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THE FUNDAMENTAL THEOREM OF TROPICAL DIFFERENTIAL ALGEBRAIC GEOMETRY

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Let I be an ideal of the ring of Laurent polynomials $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with coefficients in a real-valued field (K, v). The fundamental theorem of tropical algebraic geometry states the equality $\mathrm{trop}(V(I)) = V(\mathrm{trop}(I))$ between the tropicalization $\mathrm{trop}(V(I))$ of the closed subscheme $V(I) \subset (K^*)^n$ and the tropical variety $V(\mathrm{trop}(I))$ associated to the tropicalization of the ideal $\mathrm{trop}(I)$.

In this work we prove an analogous result for a differential ideal G of the ring of differential polynomials $K[[t]]\{x_1, \dots, x_n\}$, where K is an uncountable algebraically closed field of characteristic zero. We define the tropicalization $\mathrm{trop}(\mathrm{Sol}(G))$ of the set of solutions $\mathrm{Sol}(G) \subset K[[t]]^n$ of G, and the set of solutions $\mathrm{Sol}(\mathrm{trop}(G)) \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ associated to the tropicalization of the ideal $\mathrm{trop}(G)$. These two sets are linked by a tropicalization morphism $\mathrm{trop} : \mathrm{Sol}(G) \to \mathrm{Sol}(\mathrm{trop}(G))$.

We show the equality $\mathrm{trop}(\mathrm{Sol}(G)) = \mathrm{Sol}(\mathrm{trop}(G))$, answering a question recently raised by D. Grigoriev.

1. Introduction

The first proof of the fundamental theorem of tropical algebraic geometry appeared in 2003 in a preprint by Einsiedler, Kapranov and Lind [Einsiedler et al. 2006], and was limited to hypersurfaces. Later, the theorem was established in full generality in [Speyer and Sturmfels 2004]. Extensions to arbitrary codimension ideals and arbitrary valuations have been done subsequently; see, for example, [Aroca et al. 2010; Jensen et al. 2008; Aroca 2010].

The tropical variety of a hypersurface is dual to a subdivision of the Newton polyhedron of its defining function. The Newton polygon was introduced by Puiseux [1850] for plane algebraic curves and extended to differential polynomials by Fine [1889]. Both the extensions of the polygon and the polyhedron have served

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to prove existence theorems and to construct algorithms that compute solutions; see for example [Grigoriev and Singer 1991; Cano 1993; Aroca and Cano 2001; Aroca et al. 2003].

Grigoriev [2015] introduces the notion of *tropical linear differential equations* in n variables and designs a polynomial complexity algorithm for solving systems of tropical linear differential equations in one variable. In the same preprint, Grigoriev suggests several lines for further research. One of his questions is whether a theorem such as the fundamental theorem of tropical algebraic geometry holds in this context.

More precisely, Grigoriev notes that, for a differential ideal G in n independent variables, we have the inclusion $\mathrm{trop}(\mathrm{Sol}(G)) \subset \mathrm{Sol}(\mathrm{trop}(G))$ and asks:

> Is it true that for any differential ideal G and a family $S_1, \dots, S_n \subset \mathbb{Z}_{\geq 0}$ being a solution of the tropical differential equation $\mathrm{trop}(g)$ for any $g \in G$, there exists a power series solution of G whose tropicalization equals $S_1, \dots, S_n$?

Here, we give a positive answer to this question when G is a differential ideal of differential polynomials over the ring of formal power series $K[\![t]\!]$, K being an uncountable algebraically closed field of characteristic zero. Our proof uses techniques developed in the theory of arc spaces; see [Nash 1995].

In Section 2, the basic definitions of differential algebraic geometry are recalled. In Sections 3, 4 and 5, we explain the tropicalization morphisms. Arc spaces and their connection with sets of solutions of differential ideals are discussed in Section 6. The main result is proved in the last two sections.

2. Differential algebraic geometry

We will begin by recalling some basic definitions of differential algebraic geometry. The reference for this section is the book by J. F. Ritt [1950].

Let R be a commutative ring with unity. A *derivation* on R is a map $d : R \to R$ that satisfies $d(a+b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. The pair (R, d) is called a *differential ring*. An ideal $I \subset R$ is said to be a *differential ideal* when $d(I) \subset I$.

