THE FREE ENTROPY DIMENSION
OF HYPERFINITE VON NEUMANN ALGEBRAS

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For my parents

ABSTRACT. Suppose M is a hyperfinite von Neumann algebra with a normal, tracial state φ and $\{a_1, \ldots, a_n\}$ is a set of selfadjoint generators for M. We calculate $\delta_0(a_1, \ldots, a_n)$, the modified free entropy dimension of $\{a_1, \ldots, a_n\}$. Moreover, we show that $\delta_0(a_1, \ldots, a_n)$ depends only on M and φ. Consequently, $\delta_0(a_1, \ldots, a_n)$ is independent of the choice of generators for M. In the course of the argument we show that if $\{b_1, \ldots, b_n\}$ is a set of selfadjoint generators for a von Neumann algebra $\mathcal{R}$ with a normal, tracial state and $\{b_1, \ldots, b_n\}$ has finite-dimensional approximants, then $\delta_0(N) \leq \delta_0(b_1, \ldots, b_n)$ for any hyperfinite von Neumann subalgebra N of $\mathcal{R}$. Combined with a result by Voiculescu, this implies that if $\mathcal{R}$ has a regular diffuse hyperfinite von Neumann subalgebra, then $\delta_0(b_1, \ldots, b_n) = 1$.

1. INTRODUCTION

Suppose G is a group. Consider the Hilbert space $L^2(G)$ where G is endowed with counting measure and for each $g \in G$ write u_g for the unitary operator on $L^2(G)$ defined by $(u_g(f))(a) = f(g^{-1}a)$. Define the group von Neumann algebra $L(G)$ to be the von Neumann algebra generated by $\{u_g : g \in G\}$. It is not too hard to show that $L(G)$ is a factor (i.e, a von Neumann algebra such that if $x \in L(G)$ commutes with every other element in $L(G)$, then x is a scalar multiple of the identity function) iff every nontrivial conjugacy class of G is infinite. By definition the free group factor on m generators is $L(F_m)$ where F_m is the free group on m generators.

Almost two decades ago Dan Voiculescu began to develop a noncommutative probability theory modeling the free group factors. The theory takes the notions of classical probability and transforms them into ones suited for noncommutative analysis. Random variables become elements in von Neumann algebras, expectations turn into normal, tracial states, and in this particular probability theory, independence always immediately follows the word "free". To clarify the last statement suppose M is a von Neumann algebra with a normal, tracial state φ and $\langle A_j \rangle_{j \in J}$ is a family of unital *-subalgebras of M. $\langle A_j \rangle_{j \in J}$ is a freely independent family provided that for any $j_1, \ldots, j_p \in J$ with $j_1 \neq j_2, \ldots, j_{p-1} \neq j_p$, and $a_i \in A_{j_i}$,

$$\varphi(a_1) = \cdots = \varphi(a_p) = 0 \Rightarrow \varphi(a_1 \cdots a_p) = 0.$$

A family of subsets of M is freely independent if the corresponding family of unital $*$-subalgebras that the subsets generate is freely independent. The definition generalizes the situation in $L(F_m)$. $L(F_m)$ has a unique normal, faithful, tracial state given by $\varphi(x) = \langle x\xi, \xi \rangle$ where ξ is the characteristic function of the identity of F_m. If the generators for F_m are $g_1, \ldots, g_m$, then one easily checks that $\{u_{g_1}\}'', \ldots, \{u_{g_m}\}''$ are freely independent. The parallels between classical and free probability go far and the interested reader can consult [8] for a general introduction.

On the operator algebra side, free probability has answered some open problems in operator algebras. Developing the ideas of free entropy and free entropy dimension, Voiculescu shows in [10] that the free group factors possess no Cartan subalgebras (the first known kind with separable predual). Ge shows in [3] that the free group factors cannot be decomposed into a tensor product of two infinite-dimensional factors (again, the first known kind with separable predual) and similarly in [6] Stefan shows that the free group factors are not the 2-norm closure of the linear span of a product of abelian $*$-subalgebras.

In this paper we take a look at what free entropy dimension has in store for the most tractable kind of von Neumann algebras: those that are hyperfinite and have a tracial state. However, free entropy dimension being the nontrivial machine that it is, we review its definition and basic properties before stating our results.

1.1. **Definitions and Properties.** We recall the concepts of free entropy and modified free entropy dimension introduced in [9]. For $k, n \in \mathbb{N}$ write $M_k^{sa}(\mathbb{C})$ for the set of $k \times k$ selfadjoint matrices with complex entries and $(M_k^{sa}(\mathbb{C}))^n$ for the set of n-tuples of elements in $M_k^{sa}(\mathbb{C})$. Suppose $a_1, \ldots, a_n \in M$ are selfadjoint. Given $R > 0, m, k \in \mathbb{N}$, and $\gamma > 0$, define $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ to be the set of $(x_1, \ldots, x_n) \in (M_k^{sa}(\mathbb{C}))^n$ such that for each j, $\|x_j\| \leq R$ and for any $1 \leq p \leq m$ and $1 \leq j_1, \ldots, j_p \leq n$,

$$|tr_k(x_{j_1} \cdots x_{j_p}) - \varphi(a_{j_1} \cdots a_{j_p})| < \gamma.$$

Here tr_k denotes the tracial state on $M_k(\mathbb{C})$, the $k \times k$ matrices over $\mathbb{C}$. If $b_1, \ldots, b_l \in M$, then $\Gamma_R(a_1, \ldots, a_n : b_1, \ldots, b_l; m, k, \gamma)$ denotes the set of all $(x_1, \ldots, x_n) \in (M_k^{sa}(\mathbb{C}))^n$ such that there exists a $(y_1, \ldots, y_l) \in (M_k^{sa}(\mathbb{C}))^l$ satisfying

$$(x_1, \ldots, x_n, y_1, \ldots, y_l) \in \Gamma_R(a_1, \ldots, a_n, b_1, \ldots, b_l; m, k, \gamma).$$

For any $d \in \mathbb{N}$ denote by vol the Lebesgue measure on $(M_k^{sa}(\mathbb{C}))^d$ (or a subspace thereof) with respect to the unnormalized Hilbert-Schmidt norm $\|(z_1, \ldots, z_d)\|_2 = (\sum_{j=1}^d Tr(z_j^2))^{\frac{1}{2}}$ where Tr is the unnormalized trace. One successively defines for any $R, \gamma > 0$ and $m, k \in \mathbb{N}$,

$$\chi_R(a_1, \ldots, a_n : b_1, \ldots, b_l; m, k, \gamma)$$
$$= k^{-2} \cdot \log(\text{vol}(\Gamma_R(a_1, \ldots, a_n : b_1, \ldots, b_l; m, k, \gamma))) + \frac{n}{2} \log k,$$

$$\chi_R(a_1, \ldots, a_n : b_1, \ldots, b_l; m, \gamma) = \limsup_{k \to \infty} \chi_R(a_1, \ldots, z_n : b_1, \ldots, b_l; m, k, \gamma),$$

$$\chi_R(a_1, \ldots, a_n : b_1, \ldots, b_l) = \inf\{\chi_R(a_1, \ldots, a_n : b_1, \ldots, b_l; m, \gamma) : m \in \mathbb{N}, \gamma > 0\},$$

$$\chi(a_1, \ldots, a_n : b_1, \ldots, b_l) = \sup_{R > 0} \chi_R(a_1, \ldots, a_n : b_1, \ldots, b_l).$$

$\chi(a_1, \ldots, a_n : b_1, \ldots, b_l)$ is called the free entropy of $a_1, \ldots, a_n$ in the presence of $b_1, \ldots, b_l$. Replacing the microstate spaces above with $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ yields $\chi(a_1, \ldots, a_n)$, which is simply called the free entropy of $a_1, \ldots, a_n$.

Now suppose $\{s_1, \ldots, s_n\}$ is a set of freely independent semicircular elements in M (by this we mean that $\langle\{s_i\}\rangle_{i=1}^n$ is a family of freely independent sets such that for each $1 \leq j \leq n$, s_j is selfadjoint and for any $d \in \mathbb{N}$, $\varphi(s_j^d) = \frac{2}{\pi} \int_{-1}^1 t^d \sqrt{1 - t^2} dt$) such that the von Neumann algebra they generate is freely independent with respect to the strongly closed algebra generated by $\{a_1, \ldots, a_n\}$. Define the modified free entropy dimension of $\{a_1, \ldots, a_n\}$ by

$$\delta_0(a_1, \ldots, a_n) = n + \limsup_{\epsilon \to 0} \frac{\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)}{|\log \epsilon|}.$$

One can view the modified free entropy dimension as a noncommutative analogue of the Minkowski dimension. Given $S \subset \mathbb{R}^d$ the Minkowski dimension of S is $d + \limsup_{\epsilon \to 0} \frac{\mu(N_\epsilon(S))}{|\log \epsilon|}$ where μ is Lebesgue measure on $\mathbb{R}^d$ and $N_\epsilon(S)$ is the ϵ-neighborhood of S (technically this is the upper Minkowski dimension of S). The Minkowski dimension defined for S turns out to be the same as the metric entropy quantity $\limsup_{\epsilon \to 0} \frac{\log P_\epsilon(S)}{|\log \epsilon|}$ where $P_\epsilon(S)$ is the maximum number of points in an ϵ separated subset of S (the ϵ packing number of S). We will have more to say about the connections between Minkowski/metric entropy and free entropy dimension.

Here are a few basic properties of δ_0, all of which are proven in [10]:

- For $1 \leq j \leq n$, $\delta_0(a_1, \ldots, a_n) \leq \delta_0(a_1, \ldots, a_j) + \delta_0(a_{j+1}, \ldots, a_n)$.
- For any $a = a^* \in M$, $\delta_0(a) = 1 - \sum_{t \in sp(a)} (\lambda(\{t\}))^2$ where λ is the Borel measure on $sp(a)$ induced by φ.
- If $\chi(a_1, \ldots, a_n) > -\infty$, then $\delta_0(a_1, \ldots, a_n) = n$.
- If $a_1, \ldots, a_n$ are freely independent, then $\delta_0(a_1, \ldots, a_n) = \delta_0(a_1) + \cdots + \delta_0(a_n)$.

Unfortunately, it is not known whether δ_0 is an invariant of von Neumann algebras with tracial states, i.e., if $\{b_1, \ldots, b_p\}$ and $\{a_1, \ldots, a_n\}$ are two sets of selfadjoint generators for M (this means that each set generates a strongly closed algebra equal to M and each element of the set is selfadjoint), then does it follow that $\delta_0(a_1, \ldots, a_n) = \delta_0(b_1, \ldots, b_p)$? An affirmative answer to this question would show that for $m \neq n$, $L(F_m)$ is not $*$-isomorphic to $L(F_n)$, for it is well known that for any $m \in \mathbb{N}$ there exist m semicircular generators $s_1, \ldots, s_m$ for $L(F_m)$ that satisfy $\delta_0(s_1, \ldots, s_m) = m$.

1.2. **Results.** We show that for hyperfinite von Neumann algebras with specified tracial state, δ_0 *is* an invariant. More specifically, suppose M is a hyperfinite von Neumann algebra with normal, tracial state φ. By decomposing M over its center it follows that

$$M \simeq M_0 \oplus (\bigoplus_{i=1}^s M_{k_i}(\mathbb{C})) \oplus M_\infty,$$

$$\varphi \simeq \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^s \alpha_i tr_{k_i}) \oplus 0$$

where $s \in \mathbb{N} \cup \{0\} \cup \{\infty\}, \alpha_i > 0$ for $1 \leq i \leq s$ $(i \in \mathbb{N})$, M_0 is a diffuse von Neumann algebra or $\{0\}$, φ_0 is a faithful, tracial state on M_0 and $\alpha_0 > 0$ if $M_0 \neq \{0\}$, $\varphi_0 = 0$ and $\alpha_0 = 0$ if $M_0 = \{0\}$, and M_∞ is a von Neumann algebra or $\{0\}$. We show that

for any selfadjoint generators $a_1, \ldots, a_n$ for M,

$$\delta_0(a_1, \ldots, a_n) = 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}.$$

Because every such M has a finite set of selfadjoint generators, it makes sense to define $\delta_0(M) = \delta_0(a_1, \ldots, a_n)$ where $\{a_1, \ldots, a_n\}$ is a set of selfadjoint generators for M. It follows that for any $k \in \mathbb{N}$, $\delta_0(M_k(\mathbb{C})) = 1 - \frac{1}{k^2}$ and if M is the hyperfinite II_1-factor, then $\delta_0(M) = 1$. The calculations also show that for hyperfinite M the free entropy dimension number we obtain for M coincides with the "free dimension" number for M which appears in Dykema's work [2].

As a consequence of the arguments leading towards the above result we obtain a "hyperfinite monotonicity" property of δ_0 which says the following. Suppose M is an arbitrary von Neumann algebra with specified tracial state and selfadjoint generators $a_1, \ldots, a_n$. Assume moreover that $\{a_1, \ldots, a_n\}$ has finite-dimensional approximants, i.e., for any $m \in \mathbb{N}$, $\epsilon > 0$, and $L > \max\{\|a_i\|\}_{1 \leq i \leq n}$ there exists an $N \in \mathbb{N}$ such that for all $k \geq N$, $\Gamma_L(a_1, \ldots, a_n; m, k, \epsilon) \neq \emptyset$. If N is a hyperfinite von Neumann subalgebra of M, then

$$\delta_0(N) \leq \delta_0(a_1, \ldots, a_n).$$

Hyperfinite monotonicity of δ_0 paired with a result by Voiculescu ([10]) show that if M has a regular, diffuse, hyperfinite von Neumann subalgebra, then $\delta_0(a_1, \ldots, a_n) = 1$ (see Remark 4.8).

The gist of the argument is simple: Essentially $\delta_0(a_1, \ldots, a_n)$ is the normalized metric entropy of the unitary orbit of a well-approximating microstate for $\{a_1, \ldots, a_n\}$. Suppose M is hyperfinite with specified tracial state and selfadjoint generators $\{a_1, \ldots, a_n\}$. $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$ is more or less the normalized logarithm of the volume of the ϵ-neighborhood around the microstates of $\{a_1, \ldots, a_n\}$. M being hyperfinite any two such microstates are approximately unitarily equivalent. So $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$ is a limiting process calculated from k^{-2} times the logarithm of the volume of the ϵ-neighborhood of the unitary orbit of a single microstate for $\{a_1, \ldots, a_n\}$. Dividing this quantity by $|\log \epsilon|$ and adding n is close to k^{-2} multiplied by the Minkowski dimension of the unitary orbit of the microstate or, equivalently, the metric entropy of the set. Very roughly then, $\delta_0(a_1, \ldots, a_n)$ is the normalized metric entropy of the unitary orbit of a single well-approximating microstate for $\{a_1, \ldots, a_n\}$.

The calculations require more delicacy than we have let on, for we must first fix an ϵ and find the volume bounds/packing number bounds with respect to ϵ not merely over one microstate in one dimension but over one microstate for each dimension (because the first process in free entropy takes a limit as the dimensions go to infinity). Weak inequalities reduce this to either the investigation of uniform bounds on the packing numbers of homogeneous spaces obtained from U_k, the $k \times k$ unitaries, or to $\delta_0(a)$ where a is a selfadjoint element. In the former case we make crucial use of the results of Szarek ([7]) and Raymond ([5]). The latter situation dealing with $\delta_0(a)$ has already been discussed.

We break up the paper into calculating upper and lower bounds for $\delta_0(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are arbitrary selfadjoint generators for hyperfinite M with specified tracial state. Section 2 is a short list of notation and assumptions we make throughout the paper. Section 3 obtains the upper bound for general M. Section

4 shows that if $\{a_1, \ldots, a_n\}$ is a set of selfadjoint generators of a diffuse (arbitrary, i.e., not necessarily hyperfinite) von Neumann algebra with a tracial state and $\{a_1, \ldots, a_n\}$ has finite-dimensional approximants, then $\delta_0(a_1, \ldots, a_n) \geq 1$. In particular, this yields the desired lower bound for diffuse M. Section 5 obtains the lower bound when M is finite dimensional and Section 6 combines the results of Sections 4 and 5 to arrive at the general lower bound. Section 7 gleans immediate corollaries (including hyperfinite monotonicity of δ_0) and comments on the relation of $\delta_0(M)$ to Dykema's free dimension [2]. Section 8 is an addendum where we prove some consequences of Szarek's metric entropy bounds of homogeneous spaces.

2. Definitions and Notation

Throughout this paper we maintain the notation in the introduction. Also, we assume throughout that M is a von Neumann algebra (not necessarily hyperfinite) with separable predual, a unit I, and a normal, tracial state φ. $\{s_i : i \in \mathbb{N}\}$ is always a semicircular family free with respect to M. $\{a_1, \ldots, a_n\} \subset M$ is a set of selfadjoint generators for M with finite-dimensional approximants. $R = \max\{\|a_i\|\}_{1 \leq i \leq n}$. Lastly, $|\cdot|_2$ is the norm on $M_k(\mathbb{C})$ or the seminorm on M given by $|x|_2 = (tr_k(x^*x))^{\frac{1}{2}}$ or $|x|_2 = (\varphi(x^*x))^{\frac{1}{2}}$, respectively.

3. Upper Bound

Throughout the section assume that M is hyperfinite and that N is a finite-dimensional $*$-subalgebra of M containing I. Also assume that N has selfadjoint generators $\{b_1, \ldots, b_n\}$ such that each b_i has operator norm no larger than R. For any $k \in \mathbb{N}$ and $\epsilon > 0$, $T(b_1, \ldots, b_n; k, \epsilon)$ denotes

$$\{(x_1, \ldots, x_n) \in (M_k^{sa}(\mathbb{C}))^n : \text{ there exists a } *\text{-homomorphism } \sigma : N \to M_k(\mathbb{C})$$
$$\text{such that for all } 1 \leq i \leq n, \quad |\sigma(b_i) - x_i|_2 \leq \epsilon \text{ and } \|tr_k \circ \sigma - \varphi|_N\| < \epsilon^2\}.$$

We show that $\delta_0(a_1, \ldots, a_n) \leq 1 - \sum_{i=1}^s \frac{\alpha_i^2}{k_i^2}$ where the α_i and k_i are as in the canonical decomposition of M discussed on page 3 of the introduction. The argument proceeds in several easy steps. First, $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$ is dominated by a number calculated more or less from $\text{vol}(T(b_1, \ldots, b_n; k, \epsilon))$. Secondly, $T(b_1, \ldots, b_n; k, \epsilon)$ is contained in the neighborhood of a restricted unitary orbit of any single element in $T(b_1, \ldots, b_n : k, \epsilon)$. Szarek's packing number estimates provide appropriate upper bounds for the volume of the neighborhoods of such orbits. Finally, by approximating M by fine enough finite-dimensional $*$-subalgebras N of M, standard approximation arguments yield the promised upper bound.

The first lemma presented below is standard and we omit the proof. It amounts to saying that matricial microstates for selfadjoint generators of a finite-dimensional von Neumann algebra correspond to approximate representations.

