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The futures of transformations and metamaterials

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Abstract

Those researchers who are part of the metamaterials community stand at a fork in the road – does the maturation of metamaterial fabrication lead to a focus on applications and technology, or does it suggest an opportunity to pursue more blue-sky scientific concepts? At Metamaterials 2013 in Bordeuax, one speaker focussed explicitly on the opportunities for applications and funding on the road leading to metamaterial technology. Here, in deliberate contrast, we look instead at the interesting opportunities in curiosity-led research based around the ideas of transformation and metamaterials. The genesis of this article was the Transforming Transformation Optics meeting held at Imperial College London in December 2013.

Keywords: Discussion; Futures; Metamaterials; Transformation

1. Introduction

The discussions in this article were prompted by two events – first, the successful session at Meta 13 in Bordeaux with the same "future" theme, and, second, by the self-imposed task of running an alternate futures discussion session [1] at the "Transforming Transformation Optics" one-day meeting organized by us in 17th December 2013 (Fig. 1).

As theorists, although the points covered and discussed in Bordeaux were interesting and relevant, the very specific focus on converting transformation concepts and metamaterial ideas rather directly into applications and technology seemed somewhat limiting. Surely transformation-based theories such as transformation optics or transformation acoustics are not completely played out as far as conceptual advance goes? Indeed one of our primary interests is to merge many or all types of such theories into a unified "transformation media" theory – at least to the extent to which it is possible [2,3]. Notably, we can at least understand the design specification – the "Transformation Design" (T-Design) – of a transformation device (T-device) such as a cloak independently of the physical implementation in T-Optics (i.e. in electromagnetism, EM) or in T-Acoustics (i.e. in acoustics).

Of course, as co-creators of the concept of spacetime transformation design [3–5], we may be predisposed to a particular view, but perhaps surprisingly, the spacetime transformation concept has had rather minimal attention from other workers so far. This is despite three rather nice nonlinear optics experiments [6–8] but the concept is scarcely overdue for being subsumed into engineering. That said, as the Cornell experiment showed [6], pre-existing time lens devices based on dispersion are ideal for implementing spacetime T-devices.

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Fig. 1. Participants at the December 2013 meeting *Transforming Transformation Optics*.

Here, as a deliberate counterpoint to a current emphasis on applications, we discuss a range of other new alternative concepts that will typically need further research investigation before they develop into the status of applications. Of course the target audience of this journal *would* probably be most interested in those applications and experimental systems; and the summary of "obvious futures" in Section 2 would therefore seem most relevant. However, it is important to keep an open eye on other developments, either potential or in-progress, because of the opportunities they present. Perhaps considering the limitations of the "coordinate transformation" view of T-Optics, as discussed in Section 3, will provide a clearer intuition; or taking a bird's eye, more abstract view of transformations (Section 7) may serve the same purpose. Would you have expected nonlinear optics to be the first research field that experimentally demonstrated the T-Optics spacetime cloak? (Section 4). What other kinds of geodesic or conformal T-devices might be mooted following the summary in Section 5? – consider the strong current interest in surface wave and other 2D devices. What thought-provoking systems might we make with hyperbolic systems (Section 6) while intending to mimic spacetime like behaviour in a purely spatial system? Can we really turn such ideas into realistic metamaterial constructs without the necessary design compromises confusing the essential idea? These are important questions, and it is only when the more experimental side of the T-Optics research community come to grips with the theoretical proposals, will we know the answers.

2. Obvious futures

One might explore some of the more obvious future directions for metamaterial and transformation work simply by first selecting a recent review article [9]. We can then envisage improving the devices therein – making them work in the visible spectrum, improving the bandwidth covered, designing smaller, better, or otherwise enhanced – perhaps even chiral [10] – meta-atoms. Theoretical schemes for numerical simulation [11,12] or for homogenization [13–16] might be improved, either by increment or step advances. More investigation could be done of multifunction "Janus" devices [17], the transformation of fields, boundaries, and impedances [18–22], Poynting vectors [23], or sources [24]. Re-imagining T-Optics as a generalization of Snell's Law [25] also shows the way to provide updated intuition, as does improvement in publicity and outreach efforts [26].

