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THE FUNCTIONAL-DIFFERENTIAL EQUATION $y'(x) = ay(\lambda x) + by(x)$[1]

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ABSTRACT. The paper discusses the functional-differential equation

$$(1) \qquad y'(x) = ay(\lambda x) + by(x) \qquad (0 \leqq x < \infty),$$

where a is a possibly complex constant, b a real constant, and λ a nonnegative constant.

The paper first shows that the boundary-value problem associated with (1) and the boundary condition

$$y(0) = 1$$

is well-posed if $\lambda < 1$, but not if $\lambda > 1$.

The remainder of the paper discusses the asymptotic properties of solutions of the equation as $x \to \infty$. If $\lambda < 1$, it is possible to discuss the asymptotics of *all* solutions of the equation; if $\lambda > 1$, it is shown that, given a specific asymptotic behavior, there is one and only one solution which possesses that asymptotic behavior.

1. Introduction.

The functional-differential equation

$$(1.1) \qquad y'(x) = ay(\lambda x) + by(x) \qquad (0 \leqq x < \infty)$$

arises as a mathematical idealization and simplification of an industrial problem involving wave motion in the overhead supply line to an electrified railway system [1]. (It is curious that the particular case $b = 0$ also appears in the quite different context of a partitioning problem in the theory of numbers [2].) In the problem as it arises in practice, a and b are real constants, and y is a real-valued function, but without significant complication we can (and shall) allow complex values for a while retaining b real. The case $a = 0$ is trivial and we shall always assume $a \neq 0$.

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The quantity λ is a nonnegative constant, the cases $\lambda=0$ and $\lambda=1$ being trivial and so omitted from further consideration, and there are basic differences between the two cases $0<\lambda<1$ and $\lambda>1$. In the practical problem, if $0<\lambda<1$, then the equation (1.1) has associated with it the boundary condition

$$y(0) = 1, \tag{1.2}$$

but if $\lambda>1$, no such boundary condition seems to arise. This is in line with the first two results we prove in this paper, that the boundary-value problem (1.1) and (1.2) is well-posed if $0<\lambda<1$ (Theorem 1), but not if $\lambda>1$ (Theorem 2).

The remainder of the paper is concerned with the asymptotic behavior as $x\to\infty$ of solutions of (1.1), not necessarily also satisfying (1.2), and to see what sort of results are to be expected, we look briefly at some general properties of solutions of (1.1). By the simple change of variable $x=e^s$, $\lambda=e^c$, $y(x)=z(s)$, (1.1) becomes the difference-differential equation

$$e^{-s}z'(s) = az(s+c) + bz(s) \qquad (-\infty < s < \infty),$$

and we shall wish to use this transformation on occasions, although it does not always simplify the analysis to do so. But it does mean that certain well-known properties of solutions of difference-differential equations can be taken over (without proof) to (1.1), and this we now do.

By a solution of (1.1) we mean a complex-valued continuous function $y(x)$ defined in some subinterval of $0\leqq x<\infty$ and satisfying (1.1). If the original interval of definition includes an interval of the form $[\lambda x_0, x_0]$ $(\lambda<1)$ or $[x_0, \lambda x_0]$ $(\lambda>1)$ for some fixed $x_0>0$, then the solution can be extended uniquely to the right (increasing values of x) or to the left (decreasing values of x) by using (1.1), and *by a solution we shall always mean from now on a solution extended as far as it can be in either direction.*

To consider first the case $\lambda<1$, we extend to the right by solving (1.1) as a differential equation for $y(x)$ in the interval $[x_0, \lambda^{-1}x_0]$, $y(\lambda x)$ being known, and the solution is unique because we demand continuity at $x=x_0$. Further, even if the solution is merely continuous in $[\lambda x_0, x_0]$, it will be continuously differentiable in $[x_0, \lambda^{-1}x_0]$ because an integration is involved, and the process can be continued indefinitely to show that the solution can be extended to all of $x\geqq\lambda x_0$, becoming more and more differentiable as x increases. Thus all solutions of (1.1) are well-behaved as $x\to\infty$, and it makes sense to try to discuss the asymptotic behavior of all solutions. This proves to be possible, and is done in Theorems 3–7, the cases $b<0$, $b>0$, $b=0$ having to be considered separately. Theorems 3, 5, 7 discuss general asymptotic behavior, while Theorems 4 and 6 make some remarks about the specific solution which also satisfies (1.2).

If, still with $\lambda<1$, we extend to the left, the solution loses degrees of differentiability instead of gaining them, and if it is, in its original interval of definition, differentiable only a finite number of times, then the extension can be carried out only a finite number of times. In order that the solution can be extended to all of $x>0$, it is therefore necessary that it be infinitely differentiable.

If we turn now to $\lambda>1$, the roles of "right" and "left" are interchanged. A solution defined in $[x_0, \lambda x_0]$ can be extended to all of $x \geqq x_0$ only if it is infinitely differentiable. It is therefore impossible to discuss the asymptotic behavior of the "general" solution, and instead we show that, given a specific asymptotic behavior, there is one and only one solution of (1.1) which exhibits this. Results of this type are discussed in Theorems 8–10, the cases $b>0$, $b<0$, $b=0$ again being considered separately.

In Theorems 8–10, the asymptotic behavior that we investigate has a certain degree of smoothness. If however we consider instead asymptotic behavior given by a distribution, then a rather beautiful duality appears between the cases $\lambda<1$ and $\lambda>1$, in that the conditions (4.3) below which characterize the possible asymptotic behavior of any solution of (1.1) when $\lambda<1$ are strikingly similar to the conditions (10.5) which characterize those solutions of (1.1) which have distributional asymptotic behavior when $\lambda>1$. The discussion of this is given in Theorem 11.

The general field of functional-differential and difference-differential equations is a rapidly growing one [general references are [**3**], [**10**]], but the work closest to ours is that of de Bruijn [[**4**], [**5**], [**6**]], which was brought to our attention by Professor K. Cooke. De Bruijn's work is relevant at various points, but in particular in [**6**] he gives a very complete account of the case $0<\lambda<1$, $b=0$. (It is perhaps curious that the case $b=0$ is in many ways the most difficult one to discuss.) Some work has also been done by E. W. Bowen and G. R. Morris [**7**], their results including a demonstration of non-uniqueness (alternative to that of Theorem 2 of the present paper) for the boundary-value problem (1.1) and (1.2) when $\lambda>1$ by explicitly exhibiting an infinite family of solutions. In [**8**], Frederickson proves the existence of a solution with periodic modulus for a class of equations which includes (1.1) with $\lambda=2$, $a>0$, $b=0$; and in [**9**] he constructs families of analytic solutions of (1.1), and of a generalization of (1.1), and uses these to construct other solutions which have specific properties.

2. Existence and uniqueness for (1.1) and (1.2): the case $0<\lambda<1$.

THEOREM 1. *If* $0<\lambda<1$, *then, given any* $\delta>0$, *the problem defined by* (1.1) *and* (1.2) *in* $[0, \delta]$ *has one and only one solution, and this solution is in fact a solution for all* x *and is an integral function of* x.

PROOF. Existence and uniqueness can be established in the usual way by Picard's method of successive approximations, setting

$$y_0(x) = 1,$$

$$y_n(x) = 1 + \int_0^x \{ay_{n-1}(\lambda t) + by_{n-1}(t)\}\, dt \qquad (n = 1, 2, \cdots).$$

The details are sufficiently well-known to require no exposition.

The method produces the solution in the form of a power series, and we can obtain this form alternatively by trying a solution

$$y = 1 + \sum_{n=1}^{\infty} a_n x^n. \tag{2.1}$$

Substitution in (1.1) and equating coefficients of x^{n-1} gives the recurrence relation

$$na_n = aa_{n-1}\lambda^{n-1} + ba_{n-1},$$

so that

$$a_n/a_{n-1} = (a\lambda^{n-1} + b)/n. \tag{2.2}$$

It follows at once that the power series has infinite radius of convergence, and so the solution is an integral function of x.

3. Existence and uniqueness for (1.1) and (1.2): the case $\lambda>1$.

THEOREM 2. *If* $\lambda>1$, *there is no solution of* (1.1) *and* (1.2) *which is analytic in a neighborhood of* $x=0$. *There are however an infinite number of solutions, even of infinitely differentiable solutions, of* (1.1) *and* (1.2).

REMARK. It is easy to deduce from the proof that every solution of (1.1) tends to a limit as $x\to 0$, although we may not have $y(0)=1$. This is a result comparable to that in [5], and in fact if we make the change of independent variable $x=e^{-t}$, so that $t\to\infty$ as $x\to 0$, our equation has strong similarities to that discussed by de Bruijn, although it does not quite fit under the conditions he imposes.

PROOF. To dispose first of the possibility of an analytic solution, we try a solution of the form (2.1) and are led as in §2 to the recurrence relation (2.2). Since now $\lambda > 1$ and $a \neq 0$ (as we always suppose), the radius of convergence of the power series is 0, as required.

To show that there are an infinite number of infinitely differentiable solutions, let us suppose that a solution is defined by setting $y(x) = f(x)$ in $[x_0, \lambda x_0]$, where $f(x)$ is infinitely differentiable and $x_0 > 0$ is to be fixed later, and then extended to the left by using (1.1). The extended solution will be infinitely differentiable for all x in $0 < x \leqq \lambda x_0$, provided that, as was indicated in §1, it is infinitely differentiable in $[x_0, \lambda x_0]$, which is so since $f(x)$ is infinitely differentiable, and provided that the values of f and its derivatives at $x = x_0$ correspond with those at $x = \lambda x_0$ according to (1.1), i.e. if we have

$$f'(x_0) = af(\lambda x_0) + bf(x_0),$$

and, by successive differentiation of (1.1),

$$f^{(n)}(x_0) = a\lambda^{n-1} f^{(n-1)}(\lambda x_0) + bf^{(n-1)}(x_0) \qquad (n = 1, 2, 3, \cdots). \tag{3.1}$$

To continue the argument, we must now show that it is possible to find infinitely differentiable functions $f(x)$ which satisfy (3.1). The simplest way to do this is to use a function $f(x)$ of the form

$$f^*(x) = \exp\{-(x - \lambda x_0)^{-2} - (x - x_0)^{-2}\} g(x), \tag{3.2}$$

where $g(x)$ is any function infinitely differentiable in $[x_0, \lambda x_0]$. For the exponential factor ensures that f^*, together with all its derivatives, vanishes at both $x = x_0$ and $x = \lambda x_0$, so that (3.1) is certainly satisfied. We shall however require a slightly different form for $f(x)$, and we satisfy (3.1) by finding a function $f(x)$, infinitely differentiable in $[x_0, \lambda x_0]$, such that

$$f(x_0) = a, \qquad f^{(n)}(x_0) = 0 \qquad (n = 1, 2, \cdots),$$
$$f(\lambda x_0) = -b, \quad f^{(n)}(\lambda x_0) = 0 \qquad (n = 1, 2, \cdots).$$

We can achieve such an f by setting

$$f(x) = -b + \int_x^{\lambda x_0} f^*(t)\, dt, \tag{3.3}$$

where f^* is given by (3.2) with g chosen so that

$$\int_{x_0}^{\lambda x_0} f^*(t)\, dt = a + b.$$

This implies that in general g is complex-valued, since a may be, but we shall insist that arg g is constant. This means that, for f defined by (3.3) and x in $[x_0, \lambda x_0]$, $|f|$ does not exceed the greater of $|a|$ and $|b|$. The degree of arbitrariness in g is still very considerable, but we shall not need to specify it further.

Let us suppose that f so defined has upper bound M_0 in $[x_0, \lambda x_0]$, so that M_0 is the larger of $|a|$ and $|b|$ and is therefore independent of the choice of x_0, which we have still to make and which will be small. If we write (1.1) in the form

$$\frac{d}{dx}(ye^{-bx}) = ae^{-bx}y(\lambda x),$$

and integrate, we see that, for the solution y defined by $y(x)=f(x)$ in $[x, \lambda x_0]$, we have, for x in $[\lambda^{-1}x_0, x_0]$,

$$|y(x)e^{-bx} - y(x_0)e^{-bx_0}| \leqq |a| M_0 e^{|b|x_0}\int_x^{x_0} dt. \tag{3.4}$$

But $y(x_0)=a$, and so

$$y(x) = y(x_0)\{1 + [Kx_0]\},$$

where $[Kx_0]$ denotes a term not exceeding Kx_0 in modulus, K being a constant which is independent of the choice of x_0 if x_0 is bounded above, and we are thinking of x_0 as small. Hence we have

$$y(\lambda^{-1}x_0) = y(x_0)\{1 + [Kx_0]\}$$

and also

$$M_1 = |y(x_0)| \{1 + [Kx_0]\},$$

where M_1 is the upper bound for y in $[\lambda^{-1}x_0, x_0]$.

We can now repeat the argument for x in $[\lambda^{-2}x_0, \lambda^{-1}x_0]$, and we obtain, from the inequality corresponding to (3.4).

$$y(x) = y(\lambda^{-1}x_0)\{1 + [K\lambda^{-1}x_0]\} + [K\lambda^{-1}x_0M_1].$$

(The reader will find it easy to verify that there is a common choice of K possible throughout the argument.) Substituting the estimates for $y(\lambda^{-1}x_0)$ and M_1, we have

$$y(x) = y(x_0)\{1 + [Kx_0]\}\{1 + [2K\lambda^{-1}x_0]\},$$

and an inductive argument shows that, for x in $[\lambda^{-(n+1)}x_0, \lambda^{-n}x_0]$,

$$\begin{aligned} y(x) = y(x_0)&\{1 + [Kx_0]\}\{1 + [2K\lambda^{-1}x_0]\} \\ &\cdot\{1 + [2K\lambda^{-2}x_0]\} \cdots \{1 + [2K\lambda^{-n}x_0]\}. \end{aligned} \tag{3.5}$$

The convergence of $\prod_n^\infty (1+\lambda^{-n})$ implies that $y(x)$ is bounded as $x \to 0$, and (1.1) then shows that $y'(x)$ is bounded and so $y(x)$ tends to a finite limit, l say, as $x \to 0$; and if x_0 is chosen sufficiently small, it is clear that there will be no zero factors in (3.5), and that consequently $l \neq 0$. Indeed, by multiplying y by a constant factor (which will not alter the fact that y satisfies (1.1)) we can arrange that $l=1$, and the degree of arbitrariness in our choice of g then shows that there are infinitely many solutions y of (1.1) and (1.2), all infinitely differentiable. This completes the proof.

4. **Asymptotic form of solutions when $\lambda<1$: the case $b<0$.** We recall from §1 that in the case $\lambda<1$ all solutions are defined in an interval of the form (R, ∞), where $R \geqq 0$ may vary from solution to solution, so that we hope to discuss the asymptotic behavior of all solutions of (1.1), regardless of whether or not they also satisfy (1.2). The results depend heavily on the sign of b, but before stating these precisely, we make some intuitive remarks about what might be expected in the case $b<0$.