Lemma 3.1. *For each $\epsilon > 0$ there exist an $m \in \mathbb{N}$ and $\gamma > 0$ such that for all $k \in \mathbb{N}$,*

$$\Gamma_{R+1}(b_1, \ldots, b_n : m, k, \gamma) \subset T(b_1, \ldots, b_n; k, \epsilon).$$

Lemma 3.1 has a trivial consequence:

Lemma 3.2. *Suppose $\epsilon_0 > 0$ and $\max\{|a_i - b_i|_2 : 1 \leq i \leq n\} < \epsilon_0$. For any $1 > \epsilon > 0$, $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$ is dominated by*

$$\limsup_{k \to \infty} \left[k^{-2} \cdot \log(\mathrm{vol}(T(b_1, \ldots, b_n; k, 3\epsilon + \epsilon_0))) + \frac{n}{2} \cdot \log k \right].$$

Proof. By Lemma 3.1 for a given $\epsilon \in (0, 1)$ there exist $2 \leq m \in \mathbb{N}$ and $\gamma > 0$ such that for any $k \in \mathbb{N}$, $\Gamma_{R+1}(b_1, \ldots, b_n; m, k, \gamma) \subset T(b_1, \ldots, b_n; k, \epsilon)$. For $\gamma' > 0$ if

$$(x_1, \ldots, x_n) \in \Gamma_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : b_1, \ldots, b_n, s_1, \ldots, s_n; m, k, \gamma'),$$

then by definition there exists $(y_1, \ldots, y_n, z_1, \ldots, z_n) \in (M_k^{sa}(\mathbb{C}))^{2n}$ satisfying

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)$$
$$\in \Gamma_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n, b_1, \ldots, b_n, s_1, \ldots, s_n; m, k, \gamma').$$

By choosing $\gamma' < \gamma$ sufficiently small one can force $|x_i - y_i|_2 < 2\epsilon + \epsilon_0$ for $1 \leq i \leq n$. Since $\gamma' < \gamma$, $(y_1, \ldots, y_n) \in \Gamma_{R+1}(b_1, \ldots, b_n; m, k, \gamma) \subset T(b_1, \ldots, b_n; k, \epsilon)$. Consequently, $(x_1, \ldots, x_n) \in T(b_1, \ldots, b_n; k, 3\epsilon + \epsilon_0)$. We have just shown that

$$\Gamma_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : b_1, \ldots, b_n, s_1, \ldots, s_n; m, k, \gamma') \subset T(b_1, \ldots, b_n; k, 3\epsilon + \epsilon_0).$$

Basic properties of free entropy imply that $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$ equals

$$\chi_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : b_1, \ldots, b_n, s_1, \ldots, s_n)$$
$$\leq \chi_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : b_1, \ldots, b_n, s_1, \ldots, s_n; m, \gamma').$$

By the preceding inclusion the dominating term is less than or equal to

$$\limsup_{k \to \infty} \left[k^{-2} \cdot \log(\mathrm{vol}(T(b_1, \ldots, b_n; k, 3\epsilon + \epsilon_0))) + \frac{n}{2} \cdot \log k \right].$$

$\square$

Now for the claim which implies that $T(b_1, \ldots, b_n; k, \epsilon)$ is contained in the $2\epsilon(1 + \sqrt{2}R)$ - neighborhood of a restricted unitary orbit of any single element of $T(b_1, \ldots, b_n; k, \epsilon)$.

Because N is finite dimensional and φ is tracial, assume from now on that $N \simeq \bigoplus_{j=1}^{p} M_{n_j}(\mathbb{C})$ and $\varphi|_N \simeq \bigoplus_{j=1}^{p} \alpha_j tr_{n_j}$.

Lemma 3.3. *For any $k \in \mathbb{N}$ and $\epsilon > 0$, if $\sigma_1, \sigma_2 : N \to M_k(\mathbb{C})$ are $*$-homomorphisms such that $\|tr_k \circ \sigma_1 - tr_k \circ \sigma_2\| \leq \epsilon^2$, then there exists a $u \in U_k$ such that*

$$|u(\sigma_1(x))u^* - \sigma_2(x)|_2 \leq 2\|x\|\epsilon.$$

Proof. Without loss of generality, assume that $N = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_p}(\mathbb{C})$. For any $l_1, \ldots, l_p \in \mathbb{N} \cup \{0\}$ with $\sum_{i=1}^{p} n_i l_i \leq k$ denote by $\pi_{l_1, \ldots, l_p} : N \to M_k(\mathbb{C})$ the $*$-homomorphism

$$\pi_{l_1, \ldots, l_p}(x_1, \ldots, x_p) = \begin{bmatrix} x_1 \otimes I_{l_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & x_p \otimes I_{l_p} & 0 \\ 0 & \cdots & 0 & 0_{p+1} \end{bmatrix}$$

where $x_i \bigotimes I_{l_i}$ is the $n_i l_i \times n_i l_i$ matrix with x_i repeated l_i times on the diagonal and 0_{p+1} is the $(k - \sum_{i=1}^{p} n_i l_i) \times (k - \sum_{i=1}^{p} n_i l_i)$ 0 matrix. There exist $m_1, \ldots, m_p, n_1, \ldots, n_p \in \mathbb{N} \cup \{0\}$ such that $\sigma_1 \sim \pi_{m_1, \ldots, m_p}, \sigma_2 \sim \pi_{n_1, \ldots, n_p}$.

Set $d_i = \min\{m_i, n_i\}$ and observe that if $\pi = \pi_{d_1, \ldots, d_p}$, then $\|tr_k \circ \pi - tr_k \circ \sigma_j\| \leq \epsilon^2$ for $j = 1, 2$. If we can show that for each j there exists a $u_j \in U_k$ satisfying $|u_j(\sigma_j(x))u_j^* - \pi(x)|_2 \leq \|x\|\epsilon$, then we will be done. A moment's thought shows that for each j there exists a $u_j \in U_k$ such that $\mathrm{Ad}\, u_j \circ \sigma_j - \pi$ is a $*$-homomorphism which we will denote by ρ_j. Obviously $\|tr_k \circ \rho_j\| \leq \epsilon^2$ so that for any $x \in N$,

$$|u_j(\sigma_j(x))u_j^* - \pi(x)|_2 = ((tr_k \circ \rho_j)(x^*x))^{1/2} \leq (\epsilon^2 \cdot \|x^*x\|)^{1/2} = \|x\|\epsilon.$$

$\square$

Given $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in T(b_1, \ldots, b_n; k, \epsilon)$ there are representations σ, $\pi : N \to M_k(\mathbb{C})$ such that for $1 \leq i \leq n$, $|\sigma(b_i) - x_i|_2, |\pi(b_i) - y_i|_2 < \epsilon$ and $\|tr_k \circ \sigma - \varphi|_N\|, \|tr_k \circ \pi - \varphi|_N\| < \epsilon^2$. So $\|tr_k \circ \sigma - tr_k \circ \pi\| < 2\epsilon^2$. By Lemma 3.3 there exists a $u \in U_k$ satisfying

$$|u(\sigma(b_i))u^* - \pi(b_i)|_2 \leq 2\|b_i\|(\sqrt{2}\epsilon) \leq 2\sqrt{2}R\epsilon$$

for $1 \leq i \leq n$. Thus $|ux_iu^* - y_i|_2 \leq 2\epsilon + 2\sqrt{2}R\epsilon = 2\epsilon(1 + \sqrt{2}R)$. From now on for $z \in M_k^{sa}(\mathbb{C})$ and $\gamma > 0$ define $B(z, \gamma) = \{x \in M_k^{sa}(\mathbb{C}) : |x - z|_2 < \gamma\}$. We have just proved:

Corollary 3.4. *If* $(x_1, \ldots, x_n) \in T(b_1, \ldots, b_n; k, \epsilon)$, *then* $T(b_1, \ldots, b_n; k, \epsilon)$ *is contained in*

$$\bigcup_{u \in U_k} [B(ux_1u^*, 2\epsilon(1 + \sqrt{2}R)) \times \cdots \times B(ux_nu^*, 2\epsilon(1 + \sqrt{2}R))].$$

We remark here that Lemma 3.1, Lemma 3.3, and Corollary 3.4 also hold in the situation where N does not contain I.

We now draw out a trivial consequence of Szarek's estimates for covering numbers of homogeneous spaces. These results are the heart of the calculation of the upper bound.

For $k \in \mathbb{N}$ suppose $m, k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{N}, \sum_{i=1}^{m} k_i l_i = k$, and $H \subset U_k$ is a proper Lie subgroup of U_k consisting of all matrices of the form

$$\begin{bmatrix} u_1 \bigotimes I_{k_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_m \bigotimes I_{k_m} \end{bmatrix}$$

where $u_i \in U_{l_i}$ and $u_i \bigotimes I_{k_i}$ is the $k_i l_i \times k_i l_i$ matrix obtained by repeating u_i k_i times along the diagonal. Such Lie subgroups H of U_k will be called tractable.

A simple application of Theorem 11 in [7] yields:

Lemma 3.5. *There exist constants* $C, \beta > 0$ *such that for any* $k \in \mathbb{N}$, *any tractable Lie subgroup* H *of* U_k *and* $\epsilon \in (0, \beta)$,

$$N(X, \epsilon) \leq \left(\frac{C}{\epsilon}\right)^d$$

where X *is the manifold* U_k/H *endowed with the quotient metric induced by the operator norm,* d *is the real dimension of* X, *and* $N(X, \epsilon)$ *is the minimum number of balls of radius* ϵ *required to cover* X.

We sequester a rigorous demonstration to the Addendum.

Lemma 3.6. *If $1 > r > 0$, then there exists a $k_0 \in \mathbb{N}$ such that for each $k > k_0$ there is a corresponding $*$-homomorphism $\sigma_k : N \to M_k(\mathbb{C})$ satisfying:*

- $\|tr_k \circ \sigma_k - \varphi|_N\| < r^2$.
- *The unitaries H_k of $\sigma_k(N)'$ form a tractable Lie subgroup of U_k satisfying*

$$
k^2 \left(1 - r - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2} \right) < \dim(U_k/H_k) < k^2 \left(1 + r - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2} \right).
$$

Proof. Suppose that $1 > \varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{p+1}$. Set $k_0 = (n_0 + 1)n_1 \cdots n_p$. Suppose $k > k_0$. Find the unique $n \in \mathbb{N}$ (dependent on k) satisfying

$$
nn_1 \cdots n_p \leq k < (n+1)n_1 \cdots n_p.
$$

Set $d = nn_1 \cdots n_p$ (d dependent on k) and find $m_1, \ldots, m_p \in \mathbb{N} \cup \{0\}$ satisfying $\alpha_i - \varepsilon < \frac{m_i}{n} < \alpha_i + \varepsilon$ for each i and $\sum_{i=1}^{p} \frac{m_i}{n} = 1$ (the m_i depend on k). Set $l_i(k) = \frac{dm_i}{nn_i} \in \mathbb{N} \cup \{0\}$ and $l_{p+1}(k) = k - \sum_{i=1}^{p} l_i(k)n_i$. Assume without loss of generality that $N = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_p}(\mathbb{C})$ and $\varphi|_N = \alpha_1 tr_{n_1} \oplus \cdots \oplus \alpha_p tr_{n_p}$. Define $\sigma_k : N \to M_k(\mathbb{C})$ by

$$
\sigma_k(x_1, \ldots, x_n) = \begin{bmatrix} I_{l_1(k)} \bigotimes x_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & I_{l_p(k)} \bigotimes x_p & 0 \\ 0 & \cdots & 0 & 0_{l_{p+1}(k)} \end{bmatrix}
$$

where $0_{l_{p+1}(k)}$ is the $l_{p+1}(k) \times l_{p+1}(k)$ 0 matrix and $I_{l_i(k)} \bigotimes x_i$ is the $l_i(k)n_i \times l_i(k)n_i$ matrix obtained by taking each entry of x_i, $(x_i)_{st}$, and stretching it out into $(x_i)_{st} \cdot I_{l_i(k)}$ where $I_{l_i(k)}$ is the $l_i(k) \times l_i(k)$ identity matrix. Then

$$
(tr_k \circ \sigma)(x_1, \ldots, x_p) = \frac{1}{k} \cdot \sum_{i=1}^{p} l_i(k) \cdot Tr(x_i) = \sum_{i=1}^{p} \frac{dm_i}{kn} \cdot tr_{n_i}(x_i).
$$

Since $\frac{d}{k} > 1 - \varepsilon$, we have $\alpha_i + \varepsilon \geq \frac{d}{k} \cdot \frac{m_i}{n} > (\alpha_i - \varepsilon)(1 - \varepsilon) > \alpha_i - 2\varepsilon$. It follows that $\|tr_k \circ \sigma_k - \varphi|_N\| < 2p\varepsilon$.

H_k consists of all matrices of the form

$$
\begin{bmatrix} u_1 \bigotimes I_{n_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & u_p \bigotimes I_{n_p} & 0 \\ 0 & \cdots & 0 & u_{p+1} \end{bmatrix}
$$

where $u_i \in U_{l_i(k)}$ for $1 \leq i \leq p+1$ and $u_i \bigotimes I_{n_i}$ is the $l_i(k)n_i \times l_i(k)n_i$ matrix obtained by repeating u_i n_i times along the diagonal. H_k is obviously tractable.

For a lower bound on $\dim(U_k/H_k)$ we have the estimate:

$$
l_{p+1}(k) = k - \sum_{i=1}^{p} \frac{dm_i}{n} = k - d < n_1 \cdots n_p < \frac{k_0}{n_0} < k \cdot \varepsilon
$$

so that

$$\dim H_k = l_{p+1}(k)^2 + \sum_{i=1}^{p} l_i(k)^2 < k^2 \left(\varepsilon + \sum_{i=1}^{p} \frac{(\alpha_i + \varepsilon)^2}{n_i^2} \right) < k^2 \left(3p\varepsilon + \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2} \right).$$

Hence, $\dim(U_k/H_k) = k^2 - \dim H_k$ is bounded from below by $k^2(1 - 3p\varepsilon - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2})$.

For an upper bound on $\dim(U_k/H_k)$ observe that $\dim H_k > \sum_{i=1}^{p} l_i(k)^2$ whence

$$\dim(U_k/H_k) = k^2 - \dim H_k < k^2 - \sum_{i=1}^{p} l_i(k)^2$$
$$< k^2 \left(1 - \sum_{i=1}^{p} \frac{(\alpha_i - 2\varepsilon)^2}{n_i^2} \right)$$
$$< k^2 \left(1 + 4\varepsilon - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2} \right).$$

Set $\varepsilon = \frac{r^2}{4p}$. $\qquad\qquad\qquad\square$

We now make the key calculation on the upper bound of Lemma 3.2.

Lemma 3.7. *For $\min\{\beta, C\} > \epsilon > 0$,*

$$\limsup_{k \to \infty} \left[k^{-2} \cdot \log(\mathrm{vol}(T(b_1, \ldots, b_n; k, \epsilon))) + \frac{n}{2} \cdot \log k \right] \leq \log(\epsilon^{n-\triangle}) + \log D$$

where $\triangle = 1 - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2}$ and $D = \pi^{\frac{n}{2}} (8(R+1))^n C^{\triangle} [(2e)^{\frac{n}{2}}]$.

Proof. Suppose $\min\{\beta, C\} > \epsilon > r > 0$. By Lemma 3.6 there is a $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ there exists a $*$-homomorphism $\sigma_k : N \to M_k(\mathbb{C})$ satisfying $\|tr_k \circ \sigma_k - \varphi|_N\| < r^2$ and the additional condition that if H_k is the unitary group of $\sigma_k(N)'$, then H_k is tractable and

$$\dim(U_k/H_k) < k^2 \left(1 + r - \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2} \right).$$

Set

$$d_k = \dim(U_k/H_k) \quad \text{and} \quad m_r = -r + \sum_{i=1}^{p} \frac{\alpha_i^2}{n_i^2}.$$

There exists a set $\langle u_{k,s} \rangle_{s \in S_k}$ contained in U_k such that for each $u \in U_k$ there exists an $s \in S_k$ and $h \in H_k$ satisfying $\|u - u_{k,s}h\| < \epsilon$ and S_k has cardinality not exceeding $\left(\frac{C}{\epsilon} \right)^{d_k} < \left(\frac{C}{\epsilon} \right)^{k^2(1-m_r)}$.

Set $L = 8(R+1)$. I claim that

$$T(b_1, \ldots, b_n; k, \epsilon) \subset \bigcup_{s \in S_k} [B(u_{k,s}\sigma_k(b_1)u_{k,s}^*, L\epsilon) \times \cdots \times B(u_{k,s}\sigma_k(b_n)u_{k,s}^*, L\epsilon)].$$

Suppose that $(x_1, \ldots, x_n) \in T(b_1, \ldots, b_n; k, \epsilon)$. Clearly $(\sigma_k(b_1), \ldots, \sigma_k(b_n)) \in T(b_1, \ldots, b_n; k, \epsilon)$. By Corollary 3.4 there exists a $u \in U_k$ such that for all $1 \leq$

$i \leq n$, $|x_i - u\sigma_k(b_i)u^*|_2 \leq 2\epsilon(1 + \sqrt{2}R)$. There exists an $s \in S_k$ and $h \in H_k$ such that $\|u - u_{k,s}h\| < \epsilon$. Now

$$
\begin{aligned}
|u\sigma_k(b_i)u^* - u_{k,s}\sigma_k(b_i)u_{k,s}^*|_2 &= |u\sigma_k(b_i)u^* - u_{k,s}h\sigma_k(b_i)h^*u_{k,s}^*|_2 \\
&\leq \|u - u_{k,s}h\| \cdot |\sigma_k(b_i)u^*|_2 \\
&\quad + |u_{k,s}h\sigma_k(b_i)|_2 \cdot \|u^* - h^*u_{k,s}^*\| \\
&\leq 2\epsilon R.
\end{aligned}
$$

Hence for all $1 \leq i \leq n$, $|x_i - u_{k,s}\sigma_k(b_i)u_{k,s}^*|_2 \leq 2\epsilon(1 + \sqrt{2}R) + 2\epsilon R < L\epsilon$.

By the inclusion demonstrated in the preceding paragraph we have that $\log(\mathrm{vol}(T(b_1, \ldots, b_n; k, \epsilon)))$ is dominated by

$$
\begin{aligned}
\log\left(|S_k| \cdot \frac{\pi^{\frac{nk^2}{2}}(L\sqrt{k}\epsilon)^{nk^2}}{(\Gamma(\frac{k^2}{2} + 1))^n}\right) &= \log\left(|S_k| \cdot \frac{(\pi^{\frac{n}{2}}L^n k^{\frac{n}{2}}\epsilon^n)^{k^2}}{(\Gamma(\frac{k^2}{2} + 1))^n}\right) \\
&\leq k^2 \cdot \log\left[\left(\frac{C}{\epsilon}\right)^{1-m_r}(\pi^{\frac{n}{2}}L^n k^{\frac{n}{2}}\epsilon^n)\right] - n \cdot \log\Gamma\left(\frac{k^2}{2} + 1\right) \\
&\leq k^2 \cdot \log(\pi^{\frac{n}{2}}L^n C^{1-m_r}k^{\frac{n}{2}}\epsilon^{n-1+m_r}) - n \cdot \log\left[\left(\frac{k^2}{2e}\right)^{\frac{k^2}{2}}\right] \\
&= k^2 \cdot \log(\pi^{\frac{n}{2}}L^n C^{1-m_r}\epsilon^{n-1+m_r}) - \frac{nk^2}{2} \cdot \log k + k^2\log[(2e)^{\frac{n}{2}}]
\end{aligned}
$$

provided $k \geq k_0$. $\limsup_{k\to\infty}\left[k^{-2}\log(\mathrm{vol}(T(b_1, \ldots, b_n; k, \epsilon))) + \frac{n}{2} \cdot \log k\right]$ is therefore dominated by

$$
\begin{aligned}
\limsup_{k\to\infty}(\log(\pi^{\frac{n}{2}}L^n C^{1-m_r}\epsilon^{n-1+m_r}) &+ \log[(2e)^{\frac{n}{2}}] \\
&= \log(\epsilon^{n-1+m_r}) + \log(\pi^{\frac{n}{2}}L^n C^{1-m_r}[(2e)^{\frac{n}{2}}]).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\log(\epsilon^{n-\triangle}) + \log D &= \log(\epsilon^{n-\triangle}) + \log(\pi^{\frac{n}{2}}L^n C^{\triangle}[(2e)^{\frac{n}{2}}]) \\
&= \lim_{r\to 0}\left[(\log(\epsilon^{n-1+m_r}) + \log(\pi^{\frac{n}{2}}L^n C^{1-m_r}[(2e)^{\frac{n}{2}}]))\right] \\
&\geq \limsup_{k\to\infty}\left[k^{-2} \cdot \log(\mathrm{vol}(T(b_1, \ldots, b_n; k, \epsilon))) + \frac{n}{2} \cdot \log k\right].
\end{aligned}
$$

$\square$

If M is finite dimensional, then Lemma 3.7 yields the desired upper bound for $\delta_0(a_1, \ldots, a_n)$. With just a few more easy observations, Lemma 3.7 allows us to bootstrap the upper bound for $\delta_0(a_1, \ldots, a_m)$ in the general situation.

