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THE FUNDAMENTAL SOLUTION OF THE SPACE-TIME FRACTIONAL ADVECTION-DISPERSION EQUATION

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Abstract. A space-time fractional advection-dispersion equation (ADE) is a generalization of the classical ADE in which the first-order time derivative is replaced with Caputo derivative of order $\alpha \in (0, 1]$, and the second-order space derivative is replaced with a Riesz-Feller derivative of order $\beta \in (0, 2]$. We derive the solution of its Cauchy problem in terms of the Green functions and the representations of the Green function by applying its Fourier-Laplace transforms. The Green function also can be interpreted as a spatial probability density function (*pdf*) evolving in time. We do the same on another kind of space-time fractional advection-dispersion equation whose space and time derivatives both replacing with Caputo derivatives.

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1. Introduction

An equation commonly used to describe solute transport in aquifers is the advection-dispersion equation(ADE) (Liu et al., 2002, 2003a, 2004):

$$\frac{\partial u}{\partial t} = -\nu \frac{\partial u}{\partial x} + \mathcal{D} \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u is solute concentration, the positive constants ν, $\mathcal{D}$ are represent the average fluid velocity and the dispersion coefficient, x is the spatial domain, t is time. The ADE is a deterministic equation describing a probability function for the location of particles in a continuum. The fundamental solutions of the

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ADE over time will be Gaussian densities with means and variances based on the values of the macroscopic transport coefficients ν and $\mathcal{D}$.

The classical ADE with a local (or asymptotical constant) dispersion tensor is a very handy predictive equation, since solutions are easily gained. The fractional-order forms of the ADE are similarly useful. Some partial differential equations of space-time fractional order were successfully used for modelling relevant physical processes (Mainardi, 1997; Benson et al., 1998,2000; El-Sayed and Aly, 2002; Basu and Acharya, 2002). Numerous authors have shown the equivalence between the transport equations that used fractional-order derivatives and some heavy-tailed motions which extended the predictive capability of models built on the stochastic process of Brownian motion, which is the basis for the classical ADE. The motions can be heavy-tailed, implying extremely long-term correlation and fractional derivatives in time and/or space. For example, Benson and his collaborator have derived the application of a fractional ADE(see Benson et al., 2000; Benson, 1998; Meerschaert et al., 1999). There are some other authors who considerd the fractional ADE. A space-fractional ADE with Eulerian derivation was derived by Schumer et al.(2001), which is used to describe solute plume evolution with a large probability of particles moving significantly ahead of and behind the mean solute velocity. For example, it can be used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium (see Meerschaert and Tadjeran, where a Riemann fractional derivation instead of the Eulerian derivation and a practical numerical method to solve it was developed). Integrodifferential equation which interpolates the heat equation and the wave equation is considered by Fujita (1990). He gave the representation of the solution by the fundamental solotion and showed some properties. The solution of the bi-fractional differential equation was developed by Kilbas et al. (2004). The fundamental solution of these problems was established and its moments are calculated. Anh et al. (2001,2002,2003) considered spectral analysis of fractional kinetic equations with random data, renormalization and homogenization of fractional diffusion equations with random data, and harmonic analysis of fractional diffusion-wave equations, respectively. The time-fractional advection-dispersion equation was considered by Liu, Anh et al.(2003b) and its complete solution was obtained by using variable transformation, Mellin and Laplace transforms, and properties of H-functions. We further extend this work and derived its solutions in half-space and a bounded space domain (Huang and Liu, 2004).

This paper is a continuation of these papers. In this paper we intend to consider two kinds of space-time fractional advection-dispersion equations. For both of them, the time fractional derivative is defined in the Caputo sense, while the space fractional derivatives are Riesz-Feller derivatives and Caputo (or Riemann-Liouville) derivatives, respectively.

