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The Fundamental Logic Structure in Quantum Mechanics

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Summary. In this article we present the logical structure given by four axioms of Mackey [3] in the set of propositions of Quantum Mechanics. The equivalence relation (PropRel(Q)) in the set of propositions (Prop Q) for given Quantum Mechanics Q is considered. The main text for this article is [6] where the structure of quotient space and the properties of equivalence relations, classes and partitions are studied.

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The articles [10], [1], [4], [2], [9], [8], [7], [5], and [6] provide the notation and terminology for this paper. In the sequel x will be arbitrary, X will be a non-empty set, and X_1 will be a set. Let us consider X, and let S be a σ-field of subsets of X. The functor probabilities S yields a non-empty set and is defined by:

$x \in$ probabilities S if and only if x is a probability on S.

We now state a proposition

(1) For every σ-field S of subsets of X holds $x \in$ probabilities S if and only if x is a probability on S.

We consider quantum mechanics structures which are systems

$\langle$ observables, states, a probability $\rangle$

where the observables, the states are non-empty sets and the probability is a function from $[\!:$ the observables, the states $:\!]$ into probabilities the Borel sets. In the sequel Q denotes a quantum mechanics structure. We now define two new functors. Let us consider Q. The functor Obs Q yields a non-empty set and is defined by:

Obs Q = the observables of Q.

The functor Sts Q yields a non-empty set and is defined by:

Sts Q = the states of Q.

The following propositions are true:

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(2) $\text{Obs}\, Q =$ the observables of Q.

(3) $\text{Sts}\, Q =$ the states of Q.

We adopt the following convention: A_1, A_2 will denote elements of $\text{Obs}\, Q$, s, s_1, s_2 will denote elements of $\text{Sts}\, Q$, and E will denote an event of the Borel sets. Let us consider Q, A_1, s. The functor $\text{Meas}(A_1, s)$ yielding a probability on the Borel sets, is defined as follows:

$\text{Meas}(A_1, s) =$ (the probability of Q)($\langle A_1, s \rangle$).

One can prove the following proposition

(4) $\text{Meas}(A_1, s) =$ (the probability of Q)($\langle A_1, s \rangle$).

A quantum mechanics structure is said to be a quantum mechanics if:

(i) for all elements A_1, A_2 of Obs it such that for every element s of Sts it holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$ holds $A_1 = A_2$,

(ii) for all elements s_1, s_2 of Sts it such that for every element A of Obs it holds $\text{Meas}(A, s_1) = \text{Meas}(A, s_2)$ holds $s_1 = s_2$,

(iii) for every elements s_1, s_2 of Sts it there exists an element s of Sts it such that for every element A of Obs it and for every E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A, s)(E) = t \cdot \text{Meas}(A, s_1)(E) + (1 - t) \cdot \text{Meas}(A, s_2)(E)$.

Next we state a proposition

(5) Q is a quantum mechanics if and only if the following conditions are satisfied:

  (i) for all A_1, A_2 such that for every s holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$ holds $A_1 = A_2$,

  (ii) for all s_1, s_2 such that for every A_1 holds $\text{Meas}(A_1, s_1) = \text{Meas}(A_1, s_2)$ holds $s_1 = s_2$,

  (iii) for every s_1, s_2 there exists s such that for every A_1, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A_1, s)(E) = t \cdot \text{Meas}(A_1, s_1)(E) + (1 - t) \cdot \text{Meas}(A_1, s_2)(E)$.

We follow the rules: Q denotes a quantum mechanics, A, A_1, A_2 denote elements of $\text{Obs}\, Q$, and s, s_1, s_2 denote elements of $\text{Sts}\, Q$. We now state three propositions:

(6) If for every s holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$, then $A_1 = A_2$.

(7) If for every A holds $\text{Meas}(A, s_1) = \text{Meas}(A, s_2)$, then $s_1 = s_2$.

(8) For every s_1, s_2 there exists s such that for every A, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A, s)(E) = t \cdot \text{Meas}(A, s_1)(E) + (1 - t) \cdot \text{Meas}(A, s_2)(E)$.

We consider POI structures which are systems

⟨ a carrier, an ordering, an involution ⟩

where the carrier is a set, the ordering is a relation on the carrier, and the involution is a function from the carrier into the carrier. In the sequel x_1 will denote an element of X_1, Ord will denote a relation on X_1, and Inv will denote a function from X_1 into X_1. Let us consider X_1. A POI structure is said to be a poset with involution over X_1 if:

the carrier of it $= X_1$.

One can prove the following proposition

(9) For every poset W with involution over X_1 holds the carrier of $W = X_1$.

Let us consider X_1, Ord, Inv. The functor LOG(Ord, Inv) yielding a poset with involution over X_1, is defined by:

LOG$(Ord, Inv) = \langle X_1, Ord, Inv \rangle$.

Next we state a proposition

(10) LOG$(Ord, Inv) = \langle X_1, Ord, Inv \rangle$.

Let us consider X_1, Inv. We say that Inv is an involution in X_1 if and only if:

$Inv(Inv(x_1)) = x_1$.

We now state a proposition

(11) Inv is an involution in X_1 if and only if for every x_1 holds $Inv(Inv(x_1)) = x_1$.