We might hope most simply for algebraic solutions. If y is to behave like x^k, then since y' is presumably negligible compared with y, we determine k from the equation

$$0 = a(\lambda x)^k + bx^k,$$

so that

$$k = \log(-b/a)/\operatorname{Log} \lambda, \tag{4.1}$$

Log denoting the principal branch of log, so that $-\pi < \operatorname{im}(\operatorname{Log} z) \leqq \pi$. The equation (4.1) determines uniquely re k $(=\kappa$, say) and so $|x^k|$ (by x^k we mean $e^{k \operatorname{Log} x}$), for

$$\kappa = (\operatorname{Log}|b/a|)/\operatorname{Log} \lambda;$$

but it does not determine k; for if k_0 is any one value, then the complete family of values is given by $k=k_m$, where

$$k_m = k_0 + 2m\pi i/\operatorname{Log} \lambda \qquad (m = 0, \pm 1, \pm 2, \cdots), \tag{4.2}$$

and consequently a complete description of the asymptotic behavior of solutions is presumably given by a linear combination of terms of the form x^{k_m}, i.e. of the form $x^{k_0} \exp\{2m\pi i \operatorname{Log} x/\operatorname{Log} \lambda\}$. In view of the Fourier completeness of the complex exponential functions, this would seem to imply that the complete asymptotic behavior is given by $x^{k_0} g(\operatorname{Log} x)$, where g is periodic in $\operatorname{Log} x$ of period $|\operatorname{Log} \lambda|$, and it is essentially this that we prove in Theorem 3.

It should also be remarked that if $a>0$, then k_0 may be taken to be real, so that there is a possible asymptotic behavior which is real and nonoscillatory. (When a is real, it is of course possible to have real solutions, and indeed complex solutions have real and imaginary parts which are separately real solutions.) But if $a<0$, then k_0 must be complex, and if we are interested in real solutions with the asymptotic behavior $x^{k_0}g(\operatorname{Log} x)$, then $g(\operatorname{Log} x)$ must be complex-valued. (That it is possible for a real solution to have such an asymptotic form is best seen by considering re $y(x)$, where $y(x)$ is a solution with the asymptotic form $x^{k_0}g(\operatorname{Log} x)$. Then re $y(x)$ has the asymptotic form

$$\begin{aligned}\operatorname{re}\{x^{k_0}g(\operatorname{Log} x)\} &= \tfrac{1}{2}\{x^{k_0}g(\operatorname{Log} x) + x^{\bar{k}_0}\bar{g}(\operatorname{Log} x)\}\\ &= \tfrac{1}{2}x^{k_0}\{g(\operatorname{Log} x) + x^{\bar{k}_0-k_0}\bar{g}(\operatorname{Log} x)\},\end{aligned}$$

and $g(\operatorname{Log} x)+x^{\bar{k}_0-k_0}\bar{g}(\operatorname{Log} x)$ has period $|\operatorname{Log}\lambda|$ in $\operatorname{Log} x$, just as g has.) Also, if $a<0$, an increase in $\operatorname{Log} x$ of $|\operatorname{Log}\lambda|$ will leave $g(\operatorname{Log} x)$ unchanged and negative x^{k_0}, and so all real solutions must be oscillatory.

Finally, if $|a|<|b|$, then re $k_0<0$, and so the modulus of solutions tends to 0 as $x\to\infty$, while if $|a|>|b|$, the modulus tends to ∞.

THEOREM 3. *Let $b<0$, and let k_0 be any particular solution of* (4.1). *Let $\kappa=\operatorname{re} k_0$.*

(i) *Every solution of* (1.1) *is $O(x^\kappa)$ as $x\to\infty$.*

(ii) *No solution of* (1.1) (*apart from the identically zero solution*) *is $o(x^\kappa)$ as $x\to\infty$.*

(iii) *Given any infinitely differentiable function $g(s)$ which is periodic of period $|\operatorname{Log}\lambda|$ and satisfies the condition that, for some positive K and all n and s,*

$$|g^{(n)}(s)| \leq K^{n+1}\lambda^{-n^2/2}, \tag{4.3}$$

then there is one (*and by* (ii) *only one*) *solution of* (1.1) *which has the asymptotic behavior*

$$x^{k_0}\{g(\operatorname{Log} x)+o(1)\}.$$

Further, every solution y of (1.1) *has this asymptotic behavior for some such $g(\operatorname{Log} x)$, and we can in fact write*

$$y = x^{k_0}\left\{g(\operatorname{Log} x) + \sum_{n=1}^{\infty}\frac{x^{-n}g_n(\operatorname{Log} x)}{b^n(1-\lambda^{-1})(1-\lambda^{-2})\cdots(1-\lambda^{-n})}\right\}, \tag{4.4}$$

where the series is absolutely and uniformly convergent for x sufficiently large, and the functions g_n, all infinitely differentiable and periodic, are *obtained from* $g = g_0$ *by the recurrence formula*

$$g_n' = -(k_0 - n)g_n + g_{n+1}. \tag{4.5}$$

(*Particular functions* g *satisfying* (4.3) *are clearly given by*

$$g(s) = \exp\{2m\pi i s/\text{Log}\,\lambda\}.) \tag{4.6}$$

(iv) *If* $g(s)$ *is a function of period* $|\text{Log}\,\lambda|$ *satisfying* (4.3), *and if the Fourier coefficients of* g *with respect to the functions* (4.6) *are* $\{c_m\}$, *so that*

$$g = \sum_{m=-\infty}^{\infty} c_m \exp\{2m\pi i s/\text{Log}\,\lambda\}, \tag{4.7}$$

then if y *is the solution of* (1.1) *corresponding to* g, *and* y_m *the solution corresponding to* $\exp\{2m\pi i s/\text{Log}\,\lambda\}$, *we have* $y = \sum_{-\infty}^{\infty} c_m y_m$, *the series being absolutely and uniformly convergent for* x *sufficiently large.* (*Thus the functions* y_m *form in some sense a complete set of solutions of* (1.1).)

REMARKS. 1. There is a striking duality between the cases $\lambda < 1$ and $\lambda > 1$, brought out particularly by the equations (4.3) and (10.5) below. Note also the resemblance between (4.4) and (7.2).

2. Note that the recurrence relation (4.5) can otherwise be written

$$\frac{d}{dx}\,(x^{k_0-n} g_n(\text{Log}\,x)) = x^{k_0-n-1} g_{n+1}(\text{Log}\,x), \tag{4.8}$$

so that

$$x^{k_0-(n+1)} g_{n+1}(\text{Log}\,x) = \frac{d^{n+1}}{dx^{n+1}}\,(x^{k_0} g(\text{Log}\,x)). \tag{4.9}$$

3. By taking $s = \text{Log}\,x$ as an independent variable, we can transform (1.1) to the form (4.10) below, when it becomes the type discussed in [4]. We could in fact use de Bruijn's results to prove (i), (ii) and parts of (iii) of the present theorem, but we would not then obtain the estimates we require for the remainder of (iii), and so we prove (i) and (iii) *de novo*. But (ii) is covered by Theorem 7 of [4], and all that we shall do is to show that it is so covered.

4. Having in mind comparison with Theorem 7 below, we note that the characterization of g given by (4.3) can alternatively be expressed in terms of the coefficients c_m in the Fourier expansion of g. In fact, (4.3) is equivalent to

$$c_m = O\{\exp(-\{\text{Log}\,|m|\}^2/2\,|\text{Log}\,\lambda| + K\,\text{Log}\,|m|)\}, \tag{4.9a}$$

for some constant K.

To show that (4.3) implies (4.9a) we differentiate (4.7) p times and apply Parseval's theorem as in the proof of (iv) below. This gives

$$\sum_{m=-\infty}^{\infty} m^{2p} | c_m |^2 \leqq K^{2p+1} \lambda^{-p^2},$$

the constant K being not necessarily the same at each appearance in the argument, and in particular, for any m and all p,

$$| m^p | \; | c_m | \leqq K^{p+1} \lambda^{-p^2/2},$$

which can be written as

$$| c_m | \leqq K \exp\{ p(\operatorname{Log} K - \operatorname{Log} | m |) - \tfrac{1}{2} p^2 \operatorname{Log} \lambda \}.$$

For varying t, the expression $t(\operatorname{Log} K - \operatorname{Log} |m|) - \frac{1}{2}t^2 \operatorname{Log} \lambda$ has a minimum when $t = t_0 = \{\operatorname{Log} K - \operatorname{Log} |m| \} / \operatorname{Log} \lambda$, which may not of course be integral. But the integral value of p which gives the best estimate for $|c_m|$ will be within 1 of t_0, and it is easy to see that such a value of p leads to the estimate (4.9a).

Conversely, if c_m satisfies (4.9a), then, for all s,

$$| g^{(n)}(s) | \leqq K^n \sum_{m=-\infty}^{\infty} \exp\{ (n + K) \operatorname{Log} | m | - (\operatorname{Log} | m |)^2 / 2 | \operatorname{Log} \lambda | \}.$$

It will evidently be sufficient to consider only positive values of m, and then once again the greatest term in the sum occurs when $\operatorname{Log} m$ is within 1 of the value $(n+K)|\operatorname{Log} \lambda|$. It is easy to see that this greatest term satisfies (4.3), and since the terms in the sum decrease on either side of the greatest one, we can, considering sufficiently larger values of m, compare the sum with

$$\int_{\operatorname{Log} x = (n+K) | \operatorname{Log} \lambda |}^{\infty} \exp\{ (n + K) \operatorname{Log} x - (\operatorname{Log} x)^2 / 2 | \operatorname{Log} \lambda | \} \, dx.$$

If we make the obvious transformation $\operatorname{Log} x = u$, the integral becomes

$$\exp\{ \tfrac{1}{2}(n + K + 1)^2 | \operatorname{Log} \lambda | \} \int_{(n+K) | \operatorname{Log} \lambda |}^{\infty} \exp\{ -(2 | \operatorname{Log} |)^{-1} \cdot \{ u - (n + K + 1) | \operatorname{Log} \lambda | \}^2 \} \, du,$$

which again leads to (4.3).

Proof of Theorem 3(i). Let $y(x)$ be any particular solution of (1.1). Set

$$s = \operatorname{Log} x, \qquad c = \operatorname{Log} \lambda \quad (<0), \qquad w(s) = x^{-k_0} y(x).$$

Then w satisfies the equation

$$w'(s) = -k_0 w(s) - be^s\{w(s+c) - w(s)\}, \tag{4.10}$$

and in order to prove (i), we need to show that w is bounded.

Let I_n denote the s-interval $[s_0 - nc, s_0 - (n+1)c]$, s_0 being some fixed value of s to be specified later. Let us suppose that $|w|$ has the upper bound K_n in I_n. Then the equation (4.10) can be written in the form

$$\frac{d}{ds}\{w(s)\exp(k_0 s - be^s)\} = -be^s \exp(k_0 s - be^s) w(s+c), \tag{4.11}$$

so that for s in I_{n+1}, we have

$$\begin{aligned} &[w \exp(k_0 t - be^t)]_{s_0-(n+1)c}^{s} \\ &\qquad = -b \int_{s_0-(n+1)c}^{s} e^t \exp(k_0 t - be^t) w(t+c)\, dt, \end{aligned} \tag{4.12}$$

and using the estimate $|w| \leqq K_n$ in I_n, we obtain

$$\begin{aligned} |w(s)\exp(\kappa s - be^s)| \leqq K_n \Big\{ &\exp[\kappa(s_0 - (n+1)c) - be^{s_0-(n+1)c}] \\ &- \int_{s_0-(n+1)c}^{s} be^t \exp(\kappa t - be^t)\, dt \Big\}. \end{aligned} \tag{4.13}$$

We shall assume, as we may, that s_0 is chosen sufficiently large that $\kappa - \frac{1}{2}be^{s_0} > 0$, which implies in particular that $\kappa t - be^t$ is increasing for t in any I_n. In fact, with future applications in mind where we shall wish to operate with κ replaced by $\kappa - k$ and s_0 replaced by $s_0 - kc$, k a nonnegative integer, we shall suppose that $\kappa - k - \frac{1}{2}be^{s_0-kc} > 0$ for all integers $k \geqq 0$. Since e^{-kc} is of higher order than k as $k \to \infty$, it is clear that such a choice of s_0 is possible.

Then the integral in (4.13) can be integrated by parts to give

$$\begin{aligned} &\left[\frac{be^t}{\kappa - be^t}\exp(\kappa t - be^t)\right]_{s_0-(n+1)c}^{s} \\ &\qquad + O\left[\exp(\kappa s - be^s)\int_{s_0-(n+1)c}^{s}\left|\frac{d}{dt}\frac{be^t}{\kappa - be^t}\right| dt\right], \end{aligned} \tag{4.14}$$

the O-term being uniform in s, κ, s_0 and n, subject to the provisos that

$s \in I_{n+1}$ and that s_0 is chosen as above. Formula (4.14) can be rewritten as

$$(4.15)\qquad \begin{aligned} -\left[\left(1-\frac{\kappa}{be^t}\right)^{-1}\exp(\kappa t - be^t)\right]_{s_0-(n+1)c}^{s} &+ O[\kappa e^{-s_0+(n+1)c}\exp(\kappa s - be^s)] \\ = -\left[\exp(\kappa t - be^t)\right]_{s_0-(n+1)c}^{s} &+ O[\kappa e^{-s_0+(n+1)c}\exp(\kappa s - be^s)]. \end{aligned}$$

Substituting this back into (4.13), we obtain finally that, in I_{n+1},

$$(4.16)\qquad |w(s)| \leqq K_n\{1 + O[\kappa e^{-s_0+(n+1)c}]\},$$

so that

$$(4.17)\qquad K_{n+1} \leqq K_n\{1 + O[\kappa e^{-s_0+(n+1)c}]\}.$$

The convergence of the product $\prod_n (1+e^{-n})$ now implies that K_n is bounded for all n, and so w is bounded, as required. Also, if κe^{-s_0} is sufficiently small, it will follow that

$$\prod_{n=0}^{\infty}\{1 + O[\kappa e^{-s_0+(n+1)c}]\} \leqq 2,$$

and then $K_n \leqq 2K_0$ for all n, i.e. $|w(s)| \leqq 2K_0$ for s in any I_n. (We note for future purposes that $(\kappa - k)e^{-s_0+kc}$ can be chosen uniformly small for all integers $k \geqq 0$.)

The result (i) is now proved, but it is convenient at this point to look at some consequences of the analysis which will be useful in discussing (iii). We note first that y' satisfies the equation

$$(4.18)\qquad y''(x) = a\lambda y'(\lambda x) + by'(x),$$

which is of the same type as (1.1) except that a is replaced by $a\lambda$. The argument above shows that

$$(4.19)\qquad \begin{aligned} y' &= O\{x^{\operatorname{Log}|b/a\lambda|/\operatorname{Log}\lambda}\} \qquad \text{as } x \to \infty \\ &= O(x^{\kappa-1}), \end{aligned}$$

and in general, for the nth derivative $y^{(n)}$, we have

$$(4.20)\qquad y^{(n)} = O(x^{\kappa-n}).$$

We can also relate the estimate on y' more closely to that on y. For if we suppose as before that $|w| \leqq K_0$ in I_0, then, as we have seen, $|w| \leqq 2K_0$ in all I_n ($n \geqq 0$), and so $|y| \leqq 2K_0 x^{\kappa}$ in all I_n ($n \geqq 0$). (It will cause no confusion to use I_n interchangeably for either an s-interval or the corresponding x-interval.) From (1.1) it follows that $|y'| \leqq 4K_0|b|x^{\kappa}$ in all I_n ($n \geqq 1$), and so if $w_1(s) = x^{-k_0+1}y'(x)$, we have

$|w_1(s)| \leqq 4K_0|b|x$ in all I_n $(n \geqq 1)$. In particular, in I_1, $|w_1| \leqq 4K_0|b|\lambda^{-2}x_0$, where $x_0 = e^{s_0}$. But w_1 satisfies an equation similar to (4.10), the only change being that $k_0 \to k_0 - 1$, and we can therefore carry out on w_1 the same analysis as on w, remembering that k_0 has to be replaced by $k_0 - 1$ and that I_0 has to be replaced by I_1 as the base interval, so that s_0 is replaced by $s_0 - c$. The analysis is thus valid for w_1 because of the requirements made above that $\kappa - k - be^{s_0 - kc} > 0$ and $(\kappa - k)e^{-s_0 + kc}$ is uniformly small for all nonnegative integers k, and we conclude that $|w_1| \leqq 8K_0|b|\lambda^{-2}x_0$ in all I_n $(n \geqq 1)$.

