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**THE FOURIER-STIELTJES ALGEBRA OF A TOPOLOGICAL  
SEMIGROUP WITH INVOLUTION**

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# THE FOURIER STIELTJES ALGEBRA OF A TOPOLOGICAL SEMIGROUP WITH INVOLUTION

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Let  $S$  be a topological semigroup with a continuous involution. We study a subalgebra  $F(S)$  of the algebra of continuous weakly almost periodic functions on  $S$ .  $F(S)$  is translation invariant, closed under conjugation and contains constants. When  $S$  has an identity, then  $F(S)$  is the linear span of the cone of continuous positive definite functions on  $S$ . We show that there exists a norm  $\|\cdot\|_D$  on  $F(S)$  such that  $(F(S), \|\cdot\|_D)$  is a commutative Banach algebra which can be identified with the predual of a  $W^*$ -algebra  $W^*(S)$ . When  $S$  is a locally compact group, then  $F(S)$  is precisely the Fourier Stieltjes algebra of  $S$ . We also show that  $\sigma(F(S))$ , the spectrum of  $F(S)$ , is a  $*$ -semigroup in  $W^*(S)$ , and study the relation of  $\sigma(F(S_1))$  and  $\sigma(F(S_2))$  when  $F(S_1)$  and  $F(S_2)$  are isometric isomorphic Banach algebras.

1. Introduction. Recently, Dunkl and Ramirez [5] defined a subalgebra  $R(S)$  of the algebra  $WAP(S)$  of complex-valued continuous weakly almost periodic functions on  $S$ . The algebra  $R(S)$ , called the *representation algebra* of  $S$ , is constructed by considering continuous representations of  $S$  into the unit ball of  $L_\infty(X, \mu)$  with the weak\*-topology, where  $(X, \mu)$  is some probability measure space. They showed that  $R(S)$  is translation invariant, closed under conjugation and contains all bounded continuous semi-characters on  $S$ . Furthermore  $R(S)$ , with an appropriate norm, becomes a commutative Banach algebra and the dual of  $R(S)$  can be identified with a weak\*-closed subalgebra of a commutative  $W^*$ -algebra. If  $G$  is a commutative locally compact group, then  $R(G) = M(\hat{G})^\wedge$ , the Fourier Stieltjes transform of the measure algebra on the dual group  $\hat{G}$  (see [6, p. 80]).

Our present work deals with the study of the subalgebra  $F(S)$  of  $WAP(S)$  of a topological  $*$ -semigroup  $S$  (i.e., a topological semigroup with a continuous involution). If  $S$  has an identity, then  $F(S)$  is the linear span of continuous positive definite function on  $S$ . Also if  $S$  is a commutative, then  $F(S)$  is contained in the representation algebra  $R(S)$ . We show that  $F(S)$  can be identified with the predual of a  $W^*$ -algebra,  $W^*(S)$ . Furthermore  $F(S)$  with the predual norm is a commutative Banach algebra, called the Fourier Stieltjes algebra of  $S$ . The algebra  $F(S)$  is also translation invariant, closed under conjugation and contains all continuous  $*$ -semi-characters of

$S$ . Also there exists an ultra-weakly continuous  $*$ -representation of  $S$  into the unit ball of  $W^*(S)$  "containing" all other ultra-weakly continuous  $*$ -representations of  $S$  into the unit ball of a  $W^*$ -algebra. In particular, if  $G$  is a locally compact group (with involution  $g \rightarrow g^{-1}$ ), then  $W^*(G)$  is the big group algebra defined by John Ernest [9] (see also [8]). Furthermore, if  $S$  is commutative and has an identity, then  $F(S)$  is isometric and algebra isomorphic to a weak\*-dense subalgebra of the measure algebra of a compact topological commutative semigroup.

This paper is organized in the following way: In §2 we list some notations and preliminary properties of topological  $*$ -semigroups  $S$ ; definitions and properties of  $F(S)$  and  $W^*(S)$  as stated in the previous paragraph will be made precise in §3 and 4. Analysis of the spectrum  $\sigma(F(S))$  of  $F(S)$  is taken up in §5. We show that  $\sigma(F(S))$  is a  $*$ -semigroup in  $W^*(S)$  and study the relation of  $\sigma(F(S_1))$  and  $\sigma(F(S_2))$  when  $S_1, S_2$  are topological  $*$ -semigroups, and  $F(S_1)$  and  $F(S_2)$  are isometric isomorphic Banach algebras.

Continuous positive definite functions on topological  $*$ -semigroups  $S$  have been studied by R. J. Lindahl and P. H. Maserick [15], and more recently by C. Berg and J. Christensen [3] for commutative  $S$  with involution on  $S$  given by the identity map. Our analysis of the spectrum of  $F(S)$  is inspired and motivated by the work of Martin E. Walter in [18] and [19].

It is our pleasure to thank the referee of this paper. His many valuable suggestions have much improved the contents of the original version of our work.

**2. Preliminaries and some notations.** Let  $A$  be a subset of a linear space  $E$ , then  $\langle A \rangle$  will denote the linear span of  $A$ . If  $E$  is also a normed linear space, then the closure of  $A$  and the closed linear span of  $A$  will be denoted by  $\bar{A}$  and  $\langle A \rangle^-$  respectively if the closure is taken with respect to the norm topology, or by  $\bar{A}^\tau$  and  $\langle A \rangle^{-\tau}$  respectively if the closure is taken with respect to a topology  $\tau$  on  $E$  different from the norm topology.

The continuous dual of a normed linear space  $E$  will be denoted by  $E^*$ . If  $x \in E$  and  $\phi \in E^*$ , then the value of  $\phi$  at  $x$  will be denoted by  $\phi(x)$  or  $\langle \phi, x \rangle$ . Also if  $F \subseteq E^*$ , then  $\sigma(E, F)$  will denote the locally convex topology on  $E$  determined by the semi-norms  $\{p_\phi; \phi \in F\}$ , where  $p_\phi(x) = |\phi(x)|$  for all  $x \in E$ .

If  $M$  is a  $W^*$ -algebra, then  $M_*$  will denote its unique predual. For each  $x \in M$ , and  $\phi \in M_*$ , write  $L_x\phi$ ,  $R_x\phi$  and  $\phi^*$  as the functionals in  $M_*$  defined by  $L_x\phi(y) = \phi(xy)$ ,  $R_x\phi(y) = \phi(yx)$  and  $\phi^*(y) = \overline{\phi(y^*)}$  for each  $y \in M$ . Also the ultraweak topology on  $M$  (i.e., the  $\sigma(M, M_*)$ -topology) will often be written as the  $\sigma$ -topology.

By a *topological semigroup*  $S$ , we shall mean a semigroup  $S$  with a Hausdorff topology such that for each  $a \in S$ , the mappings  $s \rightarrow as$  and  $s \rightarrow sa$  from  $S$  into  $S$  are continuous.