2. The time-space fractional ADE with Riesz-Feller space derivative

We first consider the equation

$$ {}_tD_*^\alpha u(x,t) = -\nu D_x u(x,t) + \mathcal{D} D_\theta^\beta u(x,t), \quad x \in I\!R, \quad t \in I\!R^+, \tag{2}$$

with the following initial and boundary conditions

$$ u(x,0) = g(x), \quad x \in I\!R, \qquad u(\mp\infty, t) = 0, \quad t > 0, \tag{3}$$

where the real parameters β, θ are always restricted as follows

$$ 0 < \beta \leq 2, \tag{4}$$

$$ |\theta| \leq \min\{\beta, 2-\beta\}. \tag{5}$$

We denote ${}_tD_*^\alpha$ is the Caputo time-fractional derivative of order α (see Podlubny, 1999; or Appendix B), D_θ^β is the Riesz-Feller space-fractional derivative of order β (see Appendix A), and $D_x = \frac{\partial}{\partial x}$, $u(x,t)$ and $g(x)$ are both the (real) field variable, and sufficiently well-behaved functions.

We can consult the literature of Mainardi et al. (2001) for the properties and more details about the Riesz-Feller fractional derivative.

By the Fourier-Laplace transform, we represent the solution of the Cauchy problem by the integral formula in terms of the Green functions. The representations of the Green functions also are obtained by a composition rule.

2.1 Some properties of the Green function

Applying Laplace and Fourier transforms to Eq. (2) with initial condition (3) with respect to variable t and x, respectively, by taking into account the Laplace transform for the Caputo time fractional derivative (52), and the Fourier transform for the Riesz-Feller space fractional derivative (42), we obtain the following nonhomogeneous differential equation

$$ s^\alpha \widetilde{u}(x,s) - s^{\alpha-1} g(x) = -\nu D_x \widetilde{u}(x,s) + \mathcal{D} D_\theta^\beta \widetilde{u}(x,s), \tag{6}$$

$$ s^\alpha \widehat{\widetilde{u}}(k,s) - s^{\alpha-1} \widehat{g}(k) = i\nu k \widehat{\widetilde{u}}(k,s) - \mathcal{D}\psi_\theta^\beta(k) \widehat{\widetilde{u}}(k,s). \tag{7}$$

Where $\psi_\theta^\beta(k) = |k|^\beta e^{i(sign k)\theta\pi/2}$, and $i^2 = -1$. So we derive

$$ \widehat{\widetilde{u}}(k,s) = \frac{s^{\alpha-1}}{s^\alpha - (i\nu k - \mathcal{D}\psi_\theta^\beta(k))} \widehat{g}(k). \tag{8}$$

To invert the Laplace transform in (8), we recall the Laplace transform pair,

$$ E_\alpha(ct^\alpha) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s^{\alpha-1}}{s^\alpha - c}, \quad \Re(s) > |c|^{1/\alpha}, \tag{9}$$

with $c \in \mathbb{C}$, $0 < \alpha \leq 1$, where E_α denotes the entire transcendental function, known as the Mittag-Leffler function of order α, defined in the complex plane by the power series

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{10}$$

For detailed informations on the Mittag-Leffler-type functions and their Laplace transforms, the reader may consult *e.g.* Erdélyi, Magnus and Oberhettinger, pp.1953-1954; Djrbashian, 1966; Podlubny, 1999; Gorenflo and Mainardi, 1997.

Going back to the time domain, we have

$$\widehat{u}(k,t) = E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha]\widehat{g}(k). \tag{11}$$

Furthermore, we invert the Fourier transform in above equation to obtain (see Podlubny, 1999)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha]\widehat{g}(k)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha]\int_{-\infty}^{+\infty} e^{iky} g(y)dydk \\ &= \int_{-\infty}^{+\infty}\Big(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik(x-y)} E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha]dk\Big)g(y)dy \\ &= \int_{-\infty}^{+\infty} G_{\alpha,\beta}^\theta(x-y,t)g(y)dy, \end{aligned} \tag{12}$$

where

$$G_{\alpha,\beta}^\theta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx} E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha]dk.$$

It is the Green function (or fundamental solution), which being the formal solution of (2) corresponding to $g(x) = \delta(x)$(the Dirac delta function). We therefore obtain

$$\widehat{G}_{\alpha,\beta}^\theta(k,t) = E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha], \tag{13}$$

$$\widetilde{\widehat{G}}_{\alpha,\beta}^\theta(k,s) = \frac{s^{\alpha-1}}{s^\alpha - (i\nu k - \mathcal{D}\psi_\theta^\beta(k))}. \tag{14}$$