It is now easy to repeat the argument to prove the induction hypothesis that, if

$$w_k(s) = x^{-k_0+k}y^{(k)}(x), \tag{4.21}$$

then

$$\begin{aligned} |w_k| &\leqq 2K_0(4|b|x_0)^k\lambda^{-(2+3+\cdots+(k+1))} \\ &= 2K_0\lambda(4|b|x_0)^k\lambda^{-(k+1)(k+2)/2} \end{aligned}$$

for s in any I_n $(n \geqq k)$. By a suitable choice of the constant R, we can in fact say that

$$|w_k| \leqq R^{k+1}\lambda^{-k^2/2} \tag{4.22}$$

for s in any I_n $(n \geqq k)$.

Proof of Theorem 3(ii). As was remarked earlier, we shall show only that this is covered by Theorem 7 of [4]. This latter theorem is (with a trivial change of notation) as follows:

Consider the equation

$$u(t)f'(t) + p(t)f(t) - f(t-1) = 0. \tag{4.22a}$$

Let B and ρ be positive constants, $\rho > 1$, and suppose that for $t \geqq 1$ the functions $u^{(n)}(t)$ and $p^{(n)}(t)$ $(n = 0, 1, 2, \cdots)$ are continuous and satisfy

$$\begin{aligned} |u^{(n)}(t)| &< B^{n+1}n^n t^{-n-\rho}, \\ |\{p(t) - 1\}^{(n)}| &< B^{n+1}n^n t^{-n-\rho} \qquad (0^0 = 1). \end{aligned} \tag{4.22b}$$

Then, if $f(t)$ is a solution of (4.22a) *and* $\lim_{t\to\infty} f(t) = 0$, *we have* $f(t) \equiv 0$.

If we now look at our equation (4.10), and set

$$s = -ct, \qquad w(s) = f(t),$$

we can reduce (4.10) to

$$(bc)^{-1}e^{ct}f'(t) + (1 - k_0e^{ct}/b)f(t) - f(t-1) = 0,$$

so that, by comparison with (4.22a),

$$u(t) = (bc)^{-1}e^{ct}, \qquad p(t) = 1 - k_0e^{ct}/b.$$

To ensure that the conditions (4.22b) are satisfied (and it will be sufficient to consider $u(t)$), we must therefore establish that

$$\left| (bc)^{-1}c^n e^{ct} \right| < B^{n+1}n^n t^{-n-\rho} \qquad (t \geqq 1;\, n = 0, 1, 2, \cdots),$$

i.e.

$$\left| c^n \right| t^{n+\rho}e^{ct} < \left| bc \right| B^{n+1}n^n.$$

But the maximum of $t^{n+\rho}e^{ct}$ occurs at $t = -(n+\rho)/c$, and so it will be sufficient to show that

$$\left| c^{-\rho} \right| (n+\rho)^{(n+\rho)}e^{-(n+\rho)} < \left| bc \right| B^{n+1}n^n,$$

i.e. that

$$\left| c^{-\rho} \right| (1 + \rho/n)^n(n+\rho)^\rho e^{-(n+\rho)} < \left| bc \right| B^{n+1}.$$

Since $(1+\rho/n)^n$ is bounded for all n, we can plainly choose B sufficiently large that the last inequality holds for all n.

PROOF OF THEOREM 3(iii). We start by showing that every solution of (1.1) has the required asymptotic behavior. If with the notation of (i) we set

$$y(x) = x^{k_0}w(s), \tag{4.23}$$

substitute in (1.1) and use (4.19), we obtain

$$w(s+c) - w(s) = O(e^{-s}).$$

Writing this equation with s replaced successively by $s-c, s-2c, \cdots$, and summing, we see that the convergence of $\sum_n e^{nc}$ for $c<0$ implies the convergence of $w(s-nc)$ as $n\to\infty$, s being held fixed. If we denote the limit function by $g(s)$, then it is clear that g is a periodic function of period $|c|$, and continuous since the convergence is uniform; and the summation process shows that

$$w(s) - g(s) = O(e^{-s}),$$

so that (4.23) can be rewritten

$$y(x) = x^{k_0}g(s) + O(x^{\kappa-1}),$$

which is the required asymptotic form if we can show that g is infinitely differentiable and satisfies (4.3).

To do this, we treat y' in the same way, starting with (4.18) in place of (1.1). We set $y'(x)=x^{k_0-1}w_1(s)$, and show that $w_1(s-nc)$ has the limit $g_1(s)$, where g_1 is continuous and periodic of period $|c|$, and that

$$y'(x) = x^{k_0-1}g_1(s) + O(x^{\kappa-2}).$$

Further,

$$\begin{aligned} w'(s) &= \frac{d}{ds}\left\{e^{-k_0 s}y(e^s)\right\} \\ &= -k_0 w(s) + e^{-(k_0-1)s}y'(e^s) = -k_0 w(s) + w_1(s), \end{aligned}$$

and so in the limit we deduce that g is continuously differentiable and that

$$g'(s) = -k_0 g(s) + g_1(s).$$

The process can be repeated to show that g_1 is continuously differentiable and that g is twice continuously differentiable and so ultimately infinitely differentiable.

To show that (4.3) is satisfied, we define $w_k(s)$ by (4.21), and observe, by analogy with what we have just done, that w_k has a continuous periodic limit g_k, say, and that

$$g_k' = -(k_0-k)g_k + g_{k+1}. \tag{4.24}$$

Further, the estimate (4.22) implies that

$$|g_k| \leqq R^{k+1}\lambda^{-k^2/2},$$

and so (4.24) gives

$$\begin{aligned} |g_k'| &\leqq R^{k+2}\lambda^{-(k+1)^2/2}\{|k_0-k|R^{-1}\lambda^{k+1/2}+1\} \\ &\leqq R^{k+3}\lambda^{-(k+1)^2/2} \end{aligned}$$

for all k, provided that the constant R is chosen large enough, as we may suppose. And if we differentiate (4.24) n times, and repeat the analysis, we can set up and prove the induction hypothesis in n that, for all k,

$$|g_k^{(n)}| \leqq R^{k+2n+1}\lambda^{-(k+n)^2/2}.$$

In particular, with $k=0$, we have

$$|g_0^{(n)}| \leqq R^{2n+1}\lambda^{-n^2/2},$$

and since $g_0=g$, this at once implies (4.3).

To complete the proof of (iii), we have merely to establish that under the conditions (4.3), the series (4.4) is absolutely and uniformly convergent for x sufficiently large and that term-by-term differentiation is justified; for it is trivial that, formally at least, it satisfies (1.1). To prove the convergence, we need estimates on each g_n. We are given that, for all nonnegative integers k,

$$|g_0^{(k)}| \leqq K^{k+1}\lambda^{-k^2/2}.$$

Let us suppose, again for all k, that

$$|g_p^{(k)}| \leqq K^{k+2p+1}\lambda^{-(k+p)^2/2} \tag{4.25}$$

for $p=0, 1, 2, \cdots, n$, and prove it for $p=n+1$. Differentiate (4.5) k times, and use the induction hypothesis to obtain

$$\begin{aligned} |g_{n+1}^{(k)}| &\leqq |k_0-n|\, K^{k+2n+1}\lambda^{-(k+n)^2/2} + K^{k+2n+2}\lambda^{-(k+n+1)^2/2} \\ &\leqq K^{k+2n+2}\lambda^{-(k+n+1)^2/2}\{|k_0-n|\, K^{-1}\lambda^{k+n+1/2}+1\} \\ &\leqq K^{k+2n+3}\lambda^{-(k+n+1)^2/2} \end{aligned}$$

if K is large enough, as we may suppose. This proves (4.25) for $p=n+1$ and so for all p; and in particular, with $k=0$, we have

$$|g_p| \leqq K^{2p+1}\lambda^{-p^2/2}.$$

This estimate is easily seen to be sufficient to prove the convergence of (4.4), and to justify term-by-term differentiation.

PROOF OF THEOREM 3(iv). Since g is infinitely differentiable, we can differentiate (4.7) as often as we please, and the resulting series is always absolutely and uniformly convergent. In fact,

$$g^{(p)}(s) = \sum_{n=-\infty}^{\infty}\left(\frac{2m\pi i}{\operatorname{Log}\lambda}\right)^p c_m \exp\{2m\pi i s/\operatorname{Log}\lambda\},$$

and Parseval's theorem gives

$$\sum_{m=-\infty}^{\infty}\left(\frac{2m\pi}{\operatorname{Log}\lambda}\right)^{2p}|c_m|^2 = |\operatorname{Log}\lambda|^{-1}\int_0^{|\operatorname{Log}\lambda|}|g^{(p)}(u)|^2\,du,$$

so that, using (4.3), with a slightly different interpretation of the constant K,

$$\sum_{m=-\infty}^{\infty} m^{2p}|c_m|^2 \leqq K^{2p+1}\lambda^{-p^2}. \tag{4.26}$$

Now consider the series $\sum c_m y_m$, which, if we substitute for y_m, the series representation given by (4.4) with $g=\exp\{2m\pi i s/\text{Log }\lambda\}$ can be written as the double series

$$\sum_{m=-\infty}^{\infty} c_m x^{k_m} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} c_m \frac{k_m(k_m-1)\cdots(k_m-n+1)}{(1-\lambda)\cdots(1-\lambda^n)}\lambda^{n(n+1)/2}(-b)^{-n}x^{k_m-n}, \tag{4.27}$$

k_m being given by (4.2). Since it is formally trivial that this series satisfies (1.1) and has the correct asymptotic behavior, it is formally the solution y corresponding to g. All that is necessary is to rigorize the formalities by showing the series to be absolutely and uniformly convergent for x sufficiently large, and so far as the first term in (4.27) is concerned, there is no difficulty. The second term we rewrite as

$$\sum_{n=1}^{\infty}\left\{\sum_{|k_m|\le n-1} + \sum_{|k_m|>n-1}\right\} c_m \frac{k_m(k_m-1)\cdots(k_m-n+1)}{(1-\lambda)\cdots(1-\lambda^n)}\lambda^{n(n+1)/2}(-b)^{-n}x^{k_m-n} = S_1+S_2,$$

say. In S_1 we can majorize the typical term (apart from a common factor x^{k_0} which need not worry us) by

$$O\{c_m[2(n-1)]^n\lambda^{n(n+1)/2}|b|^{-n}x^{-n}\}, \tag{4.28}$$

and if we remember that $\sum c_m$ is certainly absolutely convergent, the result of summing (4.28) over the relevant values of m is

$$O\{[2(n-1)]^n\lambda^{n(n+1)/2}|b|^{-n}x^{-n}\}.$$

But

$$\sum_n \{2(n-1)\}^n\lambda^{n(n+1)/2}|b|^{-n}x^{-n}$$

is absolutely and uniformly convergent for x sufficiently large, and so S_1 is dealt with.

In S_2, we majorize the typical term by

$$O\{c_m k_m^n\lambda^{n(n+1)/2}|b|^{-n}x^{-n}\},$$

and summing over m and using (4.26), we have

$$O\{M^n\lambda^{-(n+1)^2/2}\lambda^{n(n+1)/2}|b|^{-n}x^{-n}\}$$

for some positive constant M. For $k_m = O(m)$ and

$$\sum_{m=-\infty}^{\infty} m^n |c_m| \leqq \left\{ \sum_{m=-\infty}^{\infty} m^{2n+2} |c_m|^2 \right\}^{1/2} \left\{ \sum_{m=-\infty;\, m\neq 0}^{\infty} m^{-2} \right\}^{1/2}.$$

But

$$\sum_n M^n \lambda^{-(n+1)/2} |b|^{-n} x^{-n}$$

is absolutely and uniformly convergent for x sufficiently large, and so the proof is complete.

We complete the discussion of the case $b < 0$ by making a few remarks about the particular solution which also satisfies (1.2).

THEOREM 4. *Let y denote the particular solution of* (1.1) *which also satisfies* (1.2). *Then*

(i) *the series $\sum c_m y_m$ for y has either just one nonzero coefficient or else an infinite number; the first possibility occurs if and only if $a\lambda^N + b = 0$ for some nonnegative integer N, and this situation is equivalent to y being a polynomial of degree N with all its zeros imaginary;*

(ii) *if $a > 0$ and $a + b < 0$, then for all x we have*

$$y > 0, \quad y' < 0, \quad y'' > 0, \cdots, y^{(2n)} > 0, \quad y^{(2n+1)} < 0, \cdots;$$

(iii) *if $a > 0$ and $a + b > 0$, and if p is the unique nonnegative integer such that $a\lambda^p + b > 0$, $a\lambda^{p+1} + b < 0$ (the case that some p should satisfy $a\lambda^p + b = 0$ being covered in* (i)), *then, for all x,*

$$y > 0, \quad y' > 0, \cdots, y^{(p+1)} > 0,$$
$$y^{(p+2)} < 0, \cdots, y^{(p+2n)} < 0, \quad y^{(p+2n+1)} > 0, \cdots.$$

(If $a < 0$, we have already remarked before the statement of Theorem 3 that all real solutions of (1.1), including the particular one with which we are now concerned, must be oscillatory, so that no results such as (ii) and (iii) can hold.)

PROOF OF THEOREM 4(i). Since y is an integral function of x, and since each y_m is analytic except at $x = 0$ and is in modulus not larger than algebraic as $|x| \to \infty$, it follows that if the series $y = \sum c_m y_m$ has only a finite number of nonzero coefficients, then y is an integral function in modulus not larger than algebraic as $|x| \to \infty$, and therefore a polynomial. But by direct substitution of a polynomial $\sum_{r=0}^{N} a_r x^r$ in (1.1), we see, comparing coefficients of x^N, that $a\lambda^N + b = 0$.

Conversely, if $a\lambda^N + b = 0$, then if we compute the power series for the solution y of (1.1) and (1.2), we see that the power series terminates and that y is a polynomial of degree N. Further, we may take

k_0 in Theorem 3 to be N, and y, which clearly has the asymptotic form $Ax^{k_0}\{1+o(1)\}$ for some constant A, is just a multiple of the solution y_0. Hence there is only one nonzero term in $\sum c_m y_m$.

That all the zeros of the polynomial are imaginary follows a similar argument to the one in (ii) below which shows that $y>0$ for all x.

PROOF OF THEOREM 4(ii). Let us suppose that $y \not> 0$, so that y vanishes, say for the first time at $x=x_0$. Then, in view of the equation

$$\frac{d}{dx}\{ye^{-bx}\} = ay(\lambda x)e^{-bx}, \tag{4.29}$$

it follows, since $\lambda<1$ and $a>0$, that ye^{-bx} is increasing at $x=x_0$. But if y changes sign from positive to negative at $x=x_0$. then so also does ye^{-bx}, and this leads to a contradiction. Hence $y>0$.

The same argument can be applied to y' using (4.18) in place of (1.1). But y' is initially negative since $a+b < 0$, and so y' remains negative, i.e. $y'<0$; and the argument can be extended to higher derivatives to complete the proof.

PROOF of THEOREM 4(iii). The reader will be able to adapt the proof in (ii) to this case.

5. Asymptotic form of solutions when $\lambda<1$: the case $b>0$.

THEOREM 5. (i) *Every solution of* (1.1) *is* $O(e^{bx})$ *as* $x\rightarrow\infty$. *In fact, for any solution* y,

$$ye^{-bx} \rightarrow L \qquad as\ x \rightarrow \infty \tag{5.1}$$

for some (possibly zero) constant L.