Another straightforward future concept centres around the potential for nonlinearity in T-Design. If using standard T-Design techniques, nonlinearity is hard to handle in all but rather abstract ways [27]. Nevertheless, it can be put to service rather more simply as a way of controlling standard T-devices by utilizing the nonlinearity as a switching or modulation mechanism – e.g. the optical fibre scheme based on speed control by means of nonlinear refractive index, as originally proposed for the spacetime cloak [4]. Of course, the issue of EM propagation techniques in metamaterials has been addressed in a substantive way – if finite difference time domain (FDTD) methods [28,29] are too detailed, then efficient uni-directional methods already exist for linear and nonlinear field and pulse propagation [30,31].

Field and pulse propagation is at the heart of very much of the work undertaken in the nonlinear optics community; however they typically use the existence of metamaterials as a justification to explore its traditional interest in pulse propagation with unusual material parameters. It is not uncommon to see publications on pulses and solitons and their variants in negative refractive index waveguides, for example; there seems rather less interest in transformation designs and metamaterials implemented as optical systems in their own right. Further, in acoustics nonlinearity is fairly ubiquitous and unavoidable [32], *something* needs to be achieved [33,34] to allow nonlinearity to be handled easily by T-Design processes. But, given the theoretical complications even in simple cases [27], the pertinent questions are very much what systems are treatable, and what interesting or practical results might actually be achieved?

One of the T-futures under active development centres on dimensionality, curvature, and geometry [35]. There has been much recent activity on metasurfaces (see e.g. [22,36–38]) and metawires (see e.g. [39–41]), although largely as a result of the community realizing the significant challenges involved in building actual *volume* metamaterials. Metasurfaces, for example, are ideal for setting interesting boundary conditions or acting as flat lenses. The recent results on surface wave transformation [42–44], including the transmutation of singularities [45,46] show great promise. Inspired by geodesic lenses, we expect these will spur many interesting developments in the use of T-Design on flat and curved surfaces. Metawires can often simply be envisaged as metamaterial waveguides, a topic which is of interest to the nonlinear optics and microwave communities[1] (see e.g. [47,48]). Of course, an alternate take on the “wire” geometry is to coil an ordinary guide up to increase the effective path length [39,40,49], as in the sound-slowing labyrinth meta-units of Frenzel et al. [40].

However, whilst reducing dimensionality can simplify discussions in T-Design, there are certain dimensionality constraints that cannot be avoided. Cloaking, for example, intrinsically requires at least two dimensions (be they space, or space plus time). As such, foundational issues, as discussed in the following section, appear to be better addressed in more abstract terms without direct or immediate consideration of dimensionality.

One significant criterion applied to metamaterials, and both their uses in negative index and T-devices is that of causality [50,51] Typically, debates regarding this are settled using the frequency domain Kramers Kronig relations [52,53], although it has been pointed out that it makes more sense to consider causality in its native time domain [54]. However, if we consider probing a metamaterial system in the ultrafast regime, the Kramers Kronig time-only conception of causality will fail, and must be replaced [54]. Further, in such a regime, the necessarily non-local notion of spatial dispersion fails [55,56], since only information about the past light cone can affect local behaviour. This means that we can no longer rely on the extended spatial layout of a metamaterial lattice to provide interesting properties, but only its *known* past.

As we can see there are many existing avenues for useful and significant work in the research and application of metamaterial and T-Optics/Acoustics. Nevertheless, alternative routes exist to be followed, and we discuss these in the sections that follow.

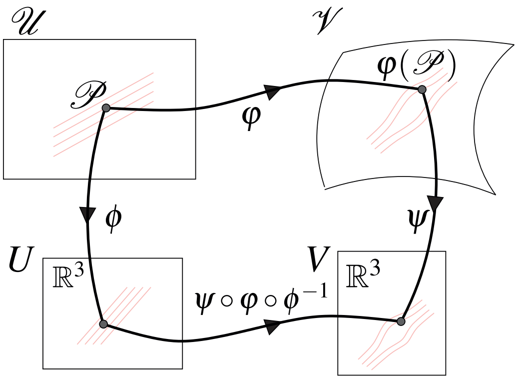

Fig. 2. (Diffeo)-morphism. Points $\mathcal{P}$ in the reference manifold $\mathcal{U}$ are mapped to points $\varphi(\mathcal{P})$ in the T-device manifold $\mathcal{V}$ by the morphism (mapping) φ. The $\mathbb{R}^3$ coordinate representation of the morphism, is $\psi \circ \varphi \circ \phi^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$. This means that the *design* of the T-device is specified by φ, but that expression of that design in coordinate terms – the “blueprint” – is $\psi \circ \varphi \circ \phi^{-1}$ in $\mathbb{R}^3$.

