Tôhoku Math. J.
47 (1995), 251–262

THE FUNCTOR OF A SMOOTH TORIC VARIETY

DAVID A. COX

(Received December 27, 1993, revised February 6, 1995)

Abstract. This paper describes the data needed to specify a map from a scheme to an arbitrary smooth toric variety. The description is in terms of a collection of line bundles and sections on the scheme which satisfy certain compatibility and nondegeneracy conditions. There is also a natural torus action on these collections. As an application, we show how homogeneous polynomials can be used to describe all maps from a projective space (or more generally a toric variety) to a smooth complete toric variety.

A map $Y \to \boldsymbol{P}_k^n$ is determined by a line bundle L on Y together with $n+1$ sections which do not vanish simultaneously. In fact, $\boldsymbol{P}_k^n$ is the variety representing the functor

$$(1) \qquad Y \mapsto \{(L, u_0, \dots, u_n) : u_i \in H^0(Y, L) \text{ do not vanish simultaneously}\}/\sim ,$$

where $\sim$ is the obvious equivalence relation. The goal of this paper is to generalize this description to the case of an arbitrary smooth toric variety.

We will work with schemes over a field k, and we will fix a smooth n-dimensional toric variety X determined by a fan Δ in $N_{\boldsymbol{R}} = \boldsymbol{R}^n$. As usual, M denotes the dual lattice of N and $\Delta(1)$ denotes the set of 1-dimensional cones of Δ. We will use $\sum_\rho$ to mean $\sum_{\rho \in \Delta(1)}$, and similarly for $\bigotimes_\rho$. Each $\rho \in \Delta(1)$ determines a divisor $D_\rho \subset X$ and a generator $n_\rho \in \rho \cap N$. Finally, let $\Delta_{\max}$ denote the set of maximal cones in Δ (i.e., those which are not proper faces of cones in Δ). Basic references for toric varieties are [3], [5] and [7].

1. Δ-collections and functors.

If a fan Δ determines a smooth toric variety X, then we can generalize the data in (1) as follows:

DEFINITION 1.1. Given a scheme Y over k, a *Δ-collection on Y* consists of line bundles L_ρ and sections $u_\rho \in H^0(Y, L_\rho)$, indexed by $\rho \in \Delta(1)$, and isomorphisms $c_m : \bigotimes_\rho L_\rho^{\otimes \langle m, n_\rho \rangle} \simeq \mathcal{O}_Y$, indexed by $m \in M$, such that:

(i) (Compatibility) $c_m \otimes c_{m'} = c_{m+m'}$ for all $m, m' \in M$.

(ii) (Nondegeneracy) For each $y \in Y$, there is $\sigma \in \Delta_{\max}$ with $u_\rho(y) \neq 0$ for all $\rho \not\subset \sigma$.

A Δ-collection on Y is written (L_ρ, u_ρ, c_m). The compatibility condition on the isomorphisms c_m implies that $\sum_\rho [L_\rho] \otimes n_\rho = 0$ in $\mathrm{Pic}(Y) \otimes_{\boldsymbol{Z}} N$. However, the triviality of this sum is not sufficient: data of the Δ-collection includes an explicit choice of trivialization (the c_m's), which is not unique. The examples given below will show why

---

1991 *Mathematics Subject Classification*. Primary 14M25; Secondary 14E99.

this is needed. As for the nondegeneracy condition, note that $u_\rho \in H^0(Y, L_\rho)$ gives $u_\rho\colon \mathcal{O}_Y \to L_\rho$, which induces $u_\rho^*\colon L_\rho^{-1} \to \mathcal{O}_Y$. Then nondegeneracy is equivalent to the surjectivity of the map

$$\sum_{\sigma\in\Delta_{\max}} \otimes_{\rho\not\subset\sigma} u_\rho^*\colon \bigoplus_{\sigma\in\Delta_{\max}} \otimes_{\rho\not\subset\sigma} L_\rho^{-1} \to \mathcal{O}_Y\,.$$

Finally, when dealing with a toric variety X determined by Δ, we will sometimes speak of X-collections rather than Δ-collections.

We get a canonical Δ-collection on the toric variety X as follows. For each ρ, the divisor D_ρ gives a line bundle $\mathcal{O}_X(D_\rho)$ since X is smooth. Furthermore, the natural inclusion $\mathcal{O}_X \subset \mathcal{O}_X(D_\rho)$ corresponds to a global section $\iota_\rho \in H^0(X, \mathcal{O}_X(D_\rho))$. Finally, given $m \in M$, the character χ^m is a rational function on X such that $\operatorname{div}(\chi^m) = \sum_\rho \langle m, n_\rho\rangle D_\rho$. Thus we get an isomorphism of sheaves

$$c_{\chi^m}\colon \otimes_\rho \mathcal{O}_X(D_\rho)^{\otimes\langle m, n_\rho\rangle} \simeq \mathcal{O}_X\left(\sum_\rho \langle m, n_\rho\rangle D_\rho\right) \simeq \mathcal{O}_X\,,$$

where the second isomorphism is induced by χ^m.

LEMMA 1.1. *$(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$ is a Δ-collection on X.*

PROOF. Compatibility is trivial since $\chi^{m+m'} = \chi^m \chi^{m'}$ for $m, m' \in M$. To prove nondegeneracy, take $x \in X$. Since $X = \bigcup_{\sigma\in\Delta_{\max}} X_\sigma$, where X_σ is the affine toric variety determined by σ, we have $x \in X_\sigma$ for some $\sigma \in \Delta_{\max}$. Then $X - X_\sigma = \sum_{\rho\not\subset\sigma} D_\rho$ shows that $\iota_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$, and nondegeneracy follows. □

The Δ-collection $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$ will be called the *universal Δ-collection.* This terminology will be justified below.

DEFINITION 1.2. An *equivalence* $(L_\rho, u_\rho, c_m) \sim (L'_\rho, u'_\rho, c'_m)$ of Δ-collections on Y consists of isomorphisms $\gamma_\rho\colon L_\rho \simeq L'_\rho$ which carry u_ρ to u'_ρ and c_m to c'_m.

