Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:4** (2011), 63–73 DOI: 10.2298/FIL1104063G

THE FUNCTION $(b^x - a^x)/x$: LOGARITHMIC CONVEXITY AND APPLICATIONS TO EXTENDED MEAN VALUES

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Abstract

In the paper, we first prove the logarithmic convexity of the elementary function $\frac{b^x-a^x}{x}$, where $x \ne 0$ and $b > a > 0$. Basing on this, we then provide a simple proof for Schur-convex properties of the extended mean values, and, finally, discover some convexity related to the extended mean values.

1 Introduction

For given numbers $b > a > 0$, let

$$g_{a,b}(t) = \begin{cases} \dfrac{b^t - a^t}{t}, & t \ne 0; \\ b - a, & t = 0. \end{cases} \tag{1}$$

This elementary and special function was first dedicated to be investigated in [34, 35]. Subsequently, it was utilized to construct Steffensen pairs in [5, 21, 22, 24] and its reciprocal was also used to generalize Bernoulli numbers and polynomials in [8, 13, 14]. It has something to do with the classical Euler gamma function Γ and the remainder of Binet's first formula for the logarithm of Γ (see, for example, [6, 9, 12, 25, 28, 32, 41] and closely related references therein). More importantly, it was employed not only to provide alternative proofs for the monotonicity of the extended mean values $E(r,s;x,y)$ in [31, 36] but also to create the logarithmic convexity and Schur-convex properties of $E(r,s;x,y)$ in [3, 7, 17, 18, 20], where the extended mean values $E(r,s;x,y)$ were defined in [11, 40] for $x, y > 0$ and $r, s \in \mathbb{R}$ as follows

$$E(r,s;x,y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, \qquad rs(r-s)(x-y) \ne 0;$$

2010 *Mathematics Subject Classifications.* Primary 26A51, 33B10; Secondary 26D07.
Key words and Phrases. Schur-convexity, Extended mean values, Lazarević's inequality.
Received: 5 November 2010
Communicated by Gradimir V. Milovanović
The second author was supported partially by the China Scholarship Council and the Science Foundation of Tianjin Polytechnic University.

$$
\begin{aligned}
E(r,0;x,y)&=\left(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, && r(x-y)\neq 0;\\
E(r,r;x,y)&=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, && r(x-y)\neq 0;\\
E(0,0;x,y)&=\sqrt{xy}, && x\neq y;\\
E(r,s;x,x)&=x, && x=y.
\end{aligned}
$$

There has been a lot of literature on the extended mean values $E(r,s;x,y)$. For more information, please refer to [2, 19] and related references therein.

In this paper, we first present the logarithmic convexity of the function $g_{a,b}(t)$. Basing on this, we then provide a concise proof for Schur-convex properties of the extended men values $E(r,s;x,y)$, and, finally, discover some monotonicity and logarithmic convexity of certain functions related to the extended mean values $E(r,s;x,y)$.

2 Logarithmic convexity of $g_{a,b}(t)$

For the sake of proceeding smoothly, we need the following definition which can be found in [30] and related references therein.

Definition 2.1. A k-times differentiable function $f(t)>0$ is said to be k-log-convex on an interval I if

$$[\ln f(t)]^{(k)}\geq 0,\quad k\in\mathbb{N} \tag{2}$$

on I; If the inequality (2) reverses then f is said to be k-log-concave on I.

Now we are in a position to state and prove the logarithmic convexity of the function $g_{a,b}(t)$ on $(-\infty,\infty)$.

Theorem 2.1. *Let $b>a>0$. Then the function $g_{a,b}(t)$ is logarithmic convex on $(-\infty,\infty)$, 3-log-convex on $(-\infty,0)$, and 3-log-concave on $(0,\infty)$. Consequently, the function*

$$h_{a,b}(t)=\begin{cases}\dfrac{b^t\ln b-a^t\ln a}{b^t-a^t}-\dfrac{1}{t}, & t\neq 0\\ \ln\sqrt{ab}\,, & t=0\end{cases} \tag{3}$$

is increasing on $(-\infty,\infty)$ and satisfies

$$\lim_{t\to-\infty}h_{a,b}(t)=\ln a\quad\text{and}\quad\lim_{t\to\infty}h_{a,b}(t)=\ln b. \tag{4}$$