3. Foundational futures

One significant task is to directly address deficiencies resulting from the various mathematical shortcuts used in early T-Design. We often hear people speak of T-Optics, T-Acoustics, etc., in the language of *coordinate* transformations, although mathematically this is incorrect. Any physical idea must be expressible in a way that is independent of coordinates, and the mathematical underpinning of all transformation theories is the idea of a *morphism*, φ, that maps a point $\mathcal{P}$ in a space $\mathcal{U}$ to another point $\varphi(\mathcal{P})$ in a space $\mathcal{V}$, as seen in Fig. 2. Coordinate maps $\phi : \mathcal{U} \to \mathbb{R}^3$ at $\mathcal{P}$, and $\psi : \mathcal{V} \to \mathbb{R}^3$ at $\varphi(\mathcal{P})$, give the coordinate representation of φ as

$$\psi \circ \varphi \circ \phi^{-1} : \mathbb{R}^3 \to \mathbb{R}^3. \tag{1}$$

However, in T-Optics, φ prescribes the electromagnetic medium in $\mathcal{V}$ given (typically) an initial vacuum $\mathcal{U}$, before any discussion of coordinates [57,58].

The main problem with the coordinate based view of T-Design can be made concrete by considering the conversion between Cartesian and polar coordinates. If for example ψ is a Cartesian coordinate map, and ϕ is a polar coordinate map, then Eq. (1) will give a mixed representation of the morphism φ. This mixed representation cannot be realized as a T-Design; the two coordinate systems are dimensioned differently, and setting φ to the identity does not yield the identity map, but rather the polar-to-Cartesian coordinate conversion at a point.

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[1] Note that very many other interesting articles can be located simply by using the phrase “metamaterial waveguide” as a search term.

Moreover, the behaviour of the geodesics in $\mathcal{V}$ as projected into the $\mathbb{R}^3$ space V can look quite distinct when compared to those from $\mathcal{U}$ as projected into their $\mathbb{R}^3$ space U.

Indeed, it is the difference in *apparent* curvature between trajectories in V and U that is the core property of a T-Design programme. Transformations or deformations that do not induce or otherwise alter the apparent curvature[2] are simply dilations, translations, rotations and boosts, i.e. consist only of those in the Poincaré group. These simple transformations cannot be seen as designing new devices, as they are straightforward re-representations of an existing situation. Consequently, we are led to assert the following definition:

> ***TransformationDesign*** is the process by which a non-Poincaré deformation (i.e. a deformation that is not an isometry, such as one whose properties change with position) is applied in order to obtain some desired device behaviour.

The motivation for this specification is in part the argument that spacetime transformations embrace the work on the artificial horizons [59–61] which form as a result of Lorentz boosting a medium characterized by refractive index n, by a speed $v > c/n$. However, this does not embrace the idea of a *deformation*, so we consider them distinct from transformation devices such as cloaks, which require non-trivial diffeomorphisms. Further, taking a metric "off the shelf" for use, even one as interesting as the Schwarzchild metric [62,63], does not meet a strict interpretation of T-*Design* either; it is simply an implementation.

Nevertheless, although some have alluded to the similarity of the machinery of General Relativity in carrying out T-Optics (e.g. [62]), few have explicitly used the machinery of *curvature* in actually designing metamaterials. The geometry of the manifold is specified by its covariant derivative as characterized by the connection coefficients $\Gamma^{\alpha}_{\beta_{\gamma}}$. In Cartesian coordinates these numbers vanish everywhere *because* Euclidean space is flat. Under the action of the morphism, however, new connection coefficients are induced according to

