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THE FUNCTIONAL EQUATION OF SOME DIRICHLET SERIES

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ABSTRACT. The functional equation for two classes of Dirichlet series is established. These Dirichlet series involve primitive characters and can be regarded as generalizations of Dirichlet's L-functions or of Epstein's zeta-functions. One class is also a generalization of some series studied by Stark.

Our objective here is to derive the functional equation for two classes of Dirichlet series. Both classes of series may be regarded as generalizations of Epstein's zeta-functions or of Dirichlet's L-functions. One class is also a generalization of Stark's L-functions [5], and so it will be convenient to use the notation in [5].

Let x denote an m-dimensional vector and $Q(x)$ a positive definite quadratic form in m variables. We let (x, y) denote the inner product of two m-dimensional vectors x and y. Write $2Q(x)=(xF, x)$, where F is a symmetric $m\times m$ matrix of real numbers. The discriminant $d>0$ of Q is defined to be the determinant of F. The inverse Q^{-1} of Q is defined by $2Q^{-1}(x)=(xdF^{-1}, x)$. The Epstein zeta-functions are defined for $\sigma=\operatorname{Re} s>m/2$ by

$$Z(s, Q; g, h) = \sum_n{}' e^{2\pi i(h,n)}\{Q(n+g)\}^{-s}, \tag{1}$$

where g and h are m-dimensional real vectors and the sum is over all integral vectors n with $Q(n+g)\neq 0$. Epstein [3, p. 207] showed that $Z(s, Q; g, h)$ satisfies the functional equation (the definitions of the discriminant and inverse differ in [3] and [5])

$$\begin{aligned}&\left(\frac{d^{1/m}}{2\pi}\right)^s \Gamma(s)Z(s, Q; g, h)\\ &\quad = e^{-2\pi i(g,h)}\left(\frac{d^{(m-1)/m}}{2\pi}\right)^{m/2-s}\Gamma(\tfrac{1}{2}m-s)Z(\tfrac{1}{2}m-s, Q^{-1}; h, -g).\end{aligned} \tag{2}$$

We now give our generalizations of (1). Throughout the sequel all characters are primitive. For $\sigma>m/2$ define

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$$Z(s, \chi, Q; g, h) = \sum_n{}' \prod_{j=1}^m \chi_j(n_j) e^{2\pi i(n,h)}\{Q(n+g)\}^{-s},$$

where $n=(n_1, \cdots, n_m)$, g and h are as above, χ_j is a character (mod k_j), $1 \leqq j \leqq m$, and k_j is any positive integer such that a primitive character (mod k_j) exists. Secondly, for $\sigma > m/2$ define

$$L(s, \chi, Q; g, h) = \sum_n{}' \chi(Q(n)) e^{2\pi i(n,h)}\{Q(n+g)\}^{-s},$$

where now Q has integral coefficients. We shall use the Gauss sum

$$G(r, \chi) = \sum_{j=1}^{k-1} \chi(j) e^{2\pi i r j/k},$$

where r is an integer and χ is a character (mod k). Furthermore, we write $G(\chi) = G(1, \chi)$.

THEOREM 1. *$Z(s, \chi, Q; g, h)$ can be continued to a meromorphic function of s that is either entire or has but one pole, a simple one at $s = m/2$. Furthermore,*

$$\prod_{j=1}^m G(\bar{\chi}_j)\left(\frac{d^{1/m}}{2\pi}\right)^s \Gamma(s) Z(s, \chi, Q; g, h)$$

$$= e^{-2\pi i(g,h)}\left(\frac{d^{(m-1)/m}}{2\pi}\right)^{m/2-s} \Gamma(\tfrac{1}{2}m - s) Z(\tfrac{1}{2}m - s, \bar{\chi}, Q^{-1}/k; h, -g/k). \tag{3}$$

Here, Q^{-1}/k denotes Q^{-1} but with x_j replaced by x_j/k_j, $1 \leqq j \leqq m$, and $g/k = (g_1/k_1, \cdots, g_m/k_m)$.

PROOF. Since χ is primitive, we have the factorization theorem [2, pp. 67–68], $G(r, \bar{\chi}) = \chi(r) G(\bar{\chi})$. Thus, for $\sigma > m/2$,

$$\prod_{j=1}^m G(\bar{\chi}_j) Z(s, \chi, Q; g, h)$$

$$= \sum_n{}' \prod_{j=1}^m G(n_j, \bar{\chi}_j) e^{2\pi i(n,h)}\{Q(n+g)\}^{-s}$$

$$= \sum_{j_1=1}^{k_1-1} \cdots \sum_{j_m=1}^{k_m-1} \bar{\chi}_1(j_1) \cdots \bar{\chi}_m(j_m) Z(s, Q; g, h + j/k), \tag{4}$$

where $j/k = (j_1/k_1, \cdots, j_m/k_m)$. Since $Z(s, Q; g, h)$ can be analytically continued to an entire function if h is nonintegral or to a function analytic everywhere except for a simple pole at $s = m/2$ if h is integral [3, p. 207], the first part of the theorem follows directly from (4). We now employ (2) to obtain, for all s,

$$\begin{aligned}\prod_{j=1}^{m} G(\bar{\chi}_j)Z(s,\chi,Q;g,h) &= d^{(m-1)/2-s}(2\pi)^{2s-m/2}\frac{\Gamma(\frac{1}{2}m-s)}{\Gamma(s)} \\ &\cdot \sum_{j_1=1}^{k_1-1}\cdots\sum_{j_m=1}^{k_m-1}\bar{\chi}_1(j_1)\cdots\bar{\chi}_m(j_m) \\ &\cdot e^{-2\pi i(g,h+j/k)}Z(\tfrac{1}{2}m-s,Q^{-1};h+j/k,-g).\end{aligned} \tag{5}$$