In view of the conjugate property of the Mittag-Leffler function($\overline{E_\alpha(z)} = E_\alpha(\overline{z})$), we have

$$\begin{aligned} \overline{\widehat{G}_{\alpha,\beta}^\theta(-k,t)} &= \overline{E_\alpha[(-i\nu k - \mathcal{D}\psi_\theta^\beta(-k))t^\alpha]} \\ &= E_\alpha[\overline{(-i\nu k - \mathcal{D}\psi_\theta^\beta(-k))t^\alpha}] \\ &= E_\alpha[(i\nu k - \mathcal{D}\psi_\theta^\beta(k))t^\alpha] \\ &= \widehat{G}_{\alpha,\beta}^\theta(k,t). \end{aligned} \tag{15}$$

Furthermore, we easily recognize

$$\widehat{G}^{\theta}_{\alpha,\beta}(0,t) = E_{\alpha}(0) = 1, \quad t \geq 0. \tag{16}$$

Provided that $G^{\theta}_{\alpha,\beta}(x,t)$ does exist as inverse Fourier transform of (13), equations (15)-(16) ensure that $G^{\theta}_{\alpha,\beta}(x,t)$ is real and normalized, *i.e.*

$$G^{\theta}_{\alpha,\beta}(x,t) \in I\!R, \quad \int_{-\infty}^{+\infty} G^{\theta}_{\alpha,\beta}(x,t) = 1.$$

It remains to prove that $G^{\theta}_{\alpha,\beta}(x,t)$ is positive which ensures that the Green function is the spatial probability density for different values of the relevant parameters α, β. It can be obtained by deriving its explicit representation as indicated below, which is also ensures that $G^{\theta}_{\alpha,\beta}(x,t)$ does exist as inverse Fourier transform .

2.2 The explicit representation of the Green function

The space fractional advection-dispersion ($\alpha = 1$)

Let us first consider $\alpha = 1, 0 < \beta \leq 2$(space fractional advection-dispersion including standard advection-dispersion for $\beta = 2$ with $\theta = 0$). In this case, reducing the MIttag-Leffer function in (13) to the exponential function. To derive the Green function in the space and time domain, we recover the characteristic function of a class of Lévy strictly stable densities $p^{\theta}_{\beta}(x)$(Mainardi, Luchko, and Pagnini, 2001), then using the notation introduced in Appendix, we write

$$\begin{aligned} \widehat{G}^{\theta}_{1,\beta}(k,t) &= e^{(i\nu k - \mathcal{D}\psi^{\beta}_{\theta}(k))t} = e^{i\nu tk} e^{-\mathcal{D}t\psi^{\beta}_{\theta}(k)} \\ &= \widehat{P}^{1}_{1}(k; -\nu t)\widehat{P}^{\theta}_{\beta}(k; \mathcal{D}t), \end{aligned} \tag{17}$$

where

$$\widehat{P}^{\theta}_{\beta}(k;c) = e^{-c\psi^{\beta}_{\theta}(k)}, \quad c \in I\!R. \tag{18}$$

By using the known scaling rule for the Fourier transform,

$$f(ax) \overset{\mathcal{F}}{\longleftrightarrow} |a|^{-1}\widehat{f}(k/a), \quad a \in I\!R, \tag{19}$$

we have

$$P^{\theta}_{\beta}(x;c) = |c|^{-1/\beta} p^{\theta}_{\beta}(x/c^{1/\beta}), \tag{20}$$

which is non negative. We can consult Appendix A for the stable probability density $p^{\theta}_{\beta}(x)$. Inverting the Fourier transform of (17), we have

$$G^{\theta}_{1,\beta}(x,t) = \int_{-\infty}^{+\infty} P^{1}_{1}(x-y; -\nu t) P^{\theta}_{\beta}(y; \mathcal{D}t) dy, \tag{21}$$

which is also non negative.