(ii) *If* $|a|<b$, *we can, for any* L, *explicitly exhibit a particular solution of* (1.1), y_L *say, possessing the property* (5.1), *viz.*

$$y_L = Le^{bx}\left\{1 + \sum_{n=1}^{\infty}(-1)^n\frac{a^n\exp\{-b(1-\lambda^n)x\}}{b^n(1-\lambda)(1-\lambda^2)\cdots(1-\lambda^n)}\right\}. \tag{5.2}$$

For other values of a *and* b, *we can argue the existence of a solution* y_L *satisfying* (5.1).

(iii) *If* y_L^* *is any other solution of* (1.1) *possessing the property* (5.1), *then*

$$v_L - y_L^* = O(x^\kappa) \qquad as\ x\rightarrow\infty,$$

where κ *is defined as in Theorem* 3, *i.e.* $\kappa=\text{re}\ k_0$, *where* k_0 *is any solution of* $a\lambda^k+b=0$.

(iv) *No solution of* (1.1), *apart from the identically zero solution, is* $o(x^\kappa)$ *as* $x\rightarrow\infty$.

(v) *Given any infinitely differentiable function* $g(\text{Log } x)$ *of the type described in* (iii) *of Theorem* 3, *there is one* (*and by* (iv) *only one*) *solution of* (1.1) *which has the asymptotic behavior*

$$x^{k_0}\{g(\text{Log } x) + o(1)\}, \tag{5.3}$$

and every solution of (1.1) *has the form*

$$y = y_L + y^*,$$

for some constant L, *where* y^* *is a solution having the asymptotic behavior* (5.3) *for some such* $g(\text{Log } x)$. *Also,* y^* *can be written in the form* $\sum c_m y_m$, *in the same notation as in Theorem* 3, *the series being absolutely and uniformly convergent for* x *sufficiently large.*

PROOF OF THEOREM 5(i). Suppose that M_n is the upper bound of $|ye^{-bx}|$ in the interval $[\lambda^{-(n-1)}, \lambda^{-n}]$, n being an integer sufficiently large that y is defined in $[\lambda^{-(n-1)}, \lambda^{-n}]$. Then for x in the interval $[\lambda^{-n}, \lambda^{-(n+1)}]$, we have

$$\begin{aligned} |[ye^{-bt}]_{\lambda^{-n}}^{x}| &= \left| a\int_{\lambda^{-n}}^{x} y(\lambda t)e^{-b\lambda t}e^{-b(1-\lambda)t}\,dt \right| \\ &\leqq |a|\, M_n \int_{\lambda^{-n}}^{x} e^{-b(1-\lambda)t}\,dt \\ &\leqq \frac{|a|\, M_n}{|b|\,(1-\lambda)}\, e^{-b(1-\lambda)\lambda^{-n}} \end{aligned}$$

and so

$$M_{n+1} \leqq M_n \left\{1 + \left|\frac{a}{b(1-\lambda)}\right| e^{-b(1-\lambda)\lambda^{-n}}\right\}.$$

In view of the convergence of the infinite product

$$\prod_n \left\{1 + \left|\frac{a}{b(1-\lambda)}\right| e^{-b(1-\lambda)\lambda^{-n}}\right\},$$

it follows that M_n is bounded for all n, i.e. $y = O(e^{bx})$ as $x \to \infty$. But then, as $X_1, X_2 \to \infty$,

$$[ye^{-bx}]_{X_1}^{X_2} = O\left\{\int_{X_1}^{X_2} e^{-b(1-\lambda)t}\,dt\right\},$$

which can be made as small as we please by taking X_1 and X_2 sufficiently large. Hence by the Cauchy principle ye^{-bx} converges as $x \to \infty$.

PROOF OF THEOREM 5(ii). If $|a| < b$, it is trivial that (5.2) converges and provides a solution of (1.1) satisfying (5.1).

If $|a| \geqq b$, the series (5.2) no longer converges, but we remember that $y^{(p)}$ satisfies a differential equation of the same type as y except that a is replaced by $a\lambda^p$, and if p is taken sufficiently large, we will certainly have $|a\lambda^p| < b$. Consider (with such a p) z_p to be a solution of

$$z_p'(x) = a\lambda^p z_p(\lambda x) + bz_p(x), \tag{5.4}$$

arranging (as we may) that

$$z_p \sim b^p L e^{bx}. \tag{5.5}$$

If $z_p(x)$ exists for $x \geqq \lambda A$, A some positive constant, we set

$$z_{p-1}(x) = \int_A^x z_p(t)\, dt + a_p$$

for some constant a_p to be fixed later and integrate (5.4) over $[A, x]$. We obtain

$$\begin{aligned} z_{p-1}'(x) &- z_p(A) \\ &= a\lambda^p \int_A^x z_p(\lambda t)\, dt + b \int_A^x z_p(t)\, dt \\ &= a\lambda^{p-1} z_{p-1}(\lambda x) + bz_{p-1}(x) + a\lambda^{p-1} \int_{\lambda A}^A z_p(u)\, du - a_p(a\lambda^{p-1} + b). \end{aligned}$$

It follows that a_p can be chosen so that z_{p-1} satisfies

$$z_{p-1}'(x) = a\lambda^{p-1} z_{p-1}(\lambda x) + bz_{p-1}(x),$$

provided that $a\lambda^{p-1} + b \neq 0$, and it is also clear, by integrating (5.5), that

$$z_{p-1} \sim b^{p-1} L e^{bx}.$$

We can thus repeat this process to obtain a solution of (1.1) satisfying (5.1), provided that $a\lambda^N + b \neq 0$ for any nonnegative integer N.

If $a\lambda^N + b = 0$ for some N, we need to proceed differently. We have to demonstrate the existence of a solution of

$$z'(x) = a\lambda^N z(\lambda x) + bz(x) \tag{5.6}$$

with the property that $z \sim b^N L e^{bx}$. We note first that $a\lambda^N + b = 0$ implies that a is real; and also that (iii) below (which we have not yet proved but whose proof does not depend on what we are now doing) tells us when applied to (5.6) that, if $z = o(e^{bx})$, then in fact $z = O(1)$. Now consider any solution of (5.6) with the property that in the interval $[\lambda, 1]$ both $z'>0$ and $z''>0$. It is then easy to see that $z''>0$ for all $x \geqq \lambda$. For if z'' vanishes for the first time at $x=x_0$, say, we must have $z'(x_0)>z'(\lambda x_0)>0$ since $z''>0$ in $[\lambda, x_0)$. But

$$z''(x_0) = a\lambda^{N+1}z'(\lambda x_0) + bz'(x_0),$$

and since the second term on the right exceeds in modulus the first, we have a contradiction to $z''(x_0)=0$. Hence $z''>0$ for all $x \geqq \lambda$, and so z' is positive increasing and z is unbounded, which implies by the remark above that $z \neq o(e^{bx})$. By multiplying by a suitable constant, we can then arrange that $z \sim b^N L e^{bx}$.

PROOF OF THEOREM 5(iii). We have to show that if a solution y of (1.1) satisfies $y=o(e^{bx})$, then $y=O(x^\kappa)$. We can integrate (1.1) to give

$$y(x) = -ae^{bx}\int_x^\infty e^{-bt}y(\lambda t)\,dt.$$

Since $|y(x)| \leqq Me^{bx}$ for $x \geqq A$, say, for suitable constants M, A, we obtain, for $x \geqq \lambda^{-1}A$,

$$|y(x)| \leqq M|a|e^{bx}\int_x^\infty e^{-b(1-\lambda)t}\,dt = \frac{M|a|e^{b\lambda x}}{b(1-\lambda)},$$

and by repetition

$$|y| \leqq \frac{M|a|^n e^{b\lambda^n x}}{b^n(1-\lambda)\cdots(1-\lambda^n)} \qquad \text{for } x \geqq \lambda^{-n}A.$$

Hence

$$|y| \leqq M'(|a|/b)^n e^{b\lambda^{-1}A} \qquad \text{for } \lambda^{-n}A \leqq x \leqq \lambda^{-n-1}A,$$

which is equivalent to $y=O(x^\kappa)$.

PROOF OF THEOREM 5(iv). This again follows from Theorem 7 of [4].

PROOF OF THEOREM 5(v). The arguments of Theorem 3 can be modified to apply, except in proving that, if $y=O(x^\kappa)$, then the periodic function g associated asymptotically with y satisfies (4.3), and this we prove now.

We first remark that if α and β are real constants, with $\beta>0$, then we can show by integration by parts that

$$(5.7) \qquad \int_x^\infty t^\alpha e^{-\beta t}\,dt \leqq \beta^{-1}(1+Ax^{-1})x^\alpha e^{-\beta x} \qquad \text{for } x \geqq 1,$$

A being a constant dependent on α and β. Further, since we can clearly take $A=0$ if $\alpha<0$, it is possible to find a uniform A as α varies, provided that the set of values of α is bounded above, as it will be in our application.

Now suppose that y is a solution of (1.1) satisfying $y=O(x^\kappa)$. Then $|y|\leqq Mx^\kappa$ for x in any I_n, where $I_n=[\lambda^{-n}x_0, \lambda^{-n-1}x_0]$ and x_0 is chosen sufficiently large that y exists in every I_n. Let us make the induction hypothesis that

$$|y^{(k)}| \leqq M(2Kbx_0)^k\lambda^{-(k+1)(k+2)/2+1}x^{\kappa-k} \tag{5.8}$$

for x in any I_n $(n\geqq k)$, the constant K being specified below. The hypothesis is evidently satisfied for $k=0$, and we prove it now for $k+1$. For (1.1) differentiated k times tells us that

$$|y^{(k+1)}| \leqq 2Mb(2Kbx_0)^k\lambda^{-(k+1)(k+2)/2+1}x^{\kappa-k} \tag{5.9}$$

for x in any I_n $(n\geqq k+1)$, and we can also differentiate (1.1) $k+1$ times and integrate once to obtain

$$y^{(k+1)}(x)e^{-bx} = a\lambda^{k+1}\int_x^\infty y^{(k+1)}(\lambda t)e^{-bt}\,dt. \tag{5.10}$$

Substituting (5.9) in (5.10) and using (5.7), we obtain, for x in I_n $(n\geqq k+2)$,

$$\begin{aligned}|y^{(k+1)}e^{-bx}| &\leqq 2Mb(2Kbx_0)^k(|a|\lambda^{k+1}/b)\lambda^{-(k+1)(k+2)/2} \\ &\quad\cdot(1+Ax^{-1})(\lambda x)^{\kappa-k}e^{-bx} \\ &\leqq 2Mb(2Kbx_0)^k(|a|\lambda^{k+1}/b)\lambda^{-(k+1)(k+2)/2+1} \\ &\quad\cdot(1+A\lambda^{k+2}x_0^{-1})(\lambda x)^{\kappa-k}e^{-bx}.\end{aligned} \tag{5.11}$$

We can repeat this, substituting (5.11) in (5.10), and eventually obtain that, for x in I_n $(n\geqq k+p,\ p\geqq 2)$,

$$\begin{aligned}|y^{(k+1)}e^{-bx}| &\leqq 2Mb(2Kbx_0)^k(|a|\lambda^{k+1}/b)^{p-1}\lambda^{-(k+1)(k+2)/2+1}(\lambda^{p-1}x)^{\kappa-k}e^{-bx} \\ &\quad\times\prod_{r=2}^{p}(1+A\lambda^{k+r}x_0^{-1}) \\ &\leqq 2KMb(2Kbx_0)^k(|a|\lambda^{k+1}/b)^{p-1}\lambda^{-(k+1)(k+2)/2+1}(\lambda^{p-1}x)^{\kappa-k}e^{-bx}\end{aligned} \tag{5.12}$$

if we take

$$K=\prod_{m=0}^{\infty}(1+A\lambda^m x_0^{-1}).$$

But

$$\kappa = \operatorname{Log}|b/a|/\operatorname{Log}\lambda, \qquad (|a|/b)^{p-1} = \lambda^{-\kappa(p-1)},$$

and so (5.12) gives, for x in I_{k+p} $(p \geqq 2)$,

$$(5.13)\qquad \begin{aligned} |y^{(k+1)}| &\leqq 2KMb(2Kbx_0)^k\lambda^{-(k+2)(k+3)/2+1}(\lambda^{k+p+1}x)x^{\kappa-k-1} \\ &\leqq M(2Kbx_0)^{k+1}\lambda^{-(k+2)(k+3)/2+1}x^{\kappa-k-1} \end{aligned}$$

since

$$\lambda^{k+p+1}x \leqq x_0 \qquad \text{for } x \text{ in } I_{k+p}.$$

But the right-hand side of (5.13) does not involve p $(p \geqq 2)$, and so we have established (5.8) with $k+1$ in place of k, at least for x in I_n $(n \geqq k+2)$; and it is easy to see that (5.9) implies (5.8) with $k+1$ in place of k and x in I_{k+1}. The induction hypothesis (5.8) is thus proved, and it is now easy to deduce (4.9a) from (5.8), and thus to obtain (4.3) as in Theorem 3.

For the particular solution of (1.1) which also satisfies (1.2) we have the following results.

THEOREM 6. *Let y denote the particular solution of* (1.1) *which also satisfies* (1.2). *Then* (i) *if $a>0$, for all x,*

$$y > 0, \qquad y' > 0, \cdots, y^{(n)} > 0, \cdots,$$

and also ye^{-bx} is positive increasing so that the case $ye^{-bx}\to 0$ cannot occur;

(ii) *if $a<0$, then the case $ye^{-bx}\to 0$ occurs if and only if $a\lambda^N+b=0$ for some nonnegative integer N, and this situation is equivalent to y being a polynomial of degree N with N distinct real zeros;*

(iii) *if $a<0$ and $|a|<b$, then for all x we have, as in* (i), *$y>0$, $y'>0, \cdots, y^{(n)}>0, \cdots$, but now ye^{-bx} is positive decreasing to a nonzero limit;*

(iv) *if $a<0$ and $|a|>b$, with $a\lambda^N+b\neq 0$ for any nonnegative integer N, then there is a unique nonnegative integer p such that $|a|\lambda^p>b$ but $|a|\lambda^{p+1}<b$, and y has then precisely $p+1$ zeros, all simple; y' has p zeros, all simple; $\cdots$; $y^{(p)}$ has one zero; $y^{(p+1)}$, $y^{(p+2)}, \cdots$ have no zeros; and ye^{-bx} tends to a nonzero limit with sign $(-1)^{p+1}$.*

PROOF OF THEOREM 6(i). The argument that $y>0$, $y'>0, \cdots$ is essentially the same as that in Theorem 4(ii). Then (1.1) in the form (4.29) shows that ye^{-bx} is positive increasing for all x.

PROOF OF THEOREM 6(ii). A comparison of the power series expression for y (remembering that $b>0$) with the exponential power series makes it quite clear that $ye^{-bx}\not\to 0$ unless the series terminates, and this happens if and only if $a\lambda^N+b=0$, when y is a polynomial. That the polynomial has N distinct real zeros is proved in the same way as the results in (iv) below.

PROOF OF THEOREM 6(iii). If $a<0$ and $|a|<b$, then initially $y'>0$, so that y is increasing. It is then impossible for y' to vanish, for if it does so for the first time at $x=x_0$, say, then $y(x_0)>y(\lambda x_0)$ and so $y'(x_0)>0$, which is a contradiction. Hence, for all x, $y'>0$, and so $y>0$. Similarly, by considering the equation for y', we see that $y''>0$, and so forth. Also, (1.1) shows that ye^{-bx} is positive decreasing, and to a nonzero limit since $a\lambda^N+b=0$ is impossible.

