62. The Fourier Transform of the Schwartz Space on a Symmetric Space

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1. Introduction. Let S be a symmetric space of the noncompact type and let $L^2(S)$ denote the space of square integrable functions on S with respect to the invariant measure. In his paper [7], S. Helgason characterized the image of $L^2(S)$ by the Fourier transform.

The purpose of this paper is to give a characterization of the image of the Harish-Chandra's Schwartz space by the Fourier transform. As an immediate consequence we obtain the above mentioned result of S. Helgason (the characterization of the image of $L^2(S)$ by the Fourier transform). The proofs of the results are given in [2].

2. Notation and preliminaries. If M is a manifold (satisfying the second countability axiom), following Schwartz $\mathcal{D}(M)$ denotes the the space of C^∞ functions on M with compact support. If V is a real vector space $\mathcal{S}(V)$ denotes the space of rapidly decreasing functions on V (see [8]) and $\boldsymbol{D}(V)$ denotes the algebra of differential operators with constant coefficients on V.

If G is a Lie group and H a closed subgroup, G/H denotes the space of left cosets gH, $g \in G$. $\boldsymbol{D}(G/H)$ denotes the algebra of differential operators on homogeneous space G/H which are invariant under left translations by G. We write $\boldsymbol{D}(G)$ for $\boldsymbol{D}(G/e)$, where e is the identity of G.

Let S be a symmetric space of the noncompact type that is a coset space $S=G/K$ where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and θ the Cartan involution which associates with the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace and $\mathfrak{a}^*$ its dual. Put $A=\exp\mathfrak{a}$. For $\lambda\in\mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda=\{X\in\mathfrak{g}\,|\,[H,X]=\lambda(H)X,\text{ for all } H\in\mathfrak{a}\}.$$

If $\lambda\neq 0$ and $\mathfrak{g}_\lambda\neq\{0\}$ then λ is called a restricted root and $m_\lambda=\dim(\mathfrak{g}_\lambda)$ is called its multiplicity. Let $\mathfrak{g}_c$ and $\mathfrak{a}_c^*$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{a}^*$ respectively. If λ, $\mu\in\mathfrak{a}_c^*$ let $H_\lambda\in\mathfrak{a}_c$ (the complex subspace of $\mathfrak{g}_c$ spanned by $\mathfrak{a}$) be determined by $\lambda(H)=\langle H_\lambda, H\rangle$ for $H\in\mathfrak{a}$ and put $\langle\lambda,\mu\rangle=\langle H_\lambda, H_\mu\rangle$. Since B is positive definite on $\mathfrak{p}$ we put $\|\lambda\|=\langle\lambda,\lambda\rangle^{1/2}$

for $\lambda \in \mathfrak{a}^*$ and $\|X\|=B(X,X)^{1/2}$ for $X \in \mathfrak{p}$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}'$ and call a restricted root α positive if it is positive on $\mathfrak{a}^+$. Let $\mathfrak{a}^*_+$ denote the corresponding Weyl chamber in $\mathfrak{a}^*$. Let $\sum$ denote the set of restricted roots, P_+ the set of positive roots. Put $\rho=(1/2)\sum_{\alpha \in P_+} \alpha$, $\mathfrak{n}=\sum_{\alpha \in P_+} \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$. Let N and $\bar{N}$ denote the corresponding analytic subgroups of G. Thus we obtain an Iwasawa decomposition $G=KAN$ of G. For each $g \in G$ can be uniquely written $g=\kappa(g)\exp H(g)n(g)$, $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Let M denote the centralizer of A in K, M' the normalizer of A in K, W the factor group M'/M, the Weyl group. The group W acts as a group of linear transformations of $\mathfrak{a}^*$ by $(s\lambda)(H)=\lambda(s^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*$ and $s \in W$. Let w denote the order of W.

Let $l=\dim \mathfrak{a}$. The Killing form induces Euclidean measures on A, $\mathfrak{a}$ and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-(1/2)l}$ we obtain invariant measures da, dH and $d\lambda$, and the inversion formula for the Fourier transform

$$f^*(\lambda)=\int_A f(a)e^{-i\lambda(\log a)}da, \qquad \lambda \in \mathfrak{a}^*, \tag{1}$$

$$f(a)=\int_{\mathfrak{a}^*} f^*(\lambda)e^{i\lambda(\log a)}d\lambda, \qquad f \in \mathcal{S}(A) \tag{2}$$

holds without multiplicative constant.

