

QUARTERLY OF APPLIED MATHEMATICS

Vol. IX

July, 1951

No. 2

THE FUNDAMENTAL THEOREM OF ELECTRICAL NETWORKS*

BY

J. L. SYNGE

School of Theoretical Physics, Dublin Institute for Advanced Studies

1. Introduction. This paper has been written to meet a need which I believe is real. The theory of electrical networks involves questions of topology, and electrical engineers cannot be expected to be expert topologists. They need only a little, but it is difficult to get to know anything at all about topology without prolonged concentrated study, for it is a closely-knit subject. Hence the necessity to pull out of the body of topology and exhibit in what the mathematician might consider a clumsy popular form, those theorems which are of fundamental importance in network theory.

Of such theorems there appears to be one outstanding, and to it this paper is devoted. Without an understanding of this theorem, every electrician who mixes mesh methods with branch methods must feel insecure; with an understanding of it, he should be able to see the essential simplicity of much that is otherwise obscure and difficult.

The thought that the present attempt must be made, even by one who is neither topologist nor electrical engineer, came to me when reading for review P. Le Corbeiller's book, *Matrix Analysis of Electrical Networks* (Harvard University Press, 1950). There the author avoids the essential topological issues, but it seemed to me that if only those issues could be discussed and understood, the whole significance of Kron's method as expounded in the book would stand out more clearly.

The matter here presented is based on the work of W. H. Ingram and C. M. Cramlet¹, expanded in some respects to make the argument easier to follow but with omission of all that does not seem to bear directly on the fundamental question.

Before proceeding to the technical arguments, I have inserted in the next section some philosophical ideas which may be obvious, but which I think need to be stated with a view to better understanding between mathematicians on the one hand and physicists and engineers on the other.

2. The meaning of "proof". The modern meaning of the words "mathematical proof" is well known: they imply a faultless logical chain which starts from undefined elements and axioms, no loop-holes or exceptions permitted. Those subjects which, like topology, have been developed in that spirit of rigor, present an almost impenetrable front to applied mathematicians or engineers. The extraction of one needed result, and an understanding of what it means, demand a long and careful study in an uncongenial atmosphere, unrelieved by interpretations in terms of natural phenomena.

It is obvious that this state of affairs cannot persist. Each mathematical subject must be treated on several levels, varying from that of extreme rigor down to simple intuitive

*Received June 20, 1950.

¹Journal of Mathematics and Physics, **23**, 134-155 (1944).

descriptions with no proof at all. We get this variety of treatment in older branches of mathematics, partly because the creators had not got the modern standards of rigor, and partly because these matters have been looked at so long by so many people and from so many different angles.

Much of the argument that passes for proof in physics and engineering is not proof in the mathematical sense, and it is unlikely that the practical subjects will ever submit to the strict mathematical discipline. There seems to be an incompatibility between those minds which excel in logic and those which are capable of dealing successfully with problems suggested by nature.

The word "proof" has such a general usage that it is inconceivable that it should be employed only in its strictest mathematical meaning. Physicists and engineers will continue to use it in a looser sense, and will regard as proved any proposition with regard to

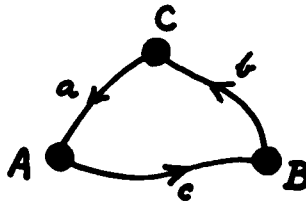


FIG. 1. Network with 3 nodes and 3 branches ($N = 3, B = 3$).

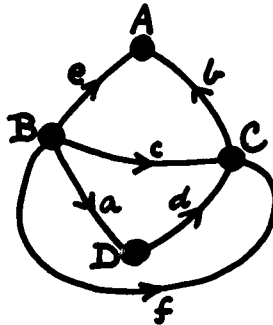


FIG. 2. Network with 4 nodes and 6 branches ($N = 4, B = 6$).

which they can assemble sufficient evidence to convince them of its truth. As Descartes pointed out long ago, to "prove" something by a series of logical steps, and to "see" or "understand" it are not the same thing; and what the physicist or engineer needs is the "seeing" and the "understanding".

The type of proof used in the present paper may appear clumsy, longwinded, and inaccurate to the pure mathematician. But that does not matter provided the proofs fulfil their purpose, which is to carry conviction to those who are willing to accept arguments with an element of intuition in them, when backed by appeal to a variety of simple and complicated special cases. All the proofs of elementary geometry carry conviction in this way; every property of the triangle is proved for a specific triangle drawn on a sheet of paper or imagined, and there is no guarantee (short of a plunge into the rigor of Hilbert or Veblen and Young) that with a different diagram the theorem may not prove false.

3. The definitions of network theory. The professional electrician is interested in physical networks, consisting of wires, generators, and so on—pieces of apparatus that

actually work. The topologist is interested in undefined elements and axioms concerning them. They both agree that they may play in common a game with pencil marks on a sheet of paper, these marks being to the electrician a representation of his apparatus, and to the topologist a representation of his undefined elements.