If B is a finite-dimensional von Neumann algebra with a positive trace ψ and $B \simeq \bigoplus_{i=1}^s M_{q_i}(\mathbb{C})$, $\psi \simeq \bigoplus_{i=1}^s r_i tr_{q_i}$, define $\triangle_\psi(B) = 1 - \sum_{i=1}^s \frac{r_i^2}{q_i^2}$. Clearly $\triangle_\psi(B)$ is well defined.

Lemma 3.8. *If $A \subset B$ is a unital inclusion of finite-dimensional von Neumann algebras, and ψ is a positive trace on B, then $\triangle_\psi(A) \leq \triangle_\psi(B)$.*

Proof. By assumption $B \simeq \bigoplus_{j=1}^s M_{q_j}(\mathbb{C})$ and $\psi \simeq \bigoplus_{j=1}^s r_j tr_{q_j}$ for some $s, q_1, \ldots, q_s, r_1, \ldots, r_s \in \mathbb{N}$. A is $*$-isomorphic to $\bigoplus_{i=1}^d M_{p_i}(\mathbb{C})$ for some $d, p_1, \ldots, p_d \in \mathbb{N}$. Denote $\langle \Lambda_{ij}\rangle_{1\leq i\leq d, 1\leq j\leq s}$ to be the inclusion matrix of A into B with respect to the

dimension vectors $\langle p_i \rangle_{i=1}^d$ and $\langle q_j \rangle_{j=1}^s$ for A and B, respectively. Since $A \subset B$ is a unital inclusion,

$$\sum_{i=1}^d \sum_{j=1}^s \frac{(\Lambda_{ij} p_i r_j)^2}{q_j^2} \cdot \frac{1}{p_i^2} = \sum_{j=1}^s \sum_{i=1}^d \frac{\Lambda_{ij}^2 r_j^2}{q_j^2} \geq \sum_{j=1}^s \frac{r_j^2}{q_j^2},$$

$$\triangle_\psi(A) = 1 - \sum_{i=1}^d \left(\left(\sum_{j=1}^s \frac{\Lambda_{ij} p_i r_j}{q_j} \right)^2 \cdot \frac{1}{p_i^2} \right) \leq 1 - \sum_{j=1}^s \frac{r_j^2}{q_j^2} = \triangle_\psi(B). \qquad \square$$

Lemma 3.9. $\delta_0(a_1, \ldots, a_n) = \delta_0(a_1, \ldots, a_n, I)$.

Proof. By Propositions 6.4 and 6.6 of [10] and Proposition 6.3 of [9],

$$\delta_0(a_1, \ldots, a_n, I) \leq \delta_0(a_1, \ldots, a_n) + \delta_0(I) = \delta_0(a_1, \ldots, a_n).$$

On the other hand, since the strongly closed $*$-algebra generated by $\{a_1, \ldots, a_n\}$ is M, by Theorem 4.3 of [10], $\delta_0(a_1, \ldots, a_n) \leq \delta_0(a_1, \ldots, a_n, I)$. $\qquad \square$

We are now in a position to calculate the upper bound for $\delta_0(a_1, \ldots, a_n)$. By decomposing M over its center it follows that

$$M \simeq M_0 \oplus (\bigoplus_{i=1}^s M_{k_i}(\mathbb{C})) \oplus M_\infty, \quad \varphi \simeq \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^s \alpha_i tr_{k_i}) \oplus 0$$

where all quantities above are as in the introduction. Write I_i for the identity of $M_{k_i}(\mathbb{C})$ for $1 \leq i \leq s$ and if $M_0 \neq \{0\}$, then write I_0 for the identity of M_0. A moment's thought shows that for the purposes of the theorem below we can neglect the M_∞ summand and assume

$$M = M_0 \oplus (\bigoplus_{i=1}^s M_{k_i}(\mathbb{C})), \quad \varphi = \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^s \alpha_i tr_{k_i}).$$

Theorem 3.10. $\delta_0(a_1, \ldots, a_n) \leq 1 - \sum_{i=1}^s \frac{\alpha_i^2}{k_i^2}$.

Proof. Set $\alpha = 1 - \sum_{i=1}^s \frac{\alpha_i^2}{k_i^2}$. There exists a nested sequence of finite-dimensional $*$-subalgebras of M, $\langle N_m \rangle_{m=1}^\infty$, such that $\bigcup_{m=1}^\infty N_m$ is strongly dense in M, for each $m \in \mathbb{N}$, $I \in N_m$, and $\lim_{m \to \infty} \triangle_\varphi(N_m) \leq \alpha$. This is clear for if $M_0 = \{0\}$, then for each m define

$$N_m = 0 \oplus (\bigoplus_{1 \leq j \leq \min\{m,s\}} M_{k_j}(\mathbb{C})) \oplus \mathbb{C} \cdot (\bigoplus_{m < j \leq s} I_j).$$

Observe that

$$\triangle_\varphi(N_m) = 1 - \sum_{j=1}^{\min\{m,s\}} \frac{\alpha_j^2}{k_j^2} - (\sum_{m < j \leq s} \alpha_j)^2.$$

$\lim_{m \to \infty} \triangle_\varphi(N_m) = \alpha$ and all the other properties required of the N_m are easily checked. If $M_0 \neq \{0\}$, then there exists a nested sequence of finite-dimensional $*$-subalgebras of M_0, $\langle A_m \rangle_{m=1}^\infty$ with $I_0 \in A_m$ for each m and $\bigcup_{m=1}^\infty A_m$ strongly dense in M_0. For each m define

$$N_m = A_m \oplus (\bigoplus_{1 \leq j \leq \min\{m,s\}} M_{k_j}(\mathbb{C})) \oplus \mathbb{C} \cdot (\bigoplus_{m < j \leq s} I_j).$$

Observe that

$$
\begin{aligned}
\triangle_\varphi(N_m) &= 1 + (\triangle_{\alpha_0\varphi_0}(A_m) - 1) - \sum_{j=1}^{\min\{m,s\}} \frac{\alpha_j^2}{k_j^2} - (\sum_{m<j\le s} \alpha_j)^2 \\
&\le 1 - \sum_{j=1}^{\min\{m,s\}} \frac{\alpha_j^2}{k_j^2} - (\sum_{m<j\le s} \alpha_j)^2.
\end{aligned}
$$

As $m \to \infty$ the dominating term above converges to α; so $\lim_{m\to\infty}\triangle_\varphi(N_m) \le \alpha$ (existence of the limit is guaranteed by Lemma 3.8 and the fact that $N_m \subset N_{m+1}$). All the other properties of the N_m are easily checked. Notice that in either case, $\lim_{m\to\infty}\triangle_\varphi(N_m) \le \alpha$ and Lemma 3.8 imply $\triangle_\varphi(N_m) \le \alpha$ for all $m \in \mathbb{N}$.

Take a sequence $\langle N_m \rangle_{m=1}^\infty$ as constructed above. Suppose $\min\{1, \beta/4, C\} > \epsilon > 0$. By Kaplansky's Density Theorem there exists an $m_0 \in \mathbb{N}$ and selfadjoint $x_1, \ldots, x_n \in N_{m_0}$ satisfying $|x_i - a_i|_2 < \epsilon$ and $\|x_i\| \le R$ for $1 \le i \le n$. Denote by B the $*$-algebra generated by $\{x_1, \ldots, x_n, I\}$. By Lemma 3.2 and Lemma 3.7, $\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n, I + \epsilon s_{n+1} : s_1, \ldots, s_{n+1})$ is dominated by

$$
\begin{aligned}
\limsup_{k\to\infty} &\left[k^{-2} \cdot \log(\mathrm{vol}(T(x_1, \ldots, x_n, I; k, 4\epsilon))) + \frac{n+1}{2} \cdot \log k \right] \\
&\le \log((4\epsilon)^{n+1-\triangle_\varphi(B)}) + \log D \\
&\le \log(\epsilon^{n+1-\triangle_\varphi(B)}) + \log(4^{n+1}D)
\end{aligned}
$$

where

$$
D = \pi^{\frac{n+1}{2}}(8(R+1))^{n+1} C^{\triangle_\varphi(B)} [(2e)^{\frac{n+1}{2}}].
$$

Set

$$
D_0 = \pi^{n+1}(8(R+1))^{n+1}(C+1)6^{n+1}.
$$

Clearly $D_0 > D$. $B \subset N_{m_0}$ is a unital inclusion and so by Lemma 3.8, $\triangle_\varphi(B) \le \triangle_\varphi(N_{m_0}) \le \alpha$. Hence, $n + 1 - \triangle_\varphi(B) \ge n + 1 - \alpha$. Since $0 < \epsilon < 1$,

$$
\log(\epsilon^{n+1-\triangle_\varphi(B)}) + \log(4^{n+1}D) \le \log(\epsilon^{n+1-\alpha}) + \log(4^{n+1}D_0).
$$

Thus,

$$
\begin{aligned}
&\frac{\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n, I + \epsilon s_{n+1} : s_1, \ldots, s_{n+1})}{|\log \epsilon|} \\
&\le -(n+1) + \alpha + \frac{\log(4^{n+1}D_0)}{|\log \epsilon|}.
\end{aligned}
$$

D_0 is independent of ϵ and so by Lemma 3.9,

$$
\begin{aligned}
\delta_0(a_1, \ldots, a_n) &= \delta_0(a_1, \ldots, a_n, I) \\
&\le (n+1) + \limsup_{\epsilon\to 0}\left(-(n+1) + \alpha + \frac{\log(4^{n+1}D_0)}{|\log \epsilon|} \right) \\
&= \alpha \\
&= 1 - \sum_{i=1}^s \frac{\alpha_i^2}{k_i^2}. \qquad \square
\end{aligned}
$$

4. Weak Hyperfinite Monotonicity

Throughout this section assume $b_1, \ldots, b_p$ are selfadjoint elements in M and the strongly closed algebra B generated by the b_j is hyperfinite. We will prove that if $\{b_1, \ldots, b_p\}$ lies in the $*$-algebra generated by $\{a_1, \ldots, a_n\}$, then

$$\delta_0(b_1, \ldots, b_p) \leq \delta_0(a_1, \ldots, a_n).$$

This "weak hyperfinite monotonic" inequality has significant implications in finding sharp lower bounds for $\delta_0(a_1, \ldots, a_n)$ when M is diffuse.

The argument is simple, despite the notation that shrouds it. Because B is hyperfinite, matricial microstates of $\{b_1, \ldots, b_p\}$ are all approximately unitarily equivalent; the proof is nothing more than a trivial generalization of Lemma 3.1. It follows that $\delta_0(b_1, \ldots, b_p)$ reflects the metric entropy of the unitary orbit of a single microstate for $\{b_1, \ldots, b_p\}$ (provided the microstate approximates well enough). Since the b_j are polynomials of the a_i (and thus images of the a_i under Lipschitz maps), the metric entropy data carries over to the microstates of $\{a_1, \ldots, a_n\}$ and yields lower bounds for the metric entropy of the unitary orbit of a microstate for $\{a_1, \ldots, a_n\}$. Stuffing this lower bound information into the modified free entropy dimension machine we arrive at the above inequality.

In addition to maintaining the conventions set forth in Section 2, we adopt the following notation in this section:

- For $r > 0$, $(M_k^{sa}(\mathbb{C}))_r$ denotes the operator norm ball of $M_k^{sa}(\mathbb{C})$ of radius r centered at the origin. For any $d \in \mathbb{N}$, $((M_k^{sa}(\mathbb{C}))_r)^d$ is the Cartesian product of d copies of $(M_k^{sa}(\mathbb{C}))_r$.
- For $d \in \mathbb{N}, K \subset (M_k^{sa}(\mathbb{C}))^d$, and $u \in U_k$ define uKu^* to be the set

$$\{(ux_1 u^*, \ldots, ux_d u^*) : (x_1, \ldots, x_d) \in K\}.$$

- For $d \in \mathbb{N}$ and $(x_1, \ldots, x_d) \in (M_k^{sa}(\mathbb{C}))^d$ define $U(x_1, \ldots, x_d) = \{(ux_1 u^*, \ldots, ux_n u^*) : u \in U_k\}$.
- For $\epsilon > 0, d \in \mathbb{N}$, and $S \subset (M_k^{sa}(\mathbb{C}))_d$ write $P_\epsilon(S)$ for the maximum number of points in an ϵ-separated subset of S and $\mathcal{N}_\epsilon(S)$ for the ϵ-neighborhood of S, both taken with respect to the metric $\rho((x_1, \ldots, x_d), (y_1, \ldots, y_d)) = \max\{|x_i - y_i|_2 : 1 \leq i \leq d\}$.

First we show that matricial microstates of $\{b_1, \ldots, b_p\}$ are approximately unitarily equivalent. We do this with the following two lemmas, the first of which makes no use of hyperfiniteness.

Lemma 4.1. If $z_1, \ldots, z_p \in B$ are selfadjoint, $\|z_j\| \leq r$ for $1 \leq j \leq p$, and $L, \gamma_0 > 0$, then there exist polynomials $f_1, \ldots, f_p$ in p noncommuting variables such that for $1 \leq j \leq p$:

- $|f_j(b_1, \ldots, b_p) - z_j|_2 < 2\gamma_0$;
- $\|f_j(b_1, \ldots, b_p)\| \leq r + 1$;
- for any $k \in \mathbb{N}$ and $(x_1, \ldots, x_p) \in ((M_k^{sa}(\mathbb{C}))_L)^p$, $f_j(x_1, ..., x_p) \in (M_k^{sa}(\mathbb{C}))_{r+1}$.

Proof. By Kaplansky's Density Theorem there exist polynomials $g_1, \ldots, g_p$ in p noncommuting variables such that for $1 \leq j \leq p$:

- $|g_j(b_1, \ldots, b_p) - z_j|_2 < \gamma_0$;
- $\|g_j(b_1, \ldots, b_p)\| \leq r$;
- $g_j(y_1, \ldots, y_p)$ is selfadjoint for any selfadjoint operators $y_1, \ldots, y_p$.

There exists an $L_1 > L + r$ such that for any $1 \le j \le p$ and $k \in \mathbb{N}$, if $(x_1, \ldots, x_p) \in ((M_k^{sa}(\mathbb{C}))_L)^p$, then $\|g_j(x_1, \ldots, x_p)\| \le L_1$. Define $f : [-L_1, L_1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} t & \text{if } |t| \le r, \\ r & \text{if } r < t \le L_1, \\ -r & \text{if } -L_1 \le t < -r. \end{cases}$$

For any $1 \le j \le p$, $f(g_j(b_1, \ldots, b_p)) = g_j(b_1, \ldots, b_p)$ and

- $|f(g_j(b_1, \ldots, b_p)) - z_j|_2 < \gamma_0$;
- $\|f(g_j(b_1, \ldots, b_p))\| \le r$;
- for any $k \in \mathbb{N}$ if $(x_1, \ldots, x_p) \in ((M_k^{sa}(\mathbb{C}))_L)^p$, then $f(g_j(x_1, \ldots, x_p)) \in (M_k^{sa}(\mathbb{C}))_r$.

Approximating f uniformly on $[-L_1, L_1]$ by a polynomial h (to within sufficiently small $\epsilon > 0$) and setting $f_j = h \circ g_j$ yields the desired result. $\square$

Lemma 4.2. *If $\epsilon > 0$ and $r \ge \max\{\|b_j\|\}_{1 \le j \le p}$, then there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that for each $k \in \mathbb{N}$ and $(x_1, \ldots, x_p), (y_1, \ldots, y_p) \in \Gamma_r(b_1, \ldots, b_p; m, k, \gamma)$ there exists a $u \in U_k$ satisfying $|u x_j u^* - y_j|_2 < \epsilon$ for $1 \le j \le p$.*

Proof. By Kaplansky's Density Theorem and the hyperfiniteness of B there exist selfadjoint elements $z_1, \ldots, z_p \in B$ that generate a finite-dimensional algebra and such that $\|z_j\| \le r$ and $|z_j - b_j|_2 < \epsilon$ for all $1 \le j \le p$. By the remark following Corollary 3.4 there exist $m_1 \in \mathbb{N}$ and $\gamma_1 > 0$ such that if $r_0 = \max\{\|z_j\|\}_{1 \le j \le p}$, $k \in \mathbb{N}$, and $(x_1, \ldots, x_p), (y_1, \ldots, y_p) \in \Gamma_{r_0+1}(z_1, \ldots, z_p; m_1, k, \gamma_1)$, then there exists a $u \in U_k$ satisfying $|u x_j u^* - y_j|_2 < \epsilon$ for $1 \le j \le p$.

By Lemma 4.1 for $\frac{\epsilon}{2} > \gamma_0 > 0$ there exist polynomials $f_1, \ldots, f_p$ in p noncommuting variables such that for $1 \le j \le p$:

- $|f_j(b_1, \ldots, b_p) - z_j|_2 < 2\gamma_0 < \epsilon$;
- $\|f_j(b_1, \ldots, b_p)\| \le r_0 + 1$;
- for any $k \in \mathbb{N}$, if $(x_1, \ldots, x_p) \in ((M_k^{sa}(\mathbb{C}))_r)^p$, then $f_j(x_1, \ldots, x_p) \in (M_k^{sa}(\mathbb{C}))_{r_0+1}$.

By making γ_0 sufficiently small it follows that for any $1 \le j \le m_1$ and $1 \le i_1, \ldots, i_j \le p$,

$$|f_{i_1}(b_1, \ldots, b_p) \cdots f_{i_j}(b_1, \ldots, b_p) - z_{i_1} \cdots z_{i_j}|_2 < \gamma_1.$$

Hence by choosing $m \in \mathbb{N}$ large enough, and $\gamma > 0$ small enough, if $k \in \mathbb{N}$ and $(x_1, \ldots, x_p) \in \Gamma_r(b_1, \ldots, b_p; m, k, \gamma)$, then

$$|f_j(x_1, \ldots, x_p) - x_j|_2 \le |f_j(b_1, \ldots, b_p) - b_j|_2 + \epsilon$$

for $1 \le j \le p$ and

$$(f_1(x_1, \ldots, x_p), \ldots, f_p(x_1, \ldots, x_p)) \in \Gamma_{r_0+1}(z_1, \ldots, z_p; m_1, k, \gamma_1).$$

Finally, suppose $k \in \mathbb{N}$ and $(x_1, \ldots, x_p), (y_1, \ldots, y_p) \in \Gamma_r(b_1, \ldots, b_p; m, k, \gamma)$. For any $1 \le j \le p$,

$$|f_j(x_1, \ldots, x_p) - x_j|_2 \le |f_j(b_1, \ldots, b_p) - b_j|_2 + \epsilon \le |z_j - b_j|_2 + 2\epsilon < 3\epsilon.$$

Similarly for $1 \le j \le p$, $|f_j(y_1, \ldots, y_p) - y_j|_2 < 3\epsilon$. By the preceding two paragraphs there exists a $u \in U_k$ such that for $1 \le j \le p$,

$$|u f_j(x_1, \ldots, x_p) u^* - f_j(y_1, \ldots, y_p)|_2 < \epsilon.$$

So for $1 \le j \le p$, $|u x_j u^* - y_j|_2 < 7\epsilon$. $\square$

In the next lemma suppose $r > \max\{\|b_j\|\}_{1 \leq j \leq p}$.

Lemma 4.3. *For each $0 < \epsilon < 1$ there exist corresponding $m_\epsilon \in \mathbb{N}$ and $\gamma_\epsilon > 0$ such that if $L > 0$ and $\langle (x_1^{(k)}, \ldots, x_p^{(k)}) \rangle_{k=1}^{\infty}$ is a sequence satisfying $(x_1^{(k)}, \ldots, x_p^{(k)}) \in (M_k^{sa}(\mathbb{C}))^p$ for all k and $(x_1^{(k)}, \ldots, x_p^{(k)}) \in \Gamma_r(b_1, \ldots, b_p; m_\epsilon, k, \gamma_\epsilon)$ for sufficiently large k, then*

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{\epsilon L}(U(x_1^{(k)}, \ldots, x_p^{(k)})))]$$

$$\geq \chi_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_0$$

where $K_0 = p \cdot \log((2 + L)\sqrt{2\pi e})$.