To better understand these definitions, let us look at some examples:

EXAMPLE 1.1. Let $X = \boldsymbol{A}_k^n$, where the n_ρ's are the standard basis $\{e_1, \dots, e_n\}$ of N. Let $\{e^1, \dots, e^n\}$ be the dual basis of M. Now suppose we have an $\boldsymbol{A}_k^n$-collection (L_i, u_i, c_m) on Y (we write L_i instead of L_{e_i} for $1 \le i \le n$). Then c_{e^i} is an isomorphism $c_{e^i}\colon L_i \simeq \mathcal{O}_Y$. This maps u_i to $v_i \in H^0(Y, \mathcal{O}_Y)$, and one can check that setting $\gamma_i = c_{e^i}$ in Definition 1.2 gives an equivalence $(L_i, u_i, c_m) \sim (\mathcal{O}_Y, v_i, 1)$. Furthermore, $(\mathcal{O}_Y, v_i, 1) \sim (\mathcal{O}_Y, v'_i, 1)$ if and only if $v_i = v'_i$ for all i. Since the nondegeneracy condition is vacuous in this case, we see that equivalence classes of $\boldsymbol{A}_k^n$-collections on Y correspond exactly to n-tuples in $H^0(Y, \mathcal{O}_Y)$. As is well-known, such n-tuples are classified by morphisms $Y \to \boldsymbol{A}_k^n$.

EXAMPLE 1.2. Let $X = \boldsymbol{P}_k^n$, where the n_ρ's are $e_1, \dots, e_n$ and $e_0 = -\sum_{i=0}^n e_i$. Now let (L_i, u_i, c_m) be a $\boldsymbol{P}_k^n$-collection on Y (where $0 \le i \le n$). Here, c_{e^i} is an isomorphism

$c_{e^i}: L_i \otimes L_0^{-1} \simeq \mathcal{O}_Y$. This induces $\gamma_i: L_i \simeq L_0$ which takes u_i to $v_i \in H^0(Y, L_0)$. One can check that the γ_i's give an equivalence $(L_i, u_i, c_m) \sim (L_0, v_i, 1)$. Thus we get the line bundle L_0 and $n+1$ sections. Since the fan for $\boldsymbol{P}_k^n$ has $n+1$ maximal cones (depending on which of $e_0, \ldots, e_n$ is omitted), the nondegeneracy condition says that the v_i never vanish simultaneously. Thus equivalence classes of $\boldsymbol{P}_k^n$-collections on Y correspond exactly to the equivalence classes in (1) and hence are classified by morphisms $Y \to \boldsymbol{P}_k^n$.

EXAMPLE 1.3. Let $X = \boldsymbol{G}_m^n$. In this case, there are no n_ρ's, and for $m \in M$, $\bigotimes_\rho L_\rho^{\otimes \langle m, n_\rho \rangle}$ reduces to $\mathcal{O}_Y$, so that a $\boldsymbol{G}_m^n$-collection on Y consists of $c_m: \mathcal{O}_Y \simeq \mathcal{O}_Y$. The notions of equivalence and nondegeneracy are vacuous in this case, so that equivalence classes of $\boldsymbol{G}_m^n$-collections on Y correspond to homomorphisms $M \to H^0(Y, \mathcal{O}_Y^*)$. Such homomorphisms are classified by morphisms $Y \to \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{G}_m) = \boldsymbol{G}_m^n$.

Returning to the general case, it is easy to see that the pull-back of a Δ-collection is again a Δ-collection. Thus we get a functor C_Δ: k-Schemes° → Sets defined by

$$C_\Delta(Y) = \{\text{all } \Delta\text{-collections } (L_\rho, u_\rho, c_m) \text{ on } Y\}/\sim .$$

Furthermore, the universal Δ-collection $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$ gives a natural transformation

$$\mathrm{Hom}_k(Y, X) \to C_\Delta(Y)$$

by sending $f: Y \to X$ to the pull-back of $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$ by f. The main result of this paper is the following theorem:

THEOREM 1.1. *If X is a smooth toric variety, then the above map* $\mathrm{Hom}_k(Y, X) \to C_\Delta(Y)$ *is a bijection for all k-schemes Y. Thus the toric variety X represents the functor C_Δ.*

PROOF. First assume that the n_ρ's span $N_{\boldsymbol{R}}$. In this case, we know by [2] that X is a geometric quotient $(\boldsymbol{A}_k^{\Delta(1)} - Z)/G$, where $\boldsymbol{A}_k^{\Delta(1)} = \mathrm{Spec}(k[x_\rho])$, Z is defined by the vanishing of $\prod_{\rho \not\subset \sigma} x_\rho$ for $\sigma \in \Delta_{\max}$, and $G = \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Pic}(X), \boldsymbol{G}_m)$.

We will construct an inverse map $C_\Delta(Y) \to \mathrm{Hom}_k(Y, X)$. Let (L_ρ, u_ρ, c_m) be a Δ-collection on Y, and let $U \subset Y$ be an open subset such that the L_ρ are trivial on U. If we choose isomorphisms $\gamma_\rho: L_{\rho|U} \to \mathcal{O}_U$, then we get an equivalence $(L_{\rho|U}, u_{\rho|U}, c_{m|U}) \sim (\mathcal{O}_U, v_\rho, c'_m)$, where $v_\rho \in H^0(U, \mathcal{O}_U)$ and $c'_m: \mathcal{O}_U \simeq \mathcal{O}_U$ can be regarded as a homomorphism $c': M \to H^0(U, \mathcal{O}_U^*)$.