In speaking of the marks made on the paper, we may use the language of geometry (points, curves, etc.) or we may (more suitably for present purposes) use the language of the electrician. However, if we take the latter course, we must exercise great care not to read into each physical term its full physical meaning. Thus, to understand the line of thought at a certain point, we must be prepared to use the word "current" without accepting as obvious that the "currents" necessarily obey Kirchhoff's law because all physical currents do. It is to avoid confusions of this sort that we have to be somewhat careful in the matter of definitions.

Let us mark some points on our paper with heavy dots; for reference we may letter them A, B, C, \dots in any order. These are *nodes* (or terminals or junction-points or vertices), and in general we shall denote their number by N . Figs. 1-4 show cases where $N = 3, 4, 13$ and 34.

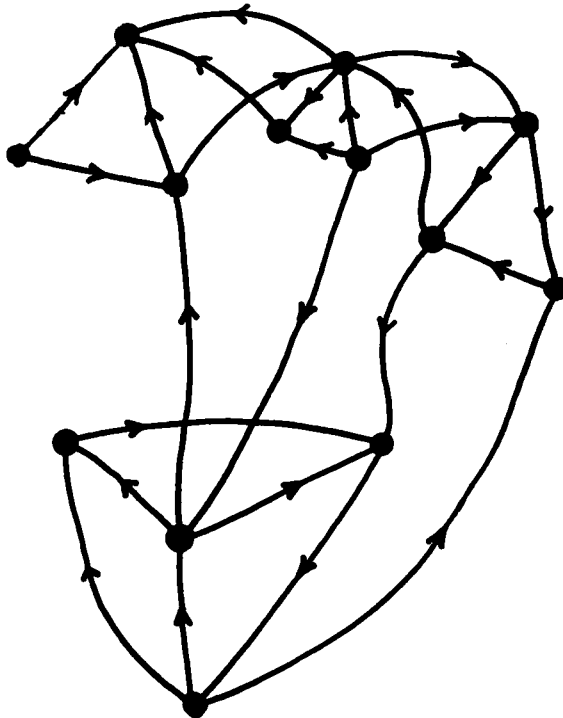


Fig. 3. Network with 13 nodes and 25 branches ($N = 13, B = 25$).

Next we join the nodes by lines or curves. Every such line or curve has a node at each end. On the lines or curves we put arrows indicating sense, one on each of them, the directions of these arrows being distributed without any plan. The line or curve, with its arrow, is called a *branch* (or directed branch if we want to emphasize that it has a sense). For reference we may attach letters a, b, c, \dots to the branches, again not according to

any plan, or we may number them in any order; we shall in general denote the number of branches by B . Figs. 1-4 show cases where $B = 3, 6, 25$ and 53 .

It is understood that, though two branches may intersect in the diagram, it is forbidden to pass from one branch to another by means of such an intersection. Such a passage can be made only through a node, and we must be careful to distinguish nodes from

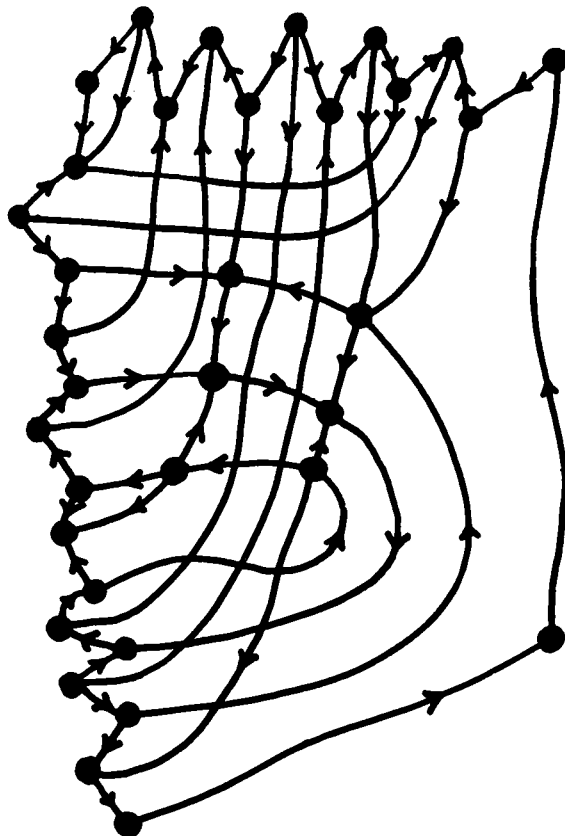


FIG. 4. Network with 34 nodes and 53 branches ($N = 34, B = 53$).

intersections by marking the former with heavy dots. This is a trivial nuisance due to the use of a representation on a sheet of paper; it would not arise if we used, as we might, a network of strings in space with knots at the nodes.

