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THE FRAMED BRAID GROUP AND 3-MANIFOLDS

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ABSTRACT. The framed braid group on n strands is defined to be a semidirect product of the braid group B_n and $\mathbf{Z}^n$. Framed braids represent 3-manifolds in a manner analogous to the representation of links by braids. Consider two framed braids equivalent if they represent homeomorphic 3-manifolds. The main result of this paper is a Markov type theorem giving moves that generate this equivalence relation.

In this paper the group of framed braids $\mathfrak{F}_n$ is introduced. This group is similar to the braid group and is an initial attempt to understand 3-manifolds in a manner analogous to the braid approach to links in the 3-sphere. The main theorem describes the equivalence relations on $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$ that yields the set of 3-manifolds.

Framed braids. Let B_n denote the braid group with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ and relations

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$;
(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

The geometric braid σ_i is shown in Figure 1.

Let Σ_n denote the symmetric group acting on $\{1, 2, \dots, n\}$. Let $\pi\colon B_n \to \Sigma_n$ be the quotient map sending σ_i to the transposition $(i, i+1)$. The kernel of π is the pure braid group denoted by P_n. B_n acts on $\{1, 2, \dots, n\}$ through π, i.e., $\sigma(i) = \pi(\sigma)(i)$ for $\sigma \in B_n$. This paper follows the convention that the symmetric group acts from the right so that $(\sigma\tau)(i) = \tau(\sigma(i))$ for $\sigma, \tau \in B_n$.

Definition. The framed braid group $\mathfrak{F}_n$ is the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}, t_1, t_2, \dots, t_n$ with the relations (1), (2) and additional relations

(3) $t_it_j = t_jt_i$ for all i, j;
(4) $\sigma_it_j = t_{\sigma_i(j)}\sigma_i$.

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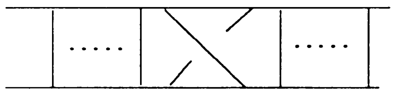


FIGURE 1

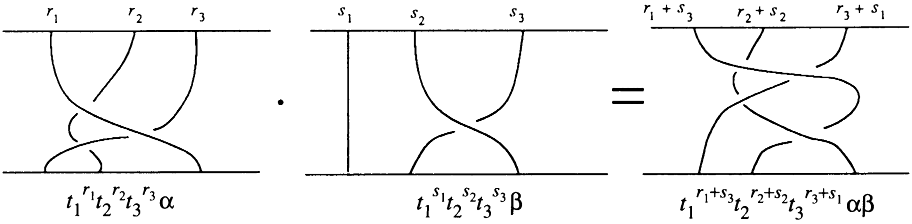


FIGURE 2

The group $\mathfrak{F}_n$ is a semidirect product $\mathbf{Z}^n \rtimes B_n$ where the action of B_n on $\mathbf{Z}^n$ is given by $\sigma(r_1, r_2, \dots, r_n) = (r_{\sigma(1)}, r_{\sigma(2)}, \dots, r_{\sigma(n)})$. If $t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha \in \mathfrak{F}_n$ with $\alpha \in B_n$ then the r_i's are called *framings.* Note that $\sigma t_i = t_{\sigma^{-1}(i)}\sigma$ for $\sigma \in B_n$. The product and the inverse in this notation are given as follows. See Figure 2 for an example.

$$(t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha)(t_1^{s_1}t_2^{s_2}\cdots t_n^{s_n}\beta) = t_1^{r_1+s_{\alpha(1)}}t_2^{r_2+s_{\alpha(2)}}\cdots t_n^{r_n+s_{\alpha(n)}}\alpha\beta$$

and

$$(t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\alpha)^{-1} = t_1^{-r_{\alpha^{-1}(1)}}t_2^{-r_{\alpha^{-1}(2)}}\cdots t_n^{-r_{\alpha^{-1}(n)}}\alpha^{-1}.$$