Let (R, d) be a differential ring and let $R\{x_1, \dots, x_n\}$ be the set of polynomials with coefficients in R in the variables $\{x_{ij} : i = 1, \dots, n,\ j \geq 0\}$. The derivation d on R can be extended to a derivation d of $R\{x_1, \dots, x_n\}$ by setting $d(x_{ij}) = x_{i(j+1)}$ for $i = 1, \dots, n$ and $j \geq 0$. The pair $(R\{x_1, \dots, x_n\}, d)$ is a differential ring called *the ring of differential polynomials in n variables with coefficients in* R.

A differential polynomial $P \in R\{x_1, \dots, x_n\}$ induces a mapping from R^n to R given by

$$P : R^n \to R, \quad (\varphi_1, \dots, \varphi_n) \mapsto P|_{x_{ij} = d^j \varphi_i}, \tag{2-1}$$

where $P|_{x_{ij}=d^j\varphi_i}$ is the element of R obtained by substituting $x_{ij} \mapsto d^j\varphi_i$ in the differential polynomial P.

The equality

$$d^k(P(\varphi)) = (d^kP)(\varphi) \tag{2-2}$$

holds for any $P \in R\{x_1, \dots, x_n\}$ and $\varphi \in R^n$.

A *zero* or a *solution* of $P \in R\{x_1, \dots, x_n\}$ is an n-tuple $\varphi \in R^n$ such that $P(\varphi)=0$. An n-tuple $\varphi \in R^n$ is a *solution* of $\Sigma \subset R\{x_1, \dots, x_n\}$ when it is a solution of every differential polynomial in Σ; that is,

$$\mathrm{Sol}(\Sigma) := \{\varphi \in R^n \ : \ P(\varphi) = 0 \text{ for all } P \in \Sigma\}.$$

The following result can be found in [Ritt 1950, p. 21].

Proposition 2.1. *The solution of any infinite system of differential polynomials*

$$\Sigma \subset F\{x_1, \dots, x_n\},$$

where F is a differential field of characteristic zero, is the solution of some finite subset of the system.

A *differential monomial* in n independent variables of order less than or equal to r is an expression of the form

$$E_M := \prod_{\substack{1\le i\le n\\ 0\le j\le r}} x_{ij}^{M_{ij}}, \tag{2-3}$$

where $M = (M_{ij})_{1\le i\le n,\, 0\le j\le r}$ is a matrix in $\mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\ge 0})$.

With this notation, a differential polynomial $P \in R\{x_1, \dots, x_n\}$ is an expression of the form

$$P = \sum_{M\in\Lambda\subset\mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\ge 0})} \psi_M E_M, \tag{2-4}$$

with $r \in \mathbb{Z}_{\ge 0}$, $\psi_M \in R$ and Λ finite.

The mapping induced by the monomial E_M is given by

$$E_M : R^n \to R, \quad (\varphi_1, \dots, \varphi_n) \mapsto \prod_{\substack{1\le i\le n\\ 0\le j\le r}} (d^j\varphi_i)^{M_{ij}},$$

and the map (2-1) induced by the differential polynomial P in (2-4) is

$$P : R^n \to R, \quad \varphi = (\varphi_1, \dots, \varphi_n) \mapsto \sum_{M\in\Lambda} \psi_M E_M(\varphi). \tag{2-5}$$

3. The differential ring of formal power series and tropicalization

In what follows, we work with the differential valued ring $R = K[[t]]$ where K is an uncountable algebraically closed field of characteristic zero. We set $F = \mathrm{Frac}(R)$.

The elements of R are expressions of the form

$$\varphi = \sum_{j \in \mathbb{Z}_{\geq 0}} a_j t^j \tag{3-1}$$

with $a_j \in K$ for $j \in \mathbb{Z}_{\geq 0}$.

The *support* of φ is the set

$$\mathrm{Supp}(\varphi) := \{i \in \mathbb{Z}_{\geq 0} : a_i \neq 0\},$$

the valuation on R is given by

$$\mathrm{val}(\varphi) = \min \mathrm{Supp}(\varphi)$$

and the derivative of φ is the element

$$d\varphi = \sum_{j \in \mathbb{Z}_{\geq 0}} j a_j t^{j-1}$$

of R. The bijection

$$\Psi : K^{\mathbb{Z}_{\geq 0}} \to R, \quad \underline{a} = (a_j)_{j \geq 0} \mapsto \sum_{j \geq 0} \frac{1}{j!} a_j t^j$$

between $K^{\mathbb{Z}_{\geq 0}}$ and R allows us to identify points of R with points of $K^{\mathbb{Z}_{\geq 0}}$. Moreover, the mapping Ψ has the following property:

$$d^s \Psi(\underline{a}) = \sum_{j \geq 0} \frac{a_{s+j}}{j!} t^j, \tag{3-2}$$

which implies

$$d^s \Psi(\underline{a})|_{t=0} = a_s, \quad s \in \mathbb{Z}_{\geq 0}$$

and then

$$\underline{a} = \left(d^j \Psi(\underline{a})|_{t=0}\right)_{j \geq 0}. \tag{3-3}$$