Any collection of nodes and branches is a *network*.

The nodes are the ends of branches. In Figs. 1-4 no case is shown where a node is the end of just one branch, although this is allowed by the definition. Such a case appears of little interest from an electrical standpoint, for no current can flow along such a branch. But the idea of a node at the end of a single branch will become important when we discuss trees in the next section.

There is another way in which we have exceeded our instructions in making the networks shown in Figs. 1-4. They are *connected* in the sense that one can travel along branches from any one node to any other. From a topological standpoint, any network

consisting of two (or more) disconnected parts is to be regarded as two (or more) networks. There may be electromagnetic connection between otherwise disconnected parts, but that will not concern us until Section 11 where the argument ceases to be purely topological. For the present we are to think only of connected networks.

We now define a *mesh* (or circuit) by the following prescription. Starting from a node, traverse branches continuously, observing the following rules:

- (i) having started to traverse a branch at one end, continue to the other end;
- (ii) when you arrive at a node, leave it by a branch other than that by which you arrived, if there is another branch;
- (iii) if you arrive at a node which terminates only one branch (that by which you arrived), stop the operation and start all over again.

If in the course of such an operation, you meet the same node (say A) twice, the set of branches you have traversed from A to A form a *mesh*.

To each mesh we assign a sense (one of two possible senses), without regard to the senses already assigned to the branches which compose the mesh. In Fig. 2 $ebda$ and $bead$ are the same mesh, taken in opposite senses.

4. Trees. A *tree* is a network of a particular type, namely a *connected network containing no meshes*. It is therefore a set of nodes and connecting branches, as shown in Figs. 5, 6, 7.



FIG. 5. Tree with one branch ($N = 2, T = 1; N = T + 1$).

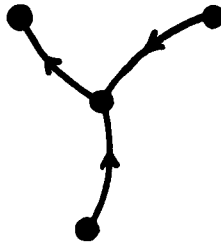


FIG. 6. Tree with 3 branches ($N = 4, T = 3; N = T + 1$).

We need certain facts about trees, and these we shall set down as theorems:

THEOREM I: *Every tree has at least one node which is an end of only one branch of the tree.*

This is certainly true of the trees shown, and is easy to prove in general. Simply put your pencil anywhere on a branch of the tree and start moving it along the branches observing the traffic rules laid down in connection with the definition of a mesh. Since there are no meshes in a tree, you cannot get back to the point from which you started. The number of nodes is finite, and so your journey must end. It can end only at a node which is the end of a single branch—the branch by which you have approached it. This proves the theorem.

Since, from any given point on a branch, you have a choice of two directions in which to set out, a tree must contain at least *two* nodes each of which is the end of a single branch. But the theorem as stated is all we need for our purposes.

THEOREM II: *In a given connected network it is possible to construct a tree containing all the nodes of the network.*

Figs. 8-11 show trees constructed in the networks of Figs. 1-4, the branches of the trees being shown by heavy wiggly lines in Figs. 8-10 and in a different way in Fig. 11.

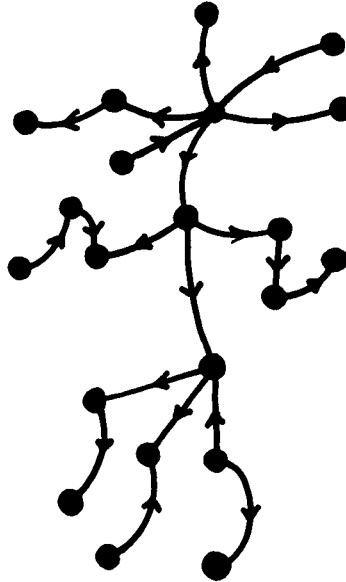


FIG. 7. Tree with 20 branches ($N = 21, T = 20; N = T + 1$).

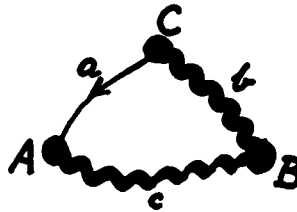


FIG. 8. Tree (heavy wiggly lines) in network of Fig. 1.
 2 branches-in-tree; 1 branch-out-of-tree; 1 basic mesh
 ($N = 3, B = 3, T = 2, M = B - T = 1; M + N = B + 1$).

In manuscript a red pencil is best. Further experimentation with networks of his own contriving should convince the reader of the truth of the theorem, but here is a proof.

Start with any branch of the network. It is either a branch of some mesh or it is not. (In Figs. 1-4, it must be a branch of a mesh.) If it is a branch of a mesh, remove the branch; its removal cannot make the network disconnected. If it is not a branch of a mesh, leave it. Testing each branch in this way, and removing it if it belongs to a mesh, we are left at each stage with a connected network. Finally all the meshes will be destroyed, and we shall have a connected network with no meshes, that is, a tree. Since we removed no nodes, the theorem is proved.