The following description of $\mathfrak{F}_n$ gives its relation to configuration spaces. The essential facts can be found in [B]. Let $F_n\mathbf{C} = \{(z_1, \dots, z_n) \in \mathbf{C}^n | z_i \neq z_j \text{ for } i \neq j\}$ then $F_n\mathbf{C}$ is a $K(P_n, 1)$ space. The symmetric group Σ_n acts freely on $F_n\mathbf{C}$ by $\sigma(z_1, \dots, z_n) = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$. Denote the orbit space by $B_n\mathbf{C}$. Then $\rho\colon F_n\mathbf{C} \to B_n\mathbf{C}$ is a regular cover with covering transformations Σ_n and $\pi_1(B_n\mathbf{C}) = B_n$. Σ_n also acts on the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ by permuting the indices and hence on $T^n \times F_n\mathbf{C}$ by the diagonal action. Let $K_n = (T^n \times F_n\mathbf{C})/\Sigma_n$ be the orbit space. For the projection $p : T^n \times F_n\mathbf{C} \to F_n\mathbf{C}$, the composite $\rho\circ p$ is a constant map on each orbit and hence there is an induced map $q\colon K_n \to B_n\mathbf{C}$.

Proposition. (1) *$q\colon K_n \to B_n\mathbf{C}$ is a locally trivial bundle with fiber T^n.*
(2) *K_n is a $K(\mathfrak{F}_n, 1)$ space.*

Proof. (1) Consider the following commutative diagram:

$$\begin{tikzcd} T^n \times F_n\mathbf{C} \arrow[r, "p"] \arrow[d, "/\Sigma_n"] & F_n\mathbf{C} \arrow[d, "\rho"] \\ K_n \arrow[r, "q"] & B_n\mathbf{C} \end{tikzcd}$$

where the vertical maps are Σ_n-coverings. If an open set $U \subset B_n\mathbf{C}$ is small enough that $\rho^{-1}U$ is homeomorphic to $\Sigma_n \times U$ then $p^{-1}(\rho^{-1}U) \approx \Sigma_n \times U \times T^n$ and so $q^{-1}U \approx U \times T^n$.

(2) Let $\widetilde{F}_n\mathbf{C}$ be the universal cover of the configuration spaces $F_n\mathbf{C}$ and $B_n\mathbf{C}$. The group $\mathfrak{F}_n$ acts on $\widetilde{F}_n\mathbf{C}$ via the projection to B_n. $\mathbf{Z}^n$ acts on $\mathbf{R}^n$ by

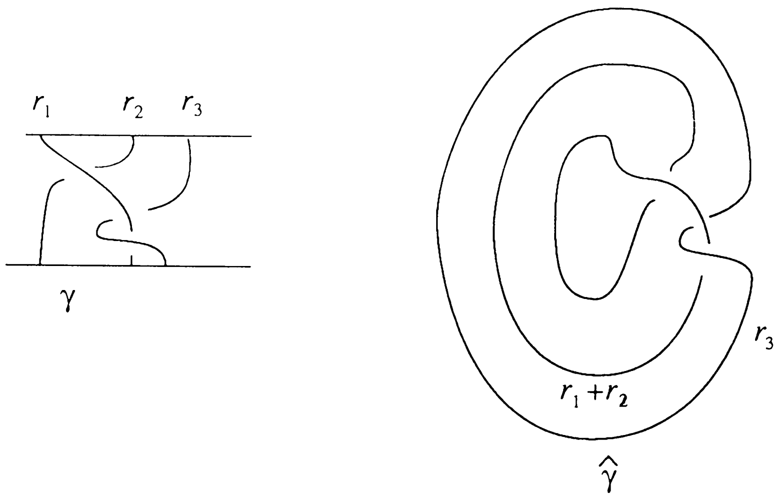


FIGURE 3

translations and so $\mathfrak{F}_n$ acts on $\mathbf{R}^n$ via the semidirect group structure. Thus $\mathfrak{F}_n$ acts on $\mathbf{R}^n \times \widetilde{F}_n\mathbf{C}$ as follows. For $\sigma \in B_n$, $(r_1, \dots, r_n) \in \mathbf{R}^n$ and $c \in \widetilde{F}_n\mathbf{C}$,

$$t_1^{x_1} \cdots t_n^{x_n} \sigma((r_1, \dots, r_n), c) = ((r_{\sigma(1)} + x_1, \dots, r_{\sigma(n)} + x_n), \sigma(c)).$$

This action is properly discontinuous since the actions of B_n on $\widetilde{F}_n\mathbf{C}$ and $\mathbf{Z}^n$ on $\mathbf{R}^n$ both are properly discontinuous. Therefore $\mathbf{R}^n \times \widetilde{F}_n\mathbf{C}$ is the universal covering space of K_n. Since $\widetilde{F}_n\mathbf{C}$ is contractible, K_n is the $K(\mathfrak{F}_n, 1)$ space.