Since the n_ρ's span $N_{\boldsymbol{R}}$ and X is smooth, we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\ \alpha\ } \boldsymbol{Z}^{\Delta(1)} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0\,, \tag{2}$$

where α is defined by $m \mapsto (\langle m, n_\rho \rangle)$. Since $\mathrm{Pic}(X)$ is torsion free, the above map $c': M \to H^0(U, \mathcal{O}_U^*)$ extends to $\tilde{c}': \boldsymbol{Z}^{\Delta(1)} \to H^0(U, \mathcal{O}_U^*)$, which means that there are $\lambda_\rho \in H^0(U, \mathcal{O}_U^*)$ such that $c'_m = \prod_\rho \lambda_\rho^{\langle m, n_\rho \rangle}$ for all $m \in M$. Then the isomorphisms $\lambda_\rho: \mathcal{O}_U \simeq \mathcal{O}_U$ give an equivalence $(\mathcal{O}_U, v_\rho, c'_m) \sim (\mathcal{O}_U, w_\rho, 1)$, where $w_\rho = \lambda_\rho v_\rho$.

Now define $\tilde{f}_U: U \to \boldsymbol{A}_k^{\Delta(1)}$ by $\tilde{f}_U(x) = (w_\rho(x))$. The nondegeneracy condition implies that $\tilde{f}_U(x)\notin Z$, so that composing with the quotient map $\pi: A^{\Delta(1)}-Z\to X$ gives $f_U=\pi\circ\tilde{f}_U: U\to X$. In the argument below, let $\mathscr{U}=A^{\Delta(1)}-Z$.

We need to see how the choices made in the above construction affect the map f_U. A different set of choices would lead to a Δ-collection $(\mathcal{O}_U, w'_\rho, 1)\sim(\mathcal{O}_U, w_\rho, 1)$. This equivalence is given by $\lambda_\rho\in H^0(U, \mathcal{O}_U)$ such that $w'_\rho=\lambda_\rho w_\rho$ and $\prod_\rho\lambda_\rho^{\langle m,n_\rho\rangle}=1$ for all $m\in M$ (because the λ_ρ's must preserve the trivializations $1: \bigotimes\mathcal{O}_U^{\langle m,n_\rho\rangle}\simeq\mathcal{O}_U$). It follows from (2) that we get a homomorphism g: $\mathrm{Pic}(X)\to H^0(U, \mathcal{O}_U^*)$ such that $g([D_\rho])=\lambda_\rho$ for all ρ. If we evaluate this at a closed point $x\in U$, we get an element $g_x\in G=\mathrm{Hom}_{\mathbf{Z}}(\mathrm{Pic}(X), \mathbf{G}_m)$. Then the points $(w_\rho(x))$ and $(w'_\rho(x))=(\lambda_\rho(x)w_\rho(x))$ are related by g_x and hence give the same point in X.

This shows that f_U: $U\to X$ depends only on the equivalence class of (L_ρ, u_ρ, c_m). From here, it follows easily that the f_U patch together to give a morphism f: $Y\to X$.

It remains to show that this map is the inverse of the map $\mathrm{Hom}_k(Y, X)\to C_\Delta(Y)$ obtained by pulling back the universal Δ-collection $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$. First suppose that (L_ρ, u_ρ, c_m) on Y determines f: $Y\to X$. We need to show that

$$(3)\qquad (L_\rho, u_\rho, c_m)\sim f^*(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})\,.$$

An easy argument shows that the natural map $C_\Delta(Y)\to\prod_{\alpha\in A}C_\Delta(U_\alpha)$ is injective whenever $\{U_\alpha\}_{\alpha\in A}$ is an open cover of Y. Thus it suffices to prove (3) on an open set $U\subset Y$ where each L_ρ is trivial on U. On such a U, we know that $(L_{\rho|U}, u_{\rho|U}, c_{m|U})\sim(\mathcal{O}_U, w_\rho, 1)$ and $f=\pi\circ\tilde{f}$, where $\tilde{f}(x)=(w_\rho(x))$. We first observe that

$$(4)\qquad \pi^*(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})\sim(\mathcal{O}_{\mathscr{U}}, x_\rho, 1)\,,$$

where $\pi: \mathscr{U}\to X$ is as above. To prove (4), note that $\pi^*(\mathcal{O}_X(D_\rho))=\mathcal{O}_{\mathscr{U}}(\mathrm{div}(x_\rho))$, so that multiplication by x_ρ gives an isomorphism $\mathcal{O}_{\mathscr{U}}(\mathrm{div}(x_\rho))\simeq\mathcal{O}_{\mathscr{U}}$. Hence $\mathcal{O}_{\mathscr{U}}\subset\mathcal{O}_{\mathscr{U}}(\mathrm{div}(x_\rho))\simeq\mathcal{O}_{\mathscr{U}}$ is multiplication by x_ρ, and since $\chi^m\circ\pi=\prod_\rho x_\rho^{\langle m,n_\rho\rangle}$, (4) follows immediately. Then, returning to $f=\pi\circ\tilde{f}$, we conclude from (4) that

$$(5)\qquad f^*(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})\sim\tilde{f}^*(\mathcal{O}_{\mathscr{U}}, x_\rho, 1)=(\mathcal{O}_U, w_\rho, 1)\,,$$

since $\tilde{f}(x)=(w_\rho(x))$, and (3) follows.

Finally, suppose we have f: $Y\to X$. This gives $f^*(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$, which in turn determines f': $Y\to X$. We need to show that $f'=f$. First suppose that f factors $f=\pi\circ\tilde{f}$ for some map $\tilde{f}$: $Y\to\mathscr{U}$. Then $\tilde{f}$ can be written $\tilde{f}(x)=(w_\rho(x))$, where $w_\rho\in H^0(Y, \mathcal{O}_Y)$. From (5) and the construction of f', it follows immediately that $f'=f$. In the general case, note that G acts freely on $\mathscr{U}$ since X is smooth (this is easy to prove), so that $\pi: \mathscr{U}\to X$ is smooth. Then standard results about smoothness imply that f: $Y\to X$ lifts locally to $\mathscr{U}$ in the étale topology. Since $\mathrm{Hom}_k(-, X)$ is a sheaf in the étale topology on Y, we obtain $f'=f$, and the theorem is proved in the case when the n_ρ's span $N_{\mathbf{R}}$.