Let  $S$  be a topological semigroup and let  $C(S)$  be the space of bounded continuous complex-valued functions on  $S$ . For each  $a \in S$ , define the left and right translation operators  $\ell_a, r_a$  on  $C(S)$  by:

$$(\ell_a f)(s) = f(as)$$

$$(r_a f)(s) = f(sa)$$

for each  $s \in S$ . A function  $f \in C(S)$  is *weakly almost periodic* if  $\{\ell_a f; a \in S\}$  is relatively compact in the weak topology of  $C(S)$ . Then, as known, the space  $WAP(S)$  of continuous weakly almost periodic functions on  $S$  is a translation invariant closed subalgebra of  $C(S)$  containing constants.

By an *involution* on a topological semigroup  $S$  we shall mean a map from  $S$  into  $S$ , denoted by  $s \rightarrow s^*$ , such that

$$(1) \quad (ab)^* = b^*a^*$$

$$(2) \quad a^{**} = a$$

for all  $a \in S$ . A topological  $*$ -semigroup is a topological semigroup with a fixed continuous involution.

REMARK 2.1. (a) Not all topological semigroups admit an involution (see [15, p. 771]).

(b) If  $S$  is commutative, then the identity map on  $S$  defines an involution on  $S$ .

(c) If  $S$  has an identity  $u$ , then  $u^* = u$ .

(d) If  $M$  is a  $W^*$ -algebra, then the unit ball of  $M$  with the  $\sigma$ -topology is a compact topological  $*$ -semigroup with the multiplication and involution of  $M$ .

If  $S$  is a topological  $*$ -semigroup,  $f \in C(S)$ , define  $f^* \in C(S)$  by  $f^*(s) = \overline{f(s^*)}$  for all  $s \in S$ . Then the map  $f \rightarrow f^*$  defines an involution on the Banach algebra  $C(S)$ .

A complex-valued function  $f$  on a topological  $*$ -semigroup  $S$  is called *positive definite* if for any complex numbers  $\lambda_1, \dots, \lambda_n$  and any  $s_1, \dots, s_n$  in  $S$ , we have

$$\sum_{i=1}^n \sum_{j=1}^n \bar{\lambda}_i \lambda_j f(s_i^* s_j) \geq 0.$$

The collection of continuous positive definite functions on  $S$  will be denoted by  $P(S)$ . The next proposition can be proved by an argument similar to that in [12, 32.9] (see also [15, Theorem 3.4]).

**PROPOSITION 2.2.** *Let  $S$  be a topological  $*$ -semigroup. Then  $P(S)$  is a cone in  $C(S)$  closed under conjugation, involution and pointwise product.*

When  $G$  is a group, then the involution on  $G$ , unless otherwise specified, will be the one defined by the inversion map  $g \rightarrow g^{-1}$ ,  $g \in G$ .

**3. The Fourier Stieltjes algebra  $F(S)$ .** Throughout this section,  $S$  will denote a topological  $*$ -semigroup.

By a representation of  $S$  we shall mean a pair  $(\omega, M)$ , where  $M$  is a  $W^*$ -algebra and  $\omega$  is a homomorphism of  $S$  into  $M_1 = \{x \in M; \|x\| \leq 1\}$  regarded as a semigroup with multiplication from  $M$  i.e.,  $\omega(ab) = \omega(a)\omega(b)$  for all  $a, b \in S$ . The representation  $(\omega, M)$  is a  $*$ -representation if  $\omega(a^*) = \omega(a)^*$  for all  $a \in S$ ; it is  $\sigma$ -continuous if  $\omega$  is continuous when  $M_1$  has the  $\sigma$ -topology.

**REMARK 3.1.** If  $S$  has an identity  $u$ , and  $(\omega, M)$  is a  $*$ -representation of  $S$ , then  $\omega(u) = p$  is a projection in  $M$ , and  $\omega(S)$  is contained in the  $W^*$ -algebra  $pMp$  for which  $p$  is the identity. Also if  $\langle \omega(S) \rangle^{-\sigma} = M$ , then  $\omega(u)$  is the identity of  $M$ .

If  $(\omega, M)$  is a  $\sigma$ -continuous  $*$ -representation of  $S$  such that  $\langle \omega(S) \rangle$  is  $\sigma$ -dense in  $M$ , then  $\text{card}(M_*) \leq c^{\text{card}(S)}$ , where  $c$  is the cardinality of the real numbers. Hence we may form the collection  $\Omega(S)$  of all  $\sigma$ -continuous  $*$ -representations  $\alpha = (\omega, M)$  of  $S$  such that  $\langle \omega(S) \rangle^\sigma = M$ . Let  $F(S)$  denote all complex-valued functions  $f$  such that  $f = \hat{\psi}$  for some  $\omega \in M_*$  and some  $\alpha = (\omega, M)$  in  $\Omega(S)$ . For each  $f \in F(S)$ , let

$$\begin{aligned} \|f\|_\infty &= \sup \{|f(s)|; s \in S\} \\ \|f\|_\sigma &= \inf \{\|\psi\|; \psi \in M_*, \hat{\psi} = f \text{ and } (\omega, M) \in \Omega(S)\}. \end{aligned}$$

**THEOREM 3.2.** (a)  $F(S)$  is a subalgebra of  $WAP(S)$  containing the constant functions. Furthermore,  $\|\cdot\|_\sigma$  is a norm on  $F(S)$  and  $(F(S), \|\cdot\|_\sigma)$  is a commutative normed algebra with unit.

(b) If  $f \in F(S)$  and  $a \in S$ , then the functions  $r_a f, \ell_a f, f^*, \bar{f}$  are all in  $F(S)$  and  $\|r_a f\|_\sigma \leq \|f\|_\sigma$ ,  $\|\ell_a f\|_\sigma \leq \|f\|_\sigma$ ,  $\|f^*\|_\sigma = \|f\|_\sigma$ ,  $\|\bar{f}\|_\sigma = \|f\|_\sigma$  and  $\|f\|_\infty \leq \|f\|_\sigma$ .

*Proof.* That  $F(S) \subseteq WAP(S)$  follows from [14, Lemma 6.3]. The remainder of the theorem can be proved quite similarly to [6, Theorem 2.1.6], we omit the details.

We shall call  $(F(S), \|\cdot\|_\sigma)$  the *Fourier Stieltjes algebra* of  $S$ .

**REMARK 3.3.** (a) The algebra  $F(S)$  cannot be enlarged and the

norm on  $F(S)$  cannot be decreased by considering a collection  $\mathcal{C}$  of  $\sigma$ -continuous  $*$ -representations of  $S$  containing  $\Omega(S)$ . Indeed, if  $f \in F(S)$  and  $\alpha = (\omega, M) \in \mathcal{C}$  such that  $f = \hat{\psi}$  for some  $\psi \in M_*$ , let  $N = \langle \omega(S) \rangle^{-\sigma}$  and  $\psi_0$  be the restriction of  $\psi$  to  $N$ . Then  $(\omega, N) \in \Omega(S)$ ,  $\hat{\psi}_0 = f$  and  $\|\psi_0\| \leq \|\psi\|$ .