Proof. Suppose $0 < \epsilon < 1$. By Lemma 4.2 there exist $m_\epsilon \in \mathbb{N}$ and $\gamma_\epsilon > 0$ such that if $k \in \mathbb{N}$,

$$(z_1, \ldots, z_p) \in \Gamma_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p; m_\epsilon, k, \gamma_\epsilon),$$

and $(x_1, \ldots, x_p) \in \Gamma_r(b_1, \ldots, b_p; m_\epsilon, k, \gamma_\epsilon)$, then there exists a $u \in U_k$ satisfying $|ux_j u^* - z_j|_2 < 2\epsilon$ for $1 \leq j \leq p$.

Assume that $\langle (x_1^{(k)}, \ldots, x_p^{(k)}) \rangle_{k=1}^{\infty}$ satisfies the hypothesis of the lemma with m_ϵ and γ_ϵ chosen according to the preceding paragraph. For sufficiently large k,

$$\Gamma_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p; m_\epsilon, k, \gamma_\epsilon) \subset \mathcal{N}_{2\epsilon}(U(x_1^{(k)}, \ldots, x_p^{(k)})).$$

Find an ϵL-separated set W_k of $U(x_1^{(k)}, \ldots, x_p^{(k)})$ (with respect to the ρ metric) of maximum cardinality $\mathcal{N}_{2\epsilon}(U(x_1^{(k)}, \ldots, x_p^{(k)})) \subset \mathcal{N}_{(2+L)\epsilon}(W_k)$. For large enough k,

$$\text{vol}(\Gamma_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p; m_\epsilon, k, \gamma_\epsilon)) \leq \text{vol}(\mathcal{N}_{(2+L)\epsilon}(W_k))$$

$$\leq |W_k| \cdot \frac{\pi^{\frac{pk^2}{2}}((2+L)\epsilon\sqrt{k})^{pk^2}}{\Gamma\left(\frac{k^2}{2}+1\right)^p}.$$

By the preceding inequality, $\chi_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p; m_\epsilon, \gamma_\epsilon)$ is dominated by

$$\limsup_{k \to \infty} \left[k^{-2} \cdot \log|W_k| + p \log((2+L)\epsilon\sqrt{\pi k}) - pk^{-2} \cdot \log\left(\Gamma\left(\frac{k^2}{2}+1\right)\right) + \frac{p}{2}\log k \right]$$

$$\leq \limsup_{k \to \infty} \left[k^{-2} \cdot \log|W_k| + p \log((2+L)\epsilon\sqrt{\pi}) - pk^{-2}\log\left(\left(\frac{k^2}{2e}\right)^{\frac{k^2}{2}}\right) + p \log k \right]$$

$$= \limsup_{k \to \infty} \left[k^{-2} \cdot \log|W_k| + p \log((2+L)\epsilon\sqrt{\pi}) - \frac{p}{2} \cdot \log\left(\frac{k^2}{2e}\right) + p \log k \right]$$

$$= \limsup_{k \to \infty}[k^{-2} \cdot \log|W_k| + p \log((2+L)\epsilon\sqrt{\pi}) + \frac{p}{2}\log(2e)]$$

$$= p \log((2+L)\sqrt{2\pi e}) + \log \epsilon + \limsup_{k \to \infty}(k^{-2} \cdot \log|W_k|)$$

$$= K_0 + p \log \epsilon + \limsup_{k \to \infty}[k^{-2} \cdot \log(P_{\epsilon L}(U(x_1^{(k)}, \ldots, x_p^{(k)})))].$$

By the above calculation, $\chi_r(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_0$ is dominated by

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{\epsilon L}(U(x_1^{(k)}, \ldots, x_p^{(k)})))].$$

$\square$

Write A for the $*$-algebra generated by $\{a_1, \ldots, a_n\}$.

Lemma 4.4. *If $\{b_1, \ldots, b_p\} \subset A$, then there exists an $L > 0$ such that for any $0 < \epsilon < 1, m \in \mathbb{N}$, and $\gamma > 0$ there is a sequence $\langle (x_1^{(k)}, \ldots, x_n^{(k)}) \rangle_{k=1}^{\infty}$ satisfying $(x_1^{(k)}, \ldots, x_n^{(k)}) \in (M_k^{sa}(\mathbb{C}))^n$ for all k, $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma)$ for sufficiently large k, and*

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)})))]$$
$$\geq \chi_\lambda(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_1$$

where $K_1 = p \cdot \log((2 + 4\sqrt{n}L)\sqrt{2\pi e})$ and $\lambda = L(R+1) + \max\{\|b_j\|\}_{1 \leq j \leq p}$.

Proof. There exist polynomials $f_1, \ldots, f_p$ in n noncommuting variables and with no constant terms such that for $1 \leq j \leq p$, $f_j(a_1, \ldots, a_n) = b_j$ and such that f_j of an n-tuple of selfadjoint operators is a selfadjoint element. There exists a constant $L > 0$ such that if $k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in (M_k^{sa}(\mathbb{C}))_{R+1}$, then for all $1 \leq j \leq p$,

$$|f_j(\xi_1, \ldots, \xi_n) - f_j(\eta_1, \ldots, \eta_n)|_2 \leq L \cdot \max\{|\xi_i - \eta_i|_2 : 1 \leq i \leq n\}$$

and $\|f_j(\xi_1, \ldots, \xi_n)\| \leq L(R+1)$.

Suppose $0 < \epsilon < 1, m \in \mathbb{N}$, and $\gamma > 0$. By Lemma 4.3 there exist an $m_\epsilon \in \mathbb{N}$ and $\gamma_\epsilon > 0$ such that if $\langle (x_1^{(k)}, \ldots, x_p^{(k)}) \rangle_{k=1}^{\infty}$ is a sequence satisfying $(x_1^{(k)}, \ldots, x_p^{(k)}) \in (M_k^{sa}(\mathbb{C}))^p$ for all k and $(x_1^{(k)}, \ldots, x_p^{(k)}) \in \Gamma_\lambda(b_1, \ldots, b_p; m_\epsilon, k, \gamma_\epsilon)$ for sufficiently large k, then

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}L}(U(x_1^{(k)}, \ldots, x_p^{(k)})))]$$
$$\geq \chi_\lambda(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_1,$$

where $K_1 = p \cdot \log((2 + 4\sqrt{n}L)\sqrt{2\pi e})$. By the assumed existence of finite-dimensional approximants for $\{a_1, \ldots, a_n\}$ there exists a $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ there is an $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma)$ satisfying

$$(f_1(x_1^{(k)}, \ldots, x_n^{(k)}), \ldots, f_p(x_1^{(k)}, \ldots, x_n^{(k)})) \in \Gamma_\lambda(b_1, \ldots, b_p; m_\epsilon, k, \gamma_\epsilon).$$

For each $k \geq k_0$ and $1 \leq j \leq p$, set $y_j^{(k)} = f_j(x_1^{(k)}, \ldots, x_n^{(k)})$. It follows that

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}L}(U(y_1^{(k)}, \ldots, y_p^{(k)})))]$$
$$\geq \chi_\lambda(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_1.$$

For $k \geq k_0$ and any $u, v \in U_k$, $1 \leq j \leq p$, observe that

$$L \cdot \max\{|ux_i^{(k)}u^* - vx_i^{(k)}v^*|_2 : 1 \leq i \leq n\}$$

dominates

$$|q_j(ux_1^{(k)}u^*, \ldots, ux_n^{(k)}u^*) - q_j(vx_1^{(k)}v^*, \ldots, vx_n^{(k)}v^*)|_2 |uy_j^{(k)}u^* - vy_j^{(k)}v^*|_2.$$

It follows that for any $k \geq k_0$,

$$k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)}))) \geq k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}L}(U(y_1^{(k)}, \ldots, y_p^{(k)}))).$$

By the last sentence of the preceding paragraph we are done. $\qquad \square$

Theorem 4.5. *(Weak Hyperfinite Monotonicity) If $\{b_1, \ldots, b_p\} \subset A$, then*

$$\delta_0(b_1, \ldots, b_p) \leq \delta_0(a_1, \ldots, a_n).$$

Proof. Consider the constants L and λ corresponding to $b_1, \ldots, b_p$ in Lemma 4.4. Suppose $0 < \gamma, \epsilon < \frac{1}{2}$ and $m \in \mathbb{N}$. By Lemma 4.4 there exists a sequence $\langle (x_1^{(k)}, \ldots, x_n^{(k)}) \rangle_{k=1}^{\infty}$ satisfying $(x_1^{(k)}, \ldots, x_n^{(k)}) \in (M_k^{sa}(\mathbb{C}))^n$ for all $k \in \mathbb{N}$, $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_{R+1}\left(a_1, \ldots, a_n; m, k, \frac{\gamma}{(8(R+2))^m}\right)$ for sufficiently large k, and

$$\limsup_{k \to \infty}[k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)})))]$$
$$\geq \chi_\lambda(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log \epsilon| - K_1$$

where $K_1 = p \cdot \log((2 + 4\sqrt{n}L)\sqrt{2\pi e})$.

By Corollary 2.14 of [11] there is an $N \in \mathbb{N}$ such that if $k \geq N$ and σ is a Radon probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^{2n}$ invariant under the action

$$(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \mapsto (\xi_1, \ldots, \xi_n, u\eta_1 u^*, \ldots, u\eta_n u^*)$$

for $u \in U_k$, then $\sigma(\omega_k) > \frac{1}{2}$ where

$$\omega_k = \{(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^{2n} :$$
$$\{\xi_1, \ldots, \xi_n\} \text{ and } \{\eta_1, \ldots, \eta_n\} \text{ are } \left(m, \frac{\gamma}{4^m}\right)\text{-free}\}.$$

For $k \in \mathbb{N}$ write ν_k for the atomic probability measure concentrated at $(x_1^{(k)}, \ldots, x_n^{(k)})$ and write m_k for the probability measure obtained by restricting vol to $\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \frac{\gamma}{8^m})$ and normalizing appropriately. $\nu_k \times m_k$ is a Radon probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^{2n}$ invariant under the U_k-action described above. Write F_k for the set of all $(z_1, \ldots, z_n) \in \Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \frac{\gamma}{8^m})$ such that $\{z_1, \ldots, z_n\}$ and $\{x_1^{(k)}, \ldots, x_n^{(k)}\}$ are $(m, \frac{\gamma}{4^m})$-free.

For $k \geq N$, $\frac{1}{2} < (\nu_k \times m_k)(\omega_k) = m_k(F_k)$. Set $E_k = (x_1^{(k)}, \ldots, x_n^{(k)}) + F_k$. For $u \in U_k$, $\mathrm{vol}(uE_k u^*) = \mathrm{vol}(E_k)$ and $uE_k u^*$ is contained in

$$\Gamma_{R+1+2\epsilon}(a_1 + \epsilon s_1, ..., a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma).$$

For each $k \in \mathbb{N}$ there exists a subset $\langle u_{k,s} \rangle_{s \in S_k}$ of U_k such that

$$|S_k| = P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)}))$$

and for $s, s' \in S_k$ with $s \neq s'$,

$$\max\{|u_{k,s}x_i^{(k)}u_{k,s}^* - u_{k,s'}x_i^{(k)}u_{k,s'}^*|_2 : 1 \leq i \leq n\} > 4\epsilon\sqrt{n}.$$

Since F_k is $\|\cdot\|_2$-bounded by $2\epsilon\sqrt{nk}$, $(u_{k,s}E_k u_{k,s}^*) \cap (u_{k,s'}E_k u_{k,s'}^*) = \emptyset$ for $s, s' \in S_k, s \neq s'$. Hence for $k \geq N$, $\mathrm{vol}(\Gamma_{R+1+2\epsilon}(a_1 + \epsilon s_1, ..., a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma))$ dominates

$$\mathrm{vol}(\bigsqcup_{s \in S_k} u_{k,s}E_k u_{k,s}^*) = |S_k| \cdot \mathrm{vol}(F_k)$$
$$= |S_k| \cdot m_k(F_k) \cdot \mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m))$$
$$> 1/2 \cdot |S_k| \cdot vol(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)).$$

By the last sentence of the preceding paragraph,

$$\chi_{R+1+2\epsilon}(a_1 + \epsilon s_1, ..., a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, \gamma)$$

$$\geq \limsup_{k\to\infty}(k^{-2} \cdot [\log(1/2 \cdot |S_k| \cdot \mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)))] + n/2 \cdot \log k)$$

$$\geq \limsup_{k\to\infty}(k^{-2} \cdot \log(|S_k|))$$
$$+ \liminf_{k\to\infty}(k^{-2} \cdot [\log(\mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, ..., \epsilon s_n; m, k, \gamma/8^m)))] + n/2 \cdot \log k)$$

$$\geq \limsup_{k\to\infty}(k^{-2} \cdot \log(|S_k|)) + \chi_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n)$$

$$= \limsup_{k\to\infty}[k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)})))] + n\log(\epsilon\sqrt{2\pi e})$$

$$\geq \chi(b_1 + \epsilon s_1, \ldots, b_p + \epsilon s_p : s_1, \ldots, s_p) + p|\log\epsilon| + n\log(\epsilon\sqrt{2\pi e}) - K_1,$$

where we used regularity of $\{\epsilon s_1, \ldots, \epsilon s_n\}$ going from the third to the fourth lines above. m and γ being arbitrary it follows that

$$\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n) \geq \chi(b_1 + \epsilon s_1, ..., b_p + \epsilon s_p : s_1, \ldots, s_p)$$
$$+ (p-n) \cdot |\log\epsilon| + n \cdot \log(\sqrt{2\pi e}) - K_1.$$

Dividing by $|\log\epsilon|$, taking lim sup's as $\epsilon \to 0$, and adding n to both sides yields

$$\delta_0(a_1, \ldots, a_n) \geq \delta_0(b_1, \ldots, b_p).$$

$\square$

Corollary 4.6. *If $a \in M$, then $\delta_0(a_1, \ldots, a_n) \geq \delta_0(a)$.*

Proof. Find a sequence $\langle z_k \rangle_{k=1}^\infty$ in A such that $z_k \to a$ strongly. By Proposition 6.14 of [9] and Corollary 6.7 of [10], $\liminf_{k\to\infty} \delta_0(z_k) = \liminf_{k\to\infty} \delta(z_k) \geq \delta(a) = \delta_0(a)$. For each k, z_k generates a hyperfinite von Neumann algebra; by Lemma 4.2, for each k, $\delta_0(a_1, \ldots, a_n) \geq \delta_0(z_k)$. So the preceding sentence yields the desired result. $\square$

Corollary 4.7. *If M has a diffuse von Neumann subalgebra, then $\delta_0(a_1, \ldots, a_n) \geq 1$.*

Proof. Find a maximal abelian subalgebra N of the diffuse von Neumann subalgebra. N has a selfadjoint generator a. N must be diffuse since it is a maximal abelian subalgebra of a diffuse von Neumann algebra. Consequently, a has no eigenvalues. Apply Corollary 4.6. $\square$

Remark 4.8. By [10] if M has a regular diffuse von Neumann subalgebra, then $\delta_0(a_1, \ldots, a_n) \leq 1$. By [4] if there exists a sequence of Haar unitaries $\langle u_j \rangle_{j=1}^s$ such that the sequence generates M as a von Neumann algebra and for each $j \in \mathbb{N}$, $u_{j+1}u_ju_{j+1}^* \in \{u_1, \ldots, u_j\}''$, then $\delta_0(a_1, \ldots, a_n) \leq 1$. Combining these results with Corollary 4.7, it follows that for any selfadjoint generators $a_1, \ldots, a_n$ for an M that satisfies either of the two conditions and that is also embeddable into the ultraproduct of the hyperfinite II_1-factor, $\delta_0(a_1, \ldots, a_n) = 1$. In other words, δ_0 is a von Neumann algebra invariant for such algebras. In particular, $\delta_0(M) = 1$ when M can be embedded into the ultraproduct of the hyperfinite II_1-factor and M has a Cartan subalgebra, $M = N_1 \otimes N_2$ for II_1-factors N_1 and N_2, or when M is a group von Neumann algebra associated to the groups $SL_n(\mathbb{Z}), n \geq 3$.

5. Lower Bound for Finite-Dimensional Algebras

In this section we calculate the lower bound for $\delta_0(a_1, \ldots, a_n)$ when M is finite dimensional. Without loss of generality, assume throughout this section that $M = \bigoplus_{i=1}^p M_{k_i}(\mathbb{C})$ and $\varphi = \bigoplus_{i=1}^p \alpha_i tr_{k_i}$ where $p \in \mathbb{N}$ and $\alpha_i > 0$ for each i. The first lemma we present is not necessary but it is convenient.

Lemma 5.1. *There exists an $x \in M$ such that the $*$-algebra generated by x is M.*

The proof is not hard and we omit it.

As in the preceding section, the calculation of the lower bound amounts to looking at the packing number of unitary orbits of microstates. We use two ingredients.

For a representation $\pi : M \to M_k(\mathbb{C})$ define H_π to be the unitary group of $(\pi(M))'$ and $X_\pi = U_k/H_\pi$. Endow X_π with the quotient metric from the $|\cdot|_2$-metric on U_k. Denote this metric on X_π by d_π. The first ingredient is a packing number estimate for certain homogeneous spaces X_π.

Lemma 5.2. *There exists a $\kappa > 0$ with the property that for every $\varepsilon > 0$ there is a corresponding sequence $\langle \sigma_k \rangle_{k=1}^\infty$ such that for each k, $\sigma_k : M \to M_k(\mathbb{C})$ is a $*$-homomorphism and for k sufficiently large:*

- $\|tr_k \circ \sigma_k - \varphi\| < \varepsilon$.
- *For each k, setting $H_k = H_{\sigma_k}$ and $X_k = X_{\sigma_k}$ we have that H_k is a tractable Lie subgroup of U_k satisfying $k^2(\triangle_\varphi(M) - \varepsilon) \le \dim(X_k)$.*
- *For any $\epsilon > 0$,*

$$\left(\frac{\kappa}{\epsilon} \right)^{\dim X_k} \le P(X_k, \epsilon)$$

where $P(X_k, \epsilon)$ is the maximum number of points in an ϵ-separated subset of X_k.

We quarantine the proof of Lemma 5.2 to the Addendum, merely noting for now that the argument will require some technical modifications to the proofs in [7].

From now on fix x as in Lemma 5.1. Given a representation $\pi : M \to M_k(\mathbb{C})$, define $U_\pi(x) = \{u\pi(x)u^* : u \in U_k\}$ and endow $U_\pi(x)$ with the inherited $|\cdot|_2$-metric. For $u \in U_k$ denote $\dot{u}$ to be the image of u in X_π and define $f_\pi : U_\pi(x) \to X_\pi$ by $f_\pi(u\pi(x)u^*) = \dot{u}$. f_π is well defined for if $u, v \in U_k, \dot{u} = \dot{v} \iff v^*u \in H_\pi \iff u\pi(x)u* = v\pi(x)v*$.

For the second ingredient recall that in Section 3 covering number estimates with respect to the induced operator norm metrics yield the desired upper bounds for $\delta_0(a_1, \ldots, a_n)$. Part of the explanation for this lies in the trivial observation that if $u, v \in U_k$ and $z \in M_k(\mathbb{C})$, then $|uzu^* - vzv^*|_2 \le 2\|u - v\| \cdot |z|_2$. The second ingredient more or less says the reverse: there exists a constant $L > 0$ such that $d_\pi(\dot{u}, \dot{v}) \le L \cdot |u\pi(x)u^* - v\pi(x)v^*|_2$.

Lemma 5.3. *If $z, p \in M_k(\mathbb{C})$ with p a projection and $zz^*, z^*z \le \|z^*z\|p$, then there exists a $y \in M_k(\mathbb{C})$ satisfying $yy^* = y^*y = p$ and*

$$|y - z|_2 \le |p - z^*z|_2 + |p - e|_2 \le 2|p - z^*z|_2$$

where e is the projection onto the range of z^*z.