It is interesting to note that constructions similar to the K_n arise in homotopy theory. Compare K_n to a construction by J. P. May with $X = S^1$ and $X = S^1 \cup \text{base point}$ (see the survey [C]).

Relationship to framed links. Given a framed braid γ one can form a framed oriented link $\hat{\gamma}$ by closing the braid. The framing of a link component is obtained by adding together the framings of all the strands that form that link component. (See Figure 3.)

Conversely given an oriented framed link one can construct a framed braid whose closure is the given oriented framed link by applying a theorem of Alexander [B]. The framings can be arbitrarily assigned so that the framed braid closes to the original framed link. There is obviously an ambiguity in assigning framings to strands. The following lemma will take care of this problem. By an isotopy between oriented framed links we mean a link isotopy that preserves framings as well as orientation.

Lemma 1. *$\hat{\alpha}$, $\hat{\beta}$ are isotopic oriented framed links if and only if framed braids α, β are equivalent via the relation generated by*

(1) *conjugation in $\mathfrak{F}_n$;*
(2) *Markov move: for $\gamma \in \mathfrak{F}_n$, $\gamma\sigma_n \sim \gamma \sim \gamma\sigma_n^{-1}$.*

Proof. This is a theorem of Markov [B] if one forgets the framings. In fact the proof of Markov's theorem shows that the components that correspond by a link isotopy can also be made to correspond when the braids are closed after conjugations and Markov moves.

Observe that moves (1) and (2) cannot change the framing of each link component after the framed braids are closed.

On the other hand, if two different framings of a braid σ give the same oriented framed link then one can fix one framing of σ to get the other via a conjugation by an element of $\mathbf{Z}^n$. Let $\pi(\sigma) = \tau_1\tau_2\cdots\tau_k$ be the decomposition of $\pi(\sigma)$ into disjoint cycles. Two different framings $t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}$ and $t_1^{s_1}t_2^{s_2}\cdots t_n^{s_n}$ give the same framed link if and only if $\sum_{i\in\tau_j} r_i = \sum_{i\in\tau_j} s_i$ for $j = 1, \ldots, k$. After conjugating $t_1^{r_1}t_2^{r_2}\cdots t_n^{r_n}\sigma$ by an element $t_1^{x_1}t_2^{x_2}\cdots t_n^{x_n}$, the framing becomes $t_1^{r_1+x_1-x_{\sigma(1)}}t_2^{r_2+x_2-x_{\sigma(2)}}\cdots t_n^{r_n+x_n-x_{\sigma(n)}}$. Thus one needs to solve the system of equations:

$$x_i - x_{\sigma(i)} = s_i - r_i \quad \text{for } i = 1, \ldots, n$$

or

$$x_i - x_{\tau_j(i)} = s_i - r_i \quad \text{for } i \in \tau_j,\ j = 1, \ldots, k$$

for $x_1, x_2, \ldots, x_n$. Suppose τ_j is an m_j cycle so $\tau_j^{m_j} = 1$. Choose $p_j \in \tau_j$ for each cycle. Each i for $i = 1, \ldots, n$ then can be uniquely written $i = \tau_j^q(p_j)$ with $1 \le q \le m_j$. Furthermore $x_{\tau_j^q(p_j)} = \sum_{i=q}^{m_j} s_{\tau_j^i(p_j)} - t_{\tau_j^i(p_j)}$ is a solution to the system of equations. Note that $x_{\tau_j(p_j)} = 0$ since $\sum_{i\in\tau_j} s_i - r_i = 0$. $\square$

Relationship to 3-manifolds. Every orientable closed 3-manifold is obtained by a framed surgery on a link in S^3 [L]. Therefore by ignoring the orientations of the closures of framed braids we have

Proposition. *Every orientable closed* 3-*manifold can be described by a framed braid.*

A theorem of Kirby [K] tells when two different framed links determine homeomorphic 3-manifolds. Since an orientation on a link is crucial to obtain a braid representation, one has to modify the theorem of Kirby to accomodate this extra structure. At the same time the moves of Kirby will be simplified so that they can be more easily described in a braid setting.