PROOF OF THEOREM 6(iv). Initially we have $y>0$, $y'<0$, $y''>0, \cdots, y^{(p)}$ of sign $(-1)^p$, and $y^{(p+1)}, y^{(p+2)} \cdots$ of sign $(-1)^{p+1}$. Since

$$y^{(p+2)} = a\lambda^{p+1}y^{(p+1)}(\lambda x) + by^{(p+1)}(x),$$

and since $|a|\lambda^{p+1}<b$, it follows by the same argument as in (iii) that $y^{(p+1)}, y^{(p+2)}, \cdots$ are all of constant sign for all x, and that sign is $(-1)^{p+1}$, and further $y^{(p+1)}\sim le^{bx}$ for a nonzero constant l of sign $(-1)^{p+1}$. Integration of this implies that $y\sim lb^{-(p+1)}e^{bx}$, so that ye^{-bx} tends to a nonzero limit of sign $(-1)^{p+1}$.

It is clear also from Rolle's theorem that, since $y^{(p+1)}$ is of constant sign, $y^{(p)}$ has at most one zero, $y^{(p-1)}$ has at most two, $\cdots$, y has at most $p+1$. It remains to show that the maximum number of zeros is in fact attained, and that the zeros are simple.

We do this by induction. Suppose that $a\lambda^{\theta+p}+b=0$ for some number θ with $0<\theta<1$. We use induction on p.

If $p=0$, we have to prove only that y has certainly one zero. Suppose not. Then, by applying (ii) to the equation satisfied by y', we see that y' is of constant sign (negative) and so y is positive decreasing, from which it follows that $ye^{-bx}\to 0$. But this implies that $a\lambda^N+b=0$, which is not true, and yields the necessary contradiction.

Suppose the result true for $p=0, 1, 2, \cdots, k$; we now prove it for $p=k+1$. By the induction hypothesis, y' (which satisfies an equation with $p=k$) has exactly $k+1$ zeros, all simple, say at $x=x_1^*, \cdots, x_{k+1}^*$. If we can show that y has zeros $x_1, \cdots, x_{k+2}$ satisfying

$$x_1 < x_1^* < x_2 < x_2^* < \cdots < x_{k+1}^* < x_{k+2},$$

then we are done.

Certainly y must vanish before $x=x_1^*$. For if not, y is positive decreasing in $[0, x_1^*]$, and so $|y(\lambda x_1^*)|>|y(x_1^*)|$. Since also $|a|>b$, this contradicts the fact that $y'(x_1^*)=0$.

Now at $x=x_1^*$, $y'=0$ and the term $by(x)$ has negative sign. Thus $ay(\lambda x)$ has positive sign and $y(\lambda x_1^*)<0$. It follows therefore that λx_1^*

lies in (x_1, x_1^*). Further, since $y'(x_1^*) = y'(x_2^*) = 0$, we have

$$\frac{y(\lambda x_1^*)}{y(x_1^*)} = \frac{y(\lambda x_2^*)}{y(x_2^*)} = -\frac{b}{a}. \tag{5.14}$$

Now suppose that y does not vanish in (x_1^*, x_2^*). Then, since y is negative increasing in (x_1^*, x_2^*),

$$| y(x_2^*) | < | y(x_1^*) |. \tag{5.15}$$

But if λx_2^* lies in $(x_1, x_1^*]$, then

$$| y(\lambda x_2^*) | > | y(\lambda x_1^*) |, \tag{5.16}$$

and (5.15) and (5.16) together contradict the first equality of (5.14); while if λx_2^* lies (x_1^*, x_2^*), then

$$| y(\lambda x_2^*) | > | y(x_2^*) |,$$

which contradicts the second equality of (5.14). Hence y does indeed vanish in (x_1^*, x_2^*).

The argument can now be repeated to obtain the full result.

6. **Asymptotic form of solutions when $\lambda < 1$: the case $b = 0$.** This case is very fully discussed by de Bruijn in [6]. For completeness and comparison purposes, we state now those of his results which are comparable with the results of Theorems 3 and 5, and of Theorem 10 to follow, but no proof will be required.

Again let $c = \text{Log}\,\lambda < 0$ and set

$$\phi(\text{Log}\,x) = x^k(\text{Log}\,x)^h \exp(-\tfrac{1}{2}c^{-1}(\text{Log}\,x - \text{Log}\,\text{Log}\,x)^2), \tag{6.1}$$

where

$$k = \tfrac{1}{2} - c^{-1} - c^{-1}\,\text{Log}(-ac), \qquad h = -1 + c^{-1}\,\text{Log}(-ac).$$

THEOREM 7. (i) *Every solution of* (1.1) *is* $O\{\phi(\text{Log}\,x)\}$ *as* $x \to \infty$.

(ii) *No solution of* (1.1) (*apart from the identically zero solution*) *is* $o\{\phi(\text{Log}\,x)\}$ *as* $x \to \infty$.

(iii) *Given any function* $g(t)$ *of the form*

$$g(t) = \sum_{n=-\infty}^{\infty} \gamma_n \exp(2n\pi i t/ |c|), \tag{6.2}$$

where, for some constant C,

$$\gamma_n = O\left\{\exp\left(-\frac{\pi^2 |n|}{|c|} - \frac{(\log |n|)^2}{2|c|} + C\,\text{Log}|n|\right)\right\}, \tag{6.3}$$

then there is one (and by (ii) *only one) solution of* (1.1) *which has the asymptotic behavior*

$$\phi(\text{Log } x)\{g(\text{Log } x - \text{Log Log } x) + o(1)\}. \tag{6.4}$$

Further, every solution of (1.1) *has this asymptotic behavior for some such* g.

REMARKS. 1. The asymptotic form (6.4) is obtained by de Bruijn in (8.3) of [6]. His notation is different from ours, but it is not difficult to carry out the routine changes necessary to show that his formula (8.3) implies (6.4). (His formula also includes an estimate of the error term.) It should perhaps be said, however, that even when the notation has been changed, de Bruijn's formula (8.3) is only equivalent to (6.4) and not identical with it, i.e. he uses in place of ϕ a function which can be shown to be $\phi\{1+o(1)\}$, and in place of g a function which can be shown to be $g+o(1)$.

2. Part (ii) of Theorem 3, and the corresponding part (iv) of Theorem 5, are real-variable results that can be established relatively easily, either by techniques in the spirit of the present paper, or, as in [4], by a Green's function technique. But part (ii) of the present theorem appears to be a much more delicate result that depends very heavily on the precise form of particular solutions of (1.1) and on their behavior as functions of a complex variable.

3. The conditions (6.2) and (6.3) on g imply at once that it is periodic of period $|c|$, and not only infinitely differentiable, but even analytic in a horizontal strip of width π. De Brui'n proves explicitly only the second half of (iii), that for any solution of (1.1) the asymptotic behavior is given by such a function g, but it is not difficult to see that the argument reverses.

4. The condition (6.3) on g can be replaced by a condition on the derivatives of g, as in Theorem 3. This condition is that, as $n\to\infty$,

$$|g^{(n)}(s)| = O\{(\tfrac{1}{2}\pi)^{-n}n!e^{-(\text{Log } n)^2/2|c|+K \text{ Log } n}\},$$

uniformly in s and for some constant K. The proof follows the same lines as that in Remark 4 following the statement of Theorem 3 and will not be given here.

5. While there is presumably a convergent series for any solution y (corresponding to (4.4)) of which $\phi(\text{Log } x)\ g(\text{Log } x-\text{Log Log } x)$ is the first term, it does not seem possible to write this down explicitly.

6. There is a complete set of solutious of (1.1) with the property that, if g is given by (6.2), then the corresponding solution y has the same coefficients γ_n when expanded in terms of the complete set. The

reader is referred to [6] for details. While the corresponding result in Theorem 3 or Theorem 5 is merely a tail-piece to the theorem, the present theorem is proved by investigating this special set of solutions and showing it to be complete.

7. **Asymptotic form of solutions when $\lambda>1$: the case $b>0$.** We recall from §1 that in the case $\lambda>1$ all solutions are defined in an interval of the form $[0, R)$, where $0<R\leqq\infty$ and R may vary from solution to solution. In what follows we are interested only in solutions for which $R=\infty$, so that we can talk about their asymptotic behavior.

We use the same notation as before, i.e. $s=\operatorname{Log} x$, $c=\operatorname{Log}\lambda$ (>0); and k_0 and κ are as they were in Theorem 3. The remarks prior to Theorem 3 are also, *mutatis mutandis*, applicable here.

THEOREM 8. *Let* $b>0$.

(i) *No solution of* (1.1) (*apart from the identically zero solution*) *is* $o(x^\kappa)$ *as* $x\to\infty$.

(ii) *Let* $g(s)$ *be periodic of period* c *and Hölder-continuous with exponent* θ, $0<\theta\leqq 1$. *Then there is one* (*and by* (i) *only one*) *solution of* (1.1) *such that*

$$y(x) = x^{k_0}g(\operatorname{Log} x) + O(x^{\kappa-\theta}) \qquad \text{as } x\to\infty. \tag{7.1}$$

(iii) *If further* $g(s)$ *has an mth derivative that is Hölder-continuous with exponent* θ, $0<\theta\leqq 1$, *then* $y(x)$ *has the asymptotic form*

$$y(x)=x^{k_0}\left\{g(\operatorname{Log} x)+\sum_{n=1}^{m}\frac{x^{-n}g_n(\operatorname{Log} x)}{b^n(1-\lambda^{-1})\cdots(1-\lambda^{-n})}+O(x^{-m-\theta})\right\}. \tag{7.2}$$

where the functions g_n *are defined recursively by* (4.5).

PROOF OF THEOREM 8(i). Suppose y is a solution of (1.1) such that $y=o(x^\kappa)$. We have to show y identically zero.

Once again, we integrate (1.1) to obtain

$$y(x)e^{-bx} = -a\int_x^\infty y(\lambda t)e^{-bt}\,dt. \tag{7.3}$$

For each $R\geqq 1$, set

$$K(R) = \sup_{x\geqq R} x^{-\kappa}|y(x)|, \tag{7.3a}$$

so that $K(R)$ decreases to 0 as $R\to\infty$. Let $x\geqq R$, so that $\lambda x\geqq\lambda R$, and (7.3) thus gives, using (5.7),

$$|y(x)|e^{-bx} \leqq (|a|/b)(\lambda x)^{\kappa}K(\lambda R)(1 + Ax^{-1})e^{-bx},$$

and so $|a|\lambda^{\kappa} = b$ implies that

$$|y(x)| \leqq x^{\kappa}K(\lambda R)(1 + Ax^{-1}),$$

so that

$$K(R) \leqq K(\lambda R)(1 + AR^{-1}).$$

Repeated application of this yields

$$K(R) \leqq K(\lambda^{n+1}R) \prod_{r=0}^{n} (1 + AR^{-1}\lambda^{-r}),$$

and if we let $n \to \infty$ and note that $K(\lambda^{n+1}R) \to 0$ and that the product converges, we have $K(R) = 0$, as required.

PROOF OF THEOREM 8(ii). We prove this by explicitly exhibiting a solution y which has the correct asymptotic behavior. We define inductively

$$y_0(x) = x^{k_0}g(\text{Log } x), \tag{7.4}$$

$$y_1(x) = be^{bx}\int_x^{\infty} e^{-bt}\{y_0(t) - y_0(x)\}\,dt, \tag{7.5}$$

$$y_{n+1}(x) = -ae^{bx}\int_x^{\infty} e^{-bt}y_n(\lambda t)\,dt \qquad (n = 1, 2, \cdots). \tag{7.6}$$

It is easy to see inductively, using (5.7), that y_n is not larger than algebraic as $x \to \infty$, and so the definition of y_{n+1} makes sense. We then prove the following two estimates:

$$|y_1(x)| \leqq Kx^{\kappa-\theta} \tag{7.7}$$

for $x \geqq 1$ and some positive constant K, and, for $n \geqq 1$,

$$|y_n(x)| \leqq K\lambda^{-(n-1)\theta}x^{\kappa-\theta}e^{\lambda A/(\lambda-1)x} \tag{7.8}$$

for $x \geqq 1$, where the constant K is the same as in (7.7), and the constant A is the same as in (5.7) with $\beta = b$, $\alpha = \kappa - \theta$.

Once (7.7) and (7.8) have been proved, and they will be proved below, we define

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

the series being absolutely and (apart from the factor x^{κ}) uniformly convergent for $x \geqq 1$ by virtue of (7.8). Further,

$$| y(x) - y_0(x) | \leqq \sum_{n=1}^{\infty} | y_n(x) | \leqq \frac{K}{1 - \lambda^{-\theta}} x^{\kappa-\theta} e^{\lambda A/(\lambda-1)x},$$

so that y has the correct asymptotic behavior. Finally, y satisfies (1.1), for

$$y'_{n+1}(x) - by_{n+1}(x) = ay_n(\lambda x), \tag{7.9}$$

while from the definition of y_1,

$$y_1(x) = be^{bx} \int_x^{\infty} e^{-bt} y_0(t)\, dt - y_0(x),$$

so that

$$\left(\frac{d}{dx} - b\right)(y_0(x) + y_1(x)) = - by_0(x) = ay_0(\lambda x), \tag{7.10}$$

the function g in (7.4) being periodic of period Log λ; and (7.9) and (7.10) together imply that y satisfies (1.1).

It remains to prove (7.7) and (7.8). To deal first with (7.7), we write

$$y_0(t) - y_0(x) = (t^{k_0} - x^{k_0}) g(\text{Log } x) + t^{k_0}\{g(\text{Log } t) - g(\text{Log } x)\} \tag{7.11}$$

on the right of (7.5). The contribution to $y_1(x)$ from the first term of (7.11) is

$$bg(\text{Log } x) \int_x^{\infty} e^{-b(t-x)}(t^{k_0} - x^{k_0})\, dt = k_0 g(\text{Log } x) \int_x^{\infty} e^{-b(t-x)} t^{k_0 - 1}\, dt,$$

by an integration by parts. Since g is bounded, this is majorized by

$$\text{const} \int_x^{\infty} e^{-b(t-x)} t^{\kappa-1}\, dt \leqq \text{const } x^{\kappa-1},$$

by use of (5.7).

Also, $|g(\sigma) - g(s)| \leqq \text{const } |\sigma - s|^{\theta}$ for all real σ, s, since g is Hölder-continuous and periodic, and so the contribution to $y_1(x)$ from the second term of (7.11) is majorized by

$$\text{const} \int_x^{\infty} e^{-b(t-x)} t^{\kappa} | \text{Log } t - \text{Log } x |^{\theta}\, dt. \tag{7.12}$$

Since

$$\text{Log } t - \text{Log } x = \text{Log}\{1 + (t - x)/x\} \leqq (t - x)/x \qquad \text{for } t \geqq x,$$

(7.12) does not exceed

$$\text{const } x^{-\theta}\int_x^\infty e^{-b(t-x)}t^\kappa(t-x)^\theta\,dt.$$

But since

$$(t-x)^\theta \leqq 1+(t-x) \qquad \text{for } t \geqq x,$$

we can use (5.7) and an integration by parts to prove that

$$\int_x^\infty e^{-b(t-x)}t^\kappa(t-x)^\theta\,dt \leqq A'x^\kappa \tag{7.13}$$

for $x \geqq 1$ and $0<\theta\leqq 1$, the constant A' being independent of x or θ. This completes the proof of (7.7).