(b) If  $S$  is commutative, then  $F(S) \subseteq R(S)$ , where  $R(S)$  is the representation algebra of  $S$  defined by Dunkl and Ramirez [5]. To see this, let  $f \in F(S)$ . Choose  $(\omega, M) \in \Omega(S)$  such that  $\hat{\phi} = f$  for some  $\phi \in M_*$ . Let  $X$  be the spectrum of  $M$ . Then  $\text{card}(X) \leq c^{\text{card}(S)}$ . By the Riesz representation theorem, there exists a probability measure  $\mu_\phi$  on  $X$  such that  $\phi(a) = \int_X a(t) d\mu_\phi(t)$  for each  $a \in M$ . Consider the mapping  $\Phi_\phi$  from  $M$  into  $L_\infty(X, \mu_\phi)$  defined by  $\Phi_\phi(a) = \hat{a}$ , where  $\hat{a}$  is the Gelfand transform of  $a$ . Then  $\Phi_\phi$  is a  $W^*$ -homomorphism of  $M$  into  $L_\infty(X, \mu_\phi)$  (see [16, p. 46]). Define a presentation  $(\omega_\phi, L_\infty(X, \mu_\phi))$  of  $S$  by  $\omega_\phi(s) = \Phi_\phi(\omega(s))$ . Then  $\hat{\phi}(s) = \langle 1, \omega_\phi(s) \rangle$  for all  $s \in S$ . Hence  $f = \hat{\phi} \in R(S)$ .

Note that the inclusion  $F(S) \subseteq R(S)$  may be proper (see Example 4.2).

(c) If  $S$  is an idempotent commutative topological semigroup with involution  $s^* = s$  for all  $s \in S$ , then any representation  $(\omega, M)$  of  $S$ , where  $M$  is a commutative  $W^*$ -algebra, is a  $*$ -representation. In particular  $F(S) = R(S)$ . Indeed, we may assume that  $M = L_\infty(X, \mu)$  for some measure space  $(X, \mu)$ . Since  $\omega(s)^2 = \omega(s^2) = \omega(s)$  for all  $s \in S$ , it follows that  $\omega(s)$  is a characteristic function on some subset of  $\Omega$ . Hence  $\omega(s)^* = \omega(s^*)$ .

(d) Let  $G$  be an abelian group. Then for any representation  $(\mu, M)$  where  $M$  is a commutative  $W^*$ -algebra, is a  $*$ -representation of  $G$ . Consequently  $F(G) = R(G)$ . Indeed, write  $M = L_\infty(X, \mu)$  for some measure space  $(X, \mu)$ . We may assume that  $\omega(u) = 1$ , where  $u$  is the identity of  $G$ . Then for each  $g \in G$ ,  $\omega(g)\omega(g^{-1}) = \omega(u) = 1$ . Hence  $|\omega(g)| = 1$  and  $\omega(g^{-1}) = \overline{\omega(g)} = \omega(g)^*$ .

(e) If  $S$  is the unit ball of a  $W^*$ -algebra  $M$ , then the restriction map is a linear isometry from  $M_*$  into  $F(S)$ .

(f) A function  $\chi: S \rightarrow C$  is called a *semi-character* if  $|\chi(s)| \leq 1$  and  $\chi(s \cdot t) = \chi(s)\chi(t)$  for all  $s, t \in S$ . A continuous semi-character  $\chi$  is in  $F(S)$  if and only if  $\chi(s^*) = \overline{\chi(s)}$  for all  $s \in S$ . In this case  $\chi \in P(S)$  and  $\|\chi\|_\Omega = 1$  whenever  $\chi$  is nonzero, (see [6, Remark 2.1.8]).

The next proposition follows easily from [15, Theorem 3.2] and Remark 3.3(a):

**PROPOSITION 3.4.** *If  $S$  has an identity, then  $F(S) = \langle P(S) \rangle$ .*

**REMARK 3.5.** (a) Let  $S_n$  denote the semigroup formed by ad-

joining to  $S$  an identity  $u$ . Equip  $S_u$  with the topology  $\eta$  that a subset  $0 \subseteq S_u$  is in  $\eta$  if and only if  $0 \cap S$  is open in  $S$ . Then  $(S_u, \eta)$  is a topological semigroup. Also the involution on  $S$  can be extended to an involution on  $S_u$  by defining  $u^* = u$ . Let  $r$  denote the restriction map from  $F(S_u)$  into  $F(S)$ . Then  $r$  is norm decreasing, onto and  $r(P(S_u)) = F(S) \cap P(S)$ .

(b) The assumption that  $S$  has an identity cannot be removed from Proposition 3.4. Indeed, let  $S$  be a set with at least two elements. Let  $z \in S$  be fixed. Define on  $S$  the multiplication  $ab = z$  for all  $a, b \in S$ . Equip  $S$  with the discrete topology and involution  $a = a^*$  for all  $a \in S$ . Pick  $w \in S, w \neq z$ . Let  $f$  be the characteristic function on the set  $\{w\}$ . Then  $f \in P(S)$ , but  $f \notin F(S)$ . Indeed, there exists no  $k$  such that

$$\left| \sum_{i=1}^n c_i f(s_i) \right|^2 \leq k \sum_{i,j=1}^n c_i \bar{c}_j f(s_i s_j^*)$$

for any  $s_1, \dots, s_n$  in  $S$  and complex numbers  $c_1, \dots, c_n$ . Hence by Corollary 1.2 in [15],  $f$  is not extendable to a function in  $P(S_u)$ . By (a),  $f \in F(S)$ .

4. The operator algebra  $W^*(S)$ . Let  $S$  be a topological  $*$ -semigroup and write  $M_\Omega = \sum \oplus M_\alpha$ , the direct summand of the  $W^*$ -algebras  $M_\alpha, \alpha \in \Omega(S)$ . (See [16, p. 2].) Define a  $*$ -homomorphism of  $S$  into  $M_\Omega$  by:  $\omega_\Omega(s)(\alpha) = \omega_\alpha(s)$  for each  $\alpha = (\omega_\alpha, M_\alpha)$  in  $\Omega(S)$ . Then

$$\|\omega_\Omega(s)\| = \sup \{ \|\omega_\alpha(s)\|; \alpha \in \Omega(S) \} \leq 1$$

for each  $s \in S$ . Also if  $s_n$  is a net in  $S$  converging to some  $s \in S$ , then the net  $\langle \omega_\Omega(s_n)(\alpha), \psi \rangle = \langle \omega_\alpha(s_n), \psi \rangle$  converges to  $\langle \omega_\Omega(s)(\alpha), \psi \rangle$  for each  $\alpha \in \Omega(S)$  and  $\psi \in (M_\alpha)_*$ . Since the  $\sigma$ -topology on  $M_\Omega$  agrees with the topology determined by the semi-norms  $\{P_{\alpha, \psi}; \alpha \in \Omega(S), \psi \in (M_\alpha)_*\}$  on the unit ball, where

$$|P_{\alpha, \psi}(x)| = |\langle x(\alpha), \psi \rangle|$$

for each  $x \in M_\Omega$ , it follows that  $(\omega_\Omega, M_\Omega)$  is a  $\sigma$ -continuous  $*$ -representation of  $S$ . Write

$$W^*(S) = \langle \omega_\Omega(S) \rangle^{-\sigma}.$$

**THEOREM 4.1.** *Let  $S$  be a topological  $*$ -semigroup. Then:*

(a) *The mapping  $\pi: W^*(S)_* \rightarrow F(S)$  defined by  $\pi(\psi) = \hat{\psi}, \psi \in W^*(S)_*$ , is a linear isometry from  $W^*(S)_*$  onto  $F(S)$ . Consequently, the normed algebra  $F(S)$  is complete. Furthermore,  $\pi(\psi)$  is positive definite if and only if  $\psi$  is positive.*

(b) If  $(\omega, M)$  is any  $\sigma$ -continuous  $*$ -representation of  $S$ , then there exists a  $W^*$ -homomorphism  $h_\omega$  from  $W^*(S)$  into  $M$  such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\omega} & W^*(S) \\ & \searrow \omega & \downarrow h_\omega \\ & & M \end{array}$$

is commutative. Also if  $\psi \in M_*$ , then  $\langle x, \hat{\psi} \rangle = \langle h_\omega(x), \psi \rangle$  for all  $x \in W^*(S)$ .