The mapping that sends each series in R to its support set (a subset of $\mathbb{Z}_{\geq 0}$) will be called the *tropicalization* map

$$\mathrm{trop} : R \to \mathcal{P}(\mathbb{Z}_{\geq 0}), \quad \varphi \mapsto \mathrm{Supp}(\varphi)$$

where $\mathcal{P}(\mathbb{Z}_{\geq 0})$ denotes the power set of $\mathbb{Z}_{\geq 0}$.

For fixed n, the mapping from R^n to the n-fold product of $\mathcal{P}(\mathbb{Z}_{\geq 0})$ will also be denoted by trop:

$$\text{trop}: R^n \to \mathcal{P}(\mathbb{Z}_{\geq 0})^n, \quad \varphi = (\varphi_1, \ldots, \varphi_n) \mapsto \text{trop}(\varphi) = (\text{Supp}(\varphi_1), \ldots, \text{Supp}(\varphi_n)).$$

Given a subset T of R^n, the *tropicalization* T is its image under the map trop:

$$\text{trop}(T) := \{\text{trop}(\varphi) \,:\, \varphi \in T\} \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n.$$

Example 3.1. Set $T := \{(a + 5t + bt^2, 2 + at - 8t^2 + ct^3) \,:\, a, b, c \in K\} \subset K[\![t]\!]^2$. We have

$$\begin{aligned}\text{trop}(T) = \big\{&(\{1\}, \{0, 2\}), (\{0, 1\}, \{0, 1, 2\}),\\ &(\{1, 2\}, \{0, 2\}), (\{1\}, \{0, 2, 3\}), (\{0, 1, 2\}, \{0, 1, 2\}),\\ &(\{0, 1\}, \{0, 1, 2, 3\}), (\{1, 2\}, \{0, 2, 3\}), (\{0, 1, 2\}, \{0, 1, 2, 3\})\big\}.\end{aligned}$$

Since K is of characteristic zero, for every $\varphi \in R$, we have

$$\text{trop}(d^j\varphi) = \{i - j \,:\, i \in \text{trop}(\varphi) \cap \mathbb{Z}_{\geq j}\}$$

then

$$\text{val}(d^j\varphi) = \min(\text{trop}(\varphi) \cap \mathbb{Z}_{\geq j}) - j.$$

The above equality justifies the following definition:

Definition. A subset $S \subseteq \mathbb{Z}_{\geq 0}$ induces a mapping $\text{Val}_S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\text{Val}_S(j) := \begin{cases} s - j & \text{with } s = \min\{\alpha \in S \,:\, \alpha \geq j\}, \\ \infty & \text{when } S \cap \mathbb{Z}_{\geq j} = \varnothing. \end{cases} \tag{3-4}$$

Example 3.2. Consider the set $S := \{1, 3, 4\}$. We have

(1) $\text{Val}_S(2) = \min\{s \in S \,:\, s \geq 2\} - 2 = 3 - 2 = 1$ and

(2) $\text{Val}_S(5) = \infty$.

4. Tropical differential polynomials

We denote by $\mathbb{T}$ the (idempotent) semiring $\mathbb{T} = (\mathbb{Z}_{\geq 0} \cup \{\infty\}, \oplus, \odot)$, with $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$.

Definition. A *tropical differential monomial* in the variables $x_1, \ldots, x_n$ of order less than or equal to r is an expression of the form

$$\varepsilon_M := x^{\odot M} = \bigodot_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r}} x_{ij}^{\odot M_{ij}}, \tag{4-1}$$

where $M = (M_{ij})_{1 \leq i \leq n,\, 0 \leq j \leq r}$ is a matrix in $\mathcal{M}_{n \times (r+1)}(\mathbb{Z}_{\geq 0})$.

Definition. A *tropical differential polynomial* in the variables $x_1, \dots, x_n$ of order less than or equal to r is an expression of the form

$$\phi = \phi(x_1, \dots, x_n) = \bigoplus_{M \in \Lambda \subset \mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\geq 0})} a_M \odot \varepsilon_M, \tag{4-2}$$

where $a_M \in \mathbb{T}$ and Λ is a finite set.