$$\tilde{\Gamma}^{\rho}_{\mu_{\nu}} = \left(\frac{\partial y^{\tilde{\rho}}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial y^{\tilde{\mu}}} \frac{\partial x^{\gamma}}{\partial y^{\tilde{\nu}}} \right) \Gamma^{\alpha}_{\gamma\beta} + \left(\frac{\partial^2 x^{\alpha}}{\partial y^{\tilde{\mu}} \partial y^{\tilde{\nu}}} \right) \frac{\partial y^{\tilde{\rho}}}{\partial x^{\alpha}} \quad , \tag{2}$$

where x^{α} are the coordinates of $\mathcal{P}$ in $\mathcal{U}$, and $y^{\tilde{\rho}}$ are the coordinates of $\varphi(\mathcal{P})$ in $\mathcal{V}$, and the summation convention has been used[3]. In general there is no global coordinate system in $\mathcal{V}$ that will make the $\tilde{\Gamma}^{\rho}_{\mu_{\nu}}$ vanish everywhere, and this failure is quantified by the Riemann curvature tensor whose coordinate components in $\mathcal{V}$ are given by

$$R^{\tilde{\rho}}_{\tilde{\mu}\tilde{\nu}\tilde{\lambda}} = \frac{\partial \Gamma^{\tilde{\rho}}_{\tilde{\mu}\tilde{\lambda}}}{\partial y^{\tilde{\nu}}} - \frac{\partial \Gamma^{\tilde{\rho}}_{\tilde{\mu}\tilde{\nu}}}{\partial y^{\tilde{\lambda}}} + \Gamma^{\tilde{\tau}}_{\tilde{\mu}\tilde{\lambda}} \Gamma^{\tilde{\rho}}_{\tilde{\tau}\tilde{\nu}} - \Gamma^{\tilde{\tau}}_{\tilde{\mu}\tilde{\nu}} \Gamma^{\tilde{\rho}}_{\tilde{\tau}\tilde{\lambda}}. \tag{3}$$

Of course, we know how a given morphism induces electromagnetic properties: for purely spatial transformation it is usually expressed using the Jacobian, so that

$$\boldsymbol{\epsilon} = \boldsymbol{\mu} = \frac{\mathbf{J}\mathbf{J}^T}{|\mathbf{J}|}, \quad \text{where} \quad J_{ij} = \frac{\partial x^i}{\partial x^j} = J^T_{ji}; \tag{4}$$

where as in spacetime we refer to how the constitutive tensor changes under a transformation L so that we have [4]

$$\chi^{\alpha'\beta'\mu'\nu'} = \Delta^{-1} L^{\alpha'}_{\alpha} L^{\beta'}_{\beta} L^{\mu'}_{\mu} L^{\nu'}_{\nu} \chi^{\alpha\beta\mu\nu}, \tag{5}$$

$$L^{\alpha'}_{\alpha} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}, \quad \text{and} \quad \Delta = \left| L^{\alpha'}_{\alpha} \right|. \tag{6}$$

However, it is also clear that we can now use Eq. (3) above as a way to relate electromagnetic properties to *curvature* rather than simply for a displacement of waves or rays.

It would be nice to go 'beyond cloaks' in T-Design. Although a cloak is the most dramatic example of field repositioning, there may be many other useful deformations. A natural subdivision of T-Design might be summarized using the following list of morphisms ("transformations") –

(a) morphisms that introduce new regions of space from which the field is excluded (i.e. $\mathcal{V} \supset \mathcal{U}$)),
(b) morphisms that excise certain regions of space (i.e. $\mathcal{V} \subset \mathcal{U}$, and
(c) morphisms that do neither (a) or (b) (i.e. $\mathcal{V} \sim \mathcal{U}$).

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[2] Note that the true mathematical curvature of $\mathcal{U}$ and $\mathcal{V}$ as given by the Riemann curvature tensor must be identical since curvature cannot be induced by a diffeomorphism. However, in $\mathbb{R}^3$ or other coordinate projection, the apparent metric is that of flat space. This means that morphed trajectories in V which approach or diverge from one another can imply an apparent curvature – regardless of the lack of any curvature in terms of the true metric on the original manifold $\mathcal{V}$.

[3] It is perhaps worth pointing out that the connection coefficients appearing in Eq. (2) are not the components of a tensor. They rather chart the twisting of coordinate bases throughout the manifold, and set the standard by which vectors are parallel transported between neighbouring points.