For $\sigma<0$, put $r_i=n_ik_i+j_i$, $1\leqq j_i\leqq k_i$, $-\infty<n_i<\infty$, $1\leqq i\leqq m$, where n is the summation index for $Z(\frac{1}{2}m-s,\ Q^{-1};\ h+j/k,\ -g)$. The right-hand side of (5) then becomes

$$d^{(m-1)/2-s}(2\pi)^{2s-m/2}e^{-2\pi i(g,h)}\frac{\Gamma(\frac{1}{2}m-s)}{\Gamma(s)} \cdot \sum_r{}' \prod_{j=1}^{m}\bar{\chi}(r_j)e^{-2\pi i(r,g/k)}\{Q^{-1}(r/k+h)\}^{s-m/2}.$$

(3) now easily follows.

If $m=1$, $g=h=0$, and $\chi(-1)=1$, we have derived the functional equation of Dirichlet's L-functions. A similar proof of the functional equation for Dirichlet L-functions originated with Hurwitz [**4**]. Slight variants of his proof can be found in [**2**, p. 73] and [**1**, pp. 232–233].

Define the character χ' by $\chi'(r)=\chi(r)\chi_1(r)^m$, where χ is a character (mod k) and χ_1 is a character defined by

$$\begin{aligned}\chi_1(r)=(k'\,|\,r),\qquad k' &= (-1)^{(k-1)/2}k, && k \text{ odd},\\ &= -k, && k\equiv 0 \pmod 4,\\ &= 4k, && k\equiv 2 \pmod 4,\end{aligned}$$

where $(k'|\,r)$ denotes the Kronecker symbol.

THEOREM 2. *Let* $(d,k)=1$ *with either* m *even or* k *odd. Suppose* χ' *is a primitive character* (mod k). *Then* $L(s,\chi,Q;g,h)$ *can be continued to a meromorphic function of* s *that is either entire or has but one pole, a simple one at* $s=m/2$. *Furthermore,*

$$\left(\frac{kd^{1/m}}{2\pi}\right)^{s}\Gamma(s)L(s,\chi,Q;g,h) = \alpha e^{-2\pi i(g,h)}\left(\frac{kd^{(m-1)/m}}{2\pi}\right)^{m/2-s}\Gamma(\tfrac{1}{2}m-s) \cdot L(\tfrac{1}{2}m-s,\bar{\chi}',Q^{-1};hk,-g/k), \tag{6}$$

where $|\alpha|=1$.

For the value of α see [5, p. 37]. If $g=h=0$, Theorem 2 is due to Stark [5]. Stark also shows [5, Lemma 4] that Q^{-1}, indeed, has integral coefficients.

PROOF. We shall let $\sum_{j=1}^{k} = \sum_{j_1=1}^{k} \cdots \sum_{j_m=1}^{k}$. From [5, Theorem 1],

$$\sum_{j=1}^{k} \chi(Q(j))e^{2\pi i(j,n)/k} = \alpha k^{m/2}\bar{\chi}'(Q^{-1}(n)).$$

Thus, for $\sigma > m/2$,

$$\begin{aligned}\alpha k^{m/2}L(s, \bar{\chi}', Q^{-1}; g, h) &= \sum_{j=1}^{k} \chi(Q(j)) \sum_{n}{}' e^{2\pi i(n,h+j/k)}\{Q^{-1}(n+g)\}^{-s} \\ &= \sum_{j=1}^{k} \chi(Q(j))Z(s, Q^{-1}; g, h+j/k).\end{aligned}$$

The first part of the theorem now follows as in the proof of Theorem 1. Now use (2) to obtain, for all s,

$$\begin{aligned}&\alpha k^{m/2}L(s, \bar{\chi}', Q^{-1}; g, h) \\ &= d^{1/2-s}(2\pi)^{2s-m/2} \frac{\Gamma(\frac{1}{2}m - s)}{\Gamma(s)} \sum_{j=1}^{k} \chi(Q(j))e^{-2\pi i(g,h+j/k)} \\ &\quad \cdot Z(\tfrac{1}{2}m - s, Q; -(h+j/k), g).\end{aligned} \tag{7}$$

For $\sigma<0$, put $r_i=n_ik-j_i$, $1\leqq j_i\leqq k$, $-\infty<n_i<\infty$, $1\leqq i\leqq m$, where n is the summation index of $Z(\frac{1}{2}m-s, Q; -(h+j/k), g)$. Since $\chi(Q(r)) =\chi(Q(j))$, the right-hand side of (7) becomes

$$d^{1/2-s}(2\pi)^{2s-m/2} \frac{\Gamma(\frac{1}{2}m - s)}{\Gamma(s)} e^{-2\pi i(g,h)}k^{m-2s}L(\tfrac{1}{2}m - s, \chi, Q; -hk, g/k).$$

If we now replace s by $\frac{1}{2}m-s$, $-hk$ by g, and g/k by h, we arrive at (6).

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