We prove (7.8) by induction, noting that it is true for $n=1$ by (7.7). Assuming it is true for n, we have

$$\begin{aligned} |y_{n+1}(x)| &\leqq |a|\int_x^\infty e^{-b(t-x)}|y_n(\lambda t)|\,dt \\ &\leqq K\lambda^{-(n-1)\theta}|a|\int_x^\infty e^{-b(t-x)}(\lambda t)^{\kappa-\theta}e^{A/(\lambda-1)t}\,dt \\ &\leqq K\lambda^{-n\theta}be^{A/(\lambda-1)x}\int_x^\infty e^{-b(t-x)}t^{\kappa-\theta}\,dt \\ &\leqq K\lambda^{-n\theta}x^{\kappa-\theta}e^{A/(\lambda-1)x}(1+Ax^{-1}), \end{aligned} \tag{7.14}$$

where we have used (5.7) and the fact that $|a|\lambda^\kappa=b$. But

$$\begin{aligned} \left(\exp\frac{A}{(\lambda-1)x}\right)(1+Ax^{-1}) &\leqq \exp\left\{\frac{A}{(\lambda-1)x}+\frac{A}{x}\right\} \\ &= \exp\frac{\lambda A}{(\lambda-1)x}, \end{aligned}$$

and so (7.14) leads to (7.8) with n replaced by $n+1$, thus completing the proof.

REMARK. The prescribed asymptotic function y_0 is by hypothesis Hölder-continuous but not necessarily differentiable, while the solution y, since it continues to exist as $x\to\infty$, is by the remarks in §1 infinitely differentiable. It is interesting that we have constructed such a function y as a sum $\sum y_n$ in which y_0 and y_1 are (in general) not differentiable separately, y_2 is differentiable only once, y_3 only twice, and so on.

PROOF OF THEOREM 8(iii). We may assume $m\geqq 1$, for if $m=0$ the result contains nothing new.

Denote by c_j the coefficient of $x^{k_0-j}g_j$ on the right of (7.2). Then

$$c_{j-1} = b(1-\lambda^{-j})c_j \qquad (j = 1, \cdots, m). \tag{7.15}$$

Set

$$y(x) = \sum_{j=0}^{m} c_j x^{k_0-j} g_j(\mathrm{Log}\, x) + z(x), \tag{7.16}$$

and we have to show that

$$z(x) = O(x^{\kappa-m-\theta}).$$

If we substitute (7.16) into (1.1), bearing in mind that $a\lambda^{k_0}+b=0$, that each g_j is periodic and that $y_0^{(m)}$ and z may not be differentiable separately, we obtain

$$\sum_{j=0}^{m-1} c_j \frac{d}{dx}\left\{x^{k_0-j}g_j(\mathrm{Log}\, x)\right\} + \frac{d}{dx}\left\{c_m x^{k_0-m}g_m(\mathrm{Log}\, x) + z(x)\right\}$$
$$= b\sum_{j=0}^{m} c_j(1-\lambda^{-j})x^{k_0-j}g_j(\mathrm{Log}\, x) + az(\lambda x) + bz(x).$$

But (4.8) and (7.15) enable us to cancel the summations on either side, and what remains can be rewritten as

$$\begin{aligned}&\frac{d}{dx}\left\{e^{-bx}(c_m x^{k_0-m}g_m(\mathrm{Log}\, x) + z(x))\right\}\\ &\qquad = e^{-bx}\left\{az(\lambda x) - bc_m x^{k_0-m}g_m(\mathrm{Log}\, x)\right\}.\end{aligned} \tag{7.17}$$

Since $y(x)=O(x^\kappa)$, the same is true of $z(x)$, and so we can integrate (7.17) to obtain

$$e^{-bx}(c_m x^{k_0-m}g_m(\mathrm{Log}\, x) + z(x))$$
$$= -\int_x^\infty e^{-bt}\left\{az(\lambda t) - bc_m t^{k_0-m}g_m(\mathrm{Log}\, t)\right\} dt,$$

which may be rewritten as

$$z(x) = z_1(x) - a\int_x^\infty e^{-b(t-x)}z(\lambda t)\, dt, \tag{7.18}$$

where

$$z_1(x) = bc_m\int_x^\infty e^{-b(t-x)}\left\{t^{k_0-m}g_m(\mathrm{Log}\, t) - x^{k_0-m}g_m(\mathrm{Log}\, x)\right\} dt.$$

Then, merely replacing k_0 by $k_0 - m$ in the proof of (7.7), we obtain

$$|z_1(x)| \leqq K' x^{\kappa - m - \theta} \tag{7.19}$$

for $x \geqq 1$ and some constant K'.

To complete the proof of (iii), we set

$$z_{n+1}(x) = -a \int_x^\infty e^{-b(t-x)} z_n(\lambda t)\, dt \qquad (n = 1, 2, \cdots),$$

and show (as in the proof of (7.8)) that

$$|z_n(x)| \leqq K' \lambda^{-(n-1)(m+\theta)} x^{\kappa - m - \theta} e^{\lambda M/(\lambda - 1)x}$$

for $x \geqq 1$ and $n \geqq 1$ and some suitable constant M. It then follows that the function

$$\hat{z}(x) = \sum_{n=1}^{\infty} z_n(x)$$

exists and satisfies (7.17) and is $O(x^{\kappa - m - \theta})$ as $x \to \infty$. But $z = o(x^\kappa)$ from its definition and part (ii) of the present theorem, and so by part (i), $z - \hat{z} \equiv 0$, so that $z = O(x^{\kappa - m - \theta})$ and the proof is complete.

8. Asymptotic form of solutions when $\lambda > 1$: the case $b < 0$.

THEOREM 9. *Let $b < 0$, with κ as in Theorem* 3.

(i) *The series in* (5.2) *converges absolutely and uniformly for all $x \geqq 0$, so that y_L is a solution of* (1.1) *decaying like Le^{bx} as $x \to \infty$.*

(ii) *There is no solution of* (1.1), *other than a constant multiple of y_L, such that $y = o(x^\kappa)$ as $x \to \infty$.*

(iii) *Let g be as in Theorem* 8(ii). *Then there is a solution y of* (1.1) *satisfying* (7.1), *and this is unique by* (ii) *up to addition of a constant multiple of y_L.*

(iv) *If further g satisfies the conditions of Theorem* 8(iii), *then y has the asymptotic form* (7.2).

REMARK. The solution y_L is analytic in x, a, b for $x > 0$, $b > 0$ and any a. In particular, $y_L(0)$ is analytic in a, b, $b > 0$, so that $y_L(0) = 0$ can occur only for exceptional pairs a, b. "In general," therefore, there is one (and only one) solution of (1.1) satisfying (1.2) and decaying exponentially as $x \to \infty$.

PROOF OF THEOREM 9(i). This is obvious.

PROOF OF THEOREM 9(ii). Let y be a solution of (1.1) such that $y = o(x^\kappa)$. We have to show that y is a multiple of y_L.

The first stage in the proof is to show that y is at least exponentially small, i.e. that

$$y = O(e^{-\epsilon x}) \quad \text{for some } \epsilon > 0, \tag{8.1}$$

and we shall need the inequality (easily proved) that

$$\int_1^x t^\alpha e^{\beta t}\, dt \leqq \beta^{-1}(1 + Mx^{-1})x^\alpha e^{\beta x} \qquad (x \geqq 1), \tag{8.2}$$

where the constant M depends on α, β, $\beta > 0$.

To prove (8.1), integrate (1.1) to obtain

$$y(x) = e^{b(x-\rho)}y(\rho) + a\int_\rho^x e^{b(x-t)}y(\lambda t)\, dt \qquad (x \geqq \rho \geqq 1). \tag{8.3}$$

Define $K(R)$ as in (7.3a), so that $K(R)$ decreases to 0 as $R \to \infty$, and then (8.3) gives

$$\begin{aligned} |y(x)| &\leqq K(\rho)\rho^\kappa e^{b(x-\rho)} + |a|\lambda^\kappa K(\lambda\rho)\int_\rho^x e^{b(x-t)}t^\kappa\, dt \qquad (x \geqq \rho \geqq 1) \\ &\leqq K(\rho)\rho^\kappa e^{b(x-\rho)} + K(\lambda\rho)x^\kappa(1 + Mx^{-1}), \end{aligned}$$

by use of (8.2) and $|a|\lambda^\kappa = -b$. Multiply the last inequality by $x^{-\kappa}$ and take sup for $x \geqq \sigma \geqq \rho \geqq 1$. Since $x^{-\kappa}e^{bx} \leqq Ce^{-b'x}$ for $x \geqq 1$, where $b' > 0$ may be arbitrarily close to $|b|$ and C depends only on κ and b' we have, since $e^{-b'x} \leqq e^{-b'\sigma}$,

$$K(\sigma) \leqq K(\rho)\rho^\kappa Ce^{-b'\sigma - b\rho} + K(\lambda\rho)(1 + M\sigma^{-1}) \qquad (\sigma \geqq \rho \geqq 1),$$

where C does not depend on ρ or σ. Setting $\rho = \sigma\lambda^{-1/2}$ and noting that $K(\rho)$ is bounded for $\rho \geqq 1$, we obtain

$$K(\sigma) \leqq C'\sigma^\kappa \exp\{-(b' + b\lambda^{-1/2})\sigma\} + K(\lambda^{1/2}\sigma)(1 + M\sigma^{-1}) \tag{8.4}$$

$(\sigma \geqq \lambda^{1/2})$, where C' does not depend on σ.

Since b' may be arbitrarily close to $|b| = -b$, and since $\lambda > 1$, we may assume

$$b' + b\lambda^{-1/2} = 2\epsilon > 0.$$

Then the first term on the right of (8.4) is $C'\sigma^\kappa e^{-2\epsilon\sigma} \leqq C''e^{-\epsilon\sigma}$ for $\sigma \geqq \lambda^{1/2}$, where C'' does not depend on σ. Hence

$$K(\sigma) \leqq C''e^{-\epsilon\sigma} + (1 + M\sigma^{-1})K(\lambda^{1/2}\sigma) \qquad (\sigma \geqq \lambda^{1/2}). \tag{8.5}$$

We now prove by induction that

$$K(\sigma) \leqq \exp\left\{\frac{\lambda^{1/2}}{\lambda^{1/2} - 1}\cdot\frac{M}{\sigma}\right\}\left[C''\sum_{n=0}^{m-1} e^{-\epsilon\lambda^{n/2}\sigma} + K(\lambda^{m/2}\sigma)\right] \tag{8.6}$$

for $\sigma \geqq \lambda^{1/2}$, $m = 1, 2, \cdots$. The result is true for $m = 1$ by (8.5). Let us assume it is true for m, replace σ by $\lambda^{1/2}\sigma$ in (8.6), and substitute the result in (8.5); we obtain

$$K(\sigma) \leqq C''e^{-\epsilon\sigma} + (1 + M\sigma^{-1}) \exp\left\{\frac{M}{(\lambda^{1/2} - 1)\sigma}\right\}$$

$$\cdot \left[C'' \sum_{n=1}^{m} \exp\{-\epsilon\lambda^{n/2}\sigma\} + K(\lambda^{(m+1)/2}\sigma)\right]$$

$$\leqq C''e^{-\epsilon\sigma} + \exp\left\{\frac{\lambda^{1/2}}{\lambda^{1/2} - 1}\cdot\frac{M}{\sigma}\right\}$$

$$\cdot \left[C'' \sum_{n=1}^{m} \exp\{-\epsilon\lambda^{n/2}\sigma\} + K(\lambda^{(m+1)/2}\sigma)\right],$$

which implies (8.6) with m replaced by $m+1$.

Now let $m \to \infty$ in (8.6). Since $K(R) \to 0$ as $R \to \infty$, we have

$$K(\sigma) \leqq \exp\left\{\frac{\lambda^{1/2}}{\lambda^{1/2} - 1}\cdot\frac{M}{\sigma}\right\} C'' \sum_{n=0}^{\infty} \exp\{-\epsilon\lambda^{n/2}\sigma\} \qquad (\sigma \geqq \lambda^{1/2}),$$

and this implies that $K(\sigma) = O(e^{-\epsilon\sigma})$ as $\sigma \to \infty$, from which (8.1) follows easily with, say, $\frac{1}{2}\epsilon$ in place of ϵ.

The next stage is to show that

$$\gamma = \lim_{x \to \infty} e^{-bx}y(x) \tag{8.7}$$

exists. Substituting (8.1) in (8.3), we have, for $x \geqq \rho$, and some constant C,

$$|y(x)| \leqq |y(\rho)| e^{b(x-\rho)} + \frac{C|a|}{|b| - \epsilon\lambda} e^{-\epsilon\lambda x}$$

provided $\epsilon\lambda < |b|$. Hence $y = O(e^{-\epsilon\lambda x})$ if $\epsilon\lambda < |b|$, and by repetition

$$y = O(e^{-\epsilon\lambda^n x}) \quad \text{if } \epsilon\lambda^n < |b|. \tag{8.8}$$

But if (8.1) is true with any particular value of ϵ, then it is true with any smaller value, and so we may choose ϵ, n so that $\epsilon\lambda^n < |b|$ while $\epsilon\lambda^{n+1} > |b|$. Then (1.1) gives

$$[e^{-bt}y(t)]_1^X = a\int_1^X e^{-bt}y(\lambda t)\,dt,$$

and since the integral on the right converges as $X \to \infty$, by (8.8) and the choice of ϵ and n, (8.7) follows.

Finally we show that

$$y(x) = \gamma L^{-1} y_L(x). \tag{8.9}$$

Set $z(x) = y(x) - \gamma L^{-1} y_L(x)$. Then $z = o(e^{bx})$, and so (1.1) gives on integration

$$e^{-bx} z(x) = -a \int_x^\infty e^{-bt} z(\lambda t) dt. \tag{8.10}$$

For each $R > 0$, define $K_1(R) = \sup_{x \geq R} e^{-bx} |z(x)|$, and then $K_1(R)$ decreases to 0 as $R \to \infty$. Further, in the usual way, (8.10) gives

$$K_1(R) \leqq \frac{|a|}{(\lambda - 1)|b|} e^{b(\lambda - 1)R} K_1(\lambda R),$$

and repetition leads to

$$K_1(R) \leqq \frac{|a|^n}{(\lambda - 1)^n |b|^n} e^{b(\lambda^n - 1)R} K_1(\lambda^n R).$$

Letting $n \to \infty$, we deduce that $K_1(R) \equiv 0$, so that $z \equiv 0$, as required.

Proof of Theorem 9(iii). The proof is similar to that of Theorem 8(ii) and will only be sketched. We define inductively

$$y_0(x) = x^{k_0} g(\operatorname{Log} x),$$

$$\begin{aligned} y_1(x) &= -b \int_\rho^x e^{b(x-t)} y_0(t) dt - y_0(x) \\ &= -b \int_\rho^x e^{b(x-t)} \{ y_0(t) - y_0(x) \} dt - e^{b(x-\rho)} y_0(x) \qquad (x \geqq \rho \geqq 1), \end{aligned}$$

$$y_{n+1}(x) = a \int_\rho^x e^{b(x-t)} y_n(\lambda t) dt \qquad (n = 1, 2, \cdots),$$

ρ being a constant to be determined below.

We first prove (7.7) for our present y_1, the proof being essentially the same as the earlier proof except that (8.2) replaces (5.7). Then we show that, for $n \geqq 1$ and $x \geqq \rho$,

$$| y_n(x) | \leqq K \lambda^{-(n-1)\theta} (1 + M\rho^{-1})^{n-1} x^{\kappa - \theta}, \tag{8.11}$$

the proof being sufficiently similar to that of (7.8) to require no repetition. The required solution y is then defined by $\sum_{n=0}^\infty y_n$, the series converging by virtue of (8.11) if $1 + M\rho^{-1} < \lambda^\theta$, which is true if ρ is chosen sufficiently large.