This proof makes the construction of a tree seem rather more complicated than it

really is. Set a red pencil on a node, and draw it along a branch to another node. Continue step by step, rejecting beforehand the reddening of any branch which would complete a mesh. A tree will result.

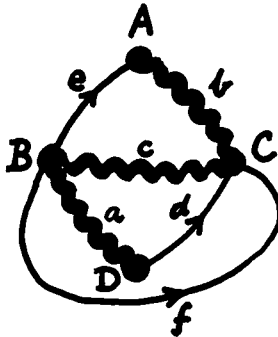


FIG. 9. Tree (heavy wiggly lines) in network of Fig. 2.
 3 branches-in-tree; 3 branches-out-of-tree; 3 basic meshes
 ($N = 4, B = 6, T = 3, M = B - T = 3; M + N = B + 1$).

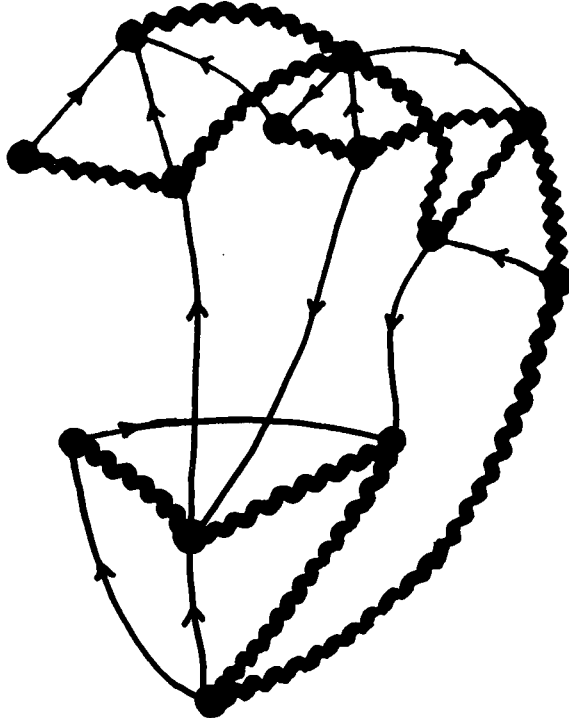


FIG. 10. Tree (heavy wiggly lines) in network of Fig. 3.
 12 branches-in-tree; 13 branches-out-of-tree; 13 basic meshes
 ($N = 13, B = 25, T = 12, M = B - T = 13; M + N = B + 1$).

If T is the number of branches of a tree and N the number of nodes, then

$$N = T + 1. \tag{4.1}$$

This is easy to see. For we can build up any tree by starting with a branch and the two

nodes at its ends, and then adding one branch and one node (at the far end of the added branch) at each step. When we start we have $N = 2, T = 1, N = T + 1$, and since in each step we add unity to T and unity to N , the relation (4.1) holds at any stage, and so for the completed tree.

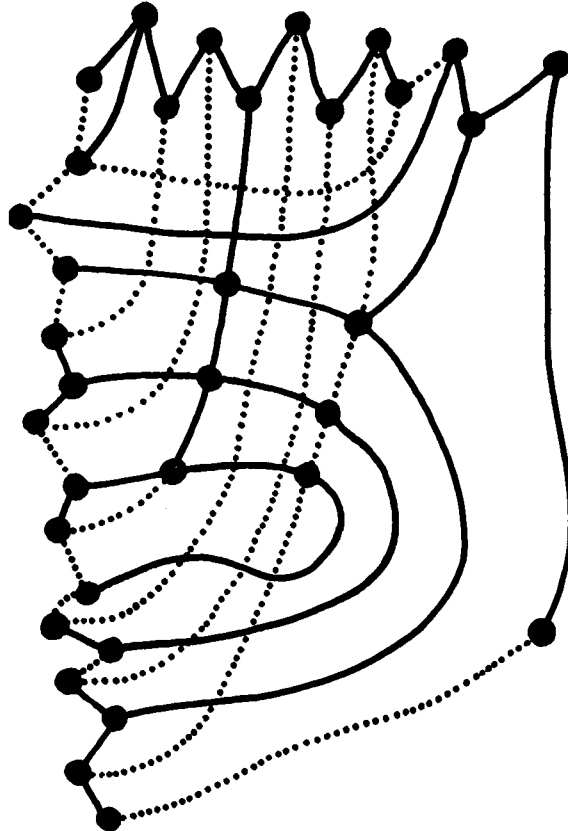


FIG. 11. Tree (tree in full lines, rest of network dotted) in network of Fig. 4.
 33 branches-in-tree; 20 branches-out-of-tree; 20 basic meshes
 ($N = 34, B = 53, T = 33, M = B - T = 20; M + N = B + 1$).