Cloaking is the canonical example of (a), and (b), the inverse of (a), may be thought of as a kind of black hole. What about (c)? Although some suggestions have been made in T-Design (e.g. [64,65]), there still seems to scope for designing new materials based on smooth deformations of space that neither avoid nor get sucked into certain regions. A example of such a medium that we have studied recently is a twisted medium [66] defined through the deformation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cos Kz & \sin Kz & 0 \\ -\sin Kz & \cos Kz & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \qquad (7)$$

The medium that results from applying the T-Design algorithm to the above morphism is one that is structurally chiral. Axial rays are deformed to helical paths and for T-Optics, polarizations are continuously rotated. However, the chirality here is quite distinct from other types of chirality and optical activity in optics, such as that in helicoidal birefringent media (e.g. [67–70]). Notably, the twisted transformation of Eq. (7) results in a medium in which light *rays* are helical.

The possibility to create a chiral medium using T-Design theory suggests a plausible inverse: can conventional chirality (e.g. optical activity) be described as a transformation? Certainly it might be done straightforwardly as a field transformation [20], but is there another strategy? Since optical activity involves magneto-electric terms in the constitutive tensor, if it is to be generated via a transformation, it will have to be one that involves space and time. However, it has been suggested [71] that it is not possible to have an isotropic magneto-electric medium where the magneto-electricity is generated via a transformation.

As a final comment, one might question whether the more secure mathematical foundations of the approach described above has any significantly greater practical utility compared to the original "coordinate transformation" conception. This is a good question – in many ways the coordinate transformation method is just as good at defining the design of most T-devices as is our morphism one. To a great extent, both are just ways of describing how we want to push, deform, or transform points and regions around in space, whilst demanding that our wave or other quantity also follows that deformation. However, in T-Designs where the original reference space is *folded up* as it is projected down onto its new location(s), the manifold/morphism approach is considerably clearer (see Fig. 3). Such examples occur both in some conformally designed T-devices (see Section 5), as well as in other cases. Notably ad hoc constructions like the "accelerating wave" spacetime cloak of Chremmos [72] as much more easily treated. In this and similar scenarios coordinate descriptions are prone to singularities and issues of coordinate uniqueness.

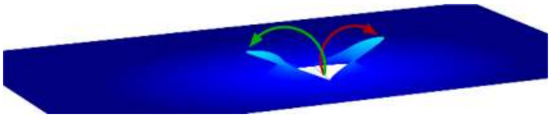

Fig. 3. A 3D depiction of a morphism intended for a 2D space creating a hole by "pulling back" the manifold away from an initial slit, as if it were a stretchy rubber sheet. Here, in any planar projection of the manifold some points in the physical plane correspond to two points on the manifold. While this might be difficult to realise physically, since the degenerate points need to discriminate between different rays, asserting directional initial conditions, and adding a relative twist between "upper" and "lower" degenerate points can achieve this – see e.g. the scheme of Chremmos [72].

4. Spacetime futures

Prior to 2011, all of the Transformation Design concepts in the literature considered only the spatial coordinates, although there may have been some consideration of on/off control of those spatial T-devices. This means that with the publication of the spacetime cloak or "history editor" [4], a whole new arena for T-Design has opened up. Even after some years this is largely unexplored as a theoretical concept, although recently we have applied spacetime transformations to a simple acoustics model and ground-plane/carpet cloaking [3], and discussed its use in a more general sense [5]; Others have also considered it in the context of an analog transformation [73] method.

Although it can be difficult to think of what kind of spacetime T-devices might find uses in technological applications, some experimental work on spacetime cloaking has been done [6–8]. It is also possible to imagine computational implementations of spacetime transformation ideas [5], as can be seen in Fig. 4. Further, since a longer discussion on possible future directions for spacetime T-devices – i.e. more than just cloaks – has been recently published [5], we will not repeat that content here.