Proof. For $t\neq 0$, taking the logarithm of $g_{a,b}(t)$ and differentiating yields

$$
\begin{aligned}
\ln g_{a,b}(t)&=\ln\left|b^t-a^t\right|-\ln|t|,\\
[\ln g_{a,b}(t)]'&=\frac{b^t\ln b-a^t\ln a}{b^t-a^t}-\frac{1}{t},
\end{aligned}
$$

$$[\ln g_{a,b}(t)]'' = \frac{1}{t^2} - \frac{a^t b^t (\ln a - \ln b)^2}{(a^t - b^t)^2},$$

and

$$\begin{aligned}
[\ln g_{a,b}(t)]''' &= \frac{a^t b^t (a^t + b^t)(\ln a - \ln b)^3}{(a^t - b^t)^3} - \frac{2}{t^3} \\
&= \frac{2(ab)^{3t/2}}{t^3} \left(\frac{t \ln a - t \ln b}{a^t - b^t} \right)^3 \left\{ \frac{(a/b)^{t/2} + (b/a)^{t/2}}{2} - \left[\frac{(a/b)^{t/2} - (b/a)^{t/2}}{(\ln a - \ln b)t} \right]^3 \right\} \\
&\triangleq \frac{2(ab)^{3t/2}}{t^3} \left(\frac{t \ln a - t \ln b}{a^t - b^t} \right)^3 Q_{a,b}(t),
\end{aligned}$$

where, by using Lazarević's inequality in [1, p. 131] and [10, p. 300],

$$Q_{a,b}\left(\frac{2t}{\ln a - \ln b} \right) = \frac{e^{-t} + e^t}{2} - \left(\frac{e^t - e^{-t}}{2t} \right)^3 = \cosh t - \left(\frac{\sinh t}{t} \right)^3 < 0.$$

Consequently,

$$[\ln g_{a,b}(t)]''' = \left[\frac{g'_{a,b}(t)}{g_{a,b}(t)} \right]'' \begin{cases} > 0, & t \in (-\infty, 0) \\ < 0, & t \in (0, \infty) \end{cases}$$

which implies that the function $[\ln g_{a,b}(t)]''$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. Since

$$\lim_{t \to -\infty} \frac{a^t b^t}{(a^t - b^t)^2} = \lim_{t \to -\infty} \frac{(a/b)^t}{[(a/b)^t - 1]^2} = \lim_{t \to -\infty} \frac{(b/a)^t}{[(b/a)^t - 1]^2} = 0$$

and the function $[\ln g_{a,b}(t)]''$ is even on $\mathbb{R}$, then $[\ln g_{a,b}(t)]'' > 0$, and so the function $[\ln g_{a,b}(t)]' = h_{a,b}(t)$ is increasing on $\mathbb{R}$. Since

$$\frac{b^t \ln b - a^t \ln a}{b^t - a^t} = \frac{(b/a)^t \ln b - \ln a}{(b/a)^t - 1} = \frac{\ln b - (a/b)^t \ln a}{1 - (a/b)^t},$$

then it follows easily that

$$\lim_{t \to -\infty} [\ln g_{a,b}(t)]' = \ln a \quad \text{and} \quad \lim_{t \to \infty} [\ln g_{a,b}(t)]' = \ln b.$$

The L'Hôspital's rule reveals that

$$\begin{aligned}
\lim_{t \to 0} \{[\ln g_{a,b}(t)]'\} &= \lim_{t \to 0} \frac{t(b^t \ln b - a^t \ln a) - (b^t - a^t)}{t(b^t - a^t)} \\
&= \lim_{t \to 0} \frac{y^t (\ln b)^2 - x^t (\ln a)^2}{(b^t - a^t)/t + (b^t \ln b - a^t \ln a)} \\
&= \frac{\ln b + \ln a}{2}.
\end{aligned}$$

The proof of Theorem 2.1 is thus completed. □

Remark 2.1. In the preprint [26], Theorem 2.1 was also verified by using the celebrated Hermite-Hadamard's integral inequality [33, 37, 38] instead of Lazarević's inequality.

Remark 2.2. Theorem 2.1 provides important supplements to the work in [34, 35].

3 A simple proof of Schur-convexity of E

Let us recall from [15, pp. 75–76] the definition of Schur-convex functions.