We next study what happens when the n_ρ's do not span $N_{\mathbf{R}}$. Let $N_1=N\cap\mathrm{Span}_{\mathbf{R}}(n_\rho)$. The fan Δ can be regarded as a fan Δ_1 in N_1, which gives a smooth toric variety X_1 of dimension $d=\mathrm{rank}(N_1)$. The inclusion $N_1\subset N$ induces an inclusion $X_1\subset X$, and the projection $N \to N/N_1$ induces a surjection $X \to T_1 = \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, \mathbf{G}_m) \simeq \mathbf{G}_m^{n-d}$, where $N_1^\perp = \mathrm{Hom}_{\mathbf{Z}}(N/N_1, \mathbf{Z}) \subset M$ is the annihilator of N_1.

Since N/N_1 is torsion free, we can write $N = N_1 \oplus N_2$ for some complement $N_2 \subset N$. Then Δ is the product fan $\Delta_1 \times \{0\}$, which implies that X is (noncanonically) the product $X_1 \times_k T_1$. If M_1 is the dual of N_1, then the projection $N \to N_1$ determines an inclusion $\alpha: M_1 \to M$ such that $M = \alpha(M_1) \oplus N_1^\perp$.

Now suppose that (L_ρ, u_ρ, c_m) is a Δ-collection on Y. Then, for every $m \in N_1^\perp$, we have $\langle m, n_\rho \rangle = 0$ for all ρ. Thus c_m is an isomorphism $c_m: \mathcal{O}_Y \simeq \mathcal{O}_Y$, which gives a homomorphism $N_1^\perp \to H^0(Y, \mathcal{O}_Y^*)$. Since this map depends only on the equivalence class of (L_ρ, u_ρ, c_m), we have a natural transformation

$$C_\Delta(Y) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, H^0(Y, \mathcal{O}_Y^*)).$$

Further, if we define $c_{m_1}^\alpha = c_{\alpha(m_1)}$ for $m_1 \in M_1$, then $(L_\rho, u_\rho, c_{m_1}^\alpha)$ is a Δ_1-collection on Y, and it follows easily that we have a natural transformation

$$C_\Delta(Y) \longrightarrow C_{\Delta_1}(Y).$$

Combining these maps, we obtain

$$C_\Delta(Y) \longrightarrow C_{\Delta_1}(Y) \times \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, H^0(Y, \mathcal{O}_Y^*)). \tag{6}$$

Since $M = \alpha(M) \oplus N_1^\perp$, it is straightforward to show that the map (6) is a bijection.

Now consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_k(Y, X) & \longrightarrow & C_\Delta(Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_k(Y, X_1) \times \mathrm{Hom}_k(Y, T_1) & \longrightarrow & C_{\Delta_1}(Y) \times \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, H^0(Y, \mathcal{O}_Y^*)). \end{array}$$

The vertical maps come from (6) and $X \simeq X_1 \times T_1$, and note that both are bijections. The map on the bottom is the product of the bijections $\mathrm{Hom}_k(Y, X_1) \simeq C_{\Delta_1}(Y)$ (since the n_ρ's span $(N_1)_{\mathbf{R}}$) and $\mathrm{Hom}_k(Y, T_1) \simeq \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, H^0(Y, \mathcal{O}_Y^*))$ (since $T_1 = \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, \mathbf{G}_m)$).

It follows that the map on top will be a bijection (and the theorem will be proved) provided the diagram commutes. By general nonsense, we only have to prove commutivity for $1_X \in \mathrm{Hom}_k(X, X)$. Going down and over, 1_X maps to $(\pi_1^*(\mathcal{O}_{X_1}(D_\rho), \iota_\rho, c_{\chi^{m_1}}), \phi)$, where $\pi_1: X \to X_1$ is the projection and $\phi: N_1^\perp \to H^0(X, \mathcal{O}_X^*)$ is defined by $m \mapsto \chi^m$ for $m \in N_1^\perp$. Going the other way, we need to study what happens to $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$ under the map

$$C_\Delta(X) \longrightarrow C_{\Delta_1}(X) \times \mathrm{Hom}_{\mathbf{Z}}(N_1^\perp, H^0(X, \mathcal{O}_X^*)).$$

Let us start with the second factor. Here, note that for $m \in N_1^\perp$, $c_{\chi^m}: \mathcal{O}_X \simeq \mathcal{O}_X$ is multiplication by χ^m. Hence the induced map $N_1^\perp \to H^0(X, \mathcal{O}_X^*)$ is exactly the above map ϕ. As for the first factor, we get $(\mathcal{O}_X(D_\rho), \iota_\rho, c'_{m_1})$, where $c'_{m_1} = c_{\chi^{\alpha(m_1)}}$ for $m_1 \in M_1$. However, since $\pi_1: X \to X_1$ is a toric map taking $n_\rho \in N$ to $n_\rho \in N_1$, it follows easily that $\pi_1^* \mathcal{O}_{X_1}(D_\rho) \simeq \mathcal{O}_X(D_\rho)$ in a way that preserves the section ι_ρ (this follows, for example, by looking at line bundles as determined by support functions and studying how π_1 affects support functions). For $m_1 \in M_1$, we have $\chi^{m_1} \circ \pi_1 = \chi^{\alpha(m_1)}$, and it follows immediately that $\pi_1^*(\mathcal{O}_{X_1}(D_\rho), \iota_\rho, c_{\chi^{m_1}}) \sim (\mathcal{O}_X(D_\rho), \iota_\rho, c'_{m_1})$. This proves commutivity, and the theorem follows. □