If we build up a general connected network like this, we might sometimes add branches **without** adding nodes. (This does not occur for a tree, for such branches would create meshes, and these a tree, by definition, must not have.) Thus for any connected network we have

$$N \leq B + 1. \tag{4.2}$$

5. Branch currents. So far we have dealt with nodes and branches, and meshes formed of branches.

With a branch we now associate a number i , in general complex. (Note that i is a complex number like $3 + 4j$, where $j = (-1)^{1/2}$.)

We recall that each branch has an arrow on it, pointing from one node (say A) to another node (say B). If i is the current in the branch, we say that a current i *flows into*

the node B , and a current $-i$ flows into the node A . (Equivalently we might say that a current $-i$ flows out of B and a current i flows out of A .)

So far the branch currents in a network are to be regarded as a set of B completely arbitrary complex numbers. The whole set may be represented by a matrix \mathbf{i} with B rows and one column.

6. Mesh currents. With a mesh of the network we associate a complex number i' , which we call a *mesh current*.

So far branch currents and mesh currents are separate things. We now set up a connection between them, so that we can say that a mesh current *generates* certain branch currents.

Consider any mesh. It contains certain branches, some agreeing in sense with the mesh and some disagreeing. Let us put in a mesh current i' . We say (as a matter of definition) that *this mesh current generates in each branch of the mesh a branch current i' if the senses agree and a branch current $-i'$ if the senses disagree; it generates no branch current in a branch not contained in the mesh.*

Thus in Fig. 2 a mesh current i' in the mesh adc generates branch currents i' , i' , $-i'$ in a , d , c respectively.

Suppose now that we assign mesh currents in a set of meshes. We say that this set of mesh currents generates a set of branch currents obtained by adding together the branch currents generated by the several mesh currents in accordance with the rule given above.

Thus if in Fig. 2 we put in mesh currents as follows:

$$i'_1 \text{ in } ebf,$$

$$i'_2 \text{ in } ebc,$$

$$i'_3 \text{ in } daeb,$$

these generate the following branch currents:

$$i_a = -i'_3,$$

$$i_b = -i'_1 - i'_2 - i'_3,$$

$$i_c = -i'_2,$$

$$i_d = -i'_3,$$

$$i_e = i'_1 + i'_2 + i'_3,$$

$$i_f = -i'_1.$$

Let us now, in any network, take R meshes and number them $1, 2, \dots, R$. We number the branches $1, 2, \dots, B$. In the meshes we put mesh currents i'_1, i'_2, \dots, i'_R . Then it is clear, from the way in which we have defined the generation of branch currents from mesh currents, that the whole system of branch currents generated by the above mesh currents may be expressed by the formula

$$i_p = \sum_{a=1}^R C_{pa} i'_a \quad (p = 1, 2, \dots, B) \quad (6.1)$$

where the coefficients are numbers defined as follows:

$$C_{pq} \begin{cases} = 1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 & \text{if branch } p \text{ is not contained in mesh } q. \end{cases} \quad (6.2)$$

We may write (6.1) in matrix notation:

$$\mathbf{i} = \mathbf{C}\mathbf{i}'. \quad (6.3)$$

Note that the meshes employed here are completely arbitrary.

7. Kirchhoff's node law.* Kirchhoff's node law states that the sum of all branch currents flowing into any node is zero.

If we assign an arbitrary set of branch currents, they will not in general satisfy this law. Indeed, until we examine the question, we cannot be sure that, for a given network, we can find *any* set of branch currents to satisfy the law (except, of course, the trivial set of zero currents).

Consider, however, the branch currents generated by a single mesh current. It is clear that Kirchhoff's node law is satisfied at each node connecting branches of the mesh, and of course at all other nodes ($0 = 0$). The law remains satisfied if we superimpose any number of mesh currents, and so we have this result:

THEOREM III: *The branch currents generated by any set of mesh currents satisfy Kirchhoff's node law at every node.*

8. Basic meshes. In any given connected network, draw a tree containing all the nodes. All branches of the network then fall into two classes:

- (i) Branches-in-tree.
- (ii) Branches-out-of-tree (also called *chords*).

The theorem which we wish to prove is this:

THEOREM IV: *Assigning arbitrary branch currents to the branches-out-of-tree, we can assign (and the assignment is unique) branch currents to the branches-in-tree so that Kirchhoff's node law is satisfied at every node.*

We saw in Theorem I that the tree has a node which is an end of only one branch-in-tree. Kirchhoff's node law may be satisfied at that node by assigning a suitable branch current to the branch-in-tree, and of course this current is uniquely determined by the known currents in the branches-out-of-tree.