Proof. Denote the polar decomposition of z by $z = u|z|$ and use the spectral theorem to write $|z| = \sum_{j=1}^m \beta_j e_j$ where the e_j are mutually orthogonal rank one

projections satisfying $e_1 + \cdots + e_m = p$ and $\beta_j \geq 0$. Now estimate:

$$|z - u|_2^2 = |u|z| - up|_2^2 \leq ||z| - p|_2^2 = \frac{1}{k} \cdot \sum_{j=1}^{m} (1 - \beta_j)^2 \leq \frac{1}{k} \cdot \sum_{j=1}^{m} (1 - \beta_j)^2 (1 + \beta_j)^2$$

$$= |p - z^* z|_2^2.$$

$|z - u|_2 \leq |p - z^* z|_2$. Since $uu^*, u^*u \leq p$ there exists a partial isometry v such that $vv^* = p - uu^*$ and $v^*v = p - u^*u$. So if $y = u + v$, then $yy^* = y^*y = p$. $u^*u = e$ whence

$$\begin{aligned} |z - y|_2 \leq |z - u|_2 + |v|_2 &\leq |p - z^* z|_2 + (tr_k(v^*v))^{\frac{1}{2}} \\ &= |p - z^* z|_2 + (tr_k(p - u^*u))^{\frac{1}{2}} \\ &= |p - z^* z|_2 + |p - e|_2 \\ &\leq 2|p - z^* z|_2. \end{aligned}$$

$\square$

Using Lemma 5.3 we obtain the second ingredient:

Lemma 5.4. $\{f_\pi : \text{for some } k \in \mathbb{N}, \ \pi : M \to M_k(\mathbb{C}) \ \text{is a representation}\}$ *is uniformly Lipschitz.*

Proof. Suppose $\pi : M \to M_k(\mathbb{C})$ is a representation. Because $|\cdot|_2$ is unitarily invariant it suffices to show that for any $u \in U_k$,

$$\inf_{h \in H_\pi} |u - h|_2 = d_\pi(f_\pi(u\pi(x)u^*), f_\pi(\pi(x))) \leq L \cdot |u\pi(x)u^* - \pi(x)|_2$$

where $L > 0$ is a constant dependent only on x.

If $u \in U_k$, then set $\epsilon = |u\pi(x)u^* - \pi(x)|_2$. Denote $\langle e_{jl}^{(i)} \rangle_{1 \leq i \leq p, 1 \leq j, l \leq k_i}$ to be the canonical system of matrix units for M. There exist polynomials in two noncommuting variables $\langle q_{jl}^{(i)} \rangle_{1 \leq i \leq p, 1 \leq j, l \leq k_i}$ such that for any i, j, and l, $q_{jl}^{(i)}(x, x^*) = e_{jl}^{(i)}$. Set $y_{jl}^{(i)} = \pi(e_{jl}^{(i)})$. There exists a constant $C > 0$ dependent only on x such that for any i, j, and l,

$$|uy_{jl}^{(i)} - y_{jl}^{(i)}u|_2 = |uy_{jl}^{(i)}u^* - y_{jl}^{(i)}|_2 \leq C\epsilon.$$

Set $K = \sum_{i=1}^{p} k_i$. By the above inequality, $|u\pi(I)u^* - \pi(I)|_2 < CK\epsilon$. Setting f to be the projection onto the orthogonal complement of the range of $\pi(I)$, $|ufu^* - f|_2 < CK\epsilon$. Now

$$\begin{aligned} |u - [(\sum_{1 \leq i \leq p, 1 \leq j \leq k_i} y_{jj}^{(i)} u y_{jj}^{(i)}) + fuf]|_2 \\ \leq (\sum_{1 \leq i \leq p, 1 \leq j \leq k_i} |uy_{jj}^{(i)} - y_{jj}^{(i)}u|_2 \cdot \|y_{jj}^{(i)}\|) + |uf - fu|_2 \cdot \|f\| \\ \leq 2CK\epsilon. \end{aligned}$$

For any $1 \leq i \leq p, 1 \leq j, l \leq k_i$,

$$\begin{aligned} |y_{jj}^{(i)} u y_{jj}^{(i)} - y_{jl}^{(i)} u y_{lj}^{(i)}|_2 &\leq |y_{jj}^{(i)} u y_{jj}^{(i)} - y_{jj}^{(i)} u|_2 + |y_{jl}^{(i)} y_{lj}^{(i)} u - y_{jl}^{(i)} u y_{lj}^{(i)}|_2 \\ &\leq |uy_{jj}^{(i)} - y_{jj}^{(i)}u|_2 + |y_{lj}^{(i)} u - uy_{lj}^{(i)}|_2 \\ &\leq 2C\epsilon. \end{aligned}$$

By Lemma 5.3 there exists for each $1 \leq i \leq p$ a $v_i \in M_k(\mathbb{C})$ such that $v_i v_i^* = v_i^* v_i = y_{11}^{(i)}$ and $|v_i - y_{11}^{(i)} u y_{11}^{(i)}|_2 \leq 2|y_{11}^{(i)} u y_{11}^{(i)} u^* y_{11}^{(i)} - y_{11}^{(i)}|_2$. So

$$|v_i - y_{11}^{(i)} u y_{11}^{(i)}|_2 \leq 2\|y_{11}^{(i)}\|^2 \cdot |u y_{11}^{(i)} u^* - y_{11}^{(i)}|_2 \leq 2C\epsilon.$$

Similarly there exists a $v \in M_k(\mathbb{C})$ such that $v^* v = v v^* = f$ and $|v - fuf|_2 \leq 2|fuf^* f - f|_2 \leq 2CK\epsilon$.

Consider $z = (\sum_{1 \leq i \leq p, 1 \leq j \leq k_i} y_{j1}^{(i)} v_i y_{1j}^{(i)}) + v$. It is easy to check that z is a unitary and, because z commutes with all the $y_{jl}^{(i)}, z \in H_\pi$. Finally by the last three inequalities of the preceding paragraph,

$$
\begin{aligned}
|u - z|_2 \quad \leq \quad & |u - [(\sum_{1 \leq i \leq p, 1 \leq j \leq k_i} y_{jj}^{(i)} u y_{jj}^{(i)}) + fuf]|_2 \\
& + \left(\sum_{1 \leq i \leq p, 1 \leq j \leq k_i} |y_{jj}^{(i)} u y_{jj}^{(i)} - y_{j1}^{(i)} v_i y_{1j}^{(i)}|_2 \right) \quad + \quad |fuf - v|_2 \\
\leq \quad & 4CK\epsilon + \sum_{1 \leq i \leq p, 1 \leq j \leq k_i} (|y_{jj}^{(i)} u y_{jj}^{(i)} - y_{j1}^{(i)} u y_{1j}^{(i)}|_2 + |y_{j1}^{(i)} u y_{1j}^{(i)} - y_{j1}^{(i)} v_i y_{1j}^{(i)}|_2) \\
\leq \quad & 4CK\epsilon + \sum_{1 \leq i \leq p, 1 \leq j \leq k_i} (2C\epsilon + |y_{11}^{(i)} u y_{11}^{(i)} - v_i|_2) \\
\leq \quad & 8CK\epsilon.
\end{aligned}
$$

Set $L = 8CK$, observe that L is dependent only on x, and that $\inf_{h \in H_\pi} |u - h|_2 \leq |u - z|_2 \leq L\epsilon$. □

Denote $L > 0$ to be the uniform Lipschitz constant of Lemma 5.4. There exists a polynomial f in n noncommuting variables satisfying $f(a_1, \ldots, a_n) = x$. There exists an $L_1 > 0$ such that for any $k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in (M_k^{sa}(\mathbb{C}))_R$,

$$|f(\xi_1, \ldots, \xi_n) - f(\eta_1, \ldots, \eta_n)|_2 \leq L_1 \cdot \max\{|\xi_i - \eta_i|_2 : 1 \leq i \leq n\}.$$

Denote $P_\epsilon(S)$ and $U(x_1, \ldots, x_d)$ to have the same meaning as in Section 4.

Lemma 5.5. *If $\rho_1, \rho_2 > 0, m \in \mathbb{N}$, and $\gamma > 0$, then there is an $N \in \mathbb{N}$ such that for each $k \geq N$ there exists an $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ satisfying for any $\frac{\kappa}{\rho_1 L L_1} > \epsilon > 0$,*

$$k^{-2} \cdot \log(P_{\epsilon \rho_1}(U(x_1^{(k)}, \ldots, x_n^{(k)}))) \geq (\triangle_\varphi(M) - \rho_2) \cdot \log\left(\frac{\kappa}{\rho_1 L L_1 \epsilon} \right).$$

Proof. By Lemma 5.2 there is an $N \in \mathbb{N}$ such that for each $k \geq N$ there exists a $*$-homomorphism $\sigma_k : M \to M_k(\mathbb{C})$ satisfying:

- $\|tr_k \circ \sigma_k - \varphi\| \leq \frac{\gamma}{(R+1)^m}$.
- For each k, setting $H_k = H_{\sigma_k}$ and $X_k = X_{\sigma_k}$ we have that H_k is a tractable Lie subgroup of U_k and $k^2(\triangle_\varphi(M) - \rho_2) \leq \dim(X_k)$.
- For $\epsilon > 0$,

$$\left(\frac{\kappa}{\epsilon} \right)^{\dim X_k} \leq P(X_k, \epsilon)$$

  where $P(X_k, \epsilon)$ is the maximum number of points in an ϵ-separated subset of X_k.

For each $1 \leq i \leq n$ and $k \geq N$, define $x_i^{(k)} = \sigma_k(a_i) \in M_k^{sa}(\mathbb{C})$. By the first condition above, $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$. To see that the second condition is fulfilled suppose ϵ satisfies the hypothesis of the lemma and $k \geq N$. $\kappa \geq \rho_1 LL_1 \epsilon$ and so

$$P(X_k, \rho_1 LL_1 \epsilon) \geq \left(\frac{\kappa}{\rho_1 LL_1 \epsilon} \right)^{\dim X_k} \geq \left(\frac{\kappa}{\rho_1 LL_1 \epsilon} \right)^{k^2(\triangle_\rho(M) - \rho_2)}.$$

For any $u, v \in U_k$,

$$
\begin{aligned}
d_{\sigma_k}(\dot{u}, \dot{v}) &\leq L \cdot |u\sigma_k(x)u^* - v\sigma_k(x)v^*|_2 \\
&= L \cdot |f(u\sigma_k(a_1)u^*, \ldots, u\sigma_k(a_n)u^*) - f(v\sigma_k(a_1)v^*, \ldots, v\sigma_k(a_n)v^*)|_2 \\
&\leq LL_1 \cdot \max\{|ux_i^{(k)}u^* - vx_i^{(k)}v^*|_2 : 1 \leq i \leq n\}.
\end{aligned}
$$

It follows that $P_{\epsilon\rho_1}(U(x_1^{(k)}, \ldots, x_n^{(k)})) \geq P(X_k, \rho_1 LL_1 \epsilon)$. Hence for $k \geq N$,

$$k^{-2} \cdot \log(P_{\epsilon\rho_1}(U(x_1^{(k)}, \ldots, x_n^{(k)}))) \geq (\triangle_\varphi(M) - \rho_2) \cdot \log\left(\frac{\kappa}{\rho_1 LL_1 \epsilon} \right).$$

$\square$

The following corollary will not be used until the next section. Suppose $r > R$.

Corollary 5.6. *If $\Omega > 0$ and $\frac{\kappa}{3\Omega LL_1} > \epsilon > 0$, then there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that if $\langle (x_1^{(k)}, \ldots, x_n^{(k)}) \rangle_{k=1}^\infty$ is a sequence satisfying $(x_1^{(k)}, \ldots, x_n^{(k)}) \in (M_k^{sa}(\mathbb{C}))^n$ for all k, and $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_r(a_1, \ldots, a_n; m, k, \gamma)$ for sufficiently large k, then*

$$k^{-2} \cdot \log(P_{\Omega\epsilon}(U(x_1^{(k)}, \ldots, x_n^{(k)}))) \geq (\triangle_\varphi(M) - \epsilon) \cdot \log\left(\frac{\kappa}{3LL_1\Omega\epsilon} \right)$$

for sufficiently large k.

Proof. By Corollary 3.4 there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that for any $k \in \mathbb{N}$ and $(y_1, \ldots, y_n), (z_1, \ldots, z_n) \in \Gamma_r(a_1, \ldots, a_n; m, k, \gamma)$ there exists a $u \in U_k$ such that for $1 \leq j \leq n$, $|uy_ju^* - z_j|_2 < \frac{\epsilon}{2}$. By Lemma 5.5 there exists an $N \in \mathbb{N}$ such that for each $k \geq N$ there exists an $(z_1^{(k)}, \ldots, z_n^{(k)}) \in \Gamma_r(a_1, \ldots, a_n; m, k, \gamma)$ satisfying

$$k^{-2} \cdot \log(P_{3\Omega\epsilon}(U(z_1^{(k)}, \ldots, z_n^{(k)}))) \geq (\triangle_{\varphi(M)} - \epsilon) \cdot \left(\frac{\kappa}{3LL_1\Omega\epsilon} \right).$$

Now merely observe that for any such sequence $\langle (x_1^{(k)}, \ldots, x_n^{(k)}) \rangle_{k=1}^\infty$ satisfying the hypothesis of the corollary with m and γ chosen above, $P_{\Omega\epsilon}(U(x_1^{(k)}, \ldots, x_n^{(k)})) \geq P_{3\Omega\epsilon}(U(z_1^{(k)}, \ldots, z_n^{(k)}))$ for $k \geq N$. $\square$

Theorem 5.7. $\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^p \frac{\alpha_i^2}{k_i^2}$.

Proof. Suppose $\min\left\{ \frac{1}{2}, \frac{\kappa}{4\sqrt{n}LL_1} \right\} > \epsilon > 0, m \in \mathbb{N}$, and $\gamma, r > 0$. Corollary 2.14 of [11] provides an $N \in \mathbb{N}$ such that if $k \geq N$ and σ is a Radon probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^{2n}$ invariant under the U_k-action $(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \mapsto (\xi_1, \ldots, \xi_n, u\eta_1 u^*, \ldots, u\eta_n u^*)$ where $u \in U_k$, then $\sigma(\omega_k) > \frac{1}{2}$ where

$$\omega_k = \{(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^{2n} :$$

$$\{\xi_1, \ldots, \xi_n\} \text{ and } \{\eta_1, \ldots, \eta_n\} \text{ are } \left(m, \frac{\gamma}{4^m} \right)\text{-free}\}.$$

Lemma 5.5 provides an $N_1 \in \mathbb{N}$ such that for each $k \geq N_1$ there exists an $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_R(a_1, \ldots, a_n; m, k, \gamma/(8(R+2))^m)$ satisfying

$$k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)}))) \geq (\triangle_\varphi(M) - r) \cdot \log\left(\frac{\kappa}{4\sqrt{n}LL_1\epsilon}\right).$$

For $k \geq N + N_1$ denote by ν_k the atomic probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^n$ concentrated at $(x_1^{(k)}, \ldots, x_n^{(k)})$ and denote by m_k the Radon probability measure obtained by restricting vol to $\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)$ and normalizing appropriately. $\nu_k \times m_k$ is a Radon probability measure on $((M_k^{sa}(\mathbb{C}))_{R+1})^{2n}$ invariant under the U_k-action described above. Write F_k for the set of all $(y_1, \ldots, y_n) \in \Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)$ such that $\{y_1, \ldots, y_n\}$ and $\{x_1^{(k)}, \ldots, x_n^{(k)}\}$ are $\left(m, \frac{\gamma}{4^m}\right)$-free.

For $k \geq N + N_1$, $\frac{1}{2} < (\nu_k \times m_k)(\omega_k) = m_k(F_k)$. Set $E_k = (x_1^{(k)}, \ldots, x_n^{(k)}) + F_k$. For $k \geq N + N_1$ and $u \in U_k, \mathrm{vol}(uE_ku^*) = \mathrm{vol}(F_k)$ (where uE_ku^* is defined as in Section 4) and uE_ku^* is contained in $\Gamma_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma)$.

For each $k \geq N + N_1$ find a subset $\langle u_{k,s}\rangle_{s\in S_k}$ of U_k such that

$$|S_k| = P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)}))$$

and for any $s, s' \in S_k, s \neq s'$,

$$\max\{|u_{k,s}x_i^{(k)}u_{k,s}^* - u_{k,s'}x_i^{(k)}u_{k,s'}^*|_2 : 1 \leq i \leq n\} > 4\epsilon\sqrt{n}.$$

$F_k \subset (M_k^{sa}(\mathbb{C}))^n$ is a $2\epsilon\sqrt{nk}$ bounded subset with respect to the $\|\cdot\|_2$-norm. Hence for any $s, s' \in S_k, s \neq s'(u_{k,s}E_ku_{k,s}^*) \cap (u_{k,s'}E_ku_{k,s'}^*) = \emptyset$. Consequently for $k \geq N + N_1$, $\mathrm{vol}(\Gamma_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma))$ dominates

$$\mathrm{vol}(\bigsqcup_{s\in S_k} u_{k,s}E_ku_{k,s}^*) = |S_k| \cdot \mathrm{vol}(F_k)$$

$$= |S_k| \cdot m_k(F_k) \cdot \mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m))$$

$$> \frac{1}{2} \cdot |S_k| \cdot \mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)).$$

By what preceded for $\min\left\{\frac{1}{2}, \frac{\kappa}{4\sqrt{n}LL_1}\right\} > \epsilon > 0, m \in \mathbb{N}$, and $\gamma, r > 0$, we have that $\chi_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : \epsilon s_1, \ldots, \epsilon s_n; m, \gamma)$ dominates

$$\limsup_{k\to\infty} k^{-2} \cdot \log\left(\frac{1}{2} \cdot |S_k| \cdot \mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)) + \frac{n}{2} \cdot \log k\right)$$

$$= \limsup_{k\to\infty} \left[k^{-2} \cdot \log\left(\mathrm{vol}(\Gamma_{2\epsilon}(\epsilon s_1, \ldots, \epsilon s_n; m, k, \gamma/8^m)) + \frac{n}{2} \cdot \log k\right)\right.$$

$$\left. + k^{-2} \cdot \log(|S_k|)\right]$$

$$\geq \chi(\epsilon s_1, \ldots, \epsilon s_n) + \log\left(\left(\frac{\kappa}{4\sqrt{n}LL_1\epsilon}\right)^{\triangle_\varphi(M)-r}\right)$$

$$= \log(\epsilon^{n+r-\triangle_\varphi(M)}) + \log\left((2\pi e)^{\frac{n}{2}}\left(\frac{\kappa}{4\sqrt{n}LL_1}\right)^{\triangle_\varphi(M)-r}\right).$$

Letting $r \to 0$ it follows that

$$\chi_{R+1}(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n; m, \gamma)$$

$$\geq \log(\epsilon^{n - \triangle_\varphi(M)}) + \log\left((2\pi e)^{\frac{n}{2}} \left(\frac{\kappa}{4\sqrt{n}LL_1}\right)^{\triangle_\varphi(M)}\right).$$

This inequality holding for all $\epsilon > 0$ sufficiently small, $m \in \mathbb{N}$, and $\gamma > 0$,

$$\chi(a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n : s_1, \ldots, s_n)$$

$$\geq \log(\epsilon^{n - \triangle_\varphi(M)}) + \log\left((2\pi e)^{\frac{n}{2}} \left(\frac{\kappa}{4\sqrt{n}LL_1}\right)^{\triangle_\varphi(M)}\right).$$

Dividing by $|\log \epsilon|$, taking lim sup's as ϵ goes to 0, and adding n to both sides above yields

$$\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^{p} \frac{\alpha_i^2}{k_i^2}.$$

$\square$

By Theorem 3.10 and Theorem 5.7 we have:

Corollary 5.8. $\delta_0(a_1, \ldots, a_n) = 1 - \sum_{i=1}^{p} \frac{\alpha_i^2}{k_i^2}$.