Proof of Theorem 4.1 is rather routine. We omit the details.

EXAMPLE 4.2. Let  $Z$  be the group of integers with addition and involution  $n \rightarrow -n$ . Then  $F(Z) = R(Z) = \langle P(Z) \rangle$  (Remark 3.3(d) and Proposition 3.4), and  $W^*(Z)$  is the commutative  $W^*$ -algebra  $C(T)^{**}$ , where  $T$  is circle group (see Remark 4.3(b)).

On the other hand, if  $Z$  has involution  $n \rightarrow n$ , then  $F(Z) = C^2$ ,  $W^*(Z) = C^2$  and hence  $F(Z)$  is a proper subset of  $R(Z)$  (see Remark 3.3(a)). To see this, consider any  $(\omega, M) \in \Omega(Z)$ . Then  $M$  is a commutative  $W^*$ -algebra. Hence  $\omega(n) = \overline{\omega(n)} = \omega(-n)$  by Remark 3.3 (d). Consequently  $\omega(n)^2 = \omega(0) = 1$  for all  $n \in Z$  and  $\omega(Z)$  has at most two elements. However, if  $M$  is the subalgebra of  $L_\infty[0, 1]$  generated by the functions  $1, h$ , where

$$h(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

then  $\omega(0) = 1$  and  $\omega(1) = \omega(-1) = h$  defines a representation of  $Z$  in  $\Omega(Z)$ , and  $\langle \omega(Z) \rangle = M$ , which is two-dimensional. Hence  $W^*(Z) = C^2$  and  $F(Z) = C^2$ .

REMARK 4.3. (a) Let  $S$  be a locally compact topological  $*$ -semi-group. Let  $M(S)$  be the Banach algebra of complex, finite, regular Borel measures on  $S$  with multiplication of two elements  $\mu, \nu$  in  $M(S)$  defined by

$$\int f d\mu * \nu = \iint f(st) d\mu(s) d\nu(t)$$

for all  $f \in C_0(S)$ , and total variation norm, where  $C_0(S)$  is the space of all functions  $f \in C(S)$  vanishing at infinity (see [11]). For each  $\mu \in M(S)$ , define  $\mu^* \in M(S)$  to be the measure representing the func-



tional  $f \rightarrow \int f^*(t)d\mu(t)$  on  $C_0(S)$ . Then as observed in [14, Lemma 6.8], the map  $\mu \rightarrow \mu^*$  is an involution on  $M(S)$  with  $\|\mu\| = \|\mu^*\|$  and  $(\varepsilon_a)^* = \varepsilon_a$  for each  $a \in S$ , where  $\varepsilon_a$  is the point measure at  $a$ .

Define

$$\langle \tilde{\omega}_\alpha(\mu), \phi \rangle = \int \hat{\phi}(t)d\mu(t)$$

for each  $\mu \in M(S)$ , and  $\phi \in W^*(S)^*$ . Clearly  $\tilde{\omega}_\alpha(\varepsilon_a) = \omega(a)$  for each  $a \in S$ . Consequently  $\tilde{\omega}_\alpha$  defines a norm-decreasing \*-homomorphism of  $M(S)$  into  $W^*(S)$  which is continuous when  $M(S)$  has the  $\sigma(M(S), F(S))$  topology and  $W^*(S)$  has the  $\sigma$ -topology. Furthermore, if  $\beta$  is any norm-decreasing \*-homomorphism from  $M(S)$  into a  $W^*$ -algebra  $M$  which is continuous when  $M(S)$  has the  $\sigma(M(S), F(S))$  topology and  $M$  has the  $\sigma$ -topology, then there exists a  $W^*$ -homomorphism  $h$  from  $W^*(S)$  into  $M$  such that the diagram

$$\begin{array}{ccc} M(S) & \xrightarrow{\tilde{\omega}_\alpha} & W^*(S) \\ & \searrow \beta & \downarrow h \\ & & M \end{array}$$

is commutative.

(b) Let  $G$  be a locally compact group. Let  $C^*(G)$  denote the completion of  $L_1(G)$  with the norm

$$\|h\|_c = \sup \{ \|T_h\| \}, \quad h \in L_1(G),$$

where the supremum is taken over all no-where trivial \*-representation  $T$  of  $L_1(G)$  as an algebra of bounded linear operators on a Hilbert space. Then as well known (see [10, Chapter 2]),  $\langle P(G) \rangle$  can be identified with the dual of  $C^*(G)$ , and  $P(G)$  is precisely the positive linear functionals on the  $C^*$ -algebra  $C^*(G)$ . In this case  $F(G) = \langle P(G) \rangle$  (see Proposition 3.4) and  $\|f\|_\alpha$  is precisely the norm of  $f$  regarded as a linear functional on  $C^*(G)$  for each  $f \in F(G)$ . Furthermore,  $W^*(G)$  is isomorphic to the second conjugate algebra of  $C^*(G)$  with the Arens product. (See [8, Remark 2.6 and Proposition 2.8].)

(c) Let  $S$  be a topological \*-semigroup and let  $C_\alpha^*(S) = \langle \omega_\alpha(S) \rangle^-$ . Then  $C_\alpha^*(S)$  is a  $C^*$ -subalgebra of  $W^*(S)$ . For each  $f \in F(S)$ ,  $x \in C_\alpha^*(S)$ , define  $f \cdot x \in W^*(S)$  by

$$\langle f \cdot x, g \rangle = \langle x, f \cdot g \rangle$$

for all  $g \in F(S)$ . Then  $f \cdot x \in C_\alpha^*(S)$ . Also if  $m \in C_\alpha^*(S)^*$  and  $x \in C_\alpha^*(S)$ , then the element  $m_L(x)$  in  $W^*(S)$  defined by

$$\langle m_L(x), f \rangle = \langle m, f \cdot x \rangle$$

for all  $f \in F(S)$  is also in  $C_D^*(S)$ . Hence we may define on  $C_D^*(S)^*$  the Arens product [1] by

$$\langle n \cdot m, x \rangle = \langle n, m_L(x) \rangle$$

for any  $n, m \in C_D^*(S)$ ,  $x \in C_D^*(S)$ . Then  $C_D^*(S)^*$  with product and the dual norm is a commutative Banach algebra containing an isometric copy of  $F(S)$ . Furthermore, multiplication in the unit ball of  $C_D^*(S)^*$  is jointly continuous with respect to the weak\*-topology. (This follows from our [14, Theorem 6.11] when  $S$  has an identity; otherwise just use an argument similar to the proof given there.)