Let us consider X_1, and let W be a poset with involution over X_1. We say that W is a quantum logic on X_1 if and only if:

there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W =$ LOG(Ord, Inv) and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Next we state a proposition

(12) Let W be a poset with involution over X_1. Then W is a quantum logic on X_1 if and only if there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W =$ LOG(Ord, Inv) and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Let us consider Q. The functor Prop Q yielding a non-empty set, is defined by:

Prop $Q = [\!:$ Obs Q, the Borel sets $:\!]$.

The following proposition is true

(13) Prop $Q = [\!:$ Obs Q, the Borel sets $:\!]$.

In the sequel p, q, r, p_1, q_1 are elements of Prop Q. Let us consider Q, p. Then $\mathbf{p_1}$ is an element of Obs Q. Then $\mathbf{p_2}$ is an event of the Borel sets.

The following propositions are true:

(14) $p = \langle \mathbf{p_1}, \mathbf{p_2} \rangle$.

(15) $(E^{c})^{c} = E$.

(16) For every E such that $E = \mathbf{p_2}^{c}$ holds Meas$(\mathbf{p_1}, s)(\mathbf{p_2}) = 1 -$ Meas$(\mathbf{p_1}, s)(E)$.

Let us consider Q, p. The functor $\neg p$ yields an element of Prop Q and is defined as follows:

$\neg p = \langle \mathbf{p_1}, \mathbf{p_2}^{c} \rangle$.

The following proposition is true

(17) $\neg p = \langle p_{\mathbf{1}}, {p_{\mathbf{2}}}^{\mathrm{c}} \rangle$.

Let us consider Q, p, q. The predicate $p \vdash q$ is defined by:
for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) \leq \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

We now state a proposition

(18) $p \vdash q$ if and only if for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) \leq \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

Let us consider Q, p, q. The predicate $p \equiv q$ is defined as follows:
$p \vdash q$ and $q \vdash p$.

One can prove the following propositions:

(19) $p \equiv q$ if and only if $p \vdash q$ and $q \vdash p$.

(20) $p \equiv q$ if and only if for every s holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

(21) $p \vdash p$.

(22) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

(23) $p \equiv p$.

(24) If $p \equiv q$, then $q \equiv p$.

(25) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(26) $(\neg p)_{\mathbf{1}} = p_{\mathbf{1}}$ and $(\neg p)_{\mathbf{2}} = {p_{\mathbf{2}}}^{\mathrm{c}}$.

(27) $\neg(\neg p) = p$.

(28) If $p \vdash q$, then $\neg q \vdash \neg p$.

Let us consider Q. The functor $\mathrm{PropRel}\, Q$ yields an equivalence relation of $\mathrm{Prop}\, Q$ and is defined as follows:
$\langle p, q \rangle \in \mathrm{PropRel}\, Q$ if and only if $p \equiv q$.

We now state a proposition

(29) $\langle p, q \rangle \in \mathrm{PropRel}\, Q$ if and only if $p \equiv q$.

In the sequel B, C will denote subsets of $\mathrm{Prop}\, Q$. Next we state a proposition

(30) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$
for all elements a, b, c, d of $\mathrm{Prop}\, Q$ such that $a \in B$ and $b \in B$ and $c \in C$ and $d \in C$ and $a \vdash c$ holds $b \vdash d$.

Let us consider Q. The functor $\mathrm{OrdRel}\, Q$ yielding a relation on $\mathrm{Classes}(\mathrm{PropRel}\, Q)$,
is defined as follows:
$\langle B, C \rangle \in \mathrm{OrdRel}\, Q$ if and only if $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$
and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

Next we state four propositions:

(31) $\langle B, C \rangle \in \mathrm{OrdRel}\, Q$ if and only if $B \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\, Q)$ and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

(32) $p \vdash q$ if and only if $\langle [p]_{\mathrm{PropRel}\, Q}, [q]_{\mathrm{PropRel}\, Q} \rangle \in \mathrm{OrdRel}\, Q$.

(33) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\,Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\,Q)$ for all p_1, q_1 such that $p_1 \in B$ and $q_1 \in B$ and $\neg p_1 \in C$ holds $\neg q_1 \in C$.

(34) For all B, C such that $B \in \mathrm{Classes}(\mathrm{PropRel}\,Q)$ and $C \in \mathrm{Classes}(\mathrm{PropRel}\,Q)$ for all p, q such that $\neg p \in C$ and $\neg q \in C$ and $p \in B$ holds $q \in B$.

Let us consider Q. The functor $\mathrm{InvRel}\,Q$ yielding a function from $\mathrm{Classes}(\mathrm{PropRel}\,Q)$ into $\mathrm{Classes}(\mathrm{PropRel}\,Q)$, is defined by:

$(\mathrm{InvRel}\,Q)([p]_{\mathrm{PropRel}\,Q}) = [\neg p]_{\mathrm{PropRel}\,Q}$.

One can prove the following two propositions:

(35) $(\mathrm{InvRel}\,Q)([p]_{\mathrm{PropRel}\,Q}) = [\neg p]_{\mathrm{PropRel}\,Q}$.

(36) For every Q holds $\mathrm{LOG}(\mathrm{OrdRel}\,Q, \mathrm{InvRel}\,Q)$ is a quantum logic on $\mathrm{Classes}(\mathrm{PropRel}\,Q)$.

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