Simulations of spacetime cloaking also present new challenges, since they require – for the less approximate implementations – exotic cross-field couplings [3]. In the case of EM spacetime cloaks, the necessary magnetoelectric effects have been robustly implemented [12]; and perhaps also there are useful and extant techniques for acoustic cloaks, since the handling of fluid flow is not uncommon in acoustics. Nevertheless, the specific fine tuning necessary for spacetime cloak implementation is

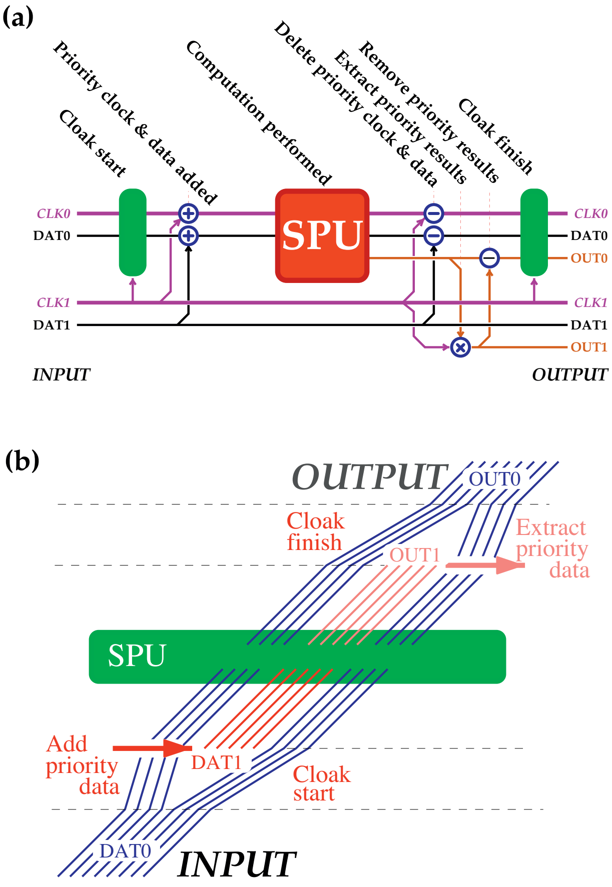

Fig. 4. A spacetime cloak re-imagined in the top panel as a digital signal processing circuit. The "background" computation, controlled by clock CLK0 and processing data DAT0, is modified by the insertion of a high-priority ("cloaked") computation on lines CLK1 & DAT1. After the signal processing unit (SPU) has processed the incoming data, the output has the extraneous CLK1 & DAT1 signals deleted, and the priority results diverted to OUT1. Lastly, the CLK0, DAT0, and OUT0 lines have their original timings restored and apparently pristine; however the priority results on OUT1 are available for use. The shifts in the signal timings are indicated in the lower panel, showing how the interior of the cloak is used to perform the priority computations.

still largely unexplored and likely requires custom coding: e.g. one cannot (yet) straightforwardly implement the necessary magneto-electric effects directly in the public domain Meep FDTD simulator [29].

One cautionary note to consider is that switching cloak properties on and off does not constitute a spacetime cloaking scheme. Although such a switchable device is time dependent, the temporal operation does not derive from a transformation. Further, and on a more general note, one should be careful to distinguish between cloaking (in the T-Optics sense) and camouflage. For example, one recently proposed scheme [74] is designed to hide or reveal data using polarization manipulation; and it does so after after establishing itself in a context of spacetime cloaking [4,6,7]. However, strictly speaking the scheme does not cloak, but instead *camouflages* data by changing its polarization, and the effect is switched on and off, rather than the being speed being modulated to create a hidden "shadow" region in which events can be (spacetime) cloaked.

It is also possible to design spacetime cloaks heuristically, a notable example being the accelerating wave packet approach [72]. In that work – in a ray interpretation – parts of the beam cross over each other. If treated as a transformation, this mechanism would require a transformation that in-effect projected multiple points on the manifold down to single spacetime points, as if the manifold were folded over onto itself.

5. Conformal futures

Conformal mapping already has a long association with T-Optics and related approaches, as we can see from its use in analysing curved waveguides as far back as 1975 [75]. More recently, Ulf Leonhardt re-publicized the concept to the transformation community with his use of conformal mapping in a range of devices, most notably the conformal cloak [76]. Conformal transformations have a very nice interpretation in terms of the refractive index of geometrical optics. Since conformal maps preserve angles, all that can change is the (local) measure of distance. Conformal transformations can therefore be described as replacing vacuum with a space filled with an inhomogeneous refractive index distribution $n(\vec{r})$. Alternatively, in a more abstract approach, we can say that they transform the vacuum metric $g \rightarrow n^2 g$.