Definition 3.1. A function f with n arguments defined on I^n is called Schur-convex if $f(x) \le f(y)$ holds for each two n-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ on I^n such that $x \prec y$, where I is an interval with nonempty interior and the relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \tag{5}$$

for $1 \le k \le n-1$, where $x_{[i]}$ denotes the i-th largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

Based on intricate conclusions in [18] and basic properties of $E(r,s;x,y)$, the following Schur-convex properties of the extended mean values $E(r,s;x,y)$ with respect to (r,s) was first obtained in [17].

Theorem 3.1. *With respect to the* 2*-tuple* (r,s)*, the extended mean values* $E(r,s;x,y)$ *are Schur-concave on* $[0,\infty)\times[0,\infty)$ *and Schur-convex on* $(-\infty,0]\times(-\infty,0]$.

The aim of this section is to demonstrate a concise proof of Theorem 3.1 with the help of Theorem 2.1.

Proof. When $y > x > 0$, the extended mean values $E(r,s;x,y)$ may be represented in terms of $g_{x,y}(t)$ as

$$E(r,s;x,y) = \begin{cases} \left[\dfrac{g_{x,y}(s)}{g_{x,y}(r)}\right]^{1/(s-r)}, & (r-s)(x-y) \ne 0; \\ \exp\left[\dfrac{g'_{x,y}(r)}{g_{x,y}(r)}\right], & r = s,\ x-y \ne 0 \end{cases}$$

and

$$\ln E(r,s;x,y) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s \dfrac{g'_{x,y}(u)}{g_{x,y}(u)}\,\mathrm{d}u, & (r-s)(x-y) \ne 0; \\ \dfrac{g'_{x,y}(r)}{g_{x,y}(r)}, & r = s,\ x-y \ne 0. \end{cases} \tag{6}$$

In virtue of Theorem 2.1, it follows that

$$[\ln g_{x,y}(t)]^{(3)} = \left[\frac{g'_{x,y}(t)}{g_{x,y}(t)}\right]'' \begin{cases} < 0, & t \in (0,\infty), \\ > 0, & t \in (-\infty,0). \end{cases} \tag{7}$$

In [4], it was obtained that the integral arithmetic mean

$$\phi(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r \neq s \\ f(r), & r = s \end{cases} \tag{8}$$

of a continuous function f on I is Schur-convex (or Schur-concave, respectively) on I^2 if and only if f is convex (or concave, respectively) on I. Consequently, by virtue of the formula (6) and Definition 3.1, it is not difficult to see that, in order that the extended mean values $E(r,s;x,y)$ are Schur-convex (or Schur-concave, respectively) with respect to (r,s), it is sufficient to show the validity of (7), which may be deduced from Theorem 2.1 straightforwardly. Theorem 3.1 is thus proved. □

Remark 3.1. In [39], an alternative proof of Theorem 3.1 was given, among other things.

4 Some logarithmic convexity related to E

In [29, Remark 6], it was pointed out that the reciprocal of the exponential mean

$$I_{s,t}(x) = \frac{1}{e}\left[\frac{(x+s)^{x+s}}{(x+t)^{x+t}}\right]^{1/(s-t)} \tag{9}$$

for $s \neq t$ is logarithmically completely monotonic on $(-\min\{s,t\},\infty)$ and that the exponential mean $I_{s,t}(x)$ for $s \neq t$ is also a completely monotonic function of first order on $(-\min\{s,t\},\infty)$.

In [16], it was remarked that the logarithmic mean

$$L_{s,t}(x) = L(x+s, x+t) \tag{10}$$

is increasing and concave on $(-\min\{s,t\},\infty)$ for $s \neq t$. In [23], the logarithmic mean $L_{s,t}(x)$ for $s \neq t$ is further proved to be a completely monotonic function of first order on $(-\min\{s,t\},\infty)$.

For $x, y > 0$ and $r, s \in \mathbb{R}$, let

$$F_{r,s;x,y}(w) = E(r+w, s+w; x, y), \quad w \in \mathbb{R}, \tag{11}$$

$$G_{r,s;x,y}(w) = E(r, s; x+w, y+w), \quad w > -\min\{x,y\} \tag{12}$$

and

$$H_{r,s;x,y}(w) = E(r+w, s+w; x+w, y+w), \quad w > -\min\{x,y\}. \tag{13}$$

By virtue of the monotonicity of the extended mean values $E(r,s;x,y)$, it is easy to see that the functions $F_{r,s;x,y}(w)$, $G_{r,s;x,y}(w)$ and $H_{r,s;x,y}(w)$ are increasing with respect to w. Furthermore, since

$$I_{s,t}(x) = E(1,1;x+s,y+t) = G_{1,1;x,y}(w)$$

and

$$L_{s,t}(x) = E(0,1; x+s, y+t) = G_{0,1;x,y}(w),$$

the following problem was posed in [23]: What about the logarithmic convexity of the functions $F_{r,s;x,y}(w)$, $G_{r,s;x,y}(w)$ and $H_{r,s;x,y}(w)$ with respect to w?