Lemma 2. *Two oriented framed links L_1 and L_2 represent homeomorphic* 3-*manifolds if and only if one can obtain L_2 from L_1 by a sequence of the following equivalences*:

(1) *isotopy* (*or combinatorial equivalence*),
(2) *adjoin or delete a split unknotted component with $+1$ or -1 framing,*
(3) *an oriented handle slide over an unknotted component with framing $+1$ and a change in framing to the old framing $+1+2$* (*linking number with the unknotted component*) (see Figure 4),
(4) *reverse the orientation of an unknotted component with framing $+1$.*

Proof. The orientation in (2) is not relevant since one can reverse the orientation of a split unknotted component with an isotopy.

Two unoriented framed links represent homeomorphic 3-manifolds if and only if they are related by isotopies and Fenn-Rourke moves [FR]. According to a simplification by Turaev (see [L1]), framed links represent homeomorphic manifolds if and only if they are related by isotopies, Fenn-Rourke moves with $+1$ framing and moves (2) with -1 framing. A Fenn-Rourke move with $+1$

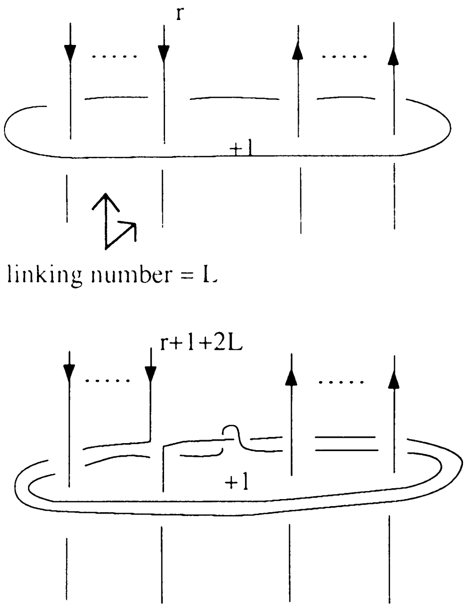


FIGURE 4a

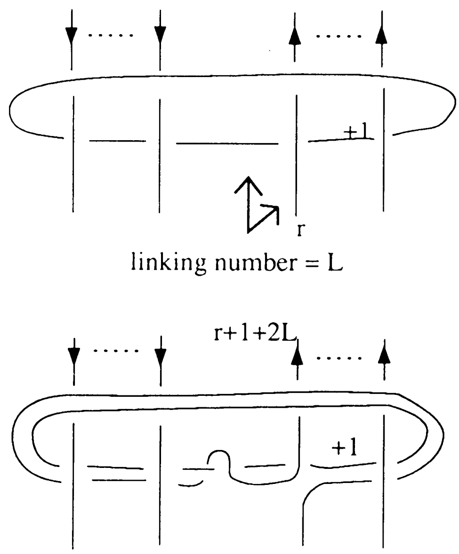


FIGURE 4b

framing can be described as a sequence of handle slides over an unknotted component with framing $+1$ as in Figure 4 and move (2) with $+1$ framing. Each of these handle slides is a move (3) or a move (4) followed by move (3). Conversely move (3) can be generated by Fenn-Rourke moves [FR].

The proof will be completed if one can reverse the orientation of any link component. First unknot the component by changing crossings with the Fenn-Rourke moves and change the framing to $+1$ again with Fenn-Rourke moves. Reverse the orientation of the now unknotted component with move (4). Then undo all the Fenn-Rourke moves to obtain the original link with the orientation of one component reversed. □

In order to obtain the moves of braids corresponding to the moves (3) and (4) of the above lemma, we need the following modification of Alexander's theorem.