The set of tropical differential polynomials will be denoted by $\mathbb{T}\{x_1, \dots, x_n\}$. A tropical differential monomial ε_M induces a mapping from $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\varepsilon_M(S_1, \dots, S_n) := \bigodot_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} \mathrm{Val}_{S_i}(j)^{\odot M_{ij}} = \sum_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} M_{ij}\cdot \mathrm{Val}_{S_i}(j),$$

where $\mathrm{Val}_{S_i}(j)$ is defined as in (3-4).

Remark 4.1. Note that $\varepsilon_M(S_1, \dots, S_n) = 0$ if and only if $j \in S_i$ for all i, j with $M_{ij} \neq 0$.

A tropical differential polynomial ϕ as in (4-2) induces a mapping from $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\phi(S) = \bigoplus_{M\in\Lambda} a_M \odot \varepsilon_M(S) = \min_{M\in\Lambda}\{a_M + \varepsilon_M(S)\}.$$

Definition. An n-tuple $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is said to be a *solution* of the tropical differential polynomial ϕ in (4-2) if either

(1) there exist $M_1, M_2 \in \Lambda$ with $M_1 \neq M_2$ such that $\phi(S) = a_{M_1} \odot \varepsilon_{M_1}(S) = a_{M_2} \odot \varepsilon_{M_2}(S)$, or

(2) $\phi(S) = \infty$.

Let $H \subset \mathbb{T}\{x_1, \dots, x_n\}$ be a family of tropical differential polynomials. An n-tuple $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is a *solution* of H when it is a solution of every tropical polynomial in H; that is,

$$\mathrm{Sol}(H) := \{S \in (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n : S \text{ is a solution of } \phi \text{ for every } \phi \in H\}.$$

Example 4.2. Consider the tropical differential polynomial

$$\phi(x) := 1 \odot x' \oplus 2 \odot x^{(3)} \oplus 3.$$

Since $\phi(S) \neq \infty$ for every $S \subset \mathcal{P}(\mathbb{Z}_{\geq 0})$, the set S is a solution of ϕ if one of the following holds:

(1) $1 + \mathrm{Val}_S(1) = 3 \leq 2 + \mathrm{Val}_S(3)$,

(2) $1 + \mathrm{Val}_S(1) = 2 + \mathrm{Val}_S(3) \leq 3$,

(3) $2 + \mathrm{Val}_S(3) = 3 \leq 1 + \mathrm{Val}_S(1)$.

The first condition never holds. The second condition holds when $S = B \cup \{2, 3\} \cup C$ and $B \subset \{0\}$, $\min C \geq 4$. The third condition holds when $S = \{4\} \cup C \cup B$ with $\min C \geq 5$ and $B \subset \{0\}$. Thus,

$$\mathrm{Sol}(P) = \big\{B \cup \{2, 3\} \cup C : B \subset \{0\},\ \min C \geq 4\big\} \cup \big\{B \cup \{4\} \cup C : \min C \geq 5,\ B \subset \{0\}\big\}.$$

5. Tropicalization of differential polynomials

Let P be a differential polynomial as in (2-4). The *tropicalization* of P is the tropical differential polynomial

$$\mathrm{trop}(P) := \bigoplus_{M \in \Lambda} \mathrm{val}(\psi_M) \odot \varepsilon_M. \tag{5-1}$$

Remark 5.1. Let P be a differential polynomial in $R\{x_1, \ldots, x_n\}$. We have that $\mathrm{trop}(tP)(S) \geq 1$ for any $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$.

Definition. Let $G \subset R\{x_1, \ldots, x_n\}$ be a differential ideal. Its *tropicalization* $\mathrm{trop}(G)$ is the set of tropical differential polynomials $\{\mathrm{trop}(P) : P \in G\}$.

Proposition 5.2. *Let G be a differential ideal in the ring of differential polynomials $R\{x_1, \ldots, x_n\}$. If $\varphi \in \mathrm{Sol}(G)$, then $\mathrm{trop}(\varphi) \in \mathrm{Sol}(\mathrm{trop}(G))$.*

Proof. Given a differential monomial E_M and $\varphi \in R^n$, we have that

$$\mathrm{val}(E_M(\varphi)) = \varepsilon_M(\mathrm{trop}(\varphi)).$$

It follows that if $\varphi \in R^n$ is a solution to the differential polynomial

$$P = \sum_{M \in \Lambda} \psi_M E_M,$$

then $\mathrm{trop}(\varphi) \in (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n$ is a solution to $\mathrm{trop}(P)$. So, if $\varphi \in R^n$ is a solution to every differential polynomial P in a differential ideal G, then $\mathrm{trop}(\varphi)$ is a solution to every tropical differential polynomial $\mathrm{trop}(P) \in \mathrm{trop}(G)$. □

We can now clearly state the question posed in [Grigoriev 2015]. The latter result allows us to define a mapping $\mathrm{trop} : \mathrm{Sol}(G) \to \mathrm{Sol}(\mathrm{trop}(G))$ for any differential ideal $G \subset R\{x_1, \ldots, x_n\}$. The question is whether or not this map is surjective.