6. The General Lower Bound

In this section we find a lower bound for $\delta_0(a_1, \ldots, a_n)$. When M is hyperfinite the lower bound will be sharp. By decomposing M over its center it follows that

$$M \simeq M_0 \oplus (\bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C})) \oplus M_\infty, \quad \varphi \simeq \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^{s} \alpha_i tr_{k_i}) \oplus 0$$

where as in the introduction $s \in \mathbb{N} \cup \{0\} \cup \{\infty\}, \alpha_i > 0$ for $1 \leq i \leq s$ $(i \in \mathbb{N})$, M_0 is a diffuse von Neumann algebra or $\{0\}$, φ_0 is a faithful, tracial state on M_0 and $\alpha_0 > 0$ if $M_0 \neq \{0\}$, $\varphi_0 = 0$ and $\alpha_0 = 0$ if $M_0 = \{0\}$, and M_∞ is a von Neumann algebra or $\{0\}$. We remark that M, hyperfinite or otherwise, always admits such a decomposition. We will show that $\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$. Again, because φ vanishes on M_∞ and our main claim concerns the calculation of a lower bound for $\delta_0(a_1, \ldots, a_n)$ assume, without loss of generality, that

$$M = M_0 \oplus (\bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C})), \quad \varphi = \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^{s} \alpha_i tr_{k_i}).$$

We proceed first by finding a suitable set of elements $\{a_1', a_2', a_3'\}$ in the $*$-algebra generated by the a_i such that the packing number of unitary orbits of *certain* microstates of $\Gamma_C(a_1', a_2', a_3'; m, k, \gamma)$ approximate (from below and in a normalized sense) $1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$. These microstates can be obtained as noncommuting polynomials of well-approximating microstates for $\{a_1, \ldots, a_n\}$ and hence the packing number of unitary orbits of such microstates of $\Gamma_C(a_1', a_2', a_3'; m, k, \gamma)$ will provide a lower bound for the packing number of unitary orbits of the microstates for $\{a_1, \ldots, a_n\}$. One can then use asymptotic freeness results to transform these metric entropy quantities into free entropy dimension quantities as in Theorem 4.5 and Theorem 5.7.

Throughout this section write A for the $*$-algebra generated by $\{a_1, \ldots, a_n\}$. We also maintain the notation introduced at the beginning of Section 4. In subsections 6.1 and 6.2 we assume that $M_0 \neq \{0\}$ (which implies $0 < \alpha_0$) and $\alpha_0 < 1$.

6.1. Construction of $\{a_1', a_2', a_3'\}$ when $M_0 \neq \{0\}$ and $\alpha_0 < 1$.

Fix $l \in \mathbb{N}$ with $l \leq s$ and $\varepsilon > 0$. Define $M_1 = \bigoplus_{j=1}^{l} M_{k_j}(\mathbb{C})$ and $M_2 = \bigoplus_{l < j \leq s} M_{k_j}(\mathbb{C})$. M_1 is a finite-dimensional C*-algebra and by Lemma 5.1 has two selfadjoint generators b_1 and b_2. A is strongly dense in M; so Ae is strongly dense in Me where $e = 0 \oplus (\bigoplus_{j=1}^{l} I_j) \oplus 0 \in Z(M)$ and the I_j are as in Section 3. Thus $Ae = Me$. Consequently there exist $a_1', a_2' \in A$ such that

$$a_i' = f_i \oplus b_i \oplus \xi_i \in M_0 \oplus M_1 \oplus M_2 = M.$$

M_0 being diffuse, there exists an $f \in M_0$ such that $\delta_0(f) = 1$ and $sp(f) = [1, 2]$ (here $\delta_0(f)$ is calculated with respect to φ_0). Ae_0 is strongly dense in Me_0 where $e_0 = I_0 \oplus 0 \in M_0 \oplus (\bigoplus_{1 \leq j \leq s} M_{k_j}(\mathbb{C}))$. By Kaplansky's Density Theorem, Proposition 6.14 of [8], and Corollary 6.7 of [9] there exists an $0 \leq a \in A$ satisfying:

- $a = g \oplus b \oplus \xi \in M_0 \oplus M_1 \oplus M_2 = M$.
- $sp(g) \subset [0, 2]$, $\delta_0(g) > 1 - \varepsilon$, and $\varphi_0(\chi_{[0,1/2]}) < \varepsilon$ where for any Borel subset $S \subset \mathbb{R}$, χ_S denotes the spectral projection of g associated to the set S.
- $\|b\| < \frac{1}{6}$.

Since $b \geq 0$ and M_1 is finite dimensional, $sp(b) = \{\beta_1, \ldots, \beta_d\} \subset \mathbb{R}$ where $0 \leq \beta_1 < \ldots < \beta_d \leq \frac{1}{6}$. Define $h : [0, 2] \to \mathbb{R}$ by $h(t) = (t - \beta_1) \cdots (t - \beta_d)$. Observe that:

- $h(b) = 0$;
- $h^{-1}(h([0, 2\beta_d])) \subset [0, 3\beta_d] \subset [0, 1/2]$;
- $h^{-1}(h((2\beta_d, 2])) \subset (\beta_d, 2]$.

The third observation implies that if $\beta \in h(2\beta_d, 2]$, then $h^{-1}(\beta)$ consists of exactly one point in $(\beta_d, 2]$ since h is strictly increasing on $(\beta_d, 2]$. Noting that for all but countably many β, $\chi_{h^{-1}\{\beta\}} = 0$,

$$
\begin{aligned}
\sum_{\beta \in sp(h(b))} \varphi_0(\chi_{h^{-1}\{\beta\}})^2 &= \sum_{\beta \in h(sp(b))} \varphi_0(\chi_{h^{-1}\{\beta\}})^2 \\
&\leq \sum_{\beta \in h([0, 2\beta_d])} \varphi_0(\chi_{h^{-1}\{\beta\}})^2 + \sum_{\beta \in h((2\beta_d, 2])} \varphi_0(\chi_{h^{-1}\{\beta\}})^2 \\
&\leq \varphi_0\Big(\sum_{\beta \in h([0, 2\beta_d])} \chi_{h^{-1}\{\beta\}}\Big) + \sum_{\beta \in [0, 2]} \varphi_0(\chi_{\{\beta\}})^2 \\
&\leq \varphi_0(\chi_{[0, 3\beta_d]}) + \varepsilon \\
&< 2\varepsilon.
\end{aligned}
$$

Define $a_3' = h(a) = h(g) \oplus 0 \oplus h(\xi)$. We have just proven:

Lemma 6.1. *If $l \in \mathbb{N}, 1 \leq l \leq s$, and $\varepsilon > 0$, then there exist $a_1', a_2', a_3' \in A$ of the form $a_i' = f_i \oplus b_i \oplus \xi_i \in M_0 \oplus M_1 \oplus M_2 = M$ satisfying:*

- *$\{b_1, b_2\}$ generates M_1 and $b_3 = 0$;*
- *$\delta_0(f_3) > 1 - \varepsilon$.*

6.2. **Lower Bounds Estimates for** $\delta_0(a_1, \ldots, a_n)$ **when** $M_0 \neq \{0\}$ **and** $\alpha_0 < 1$.
Fix $l \in \mathbb{N}$, $l \leq s$, and $\varepsilon > 0$. Find elements $a_1', a_2', a_3' \in A$ with the properties listed
in Lemma 6.1 and denote M_1, M_2 to have the same meaning as in the preceding
subsection. Suppose $C \geq \max\{\|a_i'\|\}_{i=1,2,3} + 1$. Denote by φ_1 the tracial state
on M_1 obtained by restricting φ to $0 \oplus M_1 \oplus 0$ and normalizing appropriately.
Similarly denote by φ_2 the positive trace (possibly 0) on M_2 obtained by restricting
φ to $0 \oplus 0 \oplus M_2$ and normalizing appropriately. Define $\beta_0 = \alpha_0, \beta_1 = \sum_{i=1}^l \alpha_i$,
$\beta = \min\{\beta_0, \beta_1\}$, and $\beta_2 = 1 - \beta_0 - \beta_1$. Recall the constant κ, L, and L_1 of the
previous section with respect to M_1, φ_1, and $x = b_1 + ib_2$. Finally define e_0 and
e_1 to be the projections $I_0 \oplus 0 \oplus 0$, $0 \oplus (\bigoplus_{j=1}^l I_j) \oplus 0 \in M$, respectively, and
$e_2 = I - e_0 - e_1$.

Lemma 6.2. *If* $D > 0$ *and* $\min\left\{\frac{\beta\kappa}{27DLL_1}, 1\right\} > \epsilon > 0$, *then there exist* $m_\epsilon \in \mathbb{N}$
and $\gamma_\epsilon > 0$ *(dependent on* ϵ*) such if* $\langle(z_1^{(k)}, z_2^{(k)}, z_3^{(k)})\rangle_{k=1}^\infty$ *is a sequence satisfy-*
ing $(z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \in (M_k^{sa}(\mathbb{C}))^3$ *for all* k *and* $(z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \in \Gamma_C(a_1', a_2', a_3' :$
$e_0, e_1, e_2; m_\epsilon, k, \gamma_\epsilon)$ *for sufficiently large* k, *then*

$$\limsup_{k \to \infty} k^{-2} \cdot \log(P_{D\epsilon}(U(z_1^{(k)}, z_2^{(k)}, z_3^{(k)})))$$

dominates

$$(\beta_0 + \beta_1 - \epsilon)^2 \cdot (\chi((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K)$$
$$+ (\beta_1 - \epsilon)^2 (\triangle_{\varphi_1}(M_1) - \epsilon) \cdot \log\left(\frac{\beta\kappa}{27\epsilon DLL_1}\right)$$

where $K = \log((2 + 9\beta^{-1}D)\sqrt{2\pi e})$ *and* s *is a semicircular element free with respect*
to $M_0 \oplus M_1$.

Proof. Suppose $\min\left\{\frac{\beta\kappa}{27DLL_1}, 1\right\} > \epsilon > 0$. By Lemma 4.3 there exist an $m_1 \in \mathbb{N}$ and
γ_1 (dependent on ϵ) such that if $\langle x^{(k)}\rangle_{k=1}^\infty$ is a sequence satisfying $x^{(k)} \in M_k^{sa}(\mathbb{C})$
for all $k \in \mathbb{N}$ and $x^{(k)} \in \Gamma_C(f_3 \oplus 0; m_1, k, \gamma_1)$ for sufficiently large k, then

$$\limsup_{k \to \infty}[k^{-2} \cdot \log P_{9\beta^{-1}D\epsilon}(U(x^{(k)}))] \geq \chi_C((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K$$

where $K = \log((2 + 9\beta^{-1}D)\sqrt{2\pi e})$, $f_3 \oplus 0 \in M_0 \oplus M_1$, $M_0 \oplus M_1$ is endowed with
the tracial state $(\beta_0 + \beta_1)^{-1}(\beta_0\varphi_0 \oplus \beta_1\varphi_1)$, and s is a semicircular element free with
respect to $M_0 \oplus M_1$.

By Corollary 5.6 there exist an $m_2 \in \mathbb{N}$ and $\gamma_2 > 0$ such that if $\langle(y_1^{(k)}, y_2^{(k)})\rangle_{k=1}^\infty$ is
a sequence satisfying $(y_1^{(k)}, y_2^{(k)}) \in (M_k^{sa}(\mathbb{C}))^2$ for all k and $(y_1^{(k)}, y_2^{(k)}) \in \Gamma_C(b_1, b_2;$
$m_2, k, \gamma_2)$ for sufficiently large k, then

$$k^{-2} \cdot \log P_{9\beta^{-1}D\epsilon}(U(y_1^{(k)}, y_2^{(k)})) \geq \log\left(\left(\frac{\beta\kappa}{27\epsilon DLL_1}\right)^{\triangle_{\varphi_1}(M_1) - \epsilon}\right)$$

for sufficiently large k.

By standard approximations for any $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that
for any $k \in \mathbb{N}$ if $(z_1, z_2, z_3) \in \Gamma_C(a_1', a_2', a_3' : e_0, e_1, e_2; m, k, \gamma)$, then the following
conditions are satisfied:

- There exist mutually orthogonal projections $q_0, q_1, q_2 \in M_k(\mathbb{C})$ with $n_0 + n_1 + n_2 = k$ where n_i denotes the dimension of the range of q_i and for each i, $|tr_k(q_i) - \beta_i| < \varepsilon$.
- Canonically identifying $q_0 M_k(\mathbb{C}) q_0 + q_1 M_k(\mathbb{C}) q_1 + q_2 M_k(\mathbb{C}) q_2$ with $\bigoplus_{i=0}^{2} M_{n_i}(\mathbb{C})$ $|z_i - h_i|_2 < \varepsilon$ where $h_i = x_i \oplus y_i \oplus \eta_i \in \bigoplus_{i=0}^{2} M_{n_i}(\mathbb{C}) \subset M_k(\mathbb{C})$ and $y_3 = 0$.
- $x_3 \oplus 0 \in \Gamma_C(f_3 \oplus 0; m_1, n_0 + n_1, \gamma_1)$.
- $(y_1, y_2) \in \Gamma_C(b_1, b_2; m_2, n_1, \gamma_2)$ where $b_1, b_2 \in (M, \varphi_1)$.

Secondly, given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for each $k > N$ there exists a corresponding $k < \tau(k) \in \mathbb{N}$ satisfying $|\frac{k}{\tau(k)} - (\beta_0 + \beta_1)| < \varepsilon$. Combining these two remarks, there exist $m_\epsilon \in \mathbb{N}$ and $\gamma_\epsilon > 0$ such that for any $k \in \mathbb{N}$ sufficiently large there exists a $k < \tau(k) \in \mathbb{N}$ such that if $(z_1, z_2, z_3) \in \Gamma_C(a_1', a_2', a_3' : e_0, e_1, e_2; m_\epsilon, \tau(k), \gamma_\epsilon)$, then the following conditions are satisfied:

- There exist mutually orthogonal projections $q_0, q_1, q_2 \in M_{\tau(k)}(\mathbb{C})$ with $n_0 + n_1 + n_2 = \tau(k)$ where n_i denotes the dimension of the range of q_i and for each i, $|tr_{\tau(k)}(q_i) - \beta_i| < \frac{1}{2} \cdot \min\{\epsilon, \beta\}$.
- $n_0 + n_1 = k$.
- Canonically identifying $q_0 M_{\tau(k)}(\mathbb{C}) q_0 + q_1 M_{\tau(k)}(\mathbb{C}) q_1 + q_2 M_{\tau(k)}(\mathbb{C}) q_2$ with $\bigoplus_{i=0}^{2} M_{n_i}(\mathbb{C})$, $|z_i - h_i|_2 < \frac{\epsilon}{2}$ where $h_i = x_i \oplus y_i \oplus \eta_i \in \bigoplus_{i=0}^{2} M_{n_i}(\mathbb{C}) \subset M_{\tau(k)}(\mathbb{C})$ and $y_3 = 0$.
- $x_3 \oplus 0 \in \Gamma_C(f_3 \oplus 0; m_1, k, \gamma_1)$.
- $(y_1, y_2) \in \Gamma_C(b_1, b_2; m_2, n_1, \gamma_2)$ where $b_1, b_2 \in (M, \varphi_1)$.

Now fix k and (z_1, z_2, z_3) satisfying the aforementioned conditions so that the five properties listed just above hold. Let n_i, h_i, x_i, y_i, and η_i correspond to the fixed k and (z_1, z_2, z_3). Find a set of unitaries $\langle u_s \rangle_{s \in S_k}$ of U_k such that $|S| = P_{9\beta^{-1}D\epsilon}(U(x_3 \oplus 0))$ and for any $s, s' \in S$ and $s \neq s'$,

$$|u_s(x_3 \oplus 0)u_s^* - u_{s'}(x_3 \oplus 0)u_{s'}^*|_2 > 9\beta^{-1}D\epsilon.$$

Find a set of unitaries $\langle v_g \rangle_{g \in G}$ of U_{n_1} such that $|G| = P_{9\beta^{-1}D\epsilon}(U(y_1, y_2))$ and for any $g, g' \in G$ and $g \neq g'$,

$$\max\{|v_g y_i v_g^* - v_{g'} y_i v_{g'}^*|_2 : i = 1, 2\} > 9\beta^{-1}D\epsilon.$$

For $(s, g) \in S_k \times G_k$ define $w_{s,g} \in U_{\tau(k)}$ by $w_{s,g} = (u_s(I_{n_0} \oplus v_g)) \oplus I_{n_2} \in U_{\tau(k)}$.

I claim that the family $\langle (w_{s,g} h_1 w_{s,g}^*, w_{s,g} h_2 w_{s,g}^*, w_{s,g} h_3 w_{s,g}^*) \rangle_{s,g} \in S \times G$ is a $3D\epsilon$-separated set (with respect to the ρ metric defined in section 4). Suppose $(s, g), (s', g') \in S \times G$ and $(s, g) \neq (s', g')$. Then either $s \neq s'$ or $g \neq g'$. In the former case,

$$\begin{aligned}
&|w_{s,g} h_3 w_{s,g}^* - w_{s',g'} h_3 w_{s',g'}^*|_2 \\
&= |w_{s,g}(x_3 \oplus 0 \oplus \eta_3)w_{s,g}^* - w_{s',g'}(x_3 \oplus 0 \oplus \eta_3)w_{s',g'}^*|_2 \\
&= |u_s(x_3 \oplus 0)u_s^* \oplus 0 - u_{s'}(x_3 \oplus 0)u_{s'}^* \oplus 0|_2 \\
&> \sqrt{\frac{\beta_1 + \beta_2}{2}} \cdot 9\beta^{-1}D\epsilon \\
&> 3D\epsilon.
\end{aligned}$$

Suppose $g \neq g'$. We can assume $s = s'$. For $i = 1, 2$,

$$|w_{s,g} h_i w_{s,g}^* - w_{s,g'} h_i w_{s,g'}^*|_2$$
$$= |w_{s,g}(x_i \oplus y_i \oplus \eta_i) w_{s,g}^* - w_{s,g'}(x_i \oplus y_i \oplus \eta_i) w_{s,g'}^*|_2$$
$$= |u_s(x_i \oplus v_g y_i v_g^*) u_s^* \oplus \eta_3 - u_s(x_i \oplus v_{g'} y_i v_{g'}^*) u_s^* \oplus \eta_3|_2$$
$$\geq \sqrt{\frac{\beta_1}{2}} \cdot |v_g y_i v_g^* - v_{g'} y_i v_{g'}^*|_2$$

so that

$$\max\{|w_{s,g} h_i w_{s,g}^* - w_{s,g'} h_i w_{s,g'}^*|_2 : i = 1, 2\}$$
$$\geq \sqrt{\frac{\beta_1}{2}} \cdot \max\{|v_g y_i v_g^* - v_{g'} y_i v_{g'}^*|_2 : i = 1, 2\}$$
$$> \sqrt{\frac{\beta_1}{2}} \cdot 9\beta^{-1} D\epsilon$$
$$> 3D\epsilon.$$

By the inequalities above, $P_{D\epsilon}(U(z_1, z_2, z_3)) \geq P_{3D\epsilon}(U(h_1, h_2, h_3)) \geq |S \times G|$.