PROOF OF THEOREM 9(iv). The proof is similar to that of Theorem 8(iii). Instead of (7.18), we solve the integral equation

$$\hat{z}(x) = z_1(x) + a\int_\rho^x e^{b(x-t)}\hat{z}(\lambda t)\,dt,$$

where

$$\begin{aligned} z_1(x) &= -\,bc_m\int_\rho^x e^{b(x-t)}t^{k_0-m}g_m(\operatorname{Log} t)\,dt - c_m x^{k_0-m}g_m(\operatorname{Log} x)\\ &= -\,bc_m\int_\rho^x e^{b(x-t)}\{t^{k_0-m}g_m(\operatorname{Log} t) - x^{k_0-m}g_m(\operatorname{Log} x)\}\,dt\\ &\quad - c_m e^{b(x-\rho)}x^{k_0-m}g_m(\operatorname{Log} x). \end{aligned}$$

For this $z_1(x)$ we again have the estimate (7.19), and we then define

$$z_{n+1}(x) = a\int_\rho^x e^{b(x-t)}z_n(\lambda t)\,dt \qquad (n \geqq 1)$$

and prove

$$|z_n(x)| \leqq K'\lambda^{-(n-1)(m+\theta)}(1 + M\rho^{-1})^{n-1}x^{\kappa-m-\theta} \qquad (n \geqq 1).$$

This implies the convergence of the series $\sum_1^\infty z_n$ if ρ is sufficiently large, and the argument can be completed as before.

9. Asymptotic form of solutions when $\lambda > 1$: the case $b=0$.

THEOREM 10. *We use the notation of Theorem* 7.

(i) *No solution of* (1.1) (*apart from the identically zero solution*) *is* $o\{\phi(\operatorname{Log} x)\}$ *as* $x\to\infty$.

(ii) *Let* $g(t)$ *be periodic of period* c, *and let* $g'(t)$ *be Hölder-continuous with exponent* θ, $0<\theta<1$. *Then there is one* (*and by* (i) *only one*) *solution of* (1.1) *such that*

$$y(x) = \phi(\operatorname{Log} x)\{g(\operatorname{Log} x - \operatorname{Log}\operatorname{Log} x) + O[(\operatorname{Log} x)^{-\theta}]\}.$$

REMARK. If $g(t)$ is several times differentiable, then $y(x)$ presumably has an asymptotic expansion as in Theorems 8, 9, but since the form of this would be very complicated, we do not discuss it.

PROOF OF THEOREM 10(i). Let

$$s = \operatorname{Log} x, \qquad x = e^s, \qquad Y(s) = y(x),$$

and then (1.1) becomes

$$Y'(s) = ae^sY(s + c). \tag{9.1}$$

We first collect some results on the function $\phi(s)$ which will be used in what follows, and which are all obtained by routine calculations. We have

(9.2) $\phi'(s) = c^{-1}\phi(s)\{-s + \text{Log}\, s + 1 + ck - (\text{Log}\, s)/s + hc/s\}$

(9.3) $e^s\phi(s + c) = (-ac)^{-1}\phi(s)\{s - \text{Log}\, s + (h + \frac{1}{2})c + O[(\text{Log}\, s)^2/s]\}$

as $s \to \infty$,

$$ae^s\phi(s + c) - \phi'(s) = O\{\phi(s)(\text{Log}\, s)^2/s\}. \tag{9.4}$$

Further, if $k_1 = \text{re}\, k$, $h_1 = \text{re}\, h$,

$$| \phi(s) | = \exp\{-\tfrac{1}{2} c^{-1}(s - \text{Log}\, s)^2 + k_1 s + h_1 \text{Log}\, s\}, \tag{9.5}$$

$$| \phi(s) |' = c^{-1} | \phi(s) | \{-s + \text{Log}\, s + 1 + ck_1 - (\text{Log}\, s)/s + h_1 c/s\}, \tag{9.6}$$

$$| a | e^s | \phi(s + c) | + | \phi(s) |' = O\{\phi(s)(\text{Log}\, s)^2/s\}. \tag{9.7}$$

From (9.6) we deduce the existence of a constant L such that

$$| \phi(s) | \leqq Ls^{-1}\{-| \phi(s) |'\} \tag{9.8}$$

for s sufficiently large, say $s \geqq R_0$, and (9.8) and (9.7) imply that, for $s \geqq R_0$,

$$| a | e^s | \phi(s + c) | \leqq \{1 + L(\text{Log}\, s)^2/s^2\}\{-| \phi(s) |'\}. \tag{9.9}$$

Since $|\phi(s)|$ behaves roughly like $e^{-s^2/2c}$ as $s \to \infty$, we can integrate (9.1) to give

$$Y(s) = -a \int_s^\infty e^\sigma Y(\sigma + c)\, d\sigma, \tag{9.10}$$

the integral converging absolutely. For each $R \geqq R_0$, define

$$K(R) = \sup_{s \geqq R} | \phi(s) |^{-1} Y(s),$$

so that $K(R)$ decreases to 0 as $R \to \infty$. Then (9.10) gives, for $s \geqq R$,

$$| Y(s) | \leqq | a | K(R + c) \int_s^\infty e^\sigma | \phi(\sigma + c) | \, d\sigma$$

$$\leqq K(R + c) \int_s^\infty \{1 + L(\text{Log}\, \sigma)^2/\sigma^2\}\{-| \phi(\sigma) |'\}\, d\sigma$$

by (9.9). If we suppose, as we may, that $(\text{Log}\, s)/s$ is decreasing for $s \geqq R_0$, we obtain

$$| Y(s) | \leqq K(R + c)\{1 + L(\text{Log}\, R)^2/R^2\} | \phi(s) |,$$

so that

$$K(R) \leqq K(R+c)\{1 + L(\mathrm{Log}\, R)^2/R^2\}.$$

Repeated application of this gives

$$K(R) \leqq K(R+nc) \prod_{j=0}^{n-1} \left\{1 + L\frac{\{\mathrm{Log}(R+jc)\}^2}{(R+jc)^2}\right\},$$

which implies in the usual way that $K(R) \equiv 0$, as required.

PROOF OF THEOREM 10(ii). As in Theorems 8(ii), 9(iii), we construct a solution having the required asymptotic property. We set

$$Y_0(s) = \phi(s)g(s - \mathrm{Log}\, s), \tag{9.11}$$

and then define inductively

$$F_0(s) = ae^s Y_0(s+c) - Y_0'(s), \tag{9.12}$$

$$Y_1(s) = -\int_s^\infty F_0(\sigma)\, d\sigma, \tag{9.13}$$

$$Y_{n+1}(s) = -a\int_s^\infty e^\sigma Y_n(\sigma + c)\, d\sigma \qquad (n = 1, 2, \cdots). \tag{9.14}$$

We shall prove the following estimates, which also serve to justify the convergence of the integrals in the above definitions:

$$|F_0(s)| \leqq Ks^{-\theta}|\phi(s)| \tag{9.15}$$

for sufficiently large s, say $s \geqq R_1 \geqq R_0$, and for some constant K; and

$$|Y_n(s)| \leqq KL\{s + (n-1)c\}^{-1-\theta}|\phi(s)|e^{\rho(s)} \quad (n \geqq 1,\ s \geqq R_1), \tag{9.16}$$

where $\rho(s) = L\sum_{n=0}^\infty \{\mathrm{Log}(s+nc)\}^2/(s+nc)^2$. Once (9.15) and (9.16) are proved, as they will be below, we form

$$Y(s) = \sum_{n=0}^\infty Y_n(s),$$

the series being absolutely and uniformly convergent for $s \geqq R_1$, and we have

$$\begin{aligned} |Y(s) - Y_0(s)| &\leqq \sum_{n=1}^\infty |Y_n(s)| \leqq KL|\phi(s)|e^{\rho(s)}\sum_{n=0}^\infty (s+nc)^{-1-\theta} \\ &\leqq \mathrm{const}\,|\phi(s)|\,s^{-\theta}. \end{aligned}$$

This shows that Y has the correct asymptotic behavior, and to show that Y is a solution of (9.1) for $s \geqq R_1$, it suffices to note that

$$Y_0'(s) + Y_1'(s) = Y_0'(s) + F_0(s) = ae^s Y_0(s+c),$$
$$Y_{n+1}'(s) = ae^s Y_n(s+c) \qquad (n = 1, 2, \cdots).$$

It remains only to prove (9.15) and (9.16). To prove (9.15), we use the periodicity of g and (9.4) to write

$$\begin{aligned} F_0(s) &= ae^s\phi(s+c)g(s - \mathrm{Log}(s+c)) - \phi'(s)g(s - \mathrm{Log}\, s) \\ &\quad - \phi(s)g'(s - \mathrm{Log}\, s)(1 - s^{-1}) \\ &= \phi'(s)\{g(s - \mathrm{Log}(s+c)) - g(s - \mathrm{Log}\, s)\} - \phi(s)g'(s - \mathrm{Log}\, s) \\ &\quad + O\{\phi(s)(\mathrm{Log}\, s)^2/s\}. \end{aligned}$$

The term involving $\phi'(s)$ may be written, using (9.2) and the mean value theorem,

$$\begin{aligned} -c^{-1}\phi(s)\{-s + \mathrm{Log}\, s + O(1)\}g'(\sigma)\,\mathrm{Log}(1 + c/s) \\ = s^{-1}\phi(s)\{s - \mathrm{Log}\, s + O(1)\}g'(\sigma), \end{aligned}$$

where $s - \mathrm{Log}(s+c) < \sigma < s - \mathrm{Log}\, s$. Hence

$$F_0(s) = \phi(s)\{g'(\sigma) - g'(s - \mathrm{Log}\, s) + O[(\mathrm{Log}\, s)^2/s]\},$$

and (9.8) follows from the fact that

$$\begin{aligned} |g'(\sigma) - g'(s - \mathrm{Log}\, s)| &\leqq \mathrm{const}\, |\sigma - s + \mathrm{Log}\, s|^\theta \\ &\leqq \mathrm{const}\, |\mathrm{Log}(s+c) - \mathrm{Log}\, s|^\theta \leqq \mathrm{const}(c/s)^\theta. \end{aligned}$$

To prove (9.16), we note that $\rho(s)$ is finite and decreases to 0 as $s \to \infty$.

For $n=1$, (9.16) follows from

$$\begin{aligned} |Y_1(s)| &\leqq \int_s^\infty |F_0(\sigma)|\, d\sigma \leqq K\int_s^\infty \sigma^{-\theta}|\phi(\sigma)|\, d\sigma \\ &\leqq KL\int_s^\infty \sigma^{-1-\theta}(-|\phi(\sigma)|')\, d\sigma \qquad \text{by (9.8)} \\ &\leqq KLs^{-1-\theta}|\phi(s)| \leqq KLs^{-1-\theta}|\phi(s)|\, e^{\rho(s)}. \end{aligned}$$

If (9.16) has been proved for n, then

$$\begin{aligned} |Y_{n+1}(s)| &\leqq |a|\, KL\int_s^\infty e^\sigma(\sigma+nc)^{-1-\theta}|\phi(\sigma+c)|\, e^{\rho(\sigma+c)}\, d\sigma \\ &\leqq KL\int_s^\infty (\sigma+nc)^{-1-\theta}\{1+L(\mathrm{Log}\,\sigma)^2/\sigma^2\}e^{\rho(\sigma+c)}\{-|\phi(\sigma)|'\}\, d\sigma \end{aligned}$$

by (9.9). Since the first three factors in the last integrand are decreasing, we obtain, noting that $\{1+L(\text{Log } s)^2/s^2\}e^{\rho(s+c)} \leqq e^{\rho(s)}$,

$$| Y_{n+1}(s) | \leqq KL(s + nc)^{-1-\theta} | \phi(s) | e^{\rho(s)},$$

so completing the proof of (9.16) by induction.

10. **Asymptotic form of solutions when $\lambda > 1$: distributional asymtotic behavior.** The treatment of the case $\lambda > 1$ given in §§7–9 seems unsatisfactory in two respects. In the first place the assumption of Hölder continuity on $g(s)$ (see Theorems 8–10), although necessary for technical reasons, appears to be rather unnatural. Nor did we consider the question whether those solutions which do have the asymptotic behavior discussed in Theorems 8–10 can be detected by their behavior on a finite interval, say $[x_0, \lambda x_0]$.

In order to resolve these points we shall consider in this section a larger class of solutions. Since, however, the most general solutions would be quite unmanageable (see §1), we shall still restrict ourselves to solutions having an asymptotic form such as $x^{k_0} g(\text{Log } x)$ as $x \to \infty$, but we shall now allow $g(s)$ to be a *distribution.* We therefore start by introducing the concept of a distribution in this context.

In what follows we mean by a distribution $g(s)$, a complex-valued distribution on $(-\infty, \infty)$ [see [11] for the basic definitions]. If $g(s)$ is not equal to a function, $g(s)$ is only a symbolic notation. If $\phi(s)$ is a test function (infinitely differentiable function with compact support), we use also the symbolic notation $\int g(s)\phi(s)\, ds$ for the functional $\langle g, \phi \rangle$. (Here and in the sequel, an integral without indication of the limits is understood to be taken over $(-\infty, \infty)$.)

The notation $g(s+c)$ denotes the distribution obtained from $g(s)$ by a left translation by c, i.e. $g(s+c)$ is defined by

$$\int g(s + c)\phi(s)\, ds = \int g(s)\phi(s - c)\, ds.$$

The distribution $g(s)$ is *periodic* with period c if $g(s+c) = g(s)$.

We say a sequence $\{g_n(s)\}$ of distributions converges to a distribution $g(s)$ (in symbols $g_n(s) \to g(s)$ as $n \to \infty$) if

$$\int g_n(s)\phi(s)\, ds \to \int g(s)\phi(s)\, ds \quad \text{as } n \to \infty \tag{10.1}$$

for each test function $\phi(s)$. As is well known [11, p. 74, Theorem XIII], this "weak" convergence is equivalent to "strong" convergence, i.e. it implies that the convergence is uniform on any *bounded* set of test functions. (A set $\{\phi(s)\}$ of test functions is bounded if all $\phi(s)$ have a common compact support and if, for each fixed $k=0, 1, 2, \cdots$, all $d^k\phi/ds^k$ are uniformly bounded.)

We recall that the relation $g_n(s)\to 0$ can be differentiated any number of times.

We shall say that a distribution $g(s)$ is *asymptotically zero* as $s\to\infty$, and write

(10.2) $$g(s) \overset{d}{\sim} 0 \quad \text{as } s \to \infty,$$

if the sequence $\{g(s+h_n)\}$ converges to zero for any increasing sequence $\{h_n\}$ of real numbers with $h_n\to\infty$. It is easy to see that (10.2) is true if $g(s)$ is a function such that $g(s)\to 0$ as $s\to\infty$ in the ordinary sense. The relation (10.2) can be differentiated any number of times; and it is trivial that if $g(s)$ is periodic and satisfies (10.2), then $g(s)=0$.

We write $g_1(s)\overset{d}{\sim}g_2(s)$ if $g_1(s)-g_2(s)\overset{d}{\sim}0$.

Now we can state a theorem that supplements and generalizes Theorem 8. We follow the notation used there, i.e. $c=\operatorname{Log}\lambda>0$, k_0 is a particular value of $\log(-b/a)/c$, $\kappa=\operatorname{re} k_0=\operatorname{Log}|b/a|/c$. Again we mean by a solution an infinitely differentiable function on $[0, \infty)$ that satisfies (1.1).

THEOREM 11. *Let* $b>0$, $\lambda>1$.