REMARK 1.1. When the n_ρ's span $N_{\mathbf{R}}$, we get an alternate description of the universal Δ-collection as follows. By [2], $\alpha_\rho = [D_\rho] \in \mathrm{Pic}(X)$ gives a sheaf $\mathcal{O}_X(\alpha_\rho)$ on X, which is a line bundle since X is smooth. Furthermore, [2] gives a canonical isomorphism $S_{\alpha_\rho} \simeq H^0(X, \mathcal{O}_X(\alpha_\rho))$. Since $\deg(x_\rho) = \alpha_\rho$ in $\mathrm{Pic}(X)$, we have $x_\rho \in S_{\alpha_\rho}$, so that we can write $x_\rho \in H^0(X, \mathcal{O}_X(\alpha_\rho))$. Finally, if $m \in M$, then $\sum_\rho \langle m, n_\rho \rangle \alpha_\rho = 0$ in $\mathrm{Pic}(X)$, which gives a canonical isomorphism

$$c_m: \bigotimes_\rho \mathcal{O}_X(\alpha_\rho)^{\otimes \langle m, n_\rho \rangle} \simeq \mathcal{O}_X\left(\sum_\rho \langle m, n_\rho \rangle \alpha_\rho\right) = \mathcal{O}_X .$$

Then $(\mathcal{O}_X(\alpha_\rho), x_\rho, c_m)$ is equivalent to the universal Δ-collection $(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m})$. This follows easily using the isomorphisms $\mathcal{O}_X(\alpha_\rho) \simeq \mathcal{O}_X(D_\rho)$ constructed in [2, §3].

REMARK 1.2. Using the representability criterion given in Proposition 4.5.4 from [4], one can prove directly that C_Δ is representable, without knowing the toric variety X. To see how this works, let $\sigma \in \Delta_{\max}$, and define the functor C_Δ^σ by

$$C_\Delta^\sigma(Y) = \{(L_\rho, u_\rho, c_m) \in C_\Delta(Y) : u_\rho \text{ is an isomorphism for all } \rho \not\subset \sigma\} .$$

Using the isomorphisms $u_\rho^{-1}: L_\rho \simeq \mathcal{O}_Y$ for $\rho \not\subset \sigma$, one gets an equivalence $(L_\rho, u_\rho, c_m) \sim (L'_\rho, u'_\rho, c'_m)$ where $L'_\rho = \mathcal{O}_Y$ and $u'_\rho = 1$ whenever $\rho \not\subset \sigma$. From here, the techniques of Examples 1.1 and 1.3 and Theorem 1.1 can be adapted to show that C_Δ^σ is represented by $\mathbf{A}_k^d \times_k \mathbf{G}_m^{n-d}$, where d is the dimension of σ. According to Proposition 4.5.4 of [4], C_Δ is then representable provided we can show the following:

(i) The natural transformation $C_\Delta^\sigma \to C_\Delta$ is representable by an open immersion.

(ii) The functor C_Δ is a sheaf when restricted to open subsets of Y.

(iii) C_Δ is the union (as defined in part (iii) of Proposition 4.5.4 of [4]) of the C_Δ^σ.

The proof of (ii) is completely straightforward, and (iii) follows easily from the nondegeneracy condition. For (i), we need to show that given a Δ-collection (L_ρ, u_ρ, c_m) on Z, the functor $Y \mapsto \{g \in \mathrm{Hom}_k(Y, Z) : g^*(L_\rho, u_\rho, c_m) \in C_\Delta^\sigma(Y)\}$ is representable by an open subset $Z_\sigma \subset Z$. This is easy: Z_σ is the biggest open subset of Z, where u_ρ is an isomorphism for all $\rho \not\subset \sigma$. We leave the details to the reader.

By proving that C_Δ is representable, we get an alternate construction of the smooth toric variety X. This might be useful for studying toric varieties over more general bases (for example, over the integers or over finite fields).

REMARK 1.3. Recently, other authors have used concepts similar to Δ-collections to describe maps from varieties to toric varieties over the complex numbers. For example, Oda and Sankaran (unpublished) can describe maps from a normal variety Y to a toric variety X such that the image of Y in X has nonempty intersection with the torus $T \subset X$. Every such map is determined by a group homomorphism $\varepsilon: M \to C(Y)^\times$ and Weil divisors E_ρ (one for each $\rho \in \Delta(1)$) such that

$$\operatorname{div}(\varepsilon(m)) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle E_\rho$$

for all $m \in M$ (compatibility) and that

$$E_{\rho_1} \cap \cdots \cap E_{\rho_s} = \varnothing ,$$

whenever there is no $\sigma \in \Delta$ which contains $\rho_1, \ldots, \rho_s$ (nondegeneracy). This data is uniquely determined by f, and conversely, given such data, we get a map $f: Y \to X$ such that $f(Y) \cap T$ is nonempty. This description has the advantage that it applies to all toric varieties X, not just smooth ones. On the other hand, it only works when Y is normal, and it doesn't describe all possible maps. However, this is sufficient for many applications, including those given in [6].

Another description of maps to toric varieties, due to Jaczewski in [7], uses the notion of a *vast divisor* on a complete variety Y. One starts with a divisor with normal crossings $B = \sum_\rho B_\rho$. Let $M(B) = \{\sum_\rho a_\rho B_\rho : [\sum_\rho a_\rho B_\rho] = 0 \text{ in } H^2(Y, \mathbf{Z})\}$, and let $N(B)$ be its dual. For each ρ, the map $\sum_\rho a_\rho B_\rho \to a_\rho$ determines $n_\rho \in N(B)$. For B to be vast, there needs to be a smooth complete fan Δ in $N(B)$ with the n_ρ as generators of the 1-dimensional cones. There is also a nondegeneracy condition (about the complements of certains unions of the B_ρ being an open cover of Y) and a compatibility condition (that among linear combinations of the B_ρ, homological equivalence implies linear equivalence). Then Theorem 4.5 of [7] shows that this data determines a map from Y to the smooth toric variety X determined by Δ. This theory is only stated for complete varieties and seems to require some knowledge about homological equivalence on Y.

2. The torus action. We next describe the torus action on X in terms of Δ-collections. Since $T = \operatorname{Hom}_{\mathbf{Z}}(M, \mathbf{G}_m)$ is the torus of X, we get an action of T on $C_\Delta(Y)$ as follows: a homomorphism $\phi: M \to \mathbf{G}_m$ in T acts on a Δ-collection (L_ρ, u_ρ, c_m) via

$$\phi \cdot (L_\rho, u_\rho, c_m) = (L_\rho, u_\rho, \phi(m) c_m) .$$

We still have a Δ-collection since $\phi(m)c_m \otimes \phi(m')c_{m'} = \phi(m+m')c_{m+m'}$, and this operation also preserves equivalence classes. Hence T acts on $C_\Delta(Y)$.