For the standard ADE ($\beta = 2$ with $\theta = 0$), we know

$$p_1^1(x) = \delta(x+1), \qquad p_2^0(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4},$$

then the Green function of standard ADE is

$$G_{1,2}^0(x,t) = \frac{1}{2\nu\sqrt{\mathcal{D}\pi t^3}} e^{-(x+\nu t)^2/4\mathcal{D}t} \tag{22}$$

Composition rule for the Green function with $0 < \alpha < 1$

To express the general Green function of the space-time fractional ADE for $0 < \alpha < 1, 0 < \beta \leq 2$, we note that the Fourier-Laplace transform of the Green function (14) can be re-written in integral form as in Saichev and Zaslavsky(1997); Mainardi, Luchko and Pagnini(2001)

$$\begin{aligned} \widehat{\widetilde{G}}_{\alpha,\beta}^{\theta}(k,s) &= \frac{s^{\alpha-1}}{s^\alpha - (i\nu k - \mathcal{D}\psi_\theta^\beta(k))} \\ &= s^{\alpha-1} \int_0^{+\infty} exp\{-u[s^\alpha - (i\nu k - \mathcal{D}\psi_\theta^\beta(k))]\} du \\ &= \int_0^{+\infty} exp\{u[i\nu k - \mathcal{D}\psi_\theta^\beta(k)]\}(s^{\alpha-1}e^{-us^\alpha}) du \\ &= \int_0^{+\infty} \widehat{G}_{1,\beta}^\theta(k,u) \widetilde{G}_{2\alpha}(u,s) du, \end{aligned} \tag{23}$$

where

$$\widetilde{G}_\beta(x,s) = s^{\beta/2-1} e^{-|x| s^{\beta/2}}, \quad x \in I\!R, \quad \Re(s) > 0, \tag{24}$$

with solution

$$G_\beta(x,t) = t^{\beta/2} M_{\beta/2}(|x|/t^{\beta/2}), \quad x \in I\!R, \quad t \geq 0, \tag{25}$$

where $M_{\beta/2}$ denotes the so-called M function(of the Wright type) of order $\beta/2$, whose general properties can be found in Podlubny(1999); Gorenflo, Luchko and Mainardi(2000); Mainardi, Luchko and Pagnini(2001).

Going back to the space-time domain we obtain the relation

$$G_{\alpha,\beta}^\theta(x,t) = \int_0^{+\infty} G_{1,\beta}^\theta(x,u) G_{2\alpha}(u,t) du, \tag{26}$$

then we can derive that the Green function is non negative for the non negative prosperities of $G_{1,\beta}^\theta$ and $G_{2\alpha}$.

3. The space-time fractional ADE with Caputo space derivative

Now we consider the space-time fractional AED

$${}_tD_*^\alpha u(x,t) = -\nu D_x u(x,t) + \mathcal{D} D^\beta u(x,t), \quad x \in I\!R, \quad t \in I\!R^+, \tag{27}$$

with the same initial and boundary conditions (3). Where D^β is the Riemann-Liouville or Caputo space fractional derivative of order β with the lower terminal $a = -\infty$, *i.e.* $D^\beta = {}_{-\infty}D^\beta = {}_{-\infty}{}^cD^\beta$ (see (51)).

Properties and more details about the Riemann-Liouville and Caputo fractional derivative can be found in texts on Fractional Calculus (Podlubny, 1999; Gorenflo and Mainardi, 1997 or Appendix B). We us the same above technique to obtain the following results (Podlubny, 1999)

$$u(x,t) = \int_{-\infty}^{+\infty} G_{\alpha,\beta}(x-y,t)g(y)dy, \tag{28}$$

where $G_{\alpha,\beta}(x,t)$ is the Green function (or fundamental solution), which being the formal solution of (27) corresponding to $g(x) = \delta(x)$(the Dirac delta function). The Green function is characterized by

$$\widehat{G}_{\alpha,\beta}(k,t) = E_\alpha[(\nu((ik)) + \mathcal{D}(-ik)^\beta)t^\alpha], \tag{29}$$

$$\widehat{\widetilde{G}}_{\alpha,\beta}(k,s) = \frac{s^{\alpha-1}}{s^\alpha - (\nu((ik)) + \mathcal{D}(-ik)^\beta)}. \tag{30}$$

The Green function is also real and normalized, *i.e.*

$$G_{\alpha,\beta}(x,t) \in I\!R, \quad \int\limits_{-\infty}^{+\infty} G_{\alpha,\beta}(x,t) = 1.$$

To ensure that the Green function is the spatial probability densities for different values of the relevant parameters α, β, we must further show that the Green function is non negative. In fact, we can also obtain their explicit representation.