We normalize the Haar measures dk and dm on the compact groups K and M, respectively, so that the total measure is 1. The Haar measures of the nilpotent groups N, $\bar{N}$ are normalized so that

$$\theta(dn)=d\bar{n}, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))}d\bar{n}=1.$$

The Haar measure dg on G can be normalized so that

$$\int_G f(g)dg=\int_{KAN} f(kan)e^{2\rho(\log a)}dkdadn, \qquad f \in \mathcal{D}(G).$$

Let dk_M denote the K-invariant measure on K/M of total measure 1 which satisfies

$$\int_K f(k)dk=\int_{K/M}\left(\int_M f(km)dm\right)dk_M, \qquad f \in C(K).$$

Let $dg_K=dx$ denote the G-invariant measure on G/K given by

$$\int_G f(g)dg=\int_{G/K}\left(\int_K f(gk)dk\right)dg_K, \qquad f \in C_c(G).$$

3. The Schwartz space and the spherical Fourier transform. In this section we describe some results of Harish-Chandra [4, 5] and S. Helgason [6, 7] in the form suitable for our purpose.

Let φ_λ be the spherical function given by

$$\varphi_\lambda(g)=\int_K e^{(i\lambda-\rho)(H(gk))}dk, \qquad (g \in G,\ \lambda \in \mathfrak{a}^*).$$

We put $\Xi(g)=\varphi_0(g)$ $(g \in G)$. Each $g \in G$ can be written uniquely in the form $\exp X.\,k$ $(k \in K,\ X \in \mathfrak{p})$. Then we put $\sigma(g)=\|X\|$.

Definition 1. Let $\mathcal{C}(S)$ denote the set of all complex-valued C^∞ functions f on G which satisfy the following two conditions:

(i) f is right-invariant under K,

(ii) for each $D \in \boldsymbol{D}(G)$ and each integer $q \geqq 0$

$$\tau_{D,q}(f)=\sup_{g\in G} \Xi(g)^{-1}(1+\sigma(g))^q\,|Df(g)|<+\infty.$$

Let $I(G)$ denote the set of all $f \in \mathcal{C}(S)$ which are left-invariant under K. Then $\tau_{D,q}$ is a seminorm on $I(G)$ and $\mathcal{C}(S)$. We topologize $I(G)$ and $\mathcal{C}(S)$ by means of the seminorms $\tau_{D,q}$ $(D \in \boldsymbol{D}(G), q \geqq 0)$. In this way $I(G)$ and $\mathcal{C}(S)$ become Fréchet spaces. After Harish-Chandra, we call $\mathcal{C}(S)$ the Schwartz space of S. Let $\mathcal{I}(\mathfrak{a}^*)$ denote the set of W-invariants in $\mathcal{S}(\mathfrak{a}^*)$.

For $f \in I(G)$, its spherical Fourier transform is defined by

$$\tilde{f}(\lambda)=\int_G f(g)\varphi_{-\lambda}(g)dg, \qquad (\lambda \in \mathfrak{a}^*). \tag{1}$$

The following theorem is due to Harish-Chandra [4, 5] and Helgason [6].

Theorem 1. *Let f be any function in $I(G)$. Then*

(i) $$\int_G |f(x)|^2\,dx=w^{-1}\int_{\mathfrak{a}^*}|\tilde{f}(\lambda)|^2\,|c(\lambda)|^{-2}\,d\lambda, \tag{2}$$

(ii) $$f(x)=w^{-1}\int_{\mathfrak{a}^*}\tilde{f}(\lambda)\varphi_\lambda(x)\,|c(\lambda)|^{-2}\,d\lambda. \tag{3}$$

(iii) *The mapping $f\to\tilde{f}$ is a topological isomorphism of $I(G)$ onto $\mathcal{I}(\mathfrak{a}^*)$.*

4. The Fourier transform of Schwartz space. For any function f in $\mathcal{C}(S)$, we put

$$\tilde{f}(\lambda, kM)=\tilde{f}_\lambda(kM)=\int_{AN} f(kan)e^{(-i\lambda+\rho)(\log a)}dadn, \tag{4}$$

$\lambda \in \mathfrak{a}^*$ and $kM \in K/M$. If we take into account Theorem B in [1], it can be shown without difficulty that the integral in the right-hand side of (4) converges for each λ and kM. We call the mapping $f\to\tilde{f}$ the Fourier transform.

Remark 1. If we consider only K-bi-invariant functions in $\mathcal{C}(S)$ our Fourier transform coincides with the spherical Fourier transform.

Remark 2. For any $f \in \mathcal{D}(S)$, our Fourier transform coincides with the one which was defined by S. Helgason in [6].

Using the Iwasawa decomposition, we can extend any function ϕ on $\mathfrak{a}^*\times(K/M)$ to a function on $\mathfrak{a}^*\times G$ by

$$\phi(\lambda, x)=e^{(i\lambda-\rho)(H(x))}\phi(\lambda, \kappa(x)M), \qquad x \in G. \tag{5}$$

We also write $\phi(\lambda, x)=\phi_\lambda(x)$. We put

$$\check{\phi}_\lambda(x)=\int_K \phi_\lambda(xk)dk \tag{6}$$

and call the mapping $\phi_\lambda\to\check{\phi}_\lambda$ the dual Radon transform.

Now we define the Schwartz space of $\mathfrak{a}^*\times(K/M)$ as follows.