This relates to the natural action of T on $\operatorname{Hom}_k(Y, X)$ (coming from the action of T on X) as follows:

PROPOSITION 2.1. *The natural bijection* $\operatorname{Hom}_k(Y, X) \to C_\Delta(Y)$ *from Theorem* 1.1 *is a T-equivariant map.*

PROOF. By functorality, it suffices to verify equivariance for the case $Y=X$ and for $1_X \in \mathrm{Hom}_k(X, X)$. This means finding an equivalence

$$\phi^*(\mathcal{O}_X(D_\rho), \iota_\rho, c_{\chi^m}) \sim (\mathcal{O}_X(D_\rho), \iota_\rho, \phi(m)c_{\chi^m})$$

for all $\phi \in T$. However, $\phi^{-1}(D_\rho)=D_\rho$, so that $\phi^*\mathcal{O}_X(D_\rho)=\mathcal{O}_X(D_\rho)$. Furthermore, if we represent the global sections of $\mathcal{O}_X(D_\rho)$ as $\{f \in k(X): \mathrm{div}(f)+D_\rho \geq 0\}$, then $\phi^*: H^0(X, \mathcal{O}_X(D_\rho)) \to H^0(X, \mathcal{O}_X(D_\rho))$ is the map sending map f to $f \circ \phi$. Since $\iota_\rho \in H^0(X, \mathcal{O}_X(D_\rho))$ corresponds to the constant function 1, we see that $\phi^*(\iota_\rho)=\iota_\rho$.

Finally, to see what happens to c_{χ^m} under ϕ, recall that $\chi^m: T \to \boldsymbol{G}_m$ is a homomorphism and that $\phi \in T$ acts on T by translation. Since $\chi^m(\phi)=\phi(m)$, it follows that $\chi^m \circ \phi = \phi(m)\chi^m$ as functions on T. Thus the rational functions on X given by $\chi^m \circ \phi$ and χ^m differ by the constant $\phi(m)$. Hence they have the same divisor $\sum_\rho \langle m, n_\rho \rangle D_\rho$, though the trivializations $\mathcal{O}_X(\sum_\rho \langle m, n_\rho \rangle D_\rho) \simeq \mathcal{O}_X$ they induce differ by the constant $\phi(m)$. This shows that $\phi^*(c_{\chi^m})=\phi(m)c_{\chi^m}$, which completes the proof. □

REMARK 2.1. When the n_ρ's span $N_{\boldsymbol{R}}$, there is another way to view the action of T on $C_\Delta(Y)$. If we apply $\mathrm{Hom}_{\boldsymbol{Z}}(-, G_m)$ to the exact sequence (2), we get the exact sequence

$$1 \longrightarrow G \longrightarrow \boldsymbol{G}_m^{\Delta(1)} \longrightarrow T \longrightarrow 1, \tag{7}$$

where $G=\mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Pic}(X), \boldsymbol{G}_m)$. Then $(t_\rho) \in \boldsymbol{G}_m^{\Delta(1)}$ acts on Δ-collections via $(L_\rho, u_\rho, c_m) \mapsto (L_\rho, t_\rho u_\rho, c_m)$. Since equivalence classes are preserved, $\boldsymbol{G}_m^{\Delta(1)}$ acts on $C_\Delta(Y)$.

To relate this to the action of T, note that the isomorphisms $L_\rho \simeq L_\rho$ given by multiplication by t_ρ^{-1} induce an equivalence

$$(L_\rho, t_\rho u_\rho, c_m) \sim \left(L_\rho, u_\rho, \prod_\rho t_\rho^{\langle m, n_\rho \rangle} c_m\right).$$

But $m \mapsto \prod_\rho t_\rho^{\langle m, n_\rho \rangle}$ is the element of $T=\mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{G}_m)$ which is the image of (t_ρ) under the map $\boldsymbol{G}_m^{\Delta(1)} \to T$ in (7). Hence the action of $\boldsymbol{G}_m^{\Delta(1)}$ induces the T-action on $C_\Delta(Y)$.

3. Maps between toric varieties. As an application of Theorem 1.1, we will describe all maps from $\boldsymbol{P}_k^m$ to a smooth toric variety X where the n_ρ's span $N_{\boldsymbol{R}}$. In this case, recall that X is the geometric quotient $(\boldsymbol{A}_k^{\Delta(1)}-Z)/G$.

THEOREM 3.1. *Let X be a smooth toric variety such that the n_ρ's span $N_{\boldsymbol{R}}$, and suppose we have homogeneous polynomials $P_\rho \in k[t_0, \ldots, t_m]$, indexed by $\rho \in \Delta(1)$, such that*:

(a) *If P_ρ has degree d_ρ, then $\sum_\rho d_\rho n_\rho = 0$ in N.*

(b) *$(P_\rho(t_0, \ldots, t_m)) \notin Z$ in $\boldsymbol{A}_k^{\Delta(1)}$ whenever $(t_0, \ldots, t_m) \neq 0$ in $\boldsymbol{A}_k^{m+1}$.*

If we define $\tilde{f}(t_0, \ldots, t_m)=(P_\rho(t_0, \ldots, t_m)) \in \boldsymbol{A}_k^{\Delta(1)}$, then there is a morphism $f: \boldsymbol{P}_k^m \to X$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{A}_k^{m+1}-\{0\} & \xrightarrow{\tilde{f}} & \boldsymbol{A}_k^{\Delta(1)}-Z \\ \downarrow & & \downarrow \\ \boldsymbol{P}_k^m & \xrightarrow{f} & X \end{array}$$

commutes, where the vertical maps are the quotient maps. Furthermore:

(i) *Two sets of polynomials* $\{P_\rho\}$ *and* $\{P'_\rho\}$ *determine the same morphism* $f: \boldsymbol{P}_k^m \to X$ *if and only if there is* $g \in G = \operatorname{Hom}_{\boldsymbol{Z}}(\operatorname{Pic}(X), \boldsymbol{G}_m)$ *such that* $P'_\rho = g([D_\rho])P_\rho$ *for all* ρ.