Now regard the branch-in-tree with which we have been dealing as a branch-out-of-tree, with the branch current we have just found. This leaves us with a reduced tree, and all currents in branches-out-of-tree assigned. Treat this reduced tree in the same way. Step by step, we convert branches-in-tree into branches-out-of-tree, each with an assigned current, and Kirchhoff's law is satisfied at every node which we drop from the tree. Ultimately we reach a stage where the tree consists of a single branch.

At each end of this single branch-in-tree, Kirchhoff's law determines a current for it. For a moment it may appear that our method might break down here—the two ends of

*As there is a possible confusion (Ingram and Cramlet, *loc. cit.*) as to which should be called Kirchhoff's "first" law and which his "second" law, the unambiguous words "node" and "mesh" will be used here.

the branch might demand different currents. But that cannot occur. When we started, the total current flowing into the tree at all the nodes was zero, since each branch-out-of-tree made equal and opposite contributions at its two ends. This zero total flow is preserved each time we change a branch-in-tree into a branch-out-of-tree, and so when we have reduced the tree to a single branch, the flows into its two ends are equal and opposite in sign. Thus we can get rid of the last branch of the tree, turning it into a branch-out-of-tree with a uniquely assigned current. Then Kirchhoff's node law is satisfied at every node and the theorem is proved.

For any connected network we now define *basic meshes*, using a tree in the network, containing all the nodes. This is very simple. Each branch-out-of-tree has its ends connected by a unique set of branches-in-tree (necessarily unique, since otherwise the tree would contain a mesh), and together these form a mesh; to it we assign the sense of the branch-out-of-tree contained in it. Thus, if there are M branches-out-of-tree, there are M meshes defined in this way, each mesh containing only one branch-out-of-tree, and each branch-out-of-tree being contained in just one mesh. We call this a set of *basic meshes* of the given network. Once the tree has been chosen, the basic meshes are uniquely determined; but since in general there will be several trees which may be used, so there are several sets of basic meshes.

The total number of nodes in the network being N , the number of branches-in-tree is, by (4.1), $T = N - 1$. The total number of branches in the network being B , the number of branches-out-of-tree (the same as the number of basic meshes) is thus

$$M = B - T = B - N + 1, \quad (8.1)$$

or equivalently

$$M + N = B + 1. \quad (8.2)$$

This general topological relation for a connected network is a useful check in the case of complicated networks as in Figs. 10 and 11.

Although topology (and, more generally, modern geometry) is a severe logical discipline, it seems right to use the word "topology" as we use the word "geometry" to describe properties of systems of dots and curves drawn on paper, or a net of knotted strings, or even an electrical network. In speaking of an electrical network, we try to clear our thoughts by separating its "topological" properties from its "electrical" properties. As long as the properties of which we speak are properties shared by systems of dots and curves and knotted strings, then we may say that we are in the topological domain. It is when we start connecting emfs and currents that we pass into electrical theory proper.

In this sense, then, almost all that has been done up to this point, and indeed almost all until we reach Section 11, is topology. The mesh and the tree belong to topology, and so do the basic meshes, since they are derived directly from the tree. This is not a matter for controversy regarding rival spheres of influence. It is an attempt to assist understanding by analysing and classifying the elements which, taken all together and without analysis and classification, can be bewilderingly confused.

9. Kron's transformation matrix. Suppose we have a connected network before us. We pick out a tree, as we always can, and hence a set of basic meshes. Let there be M basic meshes. We number the branches $1, 2, \dots, B$ and the basic meshes $1, 2, \dots, M$, with no attention to any particular order.

To the basic meshes we assign arbitrary mesh currents i'_1, i'_2, \dots, i'_M . Now (6.1) applies

for any meshes, and so we can express the branch currents generated by the currents in the basic meshes by the formula

$$i_p = \sum_{q=1}^M C_{pq} i'_q, \quad (p = 1, 2, \dots, B) \quad (9.1)$$

where

$$C_{pq} \begin{cases} = 1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 & \text{if branch } p \text{ is not contained in mesh } q. \end{cases} \quad (9.2)$$

In matrix form we have

$$\mathbf{i} = \mathbf{C}\mathbf{i}'; \quad (9.3)$$

here the $B \times M$ matrix \mathbf{C} is Kron's *transformation matrix*.

We note that, by Theorem III, if the mesh currents are chosen arbitrarily, the branch currents given by (9.1) or (9.3) necessarily satisfy Kirchhoff's node law at every node.

But there is another important thing which we have to prove, namely, that by proper choice of the mesh currents \mathbf{i}' (9.1) or (9.3) will yield the most general set of branch currents \mathbf{i} consistent with Kirchhoff's node law.

To prove this, let us assign any set of branch currents consistent with Kirchhoff's node law. Then consider the tree in the network which corresponds to the basic meshes involved, and think of the branches-in-tree and the branches-out-of-tree.

