2\}}(x) \in L_1(\mathbb{R}_+)$. From that, by applying the factorization properties (9) for (20), we have

$$e^{-\mu y}(F_{\{^c_s\}}f)(y)(L\varphi)(y) = e^{-\mu y}(F_{\{^c_s\}}g_1)(y)(Lg_2)(y),$$

therefore

$$(F_{\{^c_s\}}f)(y) = \frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)}. \tag{22}$$

Since $(F_{\{^c_s\}}f)(y) \in L_1(\mathbb{R}_+)$ hence $(F_{\{^c_s\}}f)(y) \in H(\mathbb{R}_+)$. From that and (22) we get $\frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)} \in H(\mathbb{R}_+)$.

Sufficiency. By the hypothesis $\frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)} \in H(\mathbb{R}_+)$, therefore there exists $f(x) \in L_1(\mathbb{R}_+)$ satisfying $(F_{\{^c_s\}}f)(y) = \frac{(F_{\{^c_s\}}g_1)(y)(Lg_2)(y)}{(L\varphi)(y)}$, hence

$$(F_{\{^c_s\}}f)(y)(L\varphi)(y) = (F_{\{^c_s\}}g_1)(y)(Lg_2)(y).$$

Therefore

$$\big(f \overset{\gamma}{*} \varphi\big)_{\{^1_2\}}(x) = g(x),$$

and we obtain (21). Theorem 5 is proved. □

(b) Consider integral equations of the second kind

$$f(x) + \int_0^\infty f(t)\theta_{\{^1_2\}}(x,t)\,dt = g(x), \quad x > 0, \tag{23}$$

where

$$\theta_{\{^1_2\}}(x,u) = \int_{\mathbb{R}_2^+} H_{\{^1_2\}}(x,u,v)\big[\psi\big(|u-x|\big) \pm \psi(u+x)\big]\varphi(v)\,du\,dv,$$

and

$$H_{\{^1_2\}}(x,u,v) = \frac{1}{\pi\sqrt{2\pi}}\left[\frac{v+\mu}{(v+\mu)^2 + (x-u)^2} \pm \frac{v+\mu}{(v+\mu)^2 + (x+u)^2}\right]. \tag{24}$$

Theorem 6 *Let $\varphi(x), \psi(x) \in L_1(\mathbb{R}_+)$. Then the necessary and sufficient condition to ensure that the equations* (23) *have unique solutions in $L_1(\mathbb{R}_+)$ for all $g(x)$ in $L_1(\mathbb{R}_+)$ is that*

$1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y) \neq 0\ \forall y > 0$. *Moreover, the solutions can be presented in closed form as follows*:

$$f(x) = g(x) - \big(g \underset{\{\substack{F_c \\ 1}\}}{*} q\big)(x), \tag{25}$$

where the convolutions $(\cdot \underset{F_c}{*} \cdot)$, $(\cdot \underset{1}{*} \cdot)$ *are defined by* (1), (3), *respectively, and* $q \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c q)(y) = \frac{F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}{1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}. \tag{26}$$

Proof Necessity. We can rewrite equation (23) in the form

$$f(x) + \Big(\big(f \underset{\{\substack{F_c \\ 1}\}}{*} \psi\big) \overset{\gamma}{*} \varphi\Big)_{\{\frac{1}{2}\}}(x) = g(x). \tag{27}$$

Assume that the integral equation (23) have unique solutions in $L_1(\mathbb{R}_+)$ for all g in $L_1(\mathbb{R}_+)$. Therefore, there exists $g \in L_1(\mathbb{R}_+)$ such that

$$(F_{\{\substack{c \\ s}\}} g)(y) \neq 0 \quad \forall y > 0. \tag{28}$$

By using factorization properties (9), (2), and (4) for (27), we get

$$(F_{\{\substack{c \\ s}\}} f)(y) + e^{-\mu y}(F_{\{\substack{c \\ s}\}} f)(y)(F_c\psi)(y)(L\varphi)(y) = (F_{\{\substack{c \\ s}\}} g)(y).$$