Example 5.3. Let $P \in R\{x\}$ be the differential polynomial

$$P := x'' - t.$$

The set of solutions of P is the same as the set of solutions of the differential ideal generated by P:

$$\mathrm{Sol}(P) = \big\{c_1 + c_2 t + \tfrac{1}{6} t^3 \ :\ c_1, c_2 \in K\big\}.$$

The tropicalization of the set of solutions of P is

$$\operatorname{trop}(\operatorname{Sol}(P))=\big\{\{0,1,3\},\{0,3\},\{1,3\},\{3\}\big\}.$$

Now, the tropicalization of P induces the mapping

$$\operatorname{trop}(P):\mathcal{P}(\mathbb{Z}_{\geq 0})\to\mathbb{Z}_{\geq 0},\quad S\mapsto\min\{\operatorname{Val}_S(2),1\}.$$

Since $\operatorname{trop}(P)(S)\neq\infty$ for every $S\subset\mathcal{P}(\mathbb{Z}_{\geq}0)$, the set of solutions of $\operatorname{trop}(P)$ is

$$\operatorname{Sol}(\operatorname{trop}(P))=\{S\subset\mathcal{P}(\mathbb{Z}_{\geq}0)\;:\;2\notin S\text{ and }3\in S\}.$$

Differentiating P, we have that $d^2P=x^{(4)}$ is in the differential ideal generated by P. Its tropicalization induces the mapping

$$\operatorname{trop}(d^2P):\mathcal{P}(\mathbb{Z}_{\geq 0})\to\mathbb{Z}_{\geq 0},\quad S\mapsto\operatorname{Val}_S(4).$$

We have that $S\subset\mathcal{P}(\mathbb{Z}_{\geq}0)$ is a solution of $\operatorname{trop}(d^2P)$ if and only if $S\subset\{0,1,2,3\}$, i.e.,

$$\operatorname{Sol}(\operatorname{trop}(d^2P))=\mathcal{P}(\{0,1,2,3\}).$$

In this example,

$$\operatorname{Sol}(\operatorname{trop}(P))\cap\operatorname{Sol}(\operatorname{trop}(d^2P))=\operatorname{trop}(\operatorname{Sol}(P)).$$

6. Arc spaces and the set of solutions of a differential ideal

The natural inclusion $K[x_{1\,0},\dots,x_{n\,0}]\subset R\{x_1,\dots,x_n\}$ lets us recognize the arc space of the variety defined by an ideal $I\subset K[x_1,\dots,x_n]$ as the space of solutions of the differential ideal generated by I in $R\{x_1,\dots,x_n\}$. In this section we extend some definitions and results developed in the theory of arc spaces; see for example [Nash 1995; Bruschek et al. 2013].

Consider the bijection

$$\Psi:\big(K^{\mathbb{Z}_{\geq 0}}\big)^n\to R^n,\quad \underline{a}=(a_{ij})_{1\leq i\leq n,\,j\geq 0}\mapsto\bigg(\sum_{j\geq 0}\frac{1}{j!}a_{1j}t^j,\dots,\sum_{j\geq 0}\frac{1}{j!}a_{nj}t^j\bigg).$$

Lemma 6.1. *Given $P\in R\{x_1,\dots,x_n\}$ and $\underline{a}\in(K^{\mathbb{Z}_{\geq 0}})^n$, we have*

$$P(\Psi(\underline{a}))=\sum_{k\geq 0}c_kt^k \tag{6-1}$$

with

$$c_k=\frac{1}{k!}(d^k(P))\big|_{t=0}(\underline{a}).$$

Proof. For $\underline{a} = (a_{ij})_{1 \le i \le n,\, j \ge 0} \in \left(K^{\mathbb{Z}_{\ge 0}}\right)^n$, write $\Psi(\underline{a}) = (\Psi(\underline{a})_1, \ldots, \Psi(\underline{a})_n)$ and $P(\Psi(\underline{a})) = \sum_{k \ge 0} c_k t^k$ for some $c_k \in K$, $k \ge 0$. Differentiating (6-1) and evaluating at zero, we have