Each basic mesh contains just one branch-out-of-tree, and the branch current in it has been assigned. To the mesh containing that branch assign a mesh current equal to that branch current. Having done this for all the basic meshes, we have a set of mesh currents which generates a set of branch currents. We have deliberately made this set of branch currents coincide with the assigned branch currents in the branches-out-of-tree, and the coincidence for the branches-in-tree (and hence for all branches) follows from Theorem IV. Thus the assigned branch currents, arbitrary except for the satisfaction of Kirchhoff's node law, can be generated by a suitably chosen set of mesh currents in the basic meshes.

Let us summarise this result. It is the central point; what has gone before was preparation for it, and what follows is easy deduction. It is the central theorem involved in Kron's formula (9.3), or indeed in any mesh-branch transition in network theory. This is the theorem:

THEOREM V: *Given a network, there exists at least one set of basic meshes such that the following statements are true, the matrix \mathbf{C} being defined as in (9.2):*

- (i) *If a set of mesh currents \mathbf{i}' is arbitrarily assigned, the branch currents given by $\mathbf{i} = \mathbf{C}\mathbf{i}'$ satisfied Kirchhoff's node law at every node.*
- (ii) *If a set of branch currents \mathbf{i} is assigned, arbitrarily except for the satisfaction of Kirchhoff's node law at every node, there exists a set of mesh currents \mathbf{i}' in the basic meshes such that $\mathbf{i} = \mathbf{C}\mathbf{i}'$.*

10. Branch emfs and mesh emfs. For present purposes a *branch emf* e is simply a complex number associated with a branch of a network, and a *mesh emf* e' is a complex number associated with a mesh.

We bring these two concepts into relation with one another in a manner complementary to that in which we brought together branch currents and mesh currents. We say

that if branch emfs are assigned in all the branches, *they generate in any mesh a mesh emf equal to the sum of the branch emfs in the branches contained in the mesh, a + or - sign being prefixed to the branch emf before adding according as the senses of the branch and the mesh agree or disagree.*

If, as usual, we number the branches 1, 2, \dots B and the meshes 1, 2, \dots M (we take a basic set, although the immediate statement holds more generally), then the above verbal statement is obviously equivalent to the following formula connecting mesh emfs e' and branch emfs e :

$$e'_q = \sum_{p=1}^B D_{qp} e_p \quad (q = 1, 2, \dots, M), \quad (10.1)$$

where the coefficients are defined by

$$D_{qp} \begin{cases} = 1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the same sense,} \\ = -1 & \text{if branch } p \text{ is contained in mesh } q \text{ and has the opposite sense,} \\ = 0 & \text{if branch } p \text{ is not contained in mesh } q. \end{cases} \quad (10.2)$$

When we compare this with (9.2), we see at once that $D_{qp} = C_{pq}$, or in matrix language $\mathbf{D} = \mathbf{C}_t$ where \mathbf{C}_t is the transpose of \mathbf{C} . Thus (10.1) may be written

$$e'_q = \sum_{p=1}^B C_{pq} e_p, \quad (10.3)$$

or

$$\mathbf{e}' = \mathbf{C}_t \mathbf{e}. \quad (10.4)$$

To sum up:

THEOREM VI: *Given a network and a basic set of meshes in it, any assigned set of branch emfs \mathbf{e} generate mesh emfs \mathbf{e}' according to the formula (10.4), where \mathbf{C}_t is the transpose of Kron's transformation matrix.*

We have then at this stage obtained, with it is hoped a proper understanding of their meanings, the two formulae

$$\mathbf{i} = \mathbf{C}_i \mathbf{i}', \quad \mathbf{e}' = \mathbf{C}_t \mathbf{e}. \quad (10.5)$$

The currents and emfs have appeared so far only as numbers associated with branches and meshes, without electrical significance except in connection with Kirchhoff's node law and the generation rules connecting quantities for branches and meshes. There has been no reference at all to an impedance matrix.

We have been thinking of a single connected network. If we have a set of disconnected networks, the formulae (10.5) hold for each of them, and it is easy to see (such is the elasticity of matrix notation) that the whole set of such relations, a pair for each connected network, may be compressed into a pair of formulae of precisely the form (10.5), now covering the whole network.

This ends what may be called the purely topological part of the paper.