The aim of this section is to supply a solution to the above problem about the function $F_{r,s;x,y}(w)$. Our main results are the following theorems.

Theorem 4.1. *The function $F_{r,s;x,y}(w)$ is logarithmically convex on $\left(-\infty, -\frac{s+r}{2}\right)$ and logarithmically concave on $\left(-\frac{s+r}{2}, \infty\right)$.*

Theorem 4.2. *The product $\mathcal{F}_{r,s;x,y}(w) = F_{r,s;x,y}(w)F_{r,s;x,y}(-w)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.*

Theorem 4.3. *If $s+r>0$, the function $w \ln F_{r,s;x,y}(w)$ is convex on $\left(-\frac{s+r}{2}, 0\right)$; If $s+r<0$, it is also convex on $\left(0, -\frac{s+r}{2}\right)$.*

Proof of Theorem 4.1. In the first place, we claim that if $f(t)$ is even on $(-\infty, \infty)$ and increasing on $(-\infty, 0)$, then the function

$$p(t) = f(t+\alpha) - f(t), \quad \alpha > 0 \tag{14}$$

is positive on $\left(-\infty, -\frac{\alpha}{2}\right)$ and negative on $\left(-\frac{\alpha}{2}, \infty\right)$. This can be verified as follows:

1. If $t+\alpha > t > 0$, since $f(t)$ is decreasing on $(0, \infty)$, then $F(t) < 0$;
2. If $t < t+\alpha < 0$, since $f(t)$ is increasing on $(-\infty, 0)$, then $F(t) > 0$;
3. If $t+\alpha > 0 > t$,
   (a) when $t+\alpha > -t > 0$, i.e., $t > -\frac{\alpha}{2}$, using the even and monotonic properties of $f(t)$ shows that $F(t) = f(t+\alpha) - f(-t)$ and it is negative;
   (b) similarly, when $-t > t+\alpha > 0$, i.e., $t < -\frac{\alpha}{2}$, the function $F(t)$ is positive.

The claim is thus proved.

From (6), it follows that if $y > x > 0$ then

$$\frac{\mathrm{d}^2 \ln F_{r,s;x,y}(w)}{\mathrm{d}w^2} = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s \frac{\mathrm{d}^2}{\mathrm{d}w^2}\left[\frac{g'_{x,y}(w+t)}{g_{x,y}(w+t)}\right]\mathrm{d}t, & (r-s)(x-y) \neq 0; \\ \dfrac{\mathrm{d}^2}{\mathrm{d}w^2}\left[\dfrac{g'_{x,y}(w+r)}{g_{x,y}(w+r)}\right], & r = s,\ x-y \neq 0. \end{cases} \tag{15}$$

As shown in the proof of Theorem 2.1, the function $[\ln g_{x,y}(t)]''$ for $y > x > 0$ is even on $\mathbb{R}$ and increasing on $(-\infty, 0)$. Substituting $f(t)$ and α by $[\ln g_{x,y}(t)]''$ and $s-r>0$ in (14) respectively and utilizing (15) demonstrates that

$$\frac{[\ln g_{x,y}(t+s-r)]''_t - [\ln g_{x,y}(t)]''}{s-r} = \frac{\mathrm{d}^2 \ln F_{r,s;x,y}(t-r)}{\mathrm{d}t^2} > 0$$

for $t < -\frac{s-r}{2}$ and that $\frac{\mathrm{d}^2 \ln F_{r,s;x,y}(t-r)}{\mathrm{d}t^2} < 0$ for $t > -\frac{s-r}{2}$. As a result,

$$\frac{\mathrm{d}^2 \ln F_{r,s;x,y}(w)}{\mathrm{d}w^2} = \frac{[\ln g_{x,y}(w+s)]''_w - [\ln g_{x,y}(w+r)]''_w}{s-r} \begin{cases} > 0, & w < -\dfrac{s+r}{2}, \\ < 0, & w > -\dfrac{s+r}{2}. \end{cases}$$