However, although the conformal approach has the advantage of an implementation that consists just of a refractive index (speed) profile, it often comes with the disadvantage of working for only a single incident direction. One might, therefore, regard conformal mapping as a pre-cursor to the true – i.e. in-principle exact and all-angle – T-Optics/T-Design in the sense the research community generally uses and applies it now, and as introduced with the electromagnetic cloak [77]. Nevertheless, some conformally mapped device designs *are* direction insensitive, such as geodesic lenses [78]; these have a remarkably diverse range of applications in fields well outside optics, especially in microwave and radar [79,80].

The idea of conformal mapping has a rather pleasing simplicity – define a complex number $z = x + \imath y$ covering a flat planar region spanned by Cartesian coordinates (x, y). Then, a complex analytic expression $w = f(z)$ is used to implement the desired T-device. This procedure

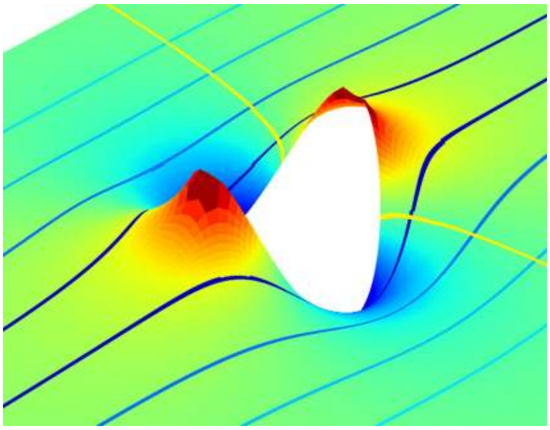

Fig. 5. Conformal cloak, showing the refractive index profile and ray deviation in the cloaked x direction. Heights and colours are coded according to the negative of the deviation of the local refractive index n from the background value. This gives the visual effect of a "landscape" through which rays travel, being pulled towards valleys (high n) and pushed away from hills (low n). Eight cloaked rays are shown, as well as a pair of uncloaked (perpendicular) rays. These are coded by colour according to the maximum n they reach, but with a colour offset to aid visibility. Any ray not parallel to the x axis at large distances will not be cloaked, as can be deduced from the lack of rotational symmetry in the index profile. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

was followed by Leonhardt when designing the original conformal cloak, which used the transformation $w = z + a/z$ to get the refractive index profile $n = |1 - a^2/z^2|$. The deformation of the parallel (cloaked) ray trajectories, the uncloaked perpendicular trajectories, and an indication of the index profile is shown in Fig. 5.

As already mentioned, whilst conformal mapping can return a T-device requiring only a customized speed profile, it nevertheless contains one rather brutal restriction. It is restricted by Liouville's Theorem, which only admits nontrivial operations in 2D – in 3D or higher, only uninteresting translations, rotations, and scalings are possible. However, at least in 2D we have the Riemann mapping theorem to tell us that all simply connected planar domains are conformally equivalent – and so *any* device becomes mathematically possible. Thus the only impediment is practical – how can we work out a conformal map that does what we want? In practice, there is a considerable difficulty in finding an analytic function that not only matches seamlessly to the outside world, but also implements a desired T-device. Nevertheless, some workers continue to achieve interesting results [81].

One area where conformal mapping can be useful – but is not required – is in the use of geodesic lenses. Again, this is an area publicized, albeit perhaps inadvertently in this case, by Leonhardt's evolving claims of "perfect imaging" in the Maxwell Fisheye lens [82–85]. Despite a perhaps inauspicious start, interesting results have been produced recently – theoretical results on generalized geodesic lenses [86,85], a transformation aquatics Maxwell's Fishpond [87] based on waves in shallow water, as well as surface T-devices [43,44,46]. Further, the so-called "perfect drain" required by the Fisheye in order to simulate perfect imaging has led directly to valuable discussions on imaging [88–90], attempts at engineering drain-like boundary conditions [91,92], and what you can – and cannot – expect to achieve in a causal manner [93].