Lemma 3. *Suppose that* L *is an oriented (polygonal) link with a link projection such that in some square region* U *the projection is already oriented counter-clockwise with respect to a base point at the center of the square. Then* L *is isotopic (combinatorially equivalent) to a closed braid and the isotopy leaves the portion in* U *unchanged.*

Look ahead to Figures 8 and 9 for an example. It does not matter what the link diagram is in Figure 8 outside of the dotted region U. One can arrange the link as in Figure 9 with the region U intact.

Proof. After small pertubations, one may assume that the projected diagram is transverse and there is no edge that runs in the radial direction from the base point, the point to be the center of the closed braid. Subdivide the edges that run clockwise so that each subdivision does not contain overcrossings and under-crossings simutaneously. Let $[a, b]$ be an edge containing only overcrossings

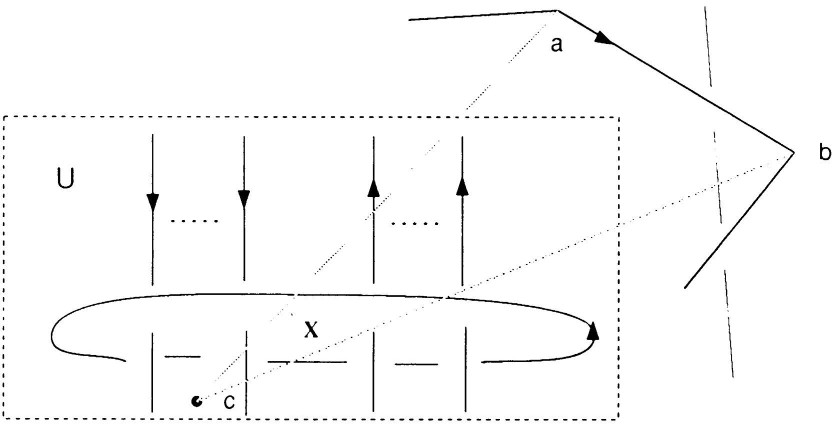


FIGURE 5a

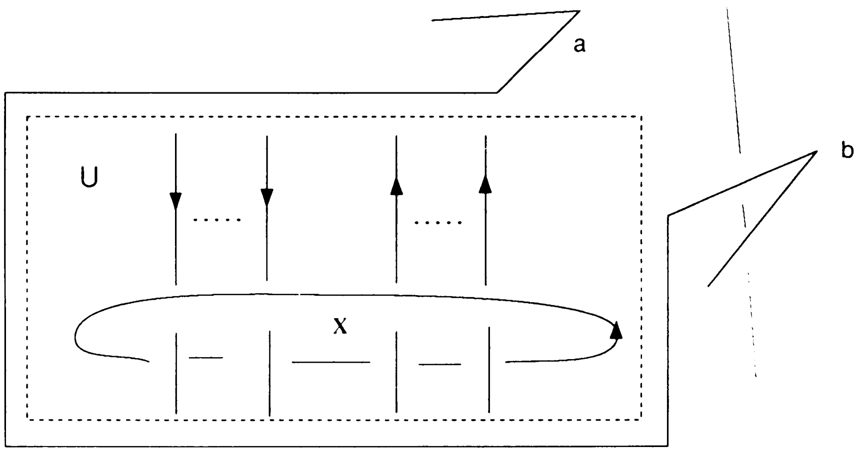


FIGURE 5b

or no crossing. Choose a point c in the square so that the triangle abc contains the base point and edges $[a, c]$ and $[b, c]$ are transverse to the diagram. This is illustrated in Figure 5a with the base point labeled X. Choose a square that is slightly larger than the given square. Replace the edge $[a, b]$ as in Figure 5b so that newly formed edges run counterclockwise over the existing diagram. If the edge $[a, b]$ contains only undercrossings do the similar construction under the existing projection. □

Let $W_{n,j} = \sigma_{n-1}\sigma_{n-2}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}\cdots\sigma_{n-2}\sigma_{n-1}$ be the braid in B_n as in Figure 6.