Combining with (9), we obtain

$$(F_{\{\substack{c \\ s}\}} f)(y)\big[1 + F_c\big(\psi \overset{\gamma}{*} \varphi\big)_1(y)\big] = (F_{\{\substack{c \\ s}\}} g)(y). \tag{29}$$

Using feedback evidence, assume that there exists $y_0 > 0$ such that $1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y_0) = 0$. Combining with (29), we get

$$(F_{\{\substack{c \\ s}\}} g)(y_0) = 0 \quad \forall g \in L_1(\mathbb{R}_+).$$

It is a contradiction to (28). Hence $1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y) \neq 0\ \forall y > 0$.

Sufficiency. From (28) and the assumption of Theorem 6, we have

$$\begin{aligned}(F_{\{\substack{c \\ s}\}} f)(y) &= \frac{(F_{\{\substack{c \\ s}\}} g)(y)}{1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y)} = (F_{\{\substack{c \\ s}\}} g)(y)\left[1 - \frac{F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}{1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}\right] \\ &= (F_{\{\substack{c \\ s}\}} g)(y) - (F_{\{\substack{c \\ s}\}} g)(y)\frac{F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}{1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y)}.\end{aligned} \tag{30}$$

With the condition $1 + F_c(\psi \overset{\gamma}{*} \varphi)_1(y) \neq 0\ \forall y > 0$, due to Wiener–Levy theorem (in [9, p. 63]), there exists a function $q \in L_1(\mathbb{R}_+)$ satisfying (26). Combining with (30), we have

$$\begin{aligned}(F_{\{\substack{c \\ s}\}} f)(y) &= (F_{\{\substack{c \\ s}\}} g)(y) - (F_{\{\substack{c \\ s}\}} g)(y)(F_c q)(y) \\ &= (F_{\{\substack{c \\ s}\}} g)(y) - F_{\{\substack{c \\ s}\}}\big(g \underset{\{\substack{F_c \\ 1}\}}{*} q\big)(y).\end{aligned}$$

Therefore we get (25). Theorem 6 is proved. □

(c) We consider the system of two integral equations

$$\begin{aligned} f(x) + \int_0^\infty g(t)M(x,t)\,dt &= p(x), \\ g(x) + \int_0^\infty f(t)N(x,t)\,dt &= q(x), \quad x > 0. \end{aligned} \tag{31}$$

Here

$$M(x,t) = \int_{\mathbb{R}_2^+} H_1(x,u,v)\big[k\big(|u-x|\big) + k(u+t)\big]\varphi(v)\,du\,dv,$$

$$N(x,t) = \int_{\mathbb{R}_2^+} H_1(x,u,v)\big[l\big(|u-x|\big) + l(u+t)\big]\psi(v)\,du\,dv$$

and H_1 is defined by (24).

Theorem 7 *Suppose that $\varphi(x), \psi(x), p(x), q(x) \in L_1(\mathbb{R}_+)$ are such that $1 - F_c((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1)(y) \neq 0\ \forall y > 0$. Then system (31) has a unique solution (f,g) in $(L_1(\mathbb{R}_+), L_1(\mathbb{R}_+))$ given by formulas*

$$f(x) = p(x) - \big(q \underset{F_c}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big)(x) + \big(p \underset{F_c}{*} \xi\big)(x) - \big(\big(q \underset{F_c}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big) \underset{F_c}{*} \xi\big)(x), \tag{32}$$

$$g(x) = q(x) - \big(p \underset{F_c}{*} \big(l \overset{\gamma}{*} \psi\big)_1\big)(x) + \big(q \underset{F_c}{*} \xi\big)(x) - \big(\big(p \underset{F_c}{*} \big(l \overset{\gamma}{*} \psi\big)_1\big) \underset{F_c}{*} \xi\big)(x). \tag{33}$$

Here $\xi \in L_1(\mathbb{R}_+)$ is such that

$$(F_c\xi)(y) = \frac{F_c((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1)(y)}{1 - F_c((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1)(y)}. \tag{34}$$

