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THE FURSTENBERG STRUCTURE THEOREM

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The Furstenberg structure theorem for minimal distal flows is proved without any countability assumptions. Thus let (X, T) be a distal flow with compact Hausdorff phase space X and phase group T. Then there exists an ordinal ν and a family of flows $(X_\alpha | \alpha \leqq \nu)$ such that X_0 is the one point flow, $X_\nu = X$, $X_{\alpha+1}$ is an almost periodic extension of X_α, and $X_\beta = \overleftarrow{\lim}_{\alpha<\beta} X_\gamma$ for all ordinals α and limit ordinals β less than or equal to ν.

0. **Introduction.** It has been fourteen years since Furstenberg proved his beautiful structure theorem for metrizable minimal distal flows [4]. Since then there have been many attempts to do without the assumption that the phase space of the flow be metrizable. These have only been partially successful; some sort of countability assumption has always seemed necessary. The purpose of this paper is to provide a proof of the Furstenberg structure theorem which avoids the use of any countability assumptions.

There are other structure theorems in the literature ([2], [3], [5]), and they too make some sort of countability assumption. Since all of these theorems are closely related it is to be hoped that the methods developed here can be applied to these others as well. Indeed the first step in the proof of the Veech structure is given in this paper (see 1.11).

The basic idea is embodied in 1.6 which states that given a topologically transitive or minimal flow (X, T), H a countable subgroup of T, and ρ a continuous pseudo-metric on X, there exists a countable subgroup $K \supset H$ with (X_K, K) topologically transitive or minimal respectively. Since K is countable, X_K is metrizable and category arguments may be used to obtain results about (X_K, K) which in turn are carried over to (X, T) by inverse limit arguments.

In the original version of this paper the basic idea was used in an entirely different way to obtain the Furstenberg theorem. I would like to thank the W. A. Veech for suggesting the present version, it is shorter and more transparent than the original.

Standing Notation 1.1. Throughout this paper (X, T) will denote a flow with compact Hausdorff phase space X and phase group T. If ρ is a pseudo-metric on X and H is a subgroup of T, then $R(H, \rho)$ or simply $R(H)$ will denote the subset, $\{(x, y) | \rho(xt, yt) = 0 (t \in H)\}$ of $X \times X$. The quotient space $X/R(H)$ will be denoted by X_H and $\pi_H: X \to X_H$, $\pi_H^K: X_K \to X_H$ will denote the cananical maps. (Here K is a subgroup of T containing H.)

LEMMA 1.2. *Let ρ be a continuous pseudo-metric on X and H a subgroup of T. Then* 1. *$R(H)$ is a closed H-invariant equivalence relation on X.* 2. *H acts on X_H.* 3. *If H is countable, X_H is metrizable.*

Proof. Statements 1 and 2 follow immediately from the definition of $R(H)$. With regards to 3 it is directly verifiable that $\sigma(a, b) = \sum_{i=1}^{\infty} 2^{-i}\rho(xt_i, yt_i)(a, b \in X_H)$ is a metric on X_H compatible with the quotient topology. Here $H = \{t_i \mid i = 1, \cdots\}$, $x, y \in X$ with $\pi_H(x) = a$ and $\pi_H(g) = b$.

DEFINITION 1.3. The flow (X, T) is *topologically transitive* if every nonnull invariant open set is dense.

REMARK 1.4. When X is metrizable, then (X, T) is topologically transitive if and only if the set of points with dense orbit is residual. (To see this consider $\cap \{UT | U \in \mathscr{U}\}$ where $\mathscr{U}$ is a countable base for the topology on X.)

LEMMA 1.5. *Let $\mathscr{S}$ be a collection of subgroups of T directed by inclusion* (i.e., *for every pair H, K of element of $\mathscr{S}$ there exists $L \in \mathscr{S}$ with $H \cup K \subset L$*) *and let ρ be a continuous pseudo-metric on X. Then* 1. *$X_S = \varprojlim (X_H, \pi_K^H)$ where $S = \bigcup \mathscr{S}$.* 2. *If the flows (X_H, H) are minimal $(H \in \mathscr{S})$ then so is (X_S, S).*

Proof. 1. Since $\pi_K^S = \pi_K^H \circ \pi_H^S(H, K \in \mathscr{S}$ with $K \subset H)$, $a = b$ if and only if $\pi_K^S(a) = \pi_K^S(b)(K \in \mathscr{S})$, $(a, b \in X_S)$.