For $i < n$ let λ_i assign to each permutation in Σ_{n-1} a function from $\{1, 2, \ldots, n-1\}$ to $\{0, 1, \ldots, n-i\}$ as follows. Suppose $\tau \in \Sigma_{n-1}$ decomposes into a product of disjoint cycles $\tau_1\tau_2\cdots\tau_k$. If p occurs in some $\tau_j = (j_1, j_2, \ldots, j_m)$ then $\lambda_i(\tau)(p)$ is defined to be the number of elements in the intersection $\{j_1, j_2, \ldots, j_m\} \cap \{i, i+1, \ldots, n-1\}$.

The following is our main theorem. It is a braid version of Kirby's theorem in [K].

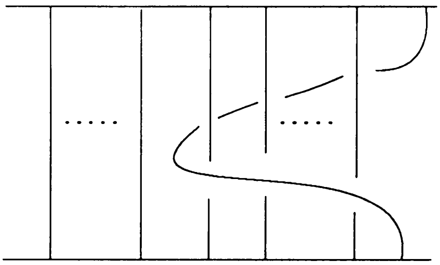


FIGURE 6

Theorem. *Two framed braids represent homeomorphic* 3*-manifolds if and only if they are related by the equivalence relation generated by the following moves*:

(1) *Markov move*: *for* $\beta \in \mathfrak{F}_n$, $\beta\sigma_n \sim \beta \sim \beta\sigma_n^{-1}$;
(2) *blow up*: *for* $\beta \in \mathfrak{F}_n$, $t_{n+1}\beta \sim \beta \sim t_{n+1}^{-1}\beta$;
(3) *handle slide*: *for* $\alpha, \beta \in \mathfrak{F}_{n-1}$,

$$t_n W_{n,j}\alpha W_{n,i}^{-1}\beta \sim t_{n+1}t_n^{2\lambda+1}(W_{n,j}\sigma_n W_{n,j}\sigma_n^{-1})\alpha(W_{n,i}^{-1}\sigma_n^{-1}W_{n,i}\sigma_n)\beta\sigma_n^{-2}\sigma_{n-1}^{-1}$$

where $\lambda = \lambda_i(\pi(\beta\alpha))(\alpha(n-1)) - \lambda_j(\pi(\alpha\beta))(n-1)$;
(4) *orientation reversing*: *for* $\alpha, \beta \in \mathfrak{F}_{n-1}$, $t_n W_{n,j}\alpha W_{n,i}^{-1}\beta \sim t_n W_{n,j}^{-1}\alpha W_{n,i}\beta$;
(5) *conjugation by framed braids.*

Figure 7 demonstrates moves (2), (3), and (4). Note that λ is the linking number between the appropriate components according to Kirby's formula.

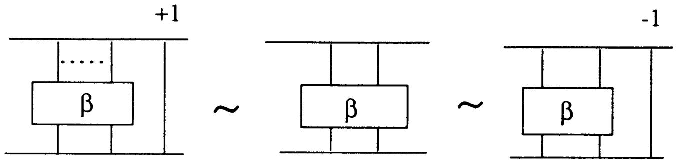
Proof. By Lemma 1, every isotopy of oriented framed links can be realized via moves (1) and (5). We will show that the moves (2), (3), and (4) between framed braids respectively correspond to the moves (2), (3), and (4) of Lemma 2 between oriented framed links. Then this theorem follows from Lemma 2.

The split unknotted component of framing ± 1 can be represented as the last strand of the framed braid so move (2) corresponds to move (2) in Lemma 2.