(ii) *All morphisms* $f: \boldsymbol{P}_k^m \to X$ *arise in this way.*

PROOF. Given the P_ρ's satisfying (a) and (b), note that for every $m \in M$ we have $\sum_\rho d_\rho \langle m, n_\rho \rangle = 0$, which gives a canonical isomorphism of sheaves

$$c_m^{\text{can}}: \otimes_\rho \mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho)^{\langle m, n_\rho \rangle} \simeq \mathcal{O}_{\boldsymbol{P}_k^m}\left(\sum_\rho d_\rho \langle m, n_\rho \rangle\right) = \mathcal{O}_{\boldsymbol{P}_k^m}.$$

Then $(\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), P_\rho, c_m^{\text{can}})$ is clearly a Δ-collection, so that we get a map $f: \boldsymbol{P}_k^m \to X$. Using the arguments from Theorem 1.1, one can show that if $\pi: \boldsymbol{A}_k^{m+1} - \{0\} \to \boldsymbol{P}_k^m$ is the quotient map, then $\pi^*(\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), P_\rho, c_m^{\text{can}}) \sim (\mathcal{O}_{\boldsymbol{A}_k^{m+1}-\{0\}}, P_\rho, 1)$. From here, the commutivity of the diagram follows easily.

Now suppose that two sets of polynomials $\{P_\rho\}$ and $\{P'_\rho\}$ give the same map f. Then, by Theorem 1.1, we know that $(\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), P_\rho, c_m^{\text{can}}) \sim (\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), P'_\rho, c_m^{\text{can}})$. This means that there are constants $\lambda_\rho \in k^*$ such that $P'_\rho = \lambda_\rho P_\rho$ and $\prod_\rho \lambda_\rho^{\langle m, n_\rho \rangle} = 1$ for all $m \in M$ because c_m^{can} is preserved. As in the proof of Theorem 1.1, this implies that there is $g \in G$ such that $g([D_\rho]) = \lambda_\rho$ for all ρ, and (i) is proved.

Finally, to prove (ii), let $f: \boldsymbol{P}_k^m \to X$ be a morphism. By Theorem 1.1, we know that f is determined by some Δ-collection (L_ρ, u_ρ, c_m). Since each $L_\rho \simeq \mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho)$ for some d_ρ, we get an equivalence $(L_\rho, u_\rho, c_m) \sim (\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), F_\rho, c'_m)$. Then $(c'_m)^{-1} \circ c_m^{\text{can}}: \mathcal{O}_{\boldsymbol{P}_k^m} \simeq \mathcal{O}_{\boldsymbol{P}_k^m}$, and thus, as in the second paragraph of the proof of Theorem 1.1, we can find $\lambda_\rho \in k^*$ such that $c_m^{\text{can}} = \prod_\rho \lambda_\rho^{\langle m, n_\rho \rangle} c'_m$ for all m (this uses our assumption that the n_ρ's span $N_{\boldsymbol{R}}$). If we set $P_\rho = \lambda_\rho F_\rho$, then $(\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), F_\rho, c'_m) \sim (\mathcal{O}_{\boldsymbol{P}_k^m}(d_\rho), P_\rho, c_m^{\text{can}})$, which shows that f is determined by the P_ρ's. It is easy to see that conditions (a) and (b) are satisfied, and the theorem is proved. □

REMARK 3.1. When $X = \boldsymbol{P}_k^n$, the n_ρ's are $e_0, \ldots, e_n$ as in Example 1.2. One can check that $\sum_{i=0}^n d_i e_i = 0$ if and only if $d_0 = \cdots = d_n$, and then Theorem 3.1 gives the usual description of maps between $\boldsymbol{P}_k^m$ and $\boldsymbol{P}_k^n$.

REMARK 3.2. Theorem 3.1 applies to all smooth *complete* toric varieties since the n_ρ's obviously span $N_{\boldsymbol{R}}$ in this case.

REMARK 3.3. When the n_ρ's do not span $N_{\boldsymbol{R}}$, then, as in the proof of Theorem 1.1, we can write $X \simeq X_1 \times_k T_1$, where $T_1 \simeq \boldsymbol{G}_m^{n-d}$. In this case, a map $\boldsymbol{P}_k^m \to X$ is determined by maps $\boldsymbol{P}_k^m \to X_1$ and $\boldsymbol{P}_k^m \to \boldsymbol{G}_m^{n-d}$. The first of these maps can be described by Theorem 3.1, and the second is obviously constant. Thus we can describe maps from $\boldsymbol{P}_k^m$ to an arbitrary smooth toric variety X.

REMARK 3.4. When X is a complete simplicial toric variety over $\boldsymbol{C}$, Morrison and Plesser [8, §3.7] indicate that Theorem 3.1 is still true, and they also describe a toric compactification of the space of all maps $\boldsymbol{P}^1 \to X$ of fixed degree.

REMARK 3.5. Over $\boldsymbol{C}$, Theorem 3.1 has been used by Guest to study the topology of the space of rational curves on X (see [6, §5]). When X is not smooth, Guest instead uses a certain configuration space to study maps $\boldsymbol{P}^1 \to X$ (see [6, Proposition 3.1]).