(i) *If* $y(x)$ *is a solution of* (1.1) *such that*

(10.3) $$e^{-k_0 s}y(e^s) \overset{d}{\sim} 0 \quad \textit{as } s \to \infty,$$

then $y(x)=0$ *identically.*

(ii) *Let* $g(s)$ *be any periodic distribution with period c. Then there is one (and by* (i) *only one) solution* $y(x)$ *of* (1.1) *such that*

(10.4) $$e^{-k_0 s}y(e^s) \overset{d}{\sim} g(s) \quad \textit{as } s \to \infty,$$

and on any interval $[x_0, \lambda x_0]$ *with* $x_0>0$, $y(x)$ *satisfies the inequalities*

(10.5) $$|y^{(n)}(x)| \leqq MK^n\lambda^{n^2/2} \qquad (n = 0, 1, 2, \cdots)$$

with some constants $M, K>0$ *which depend on* x_0.

(iii) *Let* $y(x)$ *be an infinitely differentiable function on* $[x_0, \lambda x_0]$ *satisfying* (3.1) *and* (10.5), *where* $x_0>0$ *is arbitrary but fixed, and M and K are now any given positive constants. Then* $y(x)$ *can be extended, in a unique way, to a solution on* $[0, \infty)$ *of* (1.1), *and there is a unique periodic distribution* $g(s)$ *of period c such that* (10.4) *is true. Further,* $g(s)$ *is a smooth function if K is sufficiently small.*

REMARKS. 1. There is a similar theorem for the case $b<0$, with the obvious modification due to the existence of the distinguished solution y_L (see Theorem 9); we do not state it explicitly. We have not considered the case $b=0$.

2. The uniqueness result (i) is a generalization of Theorem 8(i). The proof given below is quite different from the previous proof and depends heavily on the results for $\lambda<1$, while the previous proof is independent of the case $\lambda<1$. For this reason we have thought it worthwhile to give both proofs.

3. As we have pointed out earlier, there is a striking analogy between (10.5) and (4.3), in which the roles of $y(x)$ and $g(s)$ are interchanged.

PROOF OF THEOREM 11(i). Suppose $y(x)$ is a solution of (1.1) satisfying (10.3). It will suffice to show that $\int y(e^s)\,\phi(s)\,ds=0$ for any test function $\phi(s)$ with support smaller than c in length.

Let $\phi(s)$ be such a function, so that there is a value s_0 such that $\phi(s)=0$ unless $s_0<s<s_0+c$, and set $u(x)=x^{-1}\,\phi(\operatorname{Log} x)$ for $x_0\leqq x\leqq\lambda x_0$, where $x_0=e^{s_0}>0$, so that $u(x)$ vanishes identically near the ends of $[x_0, \lambda x_0]$. We can then extend $u(x)$ to a solution on $(0, \infty)$ of the functional equation

$$u'(x) = -a\lambda^{-2}u(\lambda^{-1}x) - b\lambda^{-1}u(x), \tag{10.6}$$

by the general arguments discussed in §1, and we note that (10.6) is of the same form as (1.1), with λ replaced by $\lambda^{-1}<1$. It is easily verified, by differentiating and using (1.1) and (10.7), that we have the identity

$$\int_x^{\lambda x_0} y(\xi)u(\xi)\,d\xi - \lambda a^{-1}y(x)u(\lambda x) = A = \text{const}, \tag{10.7}$$

and putting $x=x_0$ and using $u(\lambda x_0)=0$, we obtain

$$A = \int_{x_0}^{\lambda x_0} y(\xi)u(\xi)\,d\xi = \int y(e^s)\phi(s)\,ds. \tag{10.8}$$

Our proof is therefore complete if we can show that $A=0$.

To this end we rewrite (10.7) in terms of the variable $s=\operatorname{Log} x$, and then "mollify" it by multiplying by $\psi(s-\rho)$ and integrating with respect to s, where $\psi(s)$ is a test function such that $\psi(s)\geqq 0$ and $\int\psi(s)\,ds=1$, and where $\rho>0$ is some constant. The result is, after change of the order of integration,

$$\int y(e^s)u(e^s)\psi_1(s-\rho)e^s\,ds - \lambda a^{-1}\int y(e^s)u(e^{s+c})\psi(s-\rho)\,ds = A, \tag{10.9}$$

where $\psi_1(s) = \int_{s-c} \psi(t)\, dt$ is also a test function.

The first integral on the left of (10.9) may be written

$$\int e^{-k_0 s} y(e^s)\omega_\rho(s-\rho)\, ds = \int e^{-k_0(s+\rho)} y(e^{s+\rho})\omega_\rho(s)\, ds, \tag{10.10}$$

where $\omega_\rho(s) = e^{(k_0+1)(s+\rho)}u(e^{s+\rho})\psi_1(s)$, and the set of functions $\{\omega_\rho(s)\}$ for varying $\rho > 0$ form a bounded set of test functions, since $\psi_1(s)$ is a fixed test function and $e^{(k_0+1)s}u(e^s)$ is a bounded function together with all its derivatives as $s \to \infty$. (This is part of the proof of Theorem 3(iii), applied now to (10.6): note that $-b\lambda^{-1}/a\lambda^{-2} = -\lambda b/a = (\lambda^{-1})^{-(k_0+1)}$.) It then follows from (10.3) that (10.10) tends to 0 as $\rho \to \infty$ through any sequence; recall the remark after (10.1). Similarly we can show that the second term on the left of (10.9) tends to 0 as $\rho \to \infty$, and this gives $A = 0$, as we wished to show.

PROOF OF THEOREM 11(ii). If $g(s)$ is a periodic distribution, it is essentially a distribution on a torus and hence has a Fourier series of the form (4.7), where $c_m = O(m^N)$ for some positive integer N [**11**, p. 224].

Now define c_m^r successively by $c_m^0 = c_m$ and

$$c_m^r = c_m^{r-1}/(r + k_0 + 2\pi imc^{-1}) \qquad (m = 0, \pm 1, \pm 2, \cdots; r = 1, 2, \cdots). \tag{10.11}$$

This construction is possible unless $k_0 = -r - 2\pi imc^{-1}$ for some r and m. Assuming, for the moment, that the exceptional case does not occur, we see that $c_m^r = O(m^{-2})$ if $r \geqq N+2$, and fixing one such r, we set

$$g^r(s) = \sum_{m=-\infty}^{\infty} c_m^r \exp(2\pi ims/c). \tag{10.12}$$

The series converges and $g^r(s)$ is Hölder-continuous and periodic with period c. Also (10.11) implies that

$$e^{-k_0 s}(e^{-s}d/ds)^r e^{(k_0+r)s} g^r(s) = g(s) \tag{10.13}$$

in the distribution sense.

Since $g^r(s)$ is Hölder-continuous, there is by Theorem 8(ii) a solution $y^r(x)$ of (1.1) with a replaced by $a\lambda^{-r}$ such that

$$x^{-(k_0+r)} y^r(x) = g^r(\operatorname{Log} x) + O(x^{-\theta}) \quad \text{as } x \to \infty, \tag{10.14}$$

where $\theta > 0$, and this implies that

$$(10.15)\qquad e^{-(k_0+r)s}y^r(e^s) \overset{d}{\sim} g^s(s) \quad \text{as } s \to \infty.$$

Now $y(x)=(d/dx)^r y^r(x)$ is a solution of (1.1), and furthermore we see from (10.13) and (10.15) that $e^{-k_0 s}y(e^s)\overset{d}{\sim}g(s)$. For (10.15) can be differentiated any number of times and hence admits the application of the operator $e^{-k_0 s}(e^{-s}d/ds)^r e^{(k_0+r)s}$, which is *a differential operator with constant coefficients*. Thus $y(x)$ is the solution of (1.1) with the desired asymptotic properties.

In the exceptional case mentioned above, we may assume that $k_0=-r_0$ for some integer $r_0>0$, for k_0 is undetermined up to an integral multiple of $2\pi i c^{-1}$. We then consider $g_0(s)=g(s)-c_0$. For this $g_0(s)$ the construction (10.11) is possible if we set $c_0^r=0$ for all $r>0$, and in this way we obtain a solution $y_0(x)$ of (1.1) such that $e^{-k_0 s}\,y_0(e^s)\overset{d}{\sim}g_0(s)$. On the other hand, we know by Theorem 8(ii) that there is a solution $y_1(x)$ such that $e^{-k_0 s}\,y_1(s)\overset{d}{\sim}c_0$. Then $y(x)=y_0(x)+y_1(x)$ is the desired solution.

We next prove (10.5). Since $y(x)$ satisfies (1.1), we have

$$y(x) = e^{b(x-x_0)}y(x_0) - a\int_x^{x_0} e^{b(x-\xi)}y(\lambda\xi)\,d\xi,$$

and using (1.1) again, we obtain easily the estimate

$$|y'(x)| \leqq (2|a|+b)M \quad \text{for } \lambda^{-1}x_0 \leqq x \leqq x_0,$$

where $M=\max|y(x)|$ for $x_0\leqq x\leqq \lambda x_0$. If we denote by $\|f\|_m$ the maximum of $|f(x)|$ for $x\in[\lambda^m x_0, \lambda^{m+1}x_0]$, the above result may be written

$$\|y'\|_m \leqq (2|a| + b)\|y\|_{m+1},$$

and since $y^{(n)}(x)$ satisfies (1.1) with a replaced by $a\lambda^n$, we have

$$(10.16)\qquad \|y^{(n+1)}\|_m \leqq (2|a|\lambda^n + b)\|y^{(n)}\|_{m+1}.$$

Successive application of this inequality gives

$$\|y^{(n)}\|_0 \leqq (2|a| + b)^n\lambda^{n(n-1)/2}\|y\|_n.$$

Since $y(x)=(d/dx)^r y^r(x)$, where $y^r(x)$ is as given above (again assuming the nonexceptional case for the moment), we obtain in the same way

$$(10.17)\qquad \begin{aligned} \|y\|_n &\leqq (2|a|\lambda^{-1}+b)\cdots(2|a|\lambda^{-r}+b)\|y^r\|_{n+r} \\ &\leqq (2|a| + b)^r\|y^r\|_{n+r}. \end{aligned}$$

But we see from (10.14) that $y^r(x)=O(x^{\kappa+r})$, and hence $\|y^r\|_{n+r} \leqq M\lambda^{(\kappa+r)(n+r)}$. Altogether, therefore, we obtain

$$\|y^{(n)}\|_0 \leqq M(2|a| + b)^{n+r}\lambda^{n(n-1)/2}\lambda^{(\kappa+r)(n+r)},$$

which is equivalent to (10.5). (The exceptional case offers no further difficulty.)

PROOF OF THEOREM 11(iii). Suppose $y(x)$ satisfies the conditions stated. Then $y(x)$ can be continued to a solution on $[0, \infty)$ of (1.1) (see §1), and we have

$$y(\lambda x) = a^{-1}(D - b)y(x),$$

where $D = d/dx$, and

$$y(\lambda^2 x) = a^{-1}[y'(\lambda x) - by(\lambda x)] = a^{-1}(\lambda^{-1}D - b)y(\lambda x)$$
$$= a^{-2}(D - b)(\lambda^{-1}D - b)y(x).$$

Proceeding in the same way, we obtain

$$y(\lambda^{n+1}x) = a^{-(n+1)}(D - b)(\lambda^{-1}D - b) \cdots (\lambda^{-n}D - b)y(x)$$
$$= a^{-(n+1)} \sum_{m=0}^{n} A_m^n(-b)^{n-m}D^m y(x) \tag{10.18}$$

where $A_m^n = \sum \lambda^{-j_1-j_2-\cdots-j_m}$, the sum being taken over all indices j_r such that $0 \leqq j_1 < j_2 < \cdots < j_m \leqq n$. Writing $j_r = r - 1 + k_r$, we have

$$A_m^n = \lambda^{-m(m-1)/2} \sum \lambda^{-k_1-\cdots-k_m},$$

where the sum is to be taken over $0 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_m \leqq n - m + 1$. This sum does not exceed

$$\left(\sum_{k=0}^{\infty} \lambda^{-k}\right)^m = (1 - \lambda^{-1})^{-m},$$

so that

$$0 \leqq A_m^n \leqq \lambda^{-m(m-1)/2}(1 - \lambda^{-1})^{-m}. \tag{10.19}$$

Since (10.5) may be written $|D^n y(x)| \leqq |a| M_1 K_1^n \lambda^{n(n-1)/2}$, we obtain, from (10.18) and (10.19),

$$|y(\lambda^{n+1}x)| \leqq M_1(b/|a|)^n \sum_{m=0}^{n} b^{-m}(1 - \lambda^{-1})^{-m}K_1^m$$
$$\leqq M_1(b/|a|)^n[1 - K_1 b^{-1}(1 - \lambda^{-1})^{-1}]^{-1} \quad \text{for } x \in [x_0, \lambda x_0], \tag{10.20}$$

provided $K_1 < b(1 - \lambda^{-1})$, which is the case if K is sufficiently small.

Now (10.20) implies that $y(x) = O(x^\kappa)$. Similarly we can prove that $y'(x) = O(x^{\kappa-1})$ if K is sufficiently small. Then the same argument as in the proof of Theorem 3(iii) applies, with the result that $y(x) =$

$x^{k_0}(g(\text{Log } x)+O(x^{-1}))$, where $g(s)$ is a continuous periodic function of period c. This proves (10.4).

The general case when K is not so small can be reduced to the above case by "integration." In general one can find a solution $y^r(x)$ of (1.1) with a replaced by $a\lambda^{-r}$ such that $(d/dx)^r y^r(x)=y(x)$; the construction is given in the proof of Theorem 5(ii). (There is an exceptional case in which this fails, but the present proof can easily be modified to meet such cases.) Then (10.5) becomes

$$|D^n y^r(x)| \leqq MK^{n-r}\lambda^{(n-r)^2/2} \leqq M_2(K\lambda^{-r})^n\lambda^{n^2/2} \qquad (x \in [x_0, \lambda x_0])$$

for $n \geqq r$, and the inequality is true also for $n<r$ if M_2 is adjusted. Since $K\lambda^{-r}$ can be made as small as we please by taking r large, it follows from the result proved above that $y^r(x)$ has an asymptotic form of the type (10.4). Then the desired result for $y(x)$ follows by differentiation (cf. the computations in the proof of (ii)).

Finally, the uniqueness of $g(s)$ follows from a remark given after (10.2).

References

1. L. Fox, D. F. Mayers, J. R. Ockendon and A. B. Tayler, *On a functional differential equation*, J. Inst. Math. Appl. (to appear).

2. K. Mahler, *On a special functional equation*, J. London Math. Soc. **15** (1940), 115–123. MR **2**, 133.

3. W. A. Harris, Jr. and Y. Sibuya (editors), *Proceedings United States-Japan seminar on differential and functional equations* (University of Minnesota, Minneapolis, Minn., 1967), Benjamin, New York, 1967. MR **36** #5415.

4. N. G. de Bruijn, *The asymptotically periodic behavior of the solutions of some linear functional equations*, Amer. J. Math. **71** (1949), 313–330. MR **10**, 541.

5. ———, *On some linear functional equations*, Publ. Math. Debrecen **1** (1950), 129–134. MR **12**, 106.

6. ———, *The difference-differential equation* $F'(x)=e^{\alpha x+\beta}F(x-1)$. I, II, Nederl. Akad. Wetensch. Proc. Ser. A **56** = Indag. Math. **15** (1953), 449–464. MR **15**, 629.

7. E. W. Bowen and G. R. Morris, private communication.

8. P. O. Frederickson, *Global solutions to certain non-linear functional-differential equations*, J. Math. Anal. Appl. **33** (1971), 355–358.

9. ———, *Analytic solutions for certain functional-differential equations of advanced type* (to appear).

10. R. E. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963. MR **26** #5259.

11. L. Schwartz, *Théorie des distributions*, Publ. Math. Univ. Strasbourg, no. 9–10, Hermann, Paris, 1966. MR **35** #730.

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