11. How to solve a network. Suppose we have a network, in general consisting of several parts, disconnected topologically but connected electromagnetically. Let the branches (B in number) contain generators of emf, d.c. or a.c. A square $B \times B$ impedance

matrix \mathbf{Z} is given. The problem of solving the network consists in finding the branch currents i , given the branch emfs e , only steady states being considered.*

We cannot of course attempt to do this without some law connecting currents and emfs. To state this law, we define a set of B potential-differences \mathbf{W} in the branches by the formula

$$\mathbf{W} = \mathbf{e} - \mathbf{Z}\mathbf{i}. \quad (11.1)$$

We then accept Kirchhoff's mesh law, which asserts that *the sum of the potential differences for the branches of any mesh is zero, each potential-difference being prefixed before adding with a + or - sign according as the senses of the branch and the mesh agree or disagree.*

This holds for *any* mesh, and we could continue our discussion for a while using any meshes at all. But to avoid unnecessary generality, let us apply Kirchhoff's mesh law to a set of basic meshes. It is clear that the verbal statement made above is precisely equivalent to

$$\sum_{p=1}^B D_{op} W_p = 0, \quad (11.2)$$

where D_{op} is defined by (10.2). Thus Kirchhoff's mesh law gives

$$\mathbf{C}_t \mathbf{W} = 0, \quad (11.3)$$

since, as we saw, \mathbf{D} is the transpose of \mathbf{C} .

We now do some formal work with matrices. Multiplication of (11.1) on the left by \mathbf{C}_t gives

$$\mathbf{C}_t \mathbf{e} - \mathbf{C}_t \mathbf{Z} \mathbf{i} = 0, \quad (11.4)$$

and, by (10.5), this may be written

$$\mathbf{e}' = \mathbf{Z}' \mathbf{i}', \quad (11.5)$$

where \mathbf{Z}' is defined to be

$$\mathbf{Z}' = \mathbf{C}_t \mathbf{Z} \mathbf{C}. \quad (11.6)$$

We have not achieved our objective; we have not found the branch currents \mathbf{i} in terms of the branch emfs \mathbf{e} . We have got no further than (11.5) which expresses mesh emfs in terms of mesh currents. And we can go no further without an additional assumption which will assure us that the matrix \mathbf{Z}' is not singular.

12. The final step. The final assumption which we might make is that the network is *dissipative*, i.e. it contains resistances. We know then that if all emfs are put equal to zero, there cannot exist steady d.c. or a.c. currents, except of course zero currents. In the case of a.c. we can be more general, and say that the network may be resistanceless, but that the frequencies which we consider do not include resonant frequencies of the network; then in the absence of all emfs there can exist no periodic currents in the network, except of course the trivial zero currents.

The essential assumption is that the network (and perhaps the frequencies considered) should be such that $\mathbf{e} = 0$ implies $\mathbf{i} = 0$. This is the assumption we shall make.

* In the case of a. c., i and e are complex constants from which the physical quantities are obtained by multiplying by $\exp(jpt)$ and then taking the real part.

Consider now the equation (11.5). The matrix \mathbf{Z}' , as given by (11.6), is in part topological (through \mathbf{C}) and in part electromagnetic (through \mathbf{Z}); but it is defined by the *structure* of the network, and is independent of the values of the branch emfs.

Let us then put $\mathbf{e} = 0$. This implies, under our assumption, $\mathbf{i} = 0$. Going back to the tree from which the basic meshes were constructed in Section 8, we see that each mesh current is actually a branch current, namely, in a branch-out-of-tree. Hence $\mathbf{i} = 0$ implies $\mathbf{i}' = 0$. Now $\mathbf{e} = 0$ certainly implies $\mathbf{e}' = 0$ by (10.5). We conclude from (11.5) that the equations

$$\mathbf{Z}'\mathbf{i}' = 0 \quad (12.1)$$

are satisfied only by $\mathbf{i}' = 0$.

This means that the determinant of the matrix \mathbf{Z}' is not zero; the matrix is therefore non-singular, and \mathbf{Z}'^{-1} exists.

The rest is simple. Restoring general values to \mathbf{e} , we multiply (11.5) on the left by \mathbf{Z}'^{-1} and get

$$\mathbf{i}' = \mathbf{Z}'^{-1}\mathbf{e}'. \quad (12.2)$$

Then we multiply on the left by \mathbf{C} and use both the equations (10.5): this gives

$$\mathbf{i} = \mathbf{C}\mathbf{Z}'^{-1}\mathbf{C}_t\mathbf{e}. \quad (12.3)$$

or, inserting the value of \mathbf{Z}' from (11.6),

$$\mathbf{i} = \mathbf{C}(\mathbf{C}_t\mathbf{Z}\mathbf{C})^{-1}\mathbf{C}_t\mathbf{e}, \quad (12.4)$$

which is Kron's fundamental equation for the solution of networks.

13. Conclusion. The essential steps leading ultimately to (12.4) are as follows:

- (a) Deduction of $\mathbf{i} = \mathbf{C}\mathbf{i}'$; this is topology, oriented by Kirchhoff's node law.
- (b) Deduction of $\mathbf{e}' = \mathbf{C}_t\mathbf{e}$; very simple when we have done (a).
- (c) Deduction of $\mathbf{e}' = \mathbf{Z}'\mathbf{i}'$ from Kirchhoff's mesh law.
- (d) An assumption assuring the non-singular character of \mathbf{Z}' , so that we can solve for \mathbf{i} .