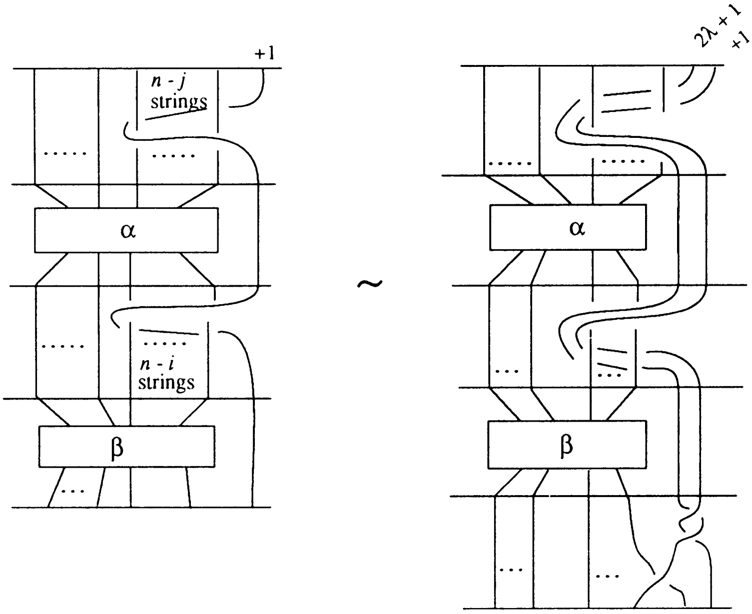
For moves (3) and (4) in Lemma 2, consider a square containing the unknotted component as in Figure 8. After applying Lemma 3 and combing the link outside the square, one can arrange the link as in Figure 9.
Then Figure 10 depicts the oriented handle slide over the unknotted component of framing $+1$ as in Figure 4(b). In each of Figures 9 and 10 there is a dashed radial line. Cut along these dashed lines to obtain an opened braid from the closed braid.

The oriented handle slide in Figure 4(a) can be obtained by rotating it by 180 degrees then reversing the orientation of the unknotted component. This gives Figure 4(b). Figure 11 depicts reversing the orientation of the unknotted component of framing $+1$. Cut along the dashed lines. □

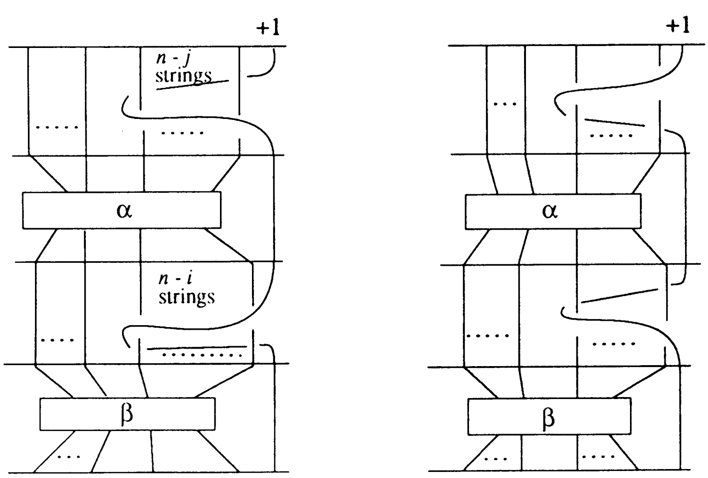
One can approach the study of 3-manifolds by studying representations of $\mathfrak{F}_n$. The particular choice of framed braid that represents a 3-manifold depends on its conjugacy class by move (1). Therefore any 3-manifold invariant is a class function. We believe the relations in Theorem 5 are reasonably simple enough so that one can do algebraic manupulations to understand some invariants of 3-manifolds. For example one may use a representation of the framed braid