2. Let $\cup$ be a nonvacuous open subset of X_S. Then there exist $H \in \mathscr{S}$ and a nonvacuous open subset V of X_H with $\pi_H^{-1}(V) \subset \cup$. Since (X_H, H) is minimal, $VH = X_H$. Hence $\cup S \supset (\pi_H^S)^{-1}(V)S \supset (\pi_H^S)^{-1}(V)H = (\pi_H^S)^{-1}(VH) = X_S$.

PROPOSITION 1.6. *Let (X, T) be topologically transitive (respectively minimal), ρ a continuous pseudo-metric on X, and H a countable subgroup of T. Then there exists a countable subgroup K of T such that $H \subset K$ and (X_K, K) is topologically transitive (respectively minimal).*

Proof. Assume (X, T) is topologically transitive. Set $H_0 = H$ and $\mathscr{U}_0$ a countable basis for the topology on $X_0 = X/R(H_0)$. Then $V_0 = \cap \{\pi_0^{-1}(\cup)T / \cup \in \mathscr{U}_0\}$ is a residual subset of X. (Here $\pi_0 \colon X \to X_0$ is the canonical map.)

Let $x_0 \in V_0$. Then $\pi_0(x_0T) \cap U \neq \emptyset (U \in \mathscr{U}_0)$. Hence there exists a countable subgroup H_1 with $H_0 \subset H_1$ and $\pi_0(x_0H_1)$ dense in X_0. Iterate the above procedure to obtain a sequence of points (x_n) of X and an increasing sequence of countable subgroups (H_n) of T such that $\pi_n(x_nH_{n+1})$ is dense in $X_n = X/R(H_n)$ (all n).

Set $K = \bigcup H_n$ and let V_1 and V_2 be two nonvacuous open subsets of X_K. Since $X_K = \overleftarrow{\lim} X_n$, there exist n and two nonvacuous open subsets U_1 and U_2 of X_n such that $\varphi^{-1}(U_i) \subset V_i (i = 1, 2)$ where $\varphi: X/R(K) \to X_n$ is the canonical map. By construction there exist $h_1, h_2 \in H_{n+1} \subset K$ with $\pi_n(x_nh_i) \in U_i (i = 1, 2)$. Since $\pi_n = \varphi \circ \pi_K$, $\pi_K(x_n)h_i = \pi_K(x_nh_i) \in \varphi^{-1}(U_i) \subset V_i (i = 1, 2)$. Thus $V_1K \cap V_2 \neq \emptyset$ and so (X_K, K) is topologically transitive as desired.

In the minimal case $\pi_0^{-1}(U)T = X (U \in \mathscr{U}_0)$ and thus since X is compact and $\mathscr{U}_0$ countable, there exists a countable subgroup H_1 of T with $H_0 \subset H_1$ and $\pi_0^{-1}(U)H_1 = X (U \in \mathscr{U}_0)$. (This implies that $\pi_0^{-1}(U)H_1 = X$ for all nonvacuous open subsets of X_0.)

Iteration now produces on increasing sequence (H_n) of countable subgroups of T such that $\pi_n^{-1}(U)H_{n+1} = X$ ($\emptyset \neq U$ open in X_n, all n). Set $K = \bigcup H_n$ and let $\emptyset \neq V$ be open in X_K. Then there exist n and a nonvacuous open subset U of X_n with $\varphi^{-1}(U) \subset V$. ($\varphi: X_K \to X_n$, canonical.) Then $X = \pi_n^{-1}(U)H_{n+1} = (\pi_K^{-1}\varphi^{-1}(U))H_{n+1} \subset \pi_K^{-1}(V)K = \pi_K^{-1}(VK)$ whence $VK = \pi_K(X) = X_K$.

The proof is completed.

REMARK 1.7. Let (X, T) be minimal, $(X \times X, T)$ topologically transitive, ρ a continuous pseudo-metric on X, and H a countable subgroup of T. Then it is clear from the above that one can find a countable subgroup K containing H with (X_K, K) minimal and $(X_K \times X_K, K)$ topologically transitive.