Finally, we will discuss a more general version of Theorem 3.1, where $\boldsymbol{P}_k^m$ is replaced by an arbitrary complete toric variety Y. If Y is determined by the fan Δ_Y, then by [2], Y has a homogeneous coordinate ring $S^Y = k[y_\tau]$, where $\tau \in \Delta_Y(1)$. The ring S^Y is graded by the Chow group $A_{n-1}(Y)$, and we denote the graded pieces by S_α^Y for $\alpha \in A_{n-1}(Y)$. Note also that $\mathrm{Pic}(Y) \subset A_{n-1}(Y)$. By [2], we can also express Y as a categorical quotient of $A_k^{\Delta_Y(1)} - Z_Y$. Then we get the following result:

THEOREM 3.2. *Let X be a smooth toric variety such that the n_ρ's span $N_{\boldsymbol{R}}$, and let Y be a complete toric variety with coordinate ring S^Y. Suppose we have homogeneous polynomials $P_\rho \in S^Y$, indexed by $\rho \in \Delta(1)$, such that:*

(a) *If $P_\rho \in S_{\beta_\rho}^Y$, then $\beta_\rho \in \mathrm{Pic}(Y)$ and $\sum_\rho \beta_\rho \otimes n_\rho = 0$ in $\mathrm{Pic}(Y) \otimes N$.*

(b) *$(P_\rho(t_\tau)) \notin Z$ in $A_k^{\Delta(1)}$ whenever $(t_\tau) \notin Z_Y$ in $A_k^{\Delta_Y(1)}$.*

If we define $\tilde f(t_\tau) = (P_\rho(t_\tau)) \in A_k^{\Delta(1)}$, then there is a morphism $f\colon Y \to X$ such that the diagram

$$\begin{array}{ccc} A_k^{\Delta_Y(1)} - Z_Y & \xrightarrow{\tilde f} & A_k^{\Delta(1)} - Z \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

commutes, where the vertical maps are the quotient maps. Furthermore:

(i) *Two sets of polynomials $\{P_\rho\}$ and $\{P'_\rho\}$ determine the same morphism $f\colon Y \to X$ if and only if there is $g \in G = \mathrm{Hom}_{\boldsymbol{Z}}(\mathrm{Pic}(X), \boldsymbol{G}_m)$ such that $P'_\rho = g([D_\rho])P_\rho$ for all ρ.*

(ii) *All morphisms $f\colon Y \to X$ arise in this way.*

PROOF. We will only sketch the proof, leaving the details to the reader. The key idea is that by [2], $\alpha \in \mathrm{Pic}(Y)$ gives a line bundle $\mathcal{O}_Y(\alpha)$ such that we have canonical isomorphisms $\mathcal{O}_Y(\alpha) \otimes \mathcal{O}_Y(\beta) \to \mathcal{O}_Y(\alpha + \beta)$ for $\alpha, \beta \in \mathrm{Pic}(Y)$. Furthermore, from [2] there is a natural isomorphism $H^0(Y, \mathcal{O}_Y(\alpha)) \simeq S_\alpha^Y$. Then it is easy to see that the P_ρ's give a Δ-collection $(\mathcal{O}_Y(\alpha_\rho), P_\rho, c_m^{\mathrm{can}})$, and from here the rest of the proof is identical to what we did in Theorem 3.1. □

4. Concluding remarks. Another description of the functor represented by a toric variety X is due to Ash, Mumford, Rapoport and Tai (see [1, Chapter I, §2]). They consider pairs $(\mathscr{S}, \pi)$ such that:

(i) $\mathscr{S}$ is a sheaf of sub-semigroups of the constant sheaf M_Y on Y determined by M.

(ii) $\pi: \mathscr{S} \to \mathscr{O}_Y$ is a semigroup homomorphism ($\mathscr{O}_Y$ is a semigroup under multiplication).

Furthermore, they assume that $(\mathscr{S}, \pi)$ has the following properties:

(iii) For $s \in \mathscr{S}$, $\pi(s)$ is invertible if and only if s is.

(iv) For each $y \in Y$, there is some $\sigma \in \Delta$ such that $\mathscr{S}_y = \sigma^\vee \cap M$.

The main result of [1, Chapter I, §2] is that for all Y, there is a natural bijection

$$\mathrm{Hom}_k(Y, X) \simeq \{\text{all pairs } (\mathscr{S}, \pi) \text{ on } Y \text{ satisfying (i)–(iv) above}\}.$$

This description of the functor represented by X is clearly related to the usual way of constructing X by patching together the affine schemes $X_\sigma = \mathrm{Spec}(k[\sigma^\vee \cap M])$.

In contrast, our description of $\mathrm{Hom}_k(Y, X)$ is more closely tied to the geometric quotient $X \simeq (A_k^{\Delta(1)} - Z)/G$. An advantage of our approach is how it generalizes the usual description of maps between projective spaces (see Theorem 3.1). The Ash-Mumford-Rapoport-Tai approach, on the other hand, has the virtue that it applies to all toric varieties, not just smooth ones. (The problem with our description in the nonsmooth case is that the sheaf $\mathscr{O}_X(D_\rho)$ need not be a line bundle, though it is reflexive.) It would be interesting to see the analog of Theorem 1.1 for the case of simplicial toric varieties.

I am grateful to Martin Guest and Stein Arild Strømme for bringing this problem to my attention. The research for this paper was supported by NSF grant DMS-9301161.

REFERENCES

[1] A. Ash, D. Mumford, M. Rapoport and Y.-S. Tai, Smooth Compactifications of Locally Symmetric Varieties, Math Sci Press, Brookline, MA, 1975.

[2] D. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), 17–50.

[3] V. Danilov, The geometry of toric varieties, Russian Math. Surveys 33 (1978), 97–154.

[4] J. Dieudonné and A. Grothendieck, Eléments de Géométrie Algébrique I, Springer-Verlag, Berlin Heidelberg, New York, 1971.

[5] W. Fulton, Introduction to Toric Varieties, Princeton University Press, Princeton, 1993.

[6] M. Guest, The topology of the space of rational curves on a toric variety, Acta Math. 174 (1995), 119–145.

[7] K. Jaczewski, Generalized Euler sequence and toric varieties, in Classification of Algebraic Varieties, edited by C. Cilberto, E. Livorni and A. Sommese, AMS, Providence, 1994, 227–247.

[8] D. Morrison and R. Plesser, Summing the instantons: quantum cohomology and miroor symmetry in toric varieties, preprint, hep-th 9412236.

[9] T. Oda, Convex Bodies and Algebraic geometry, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

Department of Mathematics and Computer Science
Amherst College
Amherst, MA 01002
U.S.A.

E-mail address: dac@cs.amherst.edu