Move 2



Move 3



Move 4

FIGURE 7

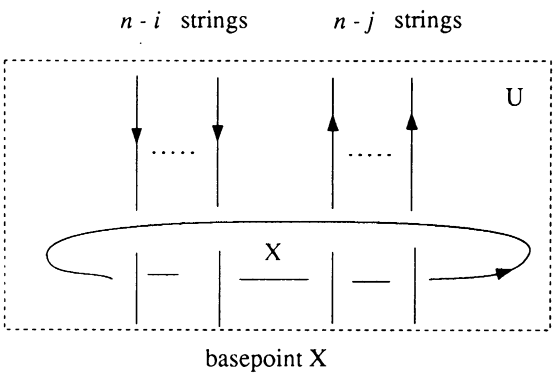


FIGURE 8

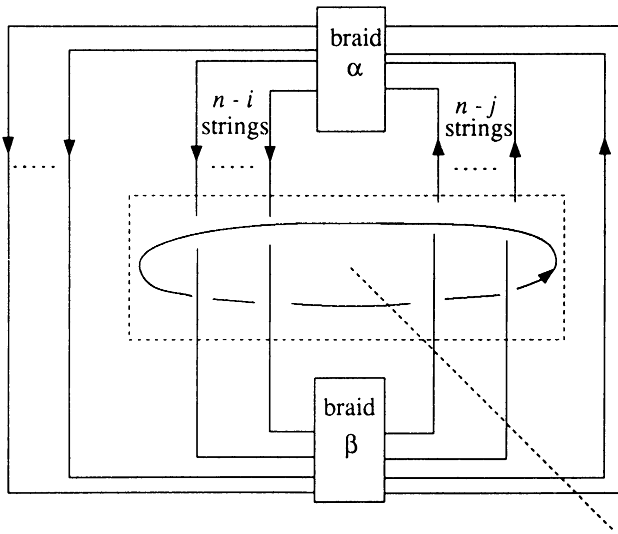


FIGURE 9

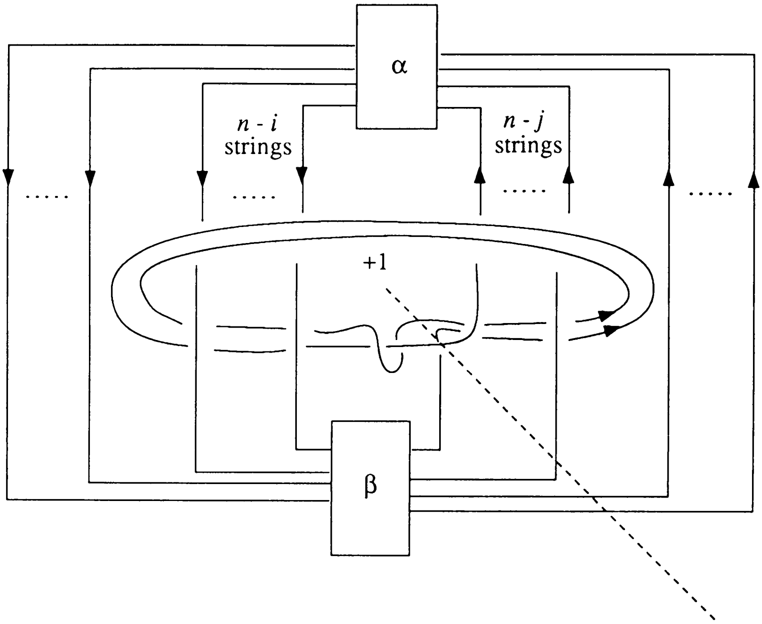


FIGURE 10

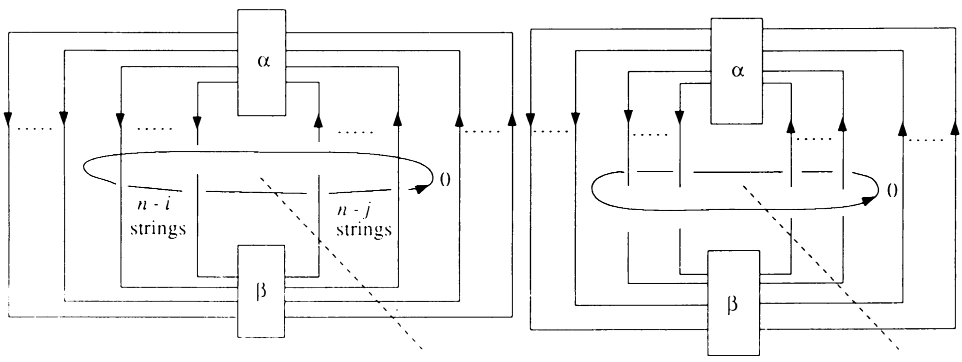


FIGURE 11

group in a deformed group algebra of the signed permutation group and concoct a trace that behaves well under the other moves in the main theorem. The braid group representation in the Hecke algebra used in [J] tends to nullify framings.

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