(d) Let  $S$  be a topological \*-semigroup and let  $S_d$  denote  $S$  with the discrete topology. For each  $m \in C_D^*(S)^*$ , define  $\hat{m}(s) = m(\omega_\rho(s))$  for all  $s \in S$ . Then clearly  $\hat{m} \in \langle P(S_d) \rangle$ . Since  $(\omega_\rho, M) \in \Omega(S)$ , where  $M$  is the enveloping  $W^*$ -algebra of  $C_D^*(S)$ , it follows that  $\hat{m} \in F(S_d)$  and  $\|\hat{m}\|_\rho \leq \|m\|$ . A simple computation shows that the map  $\tilde{\pi}: C_D^*(S)^* \rightarrow F(S_d)$  defined by  $\tilde{\pi}(m) = \hat{m}$ ,  $m \in C_D^*(S)^*$ , is norm-decreasing algebra isomorphism from  $C_D^*(S)^*$  into  $F(S_d)$ . In particular if  $S$  is discrete, then  $F(S)$  is isometric and algebra isomorphic to  $C_D^*(S)^*$ .

(e) Let  $S$  be a commutative topological \*-semigroup with an identity  $u$ , and let  $\hat{S}_d$  (resp.  $\hat{S}$ ) denote the set of all (resp. continuous) nonzero \*-semi-characters on  $S$  i.e., semi-characters  $\chi$  on  $S$  such that  $\chi(s^*) = \overline{\chi(s)}$  for all  $s \in S$ . Note that  $\chi(u) = 1$  for each  $\chi \in \hat{S}_d$ . Equip  $\hat{S}_d$  with the topology of pointwise convergence. Then  $\hat{S}_d$  with pointwise multiplication is a compact topological semigroup. Let  $\Delta(S)$  denote the spectrum of the commutative  $C^*$ -algebra  $C_D^*(S)$  and write  $\Delta(S)^\wedge = \tilde{\pi}(\Delta(S))$  where  $\tilde{\pi}$  is as defined in (d) above. Then  $\Delta(S)^\wedge$  is a compact subsemigroup of  $\hat{S}_d$  containing  $\hat{S}$ . Furthermore, it follows from [14, Theorem 6.12] that there exists a linear isometry and algebra homomorphism from  $F(S)$  into a weak\*-dense subalgebra of the measure algebra  $M(\Delta(S)^\wedge)$ . Also if  $S$  is a discrete commutative \*-semigroup with an identity, then  $F(S)$  is isometric and algebra isomorphic to  $M(\hat{S})$ .

5. The spectrum of  $F(S)$ . Throughout this section  $S$  will denote a topological \*-semigroup and  $\sigma(F(S))$  will denote the spectrum of  $F(S)$  i.e., the collection of all nonzero multiplicative linear functionals on  $F(S)$ .

Recently Martin E. Walter [18] [19] has given detailed analysis on the spectrum of the Fourier Stieltjes algebra of a locally compact group. In this section, we shall generalize some of Walter's results to the spectrum of  $F(S)$ . We begin with the following simple observations.

Let  $x \in W^*(S)$  and  $f \in F(S)$ . Define a bounded complex-valued function  $x_i(f)$  on  $S$  by

$$x_i(f)(s) = \langle x, l_s f \rangle$$

for each  $s \in S$ . Let  $\phi \in W^*(S)_*$  such that  $\hat{\phi} = f$ . Then  $(L_{\omega_\sigma(s)}\phi)^\wedge = l_s f$ . Hence

$$\begin{aligned} x_i(f)(s) &= \langle x, L_{\omega_\sigma(s)}\phi \rangle \\ &= \langle R_x \phi, \omega_\sigma(s) \rangle \\ &= (R_x \phi)^\wedge(s) \end{aligned}$$

for all  $s \in S$ . Consequently  $x_i f \in F(S)$  and  $\|x_i f\|_\sigma \leq \|x\| \|f\|_\sigma$ . Hence if  $y \in W^*(S)$ , we may define an element  $y \circ x$  in  $W^*(S)$  by

$$\langle y \circ x, f \rangle = \langle y, x_i(f) \rangle$$

for all  $f \in F(S)$ .

**LEMMA 5.1.** *If  $x, y \in W^*(S)$ , then  $y \circ x = y \cdot x$ , where  $y \cdot x$  denotes the product of  $y, x$  in  $W^*(S)$ . Consequently  $(x \cdot y)_i(f) = y_i(x_i(f))$  for each  $f \in F(S)$ .*

*Proof.* Let  $x \in W^*(S)$  be fixed. The equation  $y \circ x = y \cdot x$  clearly holds for all  $y = \omega_\sigma(s)$ ,  $s \in S$ . Hence it holds for all  $y \in \langle \omega_\sigma(S) \rangle$ . Now if  $y \in W^*(S)$  and  $y_\alpha$  is a net in  $\langle \omega_\sigma(S) \rangle$  converging to  $y$  in the  $\sigma$ -topology, then for each  $f \in F(S)$ ,

$$\begin{aligned} \langle y \circ x, f \rangle &= \langle y, x_i(f) \rangle \\ &= \lim_\alpha \langle y_\alpha, x_i(f) \rangle \\ &= \lim_\alpha \langle y_\alpha \circ x, f \rangle \\ &= \lim_\alpha \langle y_\alpha \cdot x, f \rangle \\ &= \langle y \cdot x, f \rangle \end{aligned}$$

by [16, p. 18]. The final assertions from direct computation.

**LEMMA 5.2.** (a) *If  $x \in \sigma(F(S))$ , then  $x_i$  is an algebra homomorphism from  $F(S)$  into  $F(S)$ .*

(b) *If  $S$  has an identity, and  $x$  is a nonzero element in  $W^*(S)$  such that  $x_i$  is an algebra homomorphism, then  $x \in \sigma(F(S))$ .*

*Proof.* (a) If  $f, g \in F(S)$ , then

$$x_i(f \cdot g)(s) = \langle x, l_s(f \cdot g) \rangle = \langle x, (l_s f)(l_s g) \rangle = \langle x, l_s f \rangle \langle x, l_s g \rangle$$

for all  $s \in S$ . Hence  $x_i$  is an algebra homomorphism.

(b) follows simply by evaluation at the identity.

REMARK 5.3. Note that Lemma 5.2 (b) is false when  $S$  does not have an identity as the following example shows. Let  $S = \{s_1, s_2, s_3\}$  with multiplication defined by

$$x_i \cdot x_j = \begin{cases} x_i & \text{if } i = j \\ x_1 & \text{if } i \neq j. \end{cases}$$

Then  $S$  is commutative and has no identity. Let  $S$  have the discrete topology and involution defined by the identity map. Then  $F(S)$  separate points. In fact, let  $M = L_\infty[0, 1]$  and define a  $*$ -representation  $\omega$  of  $S$  into  $M$  by  $\omega(s_1) = 0$ ,  $\omega(s_2) = 1_{[0, 1/2]}$ , and  $\omega(s_3) = 1_{[1/2, 1]}$ . Then  $F_\omega(S)$  clearly separate points and contained in  $F(S)$ . Hence  $\sigma(F(S)) = \omega_\sigma(S)$  consists of three distinct points, and the identity  $e$  of  $W^*(S)$  is not in  $\sigma(F(S))$ . However  $e_i$ , being the identity operator on  $F(S)$ , is an algebra homomorphism.