First, we consider the space fractional AED($\alpha = 1$), then

$$\begin{aligned} \widehat{G}_{1,\beta}(k,t) &= e^{[i\nu k + \mathcal{D}(-ik)^\beta]t} = e^{i\nu tk}e^{(-ik)^\beta \mathcal{D}t} \\ &= \widehat{P}_1(k;-\nu t)\widehat{P}_\beta(k;\mathcal{D}t), \end{aligned} \tag{31}$$

where

$$\widehat{P}_\beta(k;c) = e^{(-ik)^\beta c}, \quad c \in I\!R. \tag{32}$$

By the same technique as in Huang and Liu(2004b), we obtain

$$P_\beta(x,c) = \begin{cases} (|c|)^{-1/\beta} p_\beta^{-\beta}\left(\frac{x}{(-c)^{1/\beta}}\right), & 0 < \beta \le 1, \\ (|c|)^{-1/\beta} p_\beta^{2-\beta}\left(\frac{x}{c^{1/\beta}}\right), & 1 < \beta \le 2, \end{cases} \tag{33}$$

which is non negative. Where $p_\beta^{-\beta}(x)$ and $p_\beta^{2-\beta}(x)$ are the special stable probability density functions for $p_\beta^\theta(x)$(see Appendix A). So

$$G_{1,\beta}(x,t) = \int_{-\infty}^{+\infty} P_1(x-y;-\nu t)P_\beta(y;\mathcal{D}t)dy, \tag{34}$$

which is also non negative.

For the case $0 < \alpha < 1$, we can also obtain a composition rule for the Green function by using the same technique as above

$$\widehat{\widetilde{G}}_{\alpha,\beta}(k,s) = \int_0^{+\infty} \widehat{G}_{1,\beta}(k,u)\widetilde{G}_{2\alpha}(u,s)du, \tag{35}$$

where $\widetilde{G}_\beta$ is defined as (24)-(25). So

$$G_{\alpha,\beta}(x,t) = \int_0^{+\infty} G_{1,\beta}(x,u)G_{2\alpha}(u,t)du. \tag{36}$$

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Appendix: Preliminaries

Here we present an introduction to the Riesz-Feller and Caputo fractional derivatives starting from their representation in the Fourier and Laplace transform domain, respectively (Feller, 1952; Mainardi, Luchko and Pagnini, 2001; Gorenflo and Mainardi, 1997; Podlubny, 1999).

Appendix A: The Fourier transform and the Riesz-Feller fractional derivative

Let

$$\widehat{f}(k) = \mathcal{F}\{f(x);k\} = \int_{-\infty}^{+\infty} e^{+ikx} f(x)dx, \quad k \in I\!R, \tag{37}$$

be the Fourier transform of a well-behaved function $f(x)$, and let

$$f(x) = \mathcal{F}^{-1}\{\widehat{f}(k);x\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\widehat{f}(k)dk, \quad x \in I\!R, \tag{38}$$

be the inverse Fourier transform.

The Fourier transform of a convolution of two functions states

$$\mathcal{F}\{h(x) * f(x);k\} = \mathcal{F}\{\int_{-\infty}^{+\infty} h(x-y)f(y)dy;k\} = \widehat{h}(k)\widehat{f}(k). \tag{39}$$