DEFINITION 1.8. Let $x, y \in X$. Then x and y are *distal from one another* if there exists a continuous pseudo-metric ρ on X and $\varepsilon > 0$ such that $\rho(xt, yt) > \varepsilon (t \in T)$. The point x is a *distal point* if x and y are distal $(y \in X, y \neq x)$. The flow (X, T) is *point distal* if it has a distal point and *distal* if every point is a distal point.

PROPOSITION 1.9. *Let (X, T) be distal and topologically transitive. Then (X, T) is minimal.*

Proof. Let ρ be a continuous pseudo-metric on X and $\mathscr{S}$ the collection of countable subgroups H of T such that (X_H, H) is topologically transitive. Then by 1.7 $\mathscr{S}$ is directed by inclusion and $\cup \mathscr{S} = T$.

Let $H \in \mathscr{S}$. Then the canonical map π_H is a homomorphism of the flow (X, H) onto the flow (X_H, H), whence the latter is distal. Since (X_H, H) is metrizable and topologically transitive, it has a point with dense orbit whence it is minimal. Consequently $(X/R(T, \rho), T)$ is minimal by 1.5.

Now let $\mathscr{P}$ be the collection of continuous pseudo-metrics on X directed by $\leqq$. Then $(X, T) = \varprojlim_{\mathscr{P}} (X/R(T, \rho), T)$ from which it follows that (X, T) is minimal.

We are now in a position to prove the Furstenberg structure theorem without any countability assumptions. To this end it is evident from [4] or [1] that it suffices to prove the following:

PROPOSITION 1.10. *Let (X, T) be minimal distal and let $\varphi: (X, T) \to (Y, T)$ be an epimorphism which is not one-one. Then there exists a homomorphic image (Z, T) of (X, T) which in turn is a nontrivial almost periodic extension of (Y, T).*

Proof. Assume no such flow (Z, T) exists. This implies that the relation $R(\varphi)$ induced by φ coincides with the relativized equicontinuous structure relation $S(\varphi)$.

Now $R(\varphi) = \{(x_1, x_2) \mid \varphi x_1 = \varphi x_2\} \subset X \times X$ is a closed invariant subset of $X \times X$, and so we have a flow $(R(\varphi), T)$. Since (X, T) is distal so is $(R(\varphi), T)$. Consequently every point of $R(\varphi)$ is an almost periodic point of $(R(\varphi), T)$. This and the fact that that $R(\varphi) = S(\varphi)$ allow us to conclude that $(R(\varphi), T)$ is topologically transitive [6, Th. 2.6.3]. This implies that $(R(\varphi), T)$ is minimal (1.9) whence $R(\varphi) = \Delta$ the diagonal of $X \times X$, a contradiction. (φ was assumed not to be one-one.) The proof is completed.

The final result is the first step in the proof of the Veech structure theorem [2] without any countability assumptions.

PROPOSITION 1.11. *Let (X, T) be a nontrivial minimal point distal flow. Then it has a nontrivial equicontinuous factor.*

Proof. Assume the conclusion false and let x_0 be a distal point of X. Then $X \times X$ is the equicontinuous structure relation and the set $\{(x_0 s, x_0 t) \mid s, t \in T\}$ is dense in $X \times X$ and consists entirely of almost periodic points of the flow $(X \times X, T)$. Hence we may again apply [6, Th. 2.6.3] to conclude that $(X \times X, T)$ is topologically transitive.

Now let H be a countable subgroup of T and ρ a continuous pseudo-metric on X. Then by 1.7 there exists a countable subgroup K of T such that $H \subset K$, (X_K, K) is minimal, and $(X_K \times X_K, K)$ is topologically transitive.

The flow (X_K, K) is metrizable and point distal. $(\pi_K(x_0)$ is a distal point.) If it were not trivial, it would have a nontrivial equicontinuous factor (Y, K) by the metric version of the Veech structure theorem. Since $(X_K \times X_K, K)$ is topologically transitive, so is $(Y \times Y, K)$. This implies that Y must be a single point, whence (X_K, K) is trivial. This implies that X_H is a single point and so (X, T) is trivial (since H and ρ were arbitrary); a contradiction.

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