PROPOSITION 5.4. *If  $x, y \in \sigma(F(S))$ , then  $x^*$  and  $x \cdot y$  are in  $\sigma(F(S))$ .*

*Proof.* Let  $f, g \in P(S) \cap F(S)$ . Then  $f \cdot g \in P(S) \cap F(S)$  by Proposition 2.2. Hence if  $x \in \sigma(F(S))$ , then  $x^* \neq 0$  and

$$\langle x^*, f \cdot g \rangle = \overline{\langle x, f \cdot g \rangle} = \overline{\langle x, f \rangle \langle x, g \rangle} = \langle x^*, f \rangle \langle x^*, g \rangle$$

by Theorem 4.1 (b). Since  $\langle P(S) \cap F(S) \rangle = F(S)$ , it follows that  $x^* \in \sigma(F(S))$ .

If  $x, y \in \sigma(F(S))$  and  $f, g \in F(S)$ , then

$$\begin{aligned} \langle x \cdot y, f \cdot g \rangle &= \langle x, y_i(f \cdot g) \rangle = \langle x, y_i(f) y_i(g) \rangle = \langle x, y_i(f) \rangle \langle x, y_i(g) \rangle \\ &= \langle x \cdot y, f \rangle \langle x \cdot y, g \rangle \end{aligned}$$

using Lemmas 5.1 and 5.2. To see that  $x \cdot y \neq 0$ , we observe that if  $1$  is the constant one function on  $S$ , then  $\langle x, 1 \rangle = 1$ . Hence

$$\langle x \cdot y, 1 \rangle = \langle x, y_i(1) \rangle = \langle x, 1 \rangle = 1$$

using Lemma 5.1 again. Hence  $x \cdot y \in \sigma(F(S))$ .

If  $(\omega_i, M)$ ,  $i = 1, 2$ , are  $\sigma$ -continuous  $*$ -representations of  $S$ , let  $(\omega_1 \otimes \omega_2, M_1 \otimes M_2)$  denote the  $\sigma$ -continuous representation of  $S$  into the  $W^*$ -tensor product  $M_1 \otimes M_2$  by

$$(\omega_1 \otimes \omega_2)(s) = \omega_1(s) \otimes \omega_2(s)$$

for each  $s \in S$ .

PROPOSITION 5.5. *Let  $x$  be a nonzero element in  $W^*(S)$ . Then the followings are equivalent:*

- (a)  $x \in \sigma(F(S))$ .

(b)  $h_{\omega_1 \otimes \omega_2}(x) = h_{\omega_1}(x) \otimes h_{\omega_2}(x)$  for any  $\sigma$ -continuous  $*$ -representations  $(\omega_i, M_i)$ ,  $i = 1, 2$ , of  $S$ .

(c)  $h_{\omega_\rho \otimes \omega_\rho}(x) = x \otimes x$ .

*Proof.* (a)  $\Rightarrow$  (b). Let  $\phi_i \in (M_i)_*$ . Then  $(\phi_1 \otimes \phi_2)^\wedge = \hat{\phi}_1 \cdot \hat{\phi}_2$ . Hence

$$\begin{aligned} \langle h_{\omega_1 \otimes \omega_2}(x), \phi_1 \otimes \phi_2 \rangle &= \langle x, (\phi_1 \otimes \phi_2)^\wedge \rangle \\ &= \langle x, \hat{\phi}_1 \cdot \hat{\phi}_2 \rangle \\ &= \langle x, \hat{\phi}_1 \rangle \langle x, \hat{\phi}_2 \rangle \\ &= \langle h_{\omega_1}(x), \phi_1 \rangle \langle h_{\omega_2}(x), \phi_2 \rangle \\ &= \langle h_{\omega_1}(x) \otimes h_{\omega_2}(x), \phi_1 \otimes \phi_2 \rangle \end{aligned}$$

using Theorem 4.1 (b). Since  $\{\phi_1 \otimes \phi_2, \phi_i \in (M_i)_*\}$  is total in  $(M_1 \otimes M_2)_*$ , (b) follows.

(b)  $\Rightarrow$  (c) is clear.

(c)  $\Rightarrow$  (a). Let  $f_1, f_2 \in F(S)$  and  $\psi_1, \psi_2$  be the unique elements in  $W^*(S)_*$  such that  $\hat{\psi}_i = f_i$ . Then

$$\begin{aligned} \langle f_1, x \rangle \langle f_2, x \rangle &= \langle \psi_1, x \rangle \langle \psi_2, x \rangle \\ &= \langle \psi_1 \otimes \psi_2, x \otimes x \rangle \\ &= \langle \psi_1 \otimes \psi_2, h_{\omega_\rho \otimes \omega_\rho}(x) \rangle \\ &= \langle (\psi_1 \otimes \psi_2)^\wedge, x \rangle \quad (\text{by Theorem 4.1(b)}) \\ &= \langle \psi_1^\wedge \cdot \psi_2^\wedge, x \rangle \\ &= \langle f_1 \cdot f_2, x \rangle. \end{aligned}$$

Since  $x \neq 0$ ,  $x \in \sigma(F(S))$ .

REMARK 5.6. (a) Both Propositions 5.4 and 5.5 are due to Martin E. Walter ([18, Theorem 1(ii) and (iii)] and [19, Corollary to Theorem 2]) when  $S$  is a locally compact group. Our proof of Proposition 5.4 is completely different from that of Walter. However, using Proposition 5.5 and an argument similar to that in [18, Theorem 1(iii)] we can also obtain a part of Proposition 5.4, i.e., if  $x, y \in \sigma(F(S))$ , then  $x^* \in \sigma(F(S))$  and  $x \cdot y \in \sigma(F(S)) \cup \{0\}$ .

(b) It follows from Proposition 5.5 that  $\sigma(F(S))$  with the  $\sigma$ -topology is a compact topological  $*$ -semigroup. Also,  $\overline{\omega_\rho(S)}^\sigma$  is a  $*$ -subsemigroup of  $\sigma(F(S))$ ;  $\overline{\omega_\rho(S)}^\sigma$  is precisely the largest  $*$ -compactification of  $S$  as defined in [15, Theorem 5.1].

PROPOSITION 5.7. Let  $T = \overline{\omega_\rho(S)}^\sigma$ . Then there exists a linear isometry and algebra isomorphism  $U$  from  $F(T)$  onto  $F(S)$  such that  $U^*$  is a  $W^*$ -isomorphism from  $W^*(S)$  onto  $W^*(T)$ .