Let

$$0 < \beta \le 2, \qquad |\theta| \le \begin{cases} \beta, & if \quad 0 < \beta \le 1, \\ 2-\beta, & if \quad 1 < \beta \le 2, \end{cases} \tag{40}$$

and denote by $p_\beta^\theta(x)$ for $x \in R$ the stable probability density whose characteristic function(Fourier transform) is

$$\widehat{p}_\beta^\theta(k) = e^{-\psi_\theta^\beta(k)}, \tag{41}$$

where $\psi_\theta^\beta(k) = |k|^\beta e^{i(sign k)\theta\pi/2}$, (see Feller(1952) for the general theory of stable probability distributions). The explicit representations and properties of the Riesz-Feller also can be found in Feller(1952); Mainardi, Luchko and Pagnini(2001). The Riesz-Feller fractional derivative of order β and skewness θ defines as

$$\mathcal{F}\{D_\theta^\beta f(x); k\} = -\psi_\theta^\beta(k)\widehat{f}(k), \tag{42}$$

Thus, we recognize that the Riesz-Feller derivative is required to be the pseudo-differential operator whose symbol $-\psi_\theta^\beta(k)$ is the logarithm of the characteristic function of a general Lévy strictly stable probability density with index of stability β and asymmetry parameter θ(improperly called skewness) according to Feller's(1952) parameterizations, as revisited by Gorenflo and Mainardi(1997). In fact, for $0 < \beta < 2$ and $|\theta| \leq \min\{\beta, \beta - 2\}$, the Riesz-Feller derivative reads

$$\begin{aligned} D_\theta^\beta f(x) &= \frac{\Gamma(1+\beta)}{\pi}\Big\{ \sin[(\beta+\theta)\pi/2] \int_{-\infty}^{+\infty} \frac{f(x+\xi) - f(x)}{\xi^{1+\beta}} d\xi \\ &+ \sin[(\beta-\theta)\pi/2] \int_{\infty}^{+\infty} \frac{f(x-\xi) - f(x)}{\xi^{1+\beta}} d\xi \Big\}. \end{aligned} \tag{43}$$

and for $\beta = 2$, $D_0^2 = \frac{d^2}{dx^2}$.

Appendix B: The Laplace transform and the Caputo fractional derivative

The Laplace transform of a function $f(t), 0 < t < \infty$, is defined as follows (Mainardi, 1997; Podlubny, 1999):

$$\widetilde{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt, \tag{44}$$

and its inverse is given by the formula

$$f(t) = \mathcal{L}^{-1}\{\widetilde{f}(s); t\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \widetilde{f}(s) ds, \quad \gamma = \Re(s) > s_0. \tag{45}$$

It follows that

$$\mathcal{L}\{t^\nu; s\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \qquad \nu > -1. \tag{46}$$

The Laplace transform of the convolution of two functions is the product of their Laplace transforms,

$$\mathcal{L}\{h(t) * f(t)\} = \mathcal{L}\{\int_0^t h(t-\tau) f(\tau) d\tau; s\} = \widetilde{h}(s)\widetilde{f}(s). \tag{47}$$

Now if f is continuous, the fractional integral of order α of f is

$$ {}_tD_*^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0 \tag{48} $$

which is a convolution integral. From (46)-(48), we have

$$ \mathcal{L}\{{}_tD_*^{-\alpha} f(t)\} = \frac{1}{\Gamma(\alpha)} \mathcal{L}\{t^{\alpha-1}\} \mathcal{L}\{f(t)\} = s^{-\alpha} \widetilde{f}(s), \quad \alpha > 0. \tag{49} $$

The fractional derivative of order α is defined in the Caputo sense

$$ {}_tD_*^{\alpha} f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \alpha = n \in N, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, & n-1 < \alpha < n, \end{cases} \tag{50} $$

It was proposed by Caputo first in his paper(1967). For the lower terminal $a = -\infty$, we have the relation

$$ {}_{-\infty}D_t^{\alpha} f(t) = {}_{-\infty}{}^{c}D_t^{\alpha} f(t). \tag{51} $$

There are two fundamental formulas

$$ \mathcal{L}\{\frac{d^\alpha}{dt^\alpha} f(t), p\} = s^\alpha \widetilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+), \quad n-1 < \alpha \le n, \ n \in N, \tag{52} $$

$$ \mathcal{F}\{{}_{-\infty}D_t^{\alpha} f(t), k\} = \mathcal{F}\{{}_{-\infty}{}^{c}D_t^{\alpha} f(t), k\} = (-ik)^\alpha \widehat{f}(k). \tag{53} $$

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