Proof We can rewrite system of two equations (31) in the following form:

$$\begin{aligned} f(x) + \big(\big(g \underset{F_c}{*} k\big) \overset{\gamma}{*} \varphi\big)_1(x) &= p(x), \\ g(x) + \big(\big(f \underset{F_c}{*} l\big) \overset{\gamma}{*} \psi\big)_1(x) &= q(x). \end{aligned} \tag{35}$$

By using factorization properties (9), (2) for (35), we get

$$(F_cf)(y) + e^{-\mu y}(F_cg)(y)(F_ck)(y)(L\varphi)(y) = (F_cp)(y),$$
$$(F_cg)(y) + e^{-\mu y}(F_cf)(y)(F_cl)(y)(L\psi)(y) = (F_cq)(y).$$

Therefore

$$\begin{aligned} (F_cf)(y) + (F_cg)(y)F_c(k \overset{\gamma}{*} \varphi)_1(y) &= (F_cp)(y), \\ (F_cg)(y) + (F_cf)(y)F_c(l \overset{\gamma}{*} \psi)_1(y) &= (F_cq)(y). \end{aligned} \tag{36}$$

Solving the system of two linear equations (36), we get

$$\begin{aligned}(F_c f)(y) &= \frac{(F_c p)(y) - F_c(q \underset{F_c}{*} (k \overset{\gamma}{*} \varphi)_1)(y)}{1 - F_c((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1)(y)} \\ &= \big[(F_c p)(y) - F_c\big(q \underset{F_c}{*} (k \overset{\gamma}{*} \varphi)_1\big)(y)\big] \\ &\quad \times \left[1 + \frac{F_c\big((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1\big)(y)}{1 - F_c\big((k \overset{\gamma}{*} \varphi)_1 \underset{F_c}{*} (l \overset{\gamma}{*} \psi)_1\big)(y)}\right]. \end{aligned} \tag{37}$$

In virtue of Wiener–Levy theorem, there exists a function $\xi \in L_1(\mathbb{R}_+)$ satisfying (34). Combining with (37), we have

$$\begin{aligned}(F_c f)(y) &= \big[(F_c p)(y) - F_c\big(q \underset{F_c}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big)(y)\big]\big[1 + (F_c \xi)(y)\big] \\ &= (F_c p)(y) - F_c\big(q \underset{F_c}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big)(y) + F_c(p \underset{F_c}{*} \xi)(y) \\ &\quad - F_c\big(\big(q \underset{F_c}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big) \underset{F_c}{*} \xi\big)(y).\end{aligned}$$

Therefore we obtain (32). Similarly, we get (33). Theorem 7 is proved. □

We now consider the system (31) with

$$\begin{aligned}M(x,t) &= \int_{\mathbb{R}_2^+} H_2(x,u,v)\big[k\big(|u-x|\big) - k(u+t)\big]\varphi(v)\,du\,dv, \\ N(x,t) &= \int_{\mathbb{R}_2^+} H_2(x,u,v)\big[l\big(|u-x|\big) - l(u+t)\big]\psi(v)\,du\,dv,\end{aligned}$$

where H_2 is defined by (24).

Corollary 4 *Under the same hypothesis as in Theorem 7, the system (31) has unique solution (f,g) in $(L_1(\mathbb{R}_+), L_1(\mathbb{R}_+))$ given by formulas*

$$\begin{aligned}f(x) &= p(x) - \big(q \underset{1}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big)(x) + \big(p \underset{1}{*} \xi\big)(x) - \big(\big(q \underset{1}{*} \big(k \overset{\gamma}{*} \varphi\big)_1\big) \underset{1}{*} \xi\big)(x), \\ g(x) &= q(x) - \big(p \underset{1}{*} \big(l \overset{\gamma}{*} \psi\big)_1\big)(x) + \big(q \underset{1}{*} \xi\big)(x) - \big(\big(p \underset{1}{*} \big(l \overset{\gamma}{*} \psi\big)_1\big) \underset{1}{*} \xi\big)(x).\end{aligned}$$

Here $\xi \in L_1(\mathbb{R}_+)$ is defined by (34).

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