*Proof.* Let  $\omega'_\rho$  denote the  $*$ -representation of  $T$  into  $W^*(T)$  and let  $\omega'(s) = \omega'_\rho(\omega_\rho(s))$  for all  $s \in S$ . Then  $(\omega', W^*(T))$  is a  $\sigma$ -continuous  $*$ -representation of  $S$ , and  $\langle \omega'(S) \rangle^{-\sigma} = W^*(T)$ . Let  $h = h_\omega$ . (Theorem 4.1). Then  $h$  is onto and

$$h(\omega_\rho(s)) = \omega'_\rho(\omega_\rho(s))$$

for all  $s \in S$ . Consequently  $h(t) = \omega'_\rho(t)$  for all  $t \in T$ . On the other hand, if  $k = h_{\omega_0}$  where  $\omega_0$  is the injection map of  $T$  into  $W^*(S)$ , then

$$k(\omega'_\rho(t)) = t$$

for all  $t \in T$ . Hence  $k(h(t)) = t$  for all  $t \in T$ . Since  $\langle T \rangle^{-\sigma} = W^*(S)$ , it follows that  $k(h(x)) = x$  for all  $x \in W^*(S)$ . Consequently  $h$  is a  $W^*$ -isomorphism. Define  $U(\psi^\wedge) = (h^* \psi)^\wedge$  for all  $\psi \in W^*(T)_*$ . Then  $U(f)(s) = f(\omega'(s))$  for all  $s \in S, f \in F(T)$ . Hence  $U$  is a linear isometry and algebra homomorphism from  $F(T)$  onto  $F(S)$ , and  $U^* = h$  is a  $W^*$ -isomorphism from  $W^*(S)$  onto  $W^*(T)$ .

Martin Walter proved in [18] the following beautiful duality theorem: If the Fourier algebras of two locally compact groups  $G_1$  and  $G_2$  are isometric isomorphic, then  $G_1$  and  $G_2$  are topologically isomorphic. This result, as pointed out in [18, p. 18] is equivalent to B. E. Johnson's isomorphism theorem for the measure algebras of the locally compact groups when  $G_1$  and  $G_2$  are abelian. It is easy to see from Proposition 5.7 that Walter's result is no longer valid when  $G_1, G_2$  are topological  $*$ -semigroups. However we shall show in the next theorem that if  $S_1$  and  $S_2$  are topological  $*$ -semigroups with identity and  $F(S_1)$  and  $F(S_2)$  are isometric isomorphic, then the compact topological  $*$ -semigroups  $\sigma(F(S_1))$  and  $\sigma(F(S_2))$  are strongly related.

Let  $\sigma_u(F(S))$  denote all unitary elements in  $\sigma(F(S))$  and let  $\sigma_c(F(S))$  denote the centre of the semigroup  $\sigma(F(S))$ , i.e., all  $x \in \sigma(F(S))$  such that  $x \cdot y = y \cdot x$  for all  $y \in \sigma(F(S))$ . Then  $\sigma_u(F(S))$  is a group and  $\sigma_c(F(S))$  is a closed  $*$ -subsemigroup of  $\sigma(F(S))$ .

**THEOREM 5.8.** *Let  $S_1, S_2$  be topological  $*$ -semigroups with identity. If the Banach algebras  $F(S_1)$  and  $F(S_2)$  are isometric isomorphic, then there exists a homeomorphism  $\phi$  from  $\sigma(F(S_1))$  onto  $\sigma(F(S_2))$  such that*

- (a)  $\phi(x^*) = \phi(x)^*$  for all  $x \in \sigma(F(S_1))$ .
- (b) For each  $x, y \in \sigma(F(S_1))$ , either  $\phi(x \cdot y) = \phi(x)\phi(y)$  or  $\phi(x \cdot y) = \phi(y)\phi(x)$ .
- (c)  $\phi$  is a  $*$ -isomorphism from  $\sigma_c(F(S_1))$  onto  $\sigma_c(F(S_2))$ .
- (d)  $\phi$  is either a  $*$ -isomorphism or a  $*$ -anti-isomorphism from  $\sigma_u(F(S_1))$  onto  $\sigma_u(F(S_2))$ .

Furthermore, if for each  $x \in \sigma(F(S_1))$ ,

$$\begin{aligned} H_x &= \{y \in \sigma(F(S_1)); \phi(x \cdot y) = \phi(y)\phi(x)\}; \\ K_x &= \{y \in \sigma(F(S_1)); \phi(x \cdot y) = \phi(x)\phi(y)\} \end{aligned}$$

and if

$$H = \cap \{H_x; x \in \sigma_u(F(S_1))\}; \quad K = \cap \{K_x; x \in \sigma_u(F(S_2))\}$$

then

(e)  $H_x, K_x$  are  $\sigma$ -closed subsemigroups of  $\sigma(F_1(S))$  such that  $y \in H_x$  (resp.  $y \in K_x$ ) if and only if  $y^* \in K_x$  (resp.  $y^* \in H_x$ ).

(f)  $H$  and  $K$  are  $\sigma$ -closed \*-subsemigroups of  $\sigma(F(S_1))$  such that  $H \cup K = \sigma(F_1(S))$ .

*Proof.* We follow an idea Martin Walter in the proof of Theorem 2 in [18]. Let  $U$  be the isomorphism from  $F(S_2)$  onto  $F(S_1)$ . Since  $S_1$  has an identity, it follows that  $e_1$ , the identity of  $W^*(S_1)$ , is in  $\sigma(F(S_1))$ . Hence  $u = U^*(e_1)$  and  $v = u^*$  are in  $\sigma(F(S_2))$  (by Proposition 5.4) and  $v_i$  is an algebra homomorphism from  $F(S_2)$  into  $F(S_2)$  (Lemma 5.1) such that  $\|v_i(f)\|_2 \leq \|v\| \|f\|_2 = \|f\|_2$  for all  $f \in F(S_2)$ . On the other hand, since  $u$  is unitary [13, Lemma 12], it follows that

$$\|v_i(f)\| \leq \|u_i(v_i(f))\| = \|(v \cdot u)_i(f)\| = \|f\|$$

for each  $f \in F(S_2)$  by Lemma 5.1, i.e.,  $v$  is an isometry. Also if  $f \in F(S_2)$ , then  $v_i(u_i(f)) = f$ . Hence  $v_i$  is onto. Consequently  $U \circ v_i$  is also an isometric isomorphism from  $F(S_2)$  onto  $F(S_1)$ . Let  $\Phi = (U \circ v_i)^*$ . Then  $\Phi$  is an isometry from  $W^*(S_1)$  onto  $W^*(S_2)$ . Also

$$\langle \Phi(e_1), f \rangle = \langle U^*(e_1), v_i(f) \rangle = \langle u, v_i(f) \rangle = \langle e_2, f \rangle$$

for all  $f \in F(S_1)$ , where  $e_2$  is the identity of  $W^*(S_2)$ , by Lemma 5.1. Hence  $\Phi(e_1) = e_2$ . By Theorem 7 in [13],  $\Phi$  is a Jordan \*-isomorphism from  $W^*(S_1)$  onto  $W^*(S_2)$ . Let  $\phi$  be the restriction of  $\Phi$  to  $\sigma(F(S_1))$ . Then clearly  $\phi$  is a homomorphism from  $\sigma(F(S_1))$  onto  $\sigma(F(S_2))$ . We shall show that  $\phi$  has all desired properties.