Definition 2. Let $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ denote the set of all complex-valued C^∞ functions ϕ on $\mathfrak{a}^*\times(K/M)$ which satisfy the following condition: for each $E\in \boldsymbol{D}(\mathfrak{a}^*)$, $u\in \boldsymbol{D}(K/M)$ and each integer $r\geqq 0$

$$\nu_{E,u,r}(\phi)=\sup_{(\lambda,kM)\in\mathfrak{a}^*\times(K/M)}(1+\|\lambda\|)^r\,|(Eu\phi)(\lambda,kM)|<\infty.$$

Then $\nu_{E,u,r}$ is a seminorm on $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ and the collection of these seminorms, for all $E\in \boldsymbol{D}(\mathfrak{a}^*)$, $u\in \boldsymbol{D}(K/M)$ and integers $r\geqq 0$, defines a topology on $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ in the usual way so that $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ becomes a Fréchet space. And we obtain the following theorems.

Theorem 2. *Let f be any function in $\mathcal{C}(S)$. Then $\tilde{f}\in\mathcal{C}(\mathfrak{a}^*\times(K/M))$ and*

$$\check{\tilde{f}}_{s\lambda}=\check{\tilde{f}}_{\lambda} \tag{7}$$

holds for all $\lambda\in\mathfrak{a}^*$ and $s\in W$.

Theorem 3. *Let f be any function in $\mathcal{C}(S)$. Then*

(i) $\int_G |f(x)|^2\,dx=w^{-1}\int_{\mathfrak{a}^*\times(K/M)}|\tilde{f}(\lambda,kM)|^2\,|c(\lambda)|^{-2}\,dk_M d\lambda,$

(ii) $f(x)=w^{-1}\int_{\mathfrak{a}^*}\check{\tilde{f}}(\lambda,x)\,|c(\lambda)|^{-2}\,d\lambda.$

Remark 3. Theorem 3 was proved by S. Helgason [7] for the functions in $\mathcal{D}(S)$.

Remark 4. Gel'fand-Graev [3] studied the Paley-Wiener theorem for the Fourier transform on complex semisimple Lie groups G. But one should remark that their definition of the space of the infinitely differentiable rapidly decreasing functions is different from that of the Schwartz space $\mathcal{C}(G/K)$.

5. The image of the Fourier transform. In order to give a characterization of the image of $\mathcal{C}(S)$ by the Fourier transform we consider the following function space.

Defintion 3. Let $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ denote the set of all functions ϕ in $\mathcal{C}(\mathfrak{a}^*\times(K/M))$ which satisfy the following condition: $\check{\phi}_{s\lambda}=\check{\phi}_\lambda$ for all $\lambda\in\mathfrak{a}^*$ and $s\in W$.

We consider $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ with the relative topology induced from $\mathcal{C}(\mathfrak{a}^*\times(K/M))$. Then we can prove the following result.

Theorem 4. *For each $\phi\in\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$, put*

$$f_\phi(x)=w^{-1}\int_{\mathfrak{a}^*\times(K/M)}\phi(\lambda,kM)e^{-(i\lambda+\rho)(H(x^{-1}k))}\,|c(\lambda)|^{-2}\,dk_M d\lambda. \tag{8}$$

Then $f_\phi\in\mathcal{C}(S)$ and the mapping $\phi\to f_\phi$ is a one-to-one continuous linear mapping of $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$ into $\mathcal{C}(S)$.

Now we can state our main theorem as follows.

Theorem 5. *The mapping $f\to\tilde{f}$ defined by (4) is a linear topological isomorphism of $\mathcal{C}(S)$ onto $\mathcal{C}(\mathfrak{a}^*\times(K/M))_W$.*

Since $\mathcal{D}(S)\subset\mathcal{C}(S)$, $\mathcal{C}(S)$ is dense in $L^2(S)$ in the L^2-topology. Denoting $\mathcal{C}(\mathfrak{a}^*_+\times(K/M))$ the set of all restrictions $\phi|_{\mathfrak{a}^*_+\times(K/M)}$ of ϕ in

$\mathcal{C}(\mathfrak{a}^* \times (K/M))$ to $\mathfrak{a}_+^* \times (K/M)$, it is clear that $\mathcal{C}(\mathfrak{a}_+^* \times (K/M))$ is dense in $L^2(\mathfrak{a}_+^* \times (K/M))$ in the L^2-topology and that the composition of the Fourier transform and the restriction is an isometry of $\mathcal{C}(S)$ onto $\mathcal{C}(\mathfrak{a}_+^* \times (K/M))$. Hence we obtain the following result.

Corollary (S. Helgason). *The Fourier transform $f \to \tilde{f}$ defined by* (4) *extends to an isometry of $L^2(S)$ onto $L^2(\mathfrak{a}_+^* \times (K/M))$ (with the measure $|c(\lambda)|^{-2} d\lambda dk_M$ on $\mathfrak{a}_+^* \times (K/M)$).*

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