$$c_k = \frac{1}{k!}\left[d^k(P(\Psi(\underline{a})))\right]_{t=0} \overset{(2\text{-}2)}{=} \frac{1}{k!}\left[(d^kP)(\Psi(\underline{a}))\right]_{t=0} \overset{(2\text{-}1)}{=} \frac{1}{k!}\left[(d^kP)|_{x_{ij}=\Psi(\underline{a})_i^{(j)}}\right]_{t=0}$$
$$= \frac{1}{k!}\left[(d^kP)|_{x_{ij}=\Psi(\underline{a})_i^{(j)}|_{t=0}}\right]_{t=0} \overset{(3\text{-}3)}{=} \frac{1}{k!}\left[(d^kP)|_{x_{ij}=a_{ij}}\right]_{t=0} = \frac{1}{k!}(d^kP)|_{t=0}(\underline{a}). \quad \square$$

Let G be a differential ideal in $R\{x_1, \ldots, x_n\}$. We can consider G as an infinite system of differential polynomials in $F\{x_1, \ldots, x_n\}$, where $F = \mathrm{Frac}(R)$ is a field of characteristic zero. By Proposition 2.1, there exist $f_1, \ldots, f_s \in G$ such that

$$\mathrm{Sol}(G) = \bigcap_{\ell=1}^{s} \mathrm{Sol}(f_\ell).$$

For $1 \le \ell \le s$ and $k \in \mathbb{Z}_{\ge 0}$, the $(d^k f_\ell)|_{t=0}$ are polynomials in the variables x_{ij} with coefficients in K. Set

$$F_{\ell k} := (d^k f_\ell)|_{t=0} \in K[x_{ij} \,:\, 1 \le i \le n,\ j \ge 0]$$

and

$$A_\infty := V\left(\{F_{\ell k}\}_{1 \le \ell \le s,\, k \ge 0}\right) \subset \left(K^{\mathbb{Z}_{\ge 0}}\right)^n. \tag{6-2}$$

By Lemma 6.1,

$$\mathrm{Sol}(G) = \Psi(A_\infty).$$

We will now describe an extension to differential ideals of the definition of m-jet of arc spaces; see for example [Mourtada 2011].

For each $m \ge 0$, let N_m be the smallest positive integer such that

$$F_{\ell k} \in K[x_{ij} \,:\, 1 \le i \le n,\ 0 \le j \le N_m] \quad \text{for all } 1 \le \ell \le s,\ 0 \le k \le m \tag{6-3}$$

and set

$$A_m := V\left(\{F_{\ell k}\}_{1 \le \ell \le s,\, 0 \le k \le m}\right) \subset \left(K^{N_m+1}\right)^n. \tag{6-4}$$

For $m \ge m' \ge 0$, denote by $\pi_{(m,m')}$ the natural algebraic morphism

$$\pi_{(m,m')} : K^{n(N_m+1)} \to K^{n(N_{m'}+1)}.$$

Then

$$\pi_{(m,m')}(A_m) \subset A_{m'}$$

and A_∞ is the inverse limit of the system $((A_m)_{m \in \mathbb{Z}_{\ge 0}}, (\pi_{(m,m')})_{m \ge m' \in \mathbb{Z}_{\ge 0}})$:

$$A_\infty = \varprojlim A_m.$$

When $f_1, \dots, f_s$ are elements of $K[x_{10}, \dots, x_{n0}]$, the sets A_m are the m-jets of the space A_∞. Otherwise, note that the construction depends strongly on the choice of $f_1, \dots, f_s$.

7. Intersections with tori

Let $G \subset R\{x_1, \dots, x_n\}$ be a differential ideal, let $f_1, \dots, f_s \in G$ be such that $\operatorname{Sol}(G) = \bigcap_{\ell=1}^{s} \operatorname{Sol}(f_\ell)$, and let A_∞ be as in (6-2) and A_m as in (6-4).

An n-tuple $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is in $\operatorname{trop}(\operatorname{Sol}(G))$ if and only if there exists $\underline{a} \in A_\infty$ with $\operatorname{trop}(\Psi(\underline{a})) = S$, i.e., if $S_i = \{j : a_{ij} \neq 0\}$ for $i = 1, \dots, n$. Set

$$\mathbb{V}_S^* := \left\{(x_{ij})_{1\leq i\leq n,\, j\geq 0} \in \left(K^{\mathbb{Z}_{\geq 0}}\right)^n : x_{ij} = 0 \text{ if and only if } j \notin S_i\right\},$$

then $S \in \operatorname{trop}(\operatorname{Sol}(G))$ if and only if

$$(A_\infty)_S := A_\infty \cap \mathbb{V}_S^*$$

is not empty.