That (a) and (c) hold follow from Theorem 5 and Lemma 8 in [13].

To prove (b), we first note that if  $xy = yx$ , then (b) holds by [13, Theorem 5]. Otherwise, using [13, Lemma 6], we have

$$\phi(x)\phi(y) + \phi(y)\phi(x) = \phi(xy) + \phi(yx).$$

If  $\phi(xy) \neq \phi(y)\phi(x)$  and  $\phi(xy) \neq \phi(x)\phi(y)$ , then  $\phi(xy)$ ,  $\phi(yx)$ ,  $\phi(x)\phi(y)$  and  $\phi(y)\phi(x)$  are pairwise distinct elements in  $\sigma(F(S_2))$ . However, elements in  $\sigma(F(S_2))$  are linearly independent [4, p. 93], which is impossible. Hence (b) holds.

If (e) holds, then clearly  $H$  and  $K$  are  $*$ -subsemigroups of  $\sigma(F(S_1))$ . Also, if  $x \in \sigma(F(S_1))$ , then  $H_x \cap \sigma_u(F(S_1))$  and  $K_x \cap \sigma_u(F(S_1))$  are subgroups of  $\sigma_u(F(S_1))$  with union equal to  $\sigma_u(F(S_1))$  by (b). Hence either  $\sigma_u(F(S_1)) \subseteq H_x$  or  $\sigma_u(F(S_1)) \subseteq K_x$ . Hence  $H \cup K = \sigma(F(S_1))$  and (f) holds. Also a similar argument shows that either  $\sigma_u(F(S_1)) \subseteq H$  or  $\sigma_u(F(S_1)) \subseteq K$ . Hence (d) follows readily from [13, Lemma 12].

It remains to prove (e). By Theorem 10 in [13], there exists a central projections  $z_i \in W^*(S_i)_*$  such that  $\Phi$  is a  $*$ -isomorphism from  $W^*(S_1)z_1$  onto  $W^*(S_2)z_2$  and a  $*$ -anti-isomorphism from  $W^*(S_1)(e_1 - z_1)$  onto  $W^*(S_2)(e_2 - z_2)$ . Then  $\Phi(xz_1) = \Phi(x)z_1$  and  $\Phi(x(e_1 - z_1)) = \Phi(x)(e_2 - z_2)$  for all  $x \in W^*(S_1)$ . Also observe that

$$(1) \quad y \in H_x \quad \text{if and only if} \quad (xy - yx)z_1 = 0$$

and

$$(2) \quad y \in K_x \quad \text{if and only if} \quad (xy - yx)z_2 = 0.$$

To prove (1), let  $y \in H_x$ . Then

$$\begin{aligned} \Phi((xy - yx)z_1) &= \Phi(xy - yx)z_2 \\ &= \Phi(y)\Phi(x)z_2 - \Phi(yz_1)\Phi(xz_1) \\ &= 0. \end{aligned}$$

Hence  $(xy - yx)z_1 = 0$ . Conversely, if  $(xy - yx)z_1 = 0$  and  $y \notin K_x$ , then  $y \in H_x$  by (b). If  $y \in K_x$ , then  $(xy - yx)(e_1 - z_1) = 0$ . Hence  $xy = yx$ . So  $y \in H_x$  by [13, Theorem 5]. (2) can be proved similarly.

Now if  $y_1, y_2 \in H_x$ , then

$$x(y_1y_2)z_1 = (xy_1)z_1y_2 = (y_1x)z_1y_2 = y_1(xy_2)z_1 = (y_1y_2)xz_1.$$

Hence  $y_1y_2 \in H_x$  by (1). Similarly we show that  $y \in H_x$  if and only if  $y^* \in H_{x^*}$  and that  $H_x$  is  $\sigma$ -closed. The assertions on  $K_x$  can be proved by using (2).

REMARK 5.9. (a) Martin Walter [18, Theorem 1(i)] proved that if  $G$  is a locally compact group, then  $\sigma_u(F(G))$  is topologically isomorphic to  $G$ .

(b) Let  $S$  be a topological  $*$ -semigroup and  $T = \overline{\omega_\rho(S)}$ . It follows from Proposition 5.7 and its proof that there exists a homeomorphism and  $*$ -isomorphism  $\phi$  from  $\sigma(F(S))$  onto  $\sigma(F(T))$  such that  $\phi(\omega_\rho(s)) = \omega'_\rho(\omega_\rho(s))$  for all  $s \in S$ , where  $\omega_\rho$  and  $\omega'_\rho$  denote the  $*$ -representations of  $S, T$  into  $W^*(S)$  and  $W^*(T)$  respectively.

PROPOSITION 5.10. *Let  $S_1, S_2$  be topological  $*$ -semigroups with identity. If there exists a Banach algebra isomorphism  $U$  from  $F(S_2)$  onto  $F(S_1)$  such that  $U$  maps  $P(S_2)$  onto  $P(S_1)$  and  $\|Uf\|_\infty = \|f\|_\infty$*



for all  $f \in F(S_2)$ , then there exists a homeomorphism  $\phi$  from  $\overline{\omega_\rho(S_1)^\sigma}$  onto  $\overline{\omega_\rho(S_2)^\sigma}$  such that (i)  $\phi(x^*) = \phi(x)^*$  for all  $x \in \overline{\omega_\rho(S_1)^\sigma}$  and (ii) for any  $x, y \in \overline{\omega_\rho(S_1)^\sigma}$ , either  $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$  or  $\phi(x \cdot y) = \phi(y) \cdot \phi(x)$ .

*Proof.* The assumption implies that  $U^*$  takes the identity of  $W^*(S_1)$  to the identity of  $W^*(S_2)$ . Hence if  $\phi$  is the restriction of  $U^*$  to  $\overline{\omega_\rho(S_1)^\sigma}$ , then it follows from the proof of Theorem 5.8 that  $\phi$  has properties (i) and (ii). Also an argument similar to that in [7, p. 99] shows that the  $\overline{\omega_\rho(S_i)^\sigma} = \{x \in \sigma(F(S_i)); |\langle f, x \rangle| \leq \|f\|_\infty \text{ for all } f \in F(S_i)\}$ . Hence  $\phi$  is a homeomorphism mapping  $\overline{\omega_\rho(S_1)^\sigma}$  onto  $\overline{\omega_\rho(S_2)^\sigma}$ .

REMARK 5.11. (a) Theorem 5.8 remains valid when either  $S_1$  or  $S_2$  has identity. Do the conclusions of Theorem 5.8 still hold when both  $S_1$  and  $S_2$  are assumed not to have an identity?

(b) The following questions are posted to us by the referee: Do the hypotheses of Theorem 5.8 imply anything about a sup-norm isometry between  $F(S_1)$  and  $F(S_2)$ ? (It is true for groups by Walter's result.) Also can one deduce any relationship between  $\overline{\omega_\rho(S_1)^\sigma}$  and  $\overline{\omega_\rho(S_2)^\sigma}$ ? (See Proposition 5.10.)

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