Because $F_{r,s;x,y}(w) = F_{r,s;y,x}(w) = F_{s,r;x,y}(w)$, the above equation holds for all $r, s \in \mathbb{N}$ and $x, y > 0$ with $x \neq y$. Theorem 4.1 is proved. □

Proof of Theorem 4.2. It is easy to see that

$$\left[\ln \mathcal{F}_{r,s;x,y}(w)\right]' = \frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} - \frac{F'_{r,s;x,y}(-w)}{F_{r,s;x,y}(-w)}.$$

Careful computation reveals that

$$\frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} = \frac{F'_{r,s;x,y}(-w-(s+r))}{F_{r,s;x,y}(-w-(s+r))}$$

for $w \in (-\infty, \infty)$. Theorem 4.1 implies that the function

$$q(w) = \frac{F'_{r,s;x,y}(w-(s+r)/2)}{F_{r,s;x,y}(w-(s+r)/2)}$$

is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. It is also apparent that the function $q(w)$ is even, that is, $q(w) = q(-w)$ for $w \in (-\infty, \infty)$. By virtue of the claim verified in the proof of Theorem 4.1, it is easy to see that the difference $q(w + (s+r)) - q(w)$ is positive on $\left(-\infty, -\frac{s+r}{2}\right)$ and negative on $\left(-\frac{s+r}{2}, \infty\right)$, equivalently, the function

$$q\left(w + \frac{s+r}{2}\right) - q\left(w - \frac{s+r}{2}\right) = \frac{F'_{r,s;x,y}(w)}{F_{r,s;x,y}(w)} - \frac{F'_{r,s;x,y}(w-(s+r))}{F_{r,s;x,y}(w-(s+r))} \tag{16}$$

is positive on $(-\infty, 0)$ and negative on $(0, \infty)$. On the other hand, since

$$\mathcal{F}_{r,s;x,y}(w) = \frac{xyF_{r,s;x,y}(w)}{F_{r,s;x,y}(w-(s+r))}, \tag{17}$$

then the function (16) equals $\left[\ln \mathcal{F}_{r,s;x,y}(w)\right]'$. Thus, Theorem 4.2 is proved. □

Proof of Theorem 4.3. Direct calculation yields

$$\left[w \ln F_{r,s;x,y}(w)\right]'' = 2\left[\ln F_{r,s;x,y}(w)\right]' + w\left[\ln F_{r,s;x,y}(w)\right]''. \tag{18}$$

By Theorem 4.1, it follows that $\left[\ln F_{r,s;x,y}(w)\right]' > 0$ on $(-\infty, \infty)$, $\left[\ln F_{r,s;x,y}(w)\right]'' > 0$ on $\left(-\infty, -\frac{s+r}{2}\right)$, and $\left[\ln F_{r,s;x,y}(w)\right]'' < 0$ on $\left(-\frac{s+r}{2}, \infty\right)$. Therefore,

1. if $s+r<0$, then $\big[w\ln F_{r,s;x,y}(w)\big]''>0$, and so $w\ln F_{r,s;x,y}(w)$ is convex on $\big(0,-\frac{s+r}{2}\big)$;
2. if $s+r>0$, then $\big[w\ln F_{r,s;x,y}(w)\big]''>0$, and so $w\ln F_{r,s;x,y}(w)$ is convex on $\big(-\frac{s+r}{2},0\big)$.

The proof of Theorem 4.3 is complete. □

Remark 4.1. Theorem 4.1 generalizes [3, Theorem 1 and Theorem 3] and [20, Theorem 1]. Theorem 4.2 generalizes [3, Theorem 2 and Theorem 3] and [20, Theorem 2]. Theorem 4.3 generalizes [3, Theorem 5].

Remark 4.2. By the same method as in [3, Theorem 4], the function

$$(w+s-r)\big[F_{r,s;x,y}(w)\big]^{s-r}, \quad s>r \tag{19}$$

can be proved to be increasingly convex on $(-\infty,\infty)$ and logarithmically concave on $\big(-\frac{s-r}{2},\infty\big)$.

Remark 4.3. This paper is a slightly revised version of the preprint [27].

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