For $m \geq 0$, consider the finite dimensional torus

$$(\mathbb{V}_m)_S^* := \left\{(x_{ij})_{1\leq i\leq n,\, 0\leq j\leq N_m} \in K^{n(N_m+1)} : x_{ij} = 0 \text{ if and only if } j \notin S_i\right\},$$

where N_m is the minimum such that (6-3) holds. We have $(\mathbb{V}_m)_S^* \simeq (K^*)^{L_m}$, with $L_m \leq n(N_m + 1)$. Set

$$(A_m)_S := A_m \cap (\mathbb{V}_m)_S^*.$$

For $m \geq m' \geq 0$, the inclusions

$$\pi_{(m,m')}((\mathbb{V}_m)_S^*) \subset (\mathbb{V}_{m'})_S^* \quad \text{and} \quad \pi_{(m,m')}((A_m)_S) \subset (A_{m'})_S$$

hold, and $(A_\infty)_S$ is the inverse limit of $\big(((A_m)_S)_{m\in\mathbb{Z}_{\geq 0}}, (\pi_{(m,m')})_{m\geq m'\in\mathbb{Z}_{\geq 0}}\big)$:

$$(A_\infty)_S = \varprojlim (A_m)_S.$$

Set

$$(B_m)_S := \bigcap_{i=m}^{\infty} \pi_{(i,m)}((A_i)_S);$$

then

$$(A_\infty)_S = \varprojlim (B_m)_S$$

and the projections

$$\pi_{(m,m')} : (B_m)_S \to (B_{m'})_S$$

are surjective. Then (see, for example, [Bourbaki 1968, Proposition 5, p. 198]), the set $\varprojlim (B_m)_S$ is nonempty if and only if $(B_0)_S$ is nonempty. In other words, we have the following remark.

Remark 7.1. The set $(A_\infty)_S$ is nonempty if and only if $\bigcap_{i=0}^{\infty}\pi_{(i,0)}((A_i)_S)$ is nonempty.

By Chevalley's theorem (see, for example, [Mumford 1999, p. 51]), each $\pi_{(m,0)}((A_m)_S)$ is a constructible set. A constructible set is, by definition, a finite union of locally closed sets. A set is locally closed when it is an open set of its closure. The constructible sets form a Boolean algebra.

We recall the following statement about nested sequences of constructible sets:

Proposition 7.2. *Let K be an uncountable algebraically closed field of characteristic zero. Let $\{E_\alpha\}_{\alpha=1}^{\infty}$ be an increasing family of constructible sets in K^n with $K^n=\bigcup_{\alpha=1}^{\infty}E_\alpha$. Then there exists α such that $K^n=E_\alpha$.*

We are now ready to prove the result that will allow us, in the next section, to work in the noetherian ring $K[x_{ij} : 1\le i\le n,\ 0\le j\le N_m]$ instead of the nonnoetherian $K[x_{ij} : 1\le i\le n,\ 0\le j]$.

Proposition 7.3. *The set $(A_\infty)_S$ is nonempty if and only if $(A_m)_S$ is nonempty for all $m\in\mathbb{Z}_{\ge 0}$.*

Proof. Since the constructible sets form a Boolean algebra, the nested sequence of constructible sets inside $(K^*)^{L_0}\simeq(\mathbb{V}_0)^*_S$,

$$\cdots\subset\pi_{(2,0)}((A_2)_S)\subset\pi_{(1,0)}((A_1)_S)\subset(A_0)_S\subset(K^*)^{L_0}, \tag{7-1}$$

induces an increasing family of constructible sets

$$\varnothing\subset(K^*)^{L_0}\backslash(A_0)_S\subset(K^*)^{L_0}\backslash\pi_{(1,0)}((A_1)_S)\subset(K^*)^{L_0}\backslash\pi_{(2,0)}((A_2)_S)\subset\cdots. \tag{7-2}$$

The set $\bigcap_{i=0}^{\infty}\pi_{(i,0)}((A_i)_S)$ is empty if and only if $(K^*)^{L_0}\setminus\bigcap_{i=0}^{\infty}\pi_{(i,0)}((A_i)_S)$ is $(K^*)^{L_0}$; that is, if and only if

$$(K^*)^{L_0}=\bigcup_{i=0}^{\infty}(K^*)^{L_0}\setminus\pi_{(i,0)}((A_i)_S).$$

Then, by Proposition 7.2, there exists m such that $(K^*)^{L_0}\setminus\pi_{(m,0)}((A_m)_S)=(K^*)^{L_0}$. That is, there exists m such that $(A_m)_S$ is empty.