Now suppose $\langle (z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \rangle_{k=1}^\infty$ is a sequence satisfying the hypothesis of the lemma with m_ϵ and γ_ϵ as chosen on the previous page. For k sufficiently large,

$$(z_1^{(\tau(k))}, z_2^{(\tau(k))}, z_3^{(\tau(k))}) \in \Gamma_C(a_1', a_2', a_3' : e_0, e_1, e_2; m_\epsilon, \tau(k), \gamma_\epsilon).$$

Thus $(z_1^{(\tau(k))}, z_2^{(\tau(k))}, z_3^{(\tau(k))})$ satisfies the five conditions previously stated. For each k sufficiently large consider the corresponding $\tau(k) \in \mathbb{N}$ and denote $n_i(k)$, $x^{(n_i(k))}$, $y_i^{(n_i(k))}$, S_k, and G_k to be the n_i, x_i, y_i, S, and G, respectively, associated to $(z_1^{(\tau(k))}, z_2^{(\tau(k))}, z_3^{(\tau(k))})$. Then

$$\limsup_{k \to \infty} \tau(k)^{-2} \cdot \log(P_{D\epsilon}(U(z_1^{(\tau(k))}, z_2^{(\tau(k))}, z_3^{(\tau(k))})))$$
$$\geq \limsup_{k \to \infty} \tau(k)^{-2} \cdot (\log |S_k| + \log |G_k|)$$
$$\geq \limsup_{k \to \infty} (\tau(k)^{-2} \cdot \log |S_k|) + \liminf_{k \to \infty} (\tau(k)^{-2} \cdot \log |G_k|).$$

Set $\mathcal{L} = 9\beta^{-1} D\epsilon$. Then $\limsup_{k \to \infty} (\tau(k)^{-2} \cdot \log |S_k|)$ dominates

$$(\beta_0 + \beta_1 - \epsilon)^2 \cdot \limsup_{k \to \infty} \frac{1}{(n_0(k) + n_1(k))^2} \cdot \log(P_{\mathcal{L}}(U(x_3^{(n_0(k))} \oplus 0)))$$
$$= (\beta_0 + \beta_1 - \epsilon)^2 \cdot \limsup_{k \to \infty} \frac{1}{k^2} \cdot \log(P_{\mathcal{L}}(U(x_3^{(n_0(k))} \oplus 0)))$$
$$\geq (\beta_0 + \beta_1 - \epsilon)^2 \cdot \chi_C((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K$$

and since $n_1(k) \to \infty$ as $k \to \infty$,

$$\liminf_{k \to \infty} (\tau(k)^{-2} \cdot \log |G_k|)$$
$$\geq (\beta_1 - \epsilon)^2 \cdot \liminf_{k \to \infty} [n_1(k)^{-2} \cdot \log(P_{\mathcal{L}}(U(y_1^{(n_1(k))}, y_2^{(n_1(k))})))]$$
$$\geq (\beta_1 - \epsilon)^2 \cdot \log \left(\left(\frac{\beta \kappa}{27 \epsilon DLL_1} \right)^{\triangle_{\varphi_1}(M_1) - \epsilon} \right).$$

The desired result follows. $\qquad \square$

We now recycle a familiar argument. There exist polynomials p_1, p_2, p_3 in n noncommuting variables such that for $1 \leq j \leq 3$, $p_j(a_1, \ldots, a_n) = a'_j$. Without loss of generality, we can assume the p_j take n-tuples of selfadjoint elements to selfadjoint elements. Find a constant $1 < \mathcal{C}$ such that for any $(x_1, \ldots, x_n) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^n$, $\|p_j(x_1, \ldots, x_n)\| < \mathcal{C}$. Also, there exists a $D_0 > 0$ such that if $k \in \mathbb{N}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in (M_k^{sa}(\mathbb{C}))_{R+1}$, then for all j,

$$|p_j(x_1, \ldots, x_n) - p_j(y_1, \ldots, y_n)|_2 \leq D_0 \cdot \max\{|x_i - y_i|_2 : 1 \leq i \leq n\}.$$

Lemma 6.3. *For $m \in \mathbb{N}$, $\gamma > 0$, and small enough $\epsilon > 0$ there is a sequence $\langle (x_1^{(k)}, \ldots, x_n^{(k)}) \rangle_{k=1}^{\infty}$ such that $(x_1^{(k)}, \ldots, x_n^{(k)}) \in (M_k^{sa}(\mathbb{C}))^n$ for all k, $(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma)$ for sufficiently large k, and*

$$\limsup_{k \to \infty} k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)})))$$

dominates

$$(\beta_0 + \beta_1 - \epsilon)^2 \cdot (\chi((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K)$$

$$+ (\beta_1 - \epsilon)^2 (\triangle_{\varphi_1}(M_1) - \epsilon) \cdot \log\left(\frac{\beta\kappa}{108\sqrt{n}\epsilon D_0 L L_1}\right)$$

where $K = \log((2 + 36\beta^{-1}\sqrt{n}D_0)\sqrt{2\pi e})$.

Proof. Suppose $\min\left\{\frac{\beta\kappa}{32 D_0 \sqrt{n} L L_1}, 1\right\} > \epsilon > 1$. By Lemma 6.2 there exist $m_\epsilon \in \mathbb{N}$ and $\gamma_\epsilon > 0$ such that if $\langle (z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \rangle_{k=1}^{\infty}$ is a sequence satisfying $(z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \in (M_k^{sa}(\mathbb{C}))^3$ for all k and $(z_1^{(k)}, z_2^{(k)}, z_3^{(k)}) \in \Gamma_C(a'_1, a'_2, a'_3 : e_1, e_2, e_3; m_\epsilon, k, \gamma_\epsilon)$ for sufficiently large k, then

$$\limsup_{k \to \infty} k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}D_0}(U(z_1^{(k)}, z_2^{(k)}, z_3^{(k)})))$$

dominates

$$(\beta_0 + \beta_1 - \epsilon)^2 \cdot (\chi((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K)$$

$$+ (\beta_1 - \epsilon)^2 (\triangle_{\varphi_1}(M_1) - \epsilon) \cdot \log\left(\frac{\beta\kappa}{108\sqrt{n}\epsilon D_0 L L_1}\right).$$

Because $\{a_1, \ldots, a_n\}$ has finite-dimensional approximants, there exists an $N \in \mathbb{N}$ such that for each $k \geq N$ there is an $(x_1^{(k)}, \ldots, x_n^{(k)}) \in ((M_k^{sa}(\mathbb{C}))_{R+1})^n$ satisfying

$$(x_1^{(k)}, \ldots, x_n^{(k)}) \in \Gamma_{R+1}(a_1, \ldots, a_n; m, k, \gamma)$$

and

$$(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) \in \Gamma_{\mathcal{C}}(a'_1, a'_2, a'_3 : e_1, e_2, e_3; m_\epsilon, k, \gamma_\epsilon)$$

where for any $1 \leq j \leq 3$, $p_j(x_1^{(k)}, \ldots, x_n^{(k)}) = y_j^{(k)}$. Note that we can use the cutoff constant $\mathcal{C}$ by the argument of Lemma 4.1.

If $u, v \in U_k$, then $D_0 \cdot \max\{|ux_i^{(k)}u^* - vx_i^{(k)}v^*|_2 : 1 \leq i \leq n\}$ is greater than

$$|p_j(ux_1^{(k)}, \ldots, ux_n^{(k)}u^*) - p_j(vx_1^{(k)}v^*, \ldots, vx_n^{(k)}v^*)|_2 = |uy_j^{(k)}u^* - vy_j^{(k)}v^*|_2.$$

Hence,

$$\limsup_{k \to \infty} k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}}(U(x_1^{(k)}, \ldots, x_n^{(k)})))$$

$$\geq \limsup_{k \to \infty} k^{-2} \cdot \log(P_{4\epsilon\sqrt{n}D_0}(U(y_1^{(k)}, y_2^{(k)}, y_3^{(k)}))).$$

In turn the dominated term is greater than or equal to

$$(\beta_0 + \beta_1 - \epsilon)^2 \cdot (\chi((f_3 \oplus 0) + \epsilon s : s) + |\log \epsilon| - K)$$
$$+ (\beta_1 - \epsilon)^2 (\triangle_{\varphi_1}(M_1) - \epsilon) \cdot \log\left(\frac{\beta \kappa}{108\sqrt{n}\epsilon D_0 L L_1}\right).$$

$\square$

Lemma 6.3 more or less gives the lower bound. We simply use the same sort of arguments which allowed us passage from Lemma 4.4 to Theorem 4.5 and from Lemma 5.5 to Theorem 5.7. Namely Voiculescu's approximate freeness results produce an ϵ-ball, most of whose elements are semicircular microstates trying very hard to be free with respect to a matricial microstate for $\{a_1, \ldots, a_n\}$. Adding the ϵ-ball to the single microstate creates microstates for $\{a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n\}$. The ϵ packing number of the unitary orbit of the microstate yields the same number of disjoint, conjugate balls which are microstates of $\{a_1 + \epsilon s_1, \ldots, a_n + \epsilon s_n\}$. Applying log to this packing number, multiplying by k^{-2} and taking a lim sup as $k \to \infty$, dividing by $|\log \epsilon|$ and taking a lim sup as $\epsilon \to 0$ yields a lower bound for the modified free entropy dimension of the n-tuple $\{a_1, \ldots, a_n\}$.

By the discussion above $\delta_0(a_1, \ldots, a_n)$ should dominate the number obtained by taking the majorized quantity of Lemma 6.3, dividing by $|\log \epsilon|$, and taking a lim sup as $\epsilon \to 0$. The resultant quantity of these successive operations is:

$$(\beta_0 + \beta_1)^2 (\delta_0(f_3 \oplus 0)) + \beta_1^2 \cdot \triangle_{\varphi_1}(M_1)$$
$$> (\beta_0 + \beta_1)^2 \cdot [1 - (\beta_0 + \beta_1)^{-2}(\varepsilon \beta_0^2 + \beta_1^2)] + \beta_1^2 \cdot \triangle_{\varphi_1}(M_1)$$
$$> (\beta_0 + \beta_1)^2 - (\varepsilon + \beta_1^2) + \beta_1^2 \cdot \left(1 - \sum_{i=1}^{l} \frac{\beta_1^{-2}\alpha_i^2}{k_i^2}\right)$$
$$= (\beta_0 + \beta_1)^2 - \varepsilon - \sum_{i=1}^{l} \frac{\alpha_i^2}{k_i^2}.$$

Omitting the details of a familiar analysis we conclude:

Theorem 6.4. *If $M_0 \neq \{0\}$ and $\alpha_0 < 1$, then $\delta_0(a_1, \ldots, a_n) \geq (\beta_0 + \beta_1)^2 - \varepsilon - \sum_{i=1}^{l} \frac{\alpha_i^2}{k_i^2}$.*

Letting $l \to s$ and $\varepsilon \to 0$ we arrive at:

Corollary 6.5. *If $M_0 \neq \{0\}$ and $\alpha_0 < 1$, then $\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$.*

6.3. **General Lower Bound Estimates for $\delta_0(a_1, \ldots, a_n)$.** Now we deal with the situation where $M_0 = \{0\}$ or $\alpha_0 = 1$. Suppose first that $M_0 = \{0\}$. The lower estimates come even easier for, in this case, M is merely a direct sum of matrix algebras. As in section 6.1 given $l \in \mathbb{N}$ with $l \leq s$ and using the same notation, we can construct two selfadjoint elements $a_i' = b_i \oplus \xi_i \in M_1 \oplus M_2$ for $i = 1, 2$ such that they lie in the $*$-algebra generated by $\{a_1, \ldots, a_n\}$ and b_1, b_2 generate M_1. We have a similar packing number estimate in this case:

Lemma 6.6. *For small enough $\epsilon, \gamma > 0$ and any $m \in \mathbb{N}$ there is a sequence $\langle (z_1^{(\triangle_k)}, z_2^{(\triangle_k)}) \rangle_{k=1}^{\infty}$ such that $k < \triangle_k \in \mathbb{N}$ for all k, $(z_1^{(\triangle_k)}, z_2^{(\triangle_k)}) \in M_{\triangle_k}^{sa}(\mathbb{C})$ for all k,*

$$(z_1^{(\triangle_k)}, z_2^{(\triangle_k)}) \in \Gamma_C(a_1', a_2'; m, \triangle_k, \gamma)$$

for sufficiently large k, and

$$\triangle_k^{-2} \cdot \log(P_{16\epsilon}(U(z_1^{(\triangle_k)}, z_2^{(\triangle_k)}))) \geq (\beta_1 - \epsilon)^2 (\triangle_{\varphi_1}(M_1) - \epsilon) \cdot \log\left(\frac{\kappa}{16\beta^{-1}\epsilon LL_1}\right)$$

for sufficiently large k.

The proof mimics that of Lemma 6.2 but it is even easier. Lemma 6.6 requires that we find good enough microstates for a_1', a_2' and this is easy to do (in light of Lemma 3.6). One does not need to deal with the set-up concerning the f_i (they are all 0) and the estimates are similar to those of Lemma 6.2. We omit a rigorous proof of Lemma 6.6 here and leave it to the reader. We now run an argument similar to that which followed Lemma 6.2. Dividing the dominated term above by $|\log \epsilon|$ in the conclusion of the statement of Lemma 6.4 and taking a lim sup as ϵ goes to 0 yields $\beta_1^2 \cdot \triangle_{\varphi_1}(M_1)$. Using the same asymptotic freeness results in the paragraphs preceding Theorem 6.4, it follows that $\delta_0(a_1', a_2')$ is greater than or equal to this limiting process, i.e., greater than or equal to $\beta_1^2 \cdot \triangle_{\varphi_1}(M_1)$. Since M is hyperfinite (M is a direct summand of matrix algebras), by Theorem 4.5, $\delta_0(a_1, \ldots, a_n) \geq \delta_0(a_1', a_2') \geq b_1^2 - \varepsilon - \sum_{i=1}^{l} \frac{\alpha_i^2}{k_i^2}$. Then l and ε being arbitrary, $\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$.

Secondly, suppose $\alpha_0 = 1$. By Section 4, $\delta_0(a_1, \ldots, a_n) = 1$, which yields the desired lower bound.

Having considered all the cases above we have for any M and $a_1, \ldots, a_n$ as in Section 2:

Theorem 6.7. $\delta_0(a_1, \ldots, a_n) \geq 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$.

By Theorem 3.9 we also have:

Corollary 6.8. *If M is hyperfinite, then $\delta_0(a_1, \ldots, a_n) = 1 - \sum_{i=1}^{s} \frac{\alpha_i^2}{k_i^2}$.*

In light of Corollary 6.5, if M is hyperfinite, then we define $\delta_0(M) = \delta_0(a_1, \ldots, a_n)$. As in the introduction we remark that every such hyperfinite M has a finite set of selfadjoint generators.

7. Trivialities and a Final Remark

In concluding the discussion we make a few simple observations about the preceding results. The first is a strengthening of Theorem 4.5. We start with a generalization of Lemma 3.8.

Corollary 7.1. *If $N \subset M$ is a unital inclusion of hyperfinite von Neumann algebras, then $\delta_0(N) \leq \delta_0(M)$.*

Proof. Find selfadjoint generators $a_1, \ldots, a_n$ for M and $b_1, \ldots, b_p$ for N. By Theorem 4.5 and Corollary 6.8,

$$\delta_0(N) = \delta_0(b_1, \ldots, b_p) \leq \delta_0(a_1, \ldots, a_n, b_1, \ldots, b_p) = \delta_0(M).$$

$\square$

We now have:

Corollary 7.2. *(Hyperfinite Monotonicity). If $N \subset M$ is a unital inclusion of von Neumann algebras and N is hyperfinite, then $\delta_0(N) \leq \delta_0(a_1, \ldots, a_n)$.*

Proof. We may assume $M = M_0 \oplus (\bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C}))$ and $\varphi = \alpha_0 \varphi_0 \oplus (\bigoplus_{i=1}^{s} \alpha_i tr_{k_i})$ where all quantities are as in Section 6. Define $A = \mathbb{C}I_0 \oplus (\bigoplus_{i=1}^{s} M_{k_i}(\mathbb{C})) \subset M$. It is easy to see that the von Neumann algebra $\mathcal{R}$ generated by $A \cup N$ is hyperfinite. By decomposing $\mathcal{R}$ over its center and observing that the atomic projections of $Z(\mathcal{R})$ contain those of $Z(M)$, it follows from Theorem 6.7 and Corollary 6.8 that $\delta_0(\mathcal{R}) \leq \delta_0(a_1, \ldots, a_n)$. Hence by Corollary 7.1, $\delta_0(N) \leq \delta_0(\mathcal{R}) \leq \delta_0(a_1, \ldots, a_n)$. $\square$

Our second observation is a weak lower semicontinuity property for δ_0.

Lemma 7.3. *If $\langle (b_1^{(k)}, \ldots, b_n^{(k)}) \rangle_{k=1}^{\infty}$ is a sequence of n-tuples of selfadjoint elements in M such that for each $1 \leq i \leq n$, $b_i^{(k)} \to b_i$ strongly and $\{b_1, \ldots, b_n\}$ generates a diffuse von Neumann algebra, then*

$$\liminf_{k \to \infty} \delta_0(b_1^{(k)}, \ldots, b_n^{(k)}) \geq 1.$$

In particular, if $1 = \delta_0(b_1, \ldots, b_n)$, then

$$\liminf_{k \to \infty} \delta_0(b_1^{(k)}, \ldots, b_n^{(k)}) \geq \delta_0(b_1, \ldots, b_n).$$

Proof. Suppose $\varepsilon > 0$. By the proof of Corollary 4.7, it follows that there exists a $b = b^* \in W^*(b_1, \ldots, b_n)$ such that $\delta_0(b) = 1$. There exists a sequence of noncommuting polynomials in n variables $\langle q_m \rangle_{m=1}^{\infty}$ such that $q_m(b_1, \ldots, b_n)^* = q_m(b_1, \ldots, b_n) \to b$ strongly. It follows from Proposition 6.14 of [9] and Corollary 6.7 of [10] that for m sufficiently large, $\delta_0(q_m(b_1, \ldots, b_n)) > 1 - \frac{\varepsilon}{2}$. Pick one such m and call it m_0. The same proposition of [9] and corollary of [10] provide a corresponding N such that for all $k > N$, $\delta_0(q_{m_0}(b_1^{(k)}, \ldots, b_n^{(k)})) > 1 - \varepsilon$. By Corollary 4.6 for all $k > N$, $\delta_0(b_1^{(k)}, \ldots, b_n^{(k)}) \geq \delta_0(q_{m_0}(b_1^{(k)}, \ldots, b_n^{(k)})) > 1 - \varepsilon$. $\square$

Finally, we comment on the work carried out by Ken Dykema concerning free products of hyperfinite von Neumann algebras with tracial, faithful states. In [2] Dykema investigated the free product of two such algebras A and B. There it was shown that $A * B$ was isomorphic to $L(F_s) \oplus C$ where $L(F_s)$ is an interpolated free group factor and C is a finite-dimensional von Neumann algebra. Moreover, Dykema provided formulas for determining C in terms of the matricial parts of A and B and calculating s in terms of the "free dimensions" of A, B, and C. Given a hyperfinite M as above, Dykema defined the free dimension of M, fdim(M) to be

$$\alpha_0^2 + \sum_{i=1}^{s} \alpha_i^2 (1 - k_i^{-2}) + 2\alpha_0(1 - \alpha_0) + \sum_{1 \leq i,j \leq s, i \neq j} \alpha_i \alpha_j.$$

Using the identity $1 = (\sum_{i=0}^{s} \alpha_i)^2$ one finds that the number above equals $\delta_0(M)$. In other words, for a hyperfinite von Neumann algebra M with a tracial, faithful state, the quantity $\delta_0(M)$ equals the quantity fdim(M).

8. Addendum

In this final section we prove the metric entropy estimates of Lemmas 3.5 and 5.2. The proofs are essentially those of Szarek ([7]) with the addition of the explicit computations of Raymond ([5]).

Throughout H will denote a closed Lie subgroup of U_k. Define $X = U_k/H$, $|\cdot|_{\infty}$ to be the operator norm, $\mathcal{H}$ to be the Lie subalgebra of H identified in $iM_k^{sa}(\mathbb{C}) = \mathcal{G}$, and $\mathcal{X}$ to be the orthogonal complement of $\mathcal{H}$ with respect to the real inner product

on $\mathcal{G}$ generated by $Re\ Tr$. Denote d_∞ and d_2 to be the metrics on X induced by $|\cdot|_\infty$ and $|\cdot|_2$, respectively. Lastly, for a metric d on a space Ω and $\epsilon > 0$ define $N(\Omega, d, \epsilon)$ to be the minimum number of open ϵ-balls required to cover Ω with respect to d and $P(\Omega, d, \epsilon)$ to be the maximum number of points in an ϵ-separated subset of Ω with respect to d.