The result follows from Remark 7.1. □

8. The fundamental theorem of differential tropical geometry

Theorem 8.1. *Let G be a differential ideal in $K[\![t]\!]\{x_1,\dots x_n\}$, where K is an uncountable algebraically closed field of characteristic zero. The equality*

$$\mathrm{Sol}(\mathrm{trop}(G))=\mathrm{trop}(\mathrm{Sol}(G))$$

holds.

Proof. The inclusion $\operatorname{trop}(\operatorname{Sol}(G)) \subset \operatorname{Sol}(\operatorname{trop}(G))$ is just Proposition 5.2. Here we will prove

$$\operatorname{Sol}(\operatorname{trop}(G)) \subset \operatorname{trop}(\operatorname{Sol}(G)).$$

Let $S = (S_1, \ldots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ be such that there is no solution of G whose tropicalization is S. We will show that S cannot be a solution of the tropicalization of G.

Suppose that $\operatorname{Sol}(G) = \bigcap_{\ell=1}^{s} \operatorname{Sol}(f_\ell)$, for some $f_1, \ldots f_s \in G$. For $1 \leq \ell \leq s$ and $k \in \mathbb{Z}_{\geq 0}$, we write $F_{\ell k} := (d^k f_\ell)|_{t=0}$.

As we have seen above, $S \notin \operatorname{trop}(\operatorname{Sol}(G))$ implies that $(A_\infty)_S$ is empty. Then, by Proposition 7.3 there exists $m \in \mathbb{N}$ such that $(A_m)_S$ is empty.

Take $m \in \mathbb{N}$ such that $(A_m)_S$ is empty. Set $\overline{F_{\ell k}}$ to be the image of $F_{\ell k}$ in the ring

$$K[x_{ij} \,:\, 1 \leq i \leq n,\ 0 \leq j \leq N_m]/\langle x_{ij} \,:\, j \notin S_i \rangle.$$

Since $(A_m)_S$ is empty we have

$$V(\overline{F_{\ell k}} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m) \subset V\Bigg(\prod_{\{0 \leq i \leq n,\ j \in S_i \,:\, j \leq N_m\}} x_{ij} \Bigg)$$

so by the Nullstellensatz, there exists $\alpha \geq 1$ such that

$$E_M = \Bigg(\prod_{\{0 \leq i \leq n,\ j \in S_i \,:\, j \leq N_m\}} x_{ij} \Bigg)^{\alpha} \in \langle \overline{F_{\ell k}} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m \rangle.$$

Here E_M is the differential monomial induced by the matrix $M \in \mathcal{M}_{n \times (N_m+1)}(\mathbb{Z}_{\geq 0})$ with entries $M_{ij} = 0$ for $j \notin S_i$ and $M_{ij} = \alpha$ for $j \in S_i$.

It follows that there exists

$$\{G_{\ell k} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m\} \subset K[x_{ij} \,:\, 1 \leq i \leq n,\ j \in S_i,\ j \leq N_m]$$

such that

$$\sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} \overline{F_{\ell k}} = E_M. \tag{8-1}$$

Then

$$\sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} F_{\ell k} = E_M + h \tag{8-2}$$

for some $h \in \langle x_{ij} : j \notin S_i,\ j \leq N_m \rangle \subset K[x_{ij} : 1 \leq i \leq n,\ 0 \leq j \leq N_m]$.

Now, by definition of $F_{\ell k}$, there exists λ in $K[\![t]\!]\{x_0, \ldots, x_n\}$ such that

$$g := \sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} d^k f_\ell = E_M + h + t\lambda. \tag{8-3}$$

Since G is a differential ideal and $f_1, \dots f_s \in G$, the differential polynomial g is in G.

We now have:

- By Remark 4.1, $\varepsilon_M(S) = 0$ and if $h \neq 0$, then $\operatorname{trop}(h)(S) \geq 1$.
- By Remark 5.1, if $t\lambda \neq 0$, then $\operatorname{trop}(t\lambda)(S) \geq 1$.

Thus, $(\operatorname{trop}(g))(S) = 0$ and the minimum is attained only at the monomial ε_M, and hence, S is not a solution of $\operatorname{trop}(g)$. So S is not a solution of the tropicalization of G, which is what we wanted to prove. □

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