Szarek uses two essential quantities to obtain the metric entropy estimates in [7]. The first is $\kappa(M)$, the operator norm of the orthogonal projection onto $\mathcal{X}$ where the domain and range of the projection are equipped with the operator norm. The second quantity that Szarek employs are the weaving numbers of X. We will use a slightly modified version of this. The change is based on Szarek's preference to use the geodesic metric on X and my inclination to use the extrinsic norm metric. They are the same for our purposes. Given $\theta > 0$, H is $(\theta, |\cdot|_\infty)$ -woven if for $u \in H, |u - I|_\infty < \theta \Rightarrow \exists h \in \mathcal{H}$ such that $|h|_\infty < \frac{\pi}{16}$ and $e^h = u$. We define $\theta(X)$ to be the supremum over all θ satisfying the preceding condition.

We now state the main result of Szarek's ([7]), slightly altered in our new notation.

Theorem 8.1. *Suppose $\beta \in (0, 1/2]$ and $\min\{\theta(X), \kappa(X)^{-1}\} \geq \beta$. Assume that one of the following conditions holds:*

- *$\dim H \leq (1 - \beta)k^2$.*
- *There exists a subspace $E \subset \mathbb{C}^k$ invariant under H with $\dim E > \beta k$ satisfying $\beta k \leq \dim E \leq (1 - \beta)k$.*
- *There exists a subspace $E \subset \mathbb{C}^k$ invariant under H with $p = \dim E > \beta k$ such that the decomposition $\mathbb{C}^k = E \oplus E^\perp$ induces an isomorphism $H \to U(p) \times H_o$ for some subgroup H_o of U_{k-p}.*

Then for any $\epsilon \in (0, \beta/4)$,

$$\left(\frac{c}{\epsilon}\right)^{\dim X} \leq N(X, d_\infty, \epsilon) \leq \left(\frac{C}{\epsilon}\right)^{\dim X}$$

where $c, C > 0$ are constants depending only on β.

The utility of Szarek's result lies in the fact that the quantities c and C depend only on β. We now provide the proof of Lemma 3.5.

Proof of Lemma 3.5. Suppose H is tractable. Consider the conditional expectation $e : M_k(\mathbb{C}) \to H''$. $I - e$ restricted to $\mathcal{G}$ is the orthogonal projection onto $\mathcal{X}$ and since $\|e\| \leq 1$, it follows that $\kappa(X)^{-1} \geq \frac{1}{2}$. The spectral theorem shows that $\theta(X) > |e^{i\frac{\pi}{16}} - 1|$. Hence, $\min\{\theta(X), \kappa(X)^{-1}\} > \frac{1}{20}$. I claim that H satisfies one of the three conditions as stated in the theorem for $\beta = \frac{1}{20}$. Without loss of generality, assume H takes the form appearing in the definition of a tractable Lie subgroup of U_k. Suppose there exist some $1 \leq j_1, \ldots, j_q \leq m$ such that $\frac{k}{20} \leq \sum_{i=1}^q k_{j_i} l_{j_i} \leq \frac{19k}{20}$. Then H satisfies the second condition of Theorem 8.1. Otherwise there must exist some $1 \leq i \leq m$ for which $k_i l_i > \frac{19k}{20}$. If $k_i = 1$, then H satisfies the third condition of the theorem. Otherwise $k_i > 1$ and this forces there to be a reducing subspace E for H with $\frac{19k}{60} \leq \dim E \leq \frac{k}{2}$ whence H fulfills the second condition of the theorem. Theorem 8.1 now yields the desired result. $\square$

Having dealt with Lemma 3.5, let us turn to the finite-dimensional situation in Lemma 5.2. More generally first consider the viability of the lower bounds of Theorem 8.1 when X is obtained from tractable H and where instead of using d_∞ we

use d_2. Some results of [7] work for unitarily invariant norms and metrics but with the $|\cdot|_2$-norm problems arise. The quantity $\theta(X)$, properly interpreted, does not stay uniformly away from 0 even when we consider the homogeneous spaces in Section 5 associated to a finite-dimensional M. Presumably X would be $(\theta, |\cdot|_2)$-woven if for $u \in H, |u - I|_2 < \theta \Rightarrow \exists h \in \mathcal{H}$ such that $|h|_\infty < \frac{\pi}{16}$ and $e^h = u$. Unfortunately, the homogeneous spaces to which we will restrict our attention (which class is much smaller than the class of homogeneous spaces obtained from tractable H) will fail to have θ values uniformly bounded away from 0. Nevertheless, we still have the key result [7], Lemma 10, where the use of $\theta(X)$ was crucial:

Lemma 8.2. *There exist $\lambda, r > 0$ such that for any $k \in \mathbb{N}$ and tractable H of U_k, if $x, y \in \mathcal{X}$, and $|x|_\infty, |y|_\infty < r$, then*

$$d_2(q(e^x), q(e^y)) \geq \lambda |x - y|_2$$

where $q : U_k \to X$ is the quotient map.

Proof. For r (as yet to be specified) and any such x and y as above, there exists by definition an $h \in \mathcal{H}$ with $|h|_\infty \leq \pi$ satisfying

$$d_2(q(e^x), q(e^y)) = \inf_{v \in H} |e^{-y}e^x - v|_2 = |e^{-y}e^x - e^h|_2.$$

Set $u = e^{-y}e^x$. By the spectral theorem write $h = i\sum_{j=1}^d \beta_j f_j$ where the f_j are mutually orthogonal projections and the β_j are real numbers. We can arrange it so that for each j, $if_j \in \mathcal{H}$, i.e., h takes the block form of $\mathcal{H}$. Define γ_j to be $4r$ if $\beta_j > 4r$, $-4r$ if $\beta_j < -4r$, and β_j if $|\beta_j| \leq 4r$. Set $z = i\sum_{j=1}^d \gamma_j f_j \in \mathcal{H}$. $|z|_\infty \leq 4r$. Define $\Lambda_1 = \{j \in \mathbb{N} : 1 \leq j \leq d, |\beta_j| \leq 4r\}$ and $\Lambda_2 = \{1, \ldots, d\} - \Lambda_1$. Observe that $|x|_\infty, |y|_\infty < r \Rightarrow |u - I|_\infty < 2r$. Then

$$|u - e^z|_2^2 \leq \sum_{j=1}^d |uf_j - e^{i\gamma_j}f_j|_2^2 = \sum_{j \in \Lambda_1} |uf_j - e^{i\beta_j}f_j|_2^2 + \sum_{j \in \Lambda_2} |uf_j - e^{i\gamma_j}f_j|_2^2$$

$$\leq |u - e^h|_2^2 + \sum_{j \in \Lambda_2} (6r)^2 |f_j|_2^2.$$

Now for r sufficiently small, $3r \leq |1 - e^{i4r}|$ (r dependent only upon the exponential map),

$$\sum_{j \in \Lambda_2} (6r)^2 |f_j|_2^2 \leq \sum_{j \in \Lambda_2} 36(|1 - e^{i4r}| - 2r)^2 |f_j|_2^2$$

$$\leq \sum_{j \in \Lambda_2} 36(|1 - e^{i\beta_j}| - 2r)^2 |f_j|_2^2$$

$$\leq \sum_{j \in \Lambda_2} 36(|f_j - e^{i\beta_j}f_j|_2 - |f_j - uf_j|_2)^2$$

$$\leq \sum_{j \in \Lambda_2} 36|uf_j - e^{i\beta_j}f_j|_2^2$$

$$\leq 36|u - e^h|_2^2.$$

It follows that $|u - e^z|_2 \leq 7|u - e^h|_2$.

A repetition of the proof of Lemma 10 in [7] minus the parts referring to $\theta(X)$ shows that there exist $\lambda, r > 0$ (which we can make as small as we want and in particular have r satisfy $3r \leq |1 - e^{i4r}|$) independent of the tractable H such that

for any $x, y \in \mathcal{X}$ with $|x|_\infty, |y|_\infty < r$, $|e^{-y}e^x - e^z|_2 \geq \lambda |x - y|_2$. By what preceded for any $x, y \in \mathcal{X}$ with $|x|_\infty, |y|_\infty < r$,

$$d_2(q(e^x), q(e^y)) = |e^{-y}e^x - e^h|_2 \geq \frac{|e^{-y}e^x - e^z|_2}{7} \geq \frac{\lambda |x - y|_2}{7}.$$

$\square$

The result above does not quite provide the desired lower bounds for the homogeneous space associated to H. Observe that if all the normalized Hilbert-Schmidt quantities are replaced with operator norm quantities, then $P(X, d_\infty, \epsilon)$ is bounded below by the $\frac{\epsilon}{\lambda}$ packing number of the r-ball of the space $\mathcal{X}$ (endowed with $|\cdot|_\infty$). The appropriate lower bounds for packing numbers of balls in finite-dimensional spaces can be obtained through a standard volume comparison argument (see [1] for an example of how this technique yields the packing number bounds). Indeed, this is how the lower bound is achieved in Theorem 8.1. The result above is not quite the same. It shows that $P(X, d_2, \epsilon)$ dominates the $\frac{\epsilon}{\lambda}$ packing number with respect to the $|\cdot|_2$-metric of the ball of $|\cdot|_\infty$-radius r in $\mathcal{X}$. The issue is that we have a lower bound involving *two* different metrics. We want to obtain the appropriate lower bounds by using the volume comparison argument, but our task is slightly complicated by this. We must now examine the ratio of the volumes of balls of radius 1 with respect to $|\cdot|_\infty$ and $|\cdot|_2$ in the space $\mathcal{X}$ associated to H.

Despite the difficulties mentioned, Lemma 5.2 demands lower bounds on the packing numbers of a specific class of homogeneous spaces, in fact, much smaller than the class of all homogeneous spaces obtained from tractable subgroups. Hence, the task at hand is not so daunting. With Lemma 8.2 in hand we now begin the main part of the proof of Lemma 5.2. As discussed at the end of the preceding paragraph, our main objective is to examine the ratio of the volumes of balls in the orthogonal complements of certain Lie subalgebras.

Proof of Lemma 5.2. Maintain all the assumptions made on M and φ in Section 5. We assume that $M \neq \mathbb{C}I$ since Lemma 8.2 clearly holds in this situation. There exist constants $1 > \delta, c_1, c_2 > 0$ such that if $\delta > \varepsilon > 0$ and $r_1, \ldots, r_p \in \mathbb{R}$ satisfy $|r_j - \frac{\alpha_j}{n_j}| < \varepsilon$ for all j, then $c_1 < \sum_{j=1}^p r_j^2 < c_2$ ($M \neq \mathbb{C}I \Rightarrow \sum_{j=1}^p \left(\frac{\alpha_j}{n_j}\right)^2 < 1$). Now suppose $\frac{1}{2} \cdot \min\{\delta, 1 - c_1, \alpha_1, \ldots, \alpha_p\} > \varepsilon > 0$. It is a trivial consequence of Lemma 3.6 that there exists a sequence $\langle \sigma_k \rangle_{k=1}^\infty$ such that for each k, $\sigma_k : M \to M_k(\mathbb{C})$ is a $*$-homomorphism and for k sufficiently large:

- $\|tr_k \circ \sigma_k - \varphi\| < \varepsilon$.
- The set of unitaries H_k of $\sigma_k(M)'$ is a tractable Lie subgroup of U_k and setting $X_k = U_k / H_k$ we have that $k^2(\triangle_\varphi(M) - \varepsilon) \leq \dim(X_k)$.

We must demonstrate the third item in Lemma 5.2 (the lower bound packing estimate) and make sure that the constant κ obtained is independent of ε and ϵ. It can be arranged so that there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the representation σ_k takes the simple form described in the proof of Lemma 3.6. Recall from the proof of Lemma 3.6 that for each $k \geq k_0$ we have the $l_1(k), \ldots, l_{p+1}(k)$, associated to σ_k. Moreover, for such k, the construction of the σ_k in Lemma 3.6 and the bound placed on ε shows that $\frac{3}{2}\alpha_j > \frac{l_j(k)n_j}{k} > \frac{\alpha_j}{2}$ for each j, $c_1 k^2 < \sum_{j=1}^{p+1} l_j(k)^2 < c_2 k^2$, and $l_{p+1}(k) < n_1 \cdots n_p$ for $k > k_0$.

For each k denote $\mathcal{H}_k$ and $\mathcal{X}_k$ to be the spaces $\mathcal{H}$ and $\mathcal{X}$ associated to $H = H_k$. Again, we have translated the packing number problem (the third condition of

Lemma 5.2) into the problem of comparing the volumes of the balls of $\mathcal{X}_k$ with respect to the norms $|\cdot|_\infty$ and $|\cdot|_2$ and finding an appropriate relationship between the two values for sufficiently large k.

For any $r > 0$ denote by $\mathcal{G}_k^r, \mathcal{H}_k^r$, and $\mathcal{X}_k^r$ the balls centered at the origin of operator norm less than or equal to r in $iM_k^{sa}(\mathbb{C}), \mathcal{H}_k$, and $\mathcal{X}_k$, respectively. Consider the conditional expectation e for H_k''. Define $\Phi : iM_k^{sa}(\mathbb{C}) \to \mathcal{H}_k \bigoplus \mathcal{X}_k$ by $\Phi(x) = (e(x), (I - e)x)$. Since e is a contraction when its domain and range are endowed with the operator norm, it follows that $\Phi(\mathcal{G}_k^1) \subset \mathcal{H}_k^1 \times \mathcal{X}_k^2$. Φ is an isometry when its domain and range are endowed with the Hilbert-Schmidt norm (normalized or not). Thus, $\mathrm{vol}(\mathcal{G}_k^1) = \mathrm{vol}(\Phi(\mathcal{G}_k^1)) \leq \mathrm{vol}(\mathcal{H}_k^1) \cdot \mathrm{vol}(\mathcal{X}_k^2)$. Notice that here we calculate the volumes of $\mathcal{H}_k^1$ and $\mathcal{X}_k^2$ in their ambient Hilbert spaces $\mathcal{H}_k$ and $\mathcal{X}_k$ endowed with the real inner product $Re\, Tr$.

We claim that if for each d, $\Lambda_d = \mathrm{vol}(\mathcal{G}_d^1)$ and Θ_d denotes the volume of the ball of radius $\sqrt{d}$ in $\mathbb{R}^{d^2}$, then there exists some constant $1 > \zeta_1 > 0$ such that $(\zeta_1)^{d^2} \leq \frac{\Lambda_d}{\Theta_d}$ for all d. By [5] there exists a constant $c > 0$ such that $\frac{a_d}{b_d} \sim (c)^{2d^2}$ as $d \to \infty$ where a_d is the volume (with respect to the real inner product $Re\, Tr$) of the operator norm unit ball of $M_k(\mathbb{C})$ and b_d is the volume of the ball of radius $\sqrt{2d}$ in $2d^2$-dimensional real Euclidean space. We now use the same trick in the preceding paragraph. Decompose $M_k(\mathbb{C})$ as the orthogonal direct sum $M_k^{sa}(\mathbb{C}) \oplus iM_k^{sa}(\mathbb{C})$. It follows that the operator norm unit ball of $M_k(\mathbb{C})$ is contained in the direct sum of the operator norm unit ball of $M_k^{sa}(\mathbb{C})$ and the operator norm unit ball of $\mathcal{G}_d^1$. The volume of this latter set is $(\Lambda_d)^2$. So $\frac{a_d}{b_d} < \frac{(\Lambda_d)^2}{b_d} \sim \frac{(\Lambda_d)^2}{(\Theta_d)^2}$ and this yields the desired result.

Now $\mathrm{vol}(\mathcal{H}_k^1) < \Theta_{l_1(k)} \cdots \Theta_{l_{p+1}(k)} \cdot \sqrt{n_1 \cdots n_p}$ and because $\dim \mathcal{X}_k > (1 - c_2) \cdot k^2$ for sufficiently large k, it follows that there exists a $\zeta_2 > 0$ (dependent only on $n_1, \ldots, n_p, \zeta_1$ and c_1) such that for sufficiently large k,

$$\mathrm{vol}(\mathcal{X}_k^2) \geq \frac{\mathrm{vol}(\mathcal{G}_k^1)}{\mathrm{vol}(\mathcal{H}_k^1)} \geq \frac{\Theta_k}{\Theta_{l_1(k)} \cdots \Theta_{l_{p+1}(k)}} \cdot \zeta_2^{\dim \mathcal{X}_k},$$

$\frac{l_j(k)}{k} > \frac{\alpha_j}{2n_j}$, $l_{p+1}(k) < n_1 \cdots n_p$ for $k > k_0$, and $(1 - c_2) \cdot k^2 < \dim \mathcal{X}_k < (1 - c_1) \cdot k^2$ for $k > k_0$. These three facts and Stirling's formula show that there exists a constant $\zeta_3 > 0$ (dependent only on the α_i, n_i, c_1, and c_2) such that for sufficiently large k the dominated term above is greater than or equal to $(\zeta_3)^{\dim \mathcal{X}_k} \cdot C_{\dim \mathcal{X}_k}$ where $C_{\dim \mathcal{X}_k}$ is the volume of the ball of radius $\sqrt{k}$ in $\mathbb{R}^{\dim \mathcal{X}_k}$. Notice that this quantity is the volume of the ball of $\mathcal{X}_k$ of $|\cdot|_2$ radius 1. In other words there exists a $\zeta > 0$ (again dependent only on the α_i, n_i, c_1, and c_2) such that for sufficiently large k,

$$\frac{\mathrm{vol}(\mathcal{X}_k^1)}{C_{\dim \mathcal{X}_k}} > (\zeta)^{\dim \mathcal{X}_k}.$$

The standard volume comparison method (for an example of how this method is used see [1]) shows that for such k and any $\epsilon > 0$, $P(\mathcal{X}_k^r, |\cdot|_2, \epsilon) > \left(\frac{r\zeta}{\epsilon}\right)^{\dim \mathcal{X}_k}$. Using Lemma 8.2 it follows that for k sufficently large, if $\epsilon > 0$, then

$$P(X_k, d_2, \epsilon) > \left(\frac{\lambda r \zeta}{\epsilon}\right)^{\dim X_k}.$$

Set $\kappa = \lambda r \zeta$. Then κ depends only on α_i, n_i, c_1, c_2, and the upper bound placed on ε. The upper bound can be relaxed for the purposes of Lemma 5.2. We have the third and final condition of Lemma 5.2. □

Acknowledgements

References

[1] Carl, B. and Stephani, I. *Entropy, Compactness, and the Approximation of Operators*, Cambridge Tracts in Mathematics, No. 98, Cambridge University Press, Cambridge, 1990. MR **92e:**47002

[2] Dykema, K. J., *Free products of hyperfinite von Neumann algebras and free dimension*, Duke Math. J. 69 (1993), 97-119. MR **93m:**46071

[3] Ge, Liming, *Applications of free entropy to finite von Neumann algebras, II*, Annals of Mathematics, 147 (1998), 143-157. MR **99c:**46068

[4] Ge, Liming and Shen, Junhao, *On free entropy dimension of finite von Neumann algebras*, Geometric and Functional Analysis, Vol. 12, (2002), 546-566.

[5] Raymond, Jean Saint, *Le volume des idéaux d'opérateurs classiques*, Studia Mathematica, vol. LXXX (1984), 63-75.

[6] Stefan, M., *The indecomposability of free group factors over nonprime subfactors and abelian subalgebras*, preprint.

[7] Szarek, S. *Metric entropy of homogeneous spaces*, Quantum Probability (Gdansk, 1997), Banach Center Publications vol. 43, Polish Academy of Science, Warsaw, 1998, 395-410. MR **2000c:**53097

[8] Voiculescu, D., Dykema, K.J., and Nica, A. *Free Random Variables*, CRM Monograph Series, v. 1, American Mathematical Society, Providence, RI, 1992. MR **94c:**46133

[9] Voiculescu, D., *The analogues of entropy and of Fisher's information measure in free probability theory, II*, Inventiones Mathematicae 118, (1994), 411-440. MR **96a:**46117

[10] Voiculescu, D., *The analogues of entropy and of Fisher's information measure in free probability theory III: The Absence of Cartan Subalgebras*, Geometric and Functional Analysis, Vol. 6, No.1 (1996), 172-199. MR **96m:**46119

[11] Voiculescu, D., *A strengthened asymptotic freeness result for random matrices with applications to free entropy*, Internat. Math. Res. Notices 1 (1998), 41-63. MR **2000d:**46080

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