We can therefore usefully advance the question: Does conformal mapping have – or need – future development in the field of T-Design ? We can stretch Liouville's straitjacket to an extent by resorting to quasi-conformal mapping [94,95], but is there anything else? For example, Liouville's Theorem is not guaranteed to hold in non-Euclidean (curved) spaces, so that may be a means of escape. However, we should probably not expect to avoid the restrictions of Liouville's Theorem simply by deciding – as in the Maxwell's fisheye – to use an n-sphere as our starting space. *Especially* since the sphere can already be conformally mapped to the plane.

Perhaps the main point to note is that since the 2D freedom allowed for conformal mapping is already useful enough, trying too hard to force conformal mapping to be useful in 3D when other techniques are available (see Section 3) may not be a good use of research effort.

6. Exotic futures

Perhaps the most exotic T-Design inspired ideas are those with a cosmological, general relativistic, or quantum aspect. Some work which has been combining these themes for some time is the attempt to make laboratory analogs of Hawking radiation and event horizons; either in optics [96–98], or using fluid flow and water waves [99]. More straightforward black hole devices have also been of interest – mostly being attempts at a qualitative representation, but some have made an attempt at designing a more direct comparator [62]. There are even early suggestions for using metamaterials to simulate a spinning cosmic string [100] or de Sitter spacetimes [101,102]. Sadly, although not unexpectedly, attempts to find negative refraction in curved spacetimes necessarily failed, and were comprehensively demolished [103].

In one proposal, Smolyaninov suggested a scheme using systems with ordinary dispersion along one spatial axis, but hyperbolic along another. In the operating regime around which this criterion holds the "hyperbolic axis" behaves like a time coordinate laid out in space, thus a 2D spatial region can act like a 1+1D spacetime

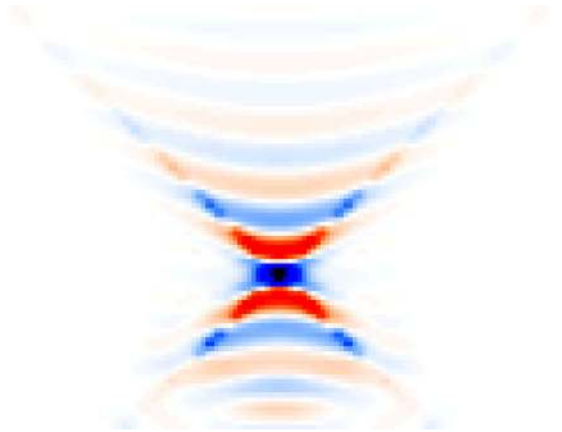

Fig. 6. Field pattern for a 2D geometry where the x (horizontal) direction has a hyperbolic dispersion. This figure is generated using a time-domain FDTD simulation [29] run until it reaches the steady state. The material dispersion was constructed by means of a fixed high-contrast birefringent background (with $\epsilon_{y0}/\epsilon_0 = 3$, and $\epsilon_{x0}/\epsilon_0 = 1$) overlaid with a single Lorentzian oscillator tuned to give $\delta\epsilon/\epsilon_0 = -2$; so that the operating permittivity values were $\epsilon_y/\epsilon_0 = 1$ and $\epsilon_x/\epsilon_0 = -1$. Although the light cone like structure did indeed show up in our simulation results, it nevertheless required some careful rescaling and normalization of the data to display it clearly.

region [104,105]. Fig. 6 shows the field pattern for a point source in such a material, with its characteristic double light cone structure; it is worth noting that this figure results from a time domain simulation of Maxwell's equations. Further, it is possible for careful engineering to make propagation along the hyperbolic "time" axis unidirectional. We expect that this clever trick will allow a variety of novel investigations of causal issues and models in systems where the whole "spacetime" history is laid out in 2D space; although of course hyperbolic materials can be interesting for other reasons (e.g. [106]).

Other ideas beyond that of the "space as time" also exist. With co-authors [107], Smolyaninov, also proposed a multiverse scenario implemented by thermal fluctuations in a ferrofluid. Further, he also suggested an analogue to gravitational force [108] as well as – rather more speculatively – a more complicated cosmological model [109] involving interacting tachyons and sub-luminal particles, again based on a spatially laid out spacetime described above. More recently, we have suggested the construction of dynamic and/or expanding fisheye-based "bubbleverses" embedded in a background spacetime [5].

Other exotic ideas include, for example, proposals for using metamaterial engineering with low permitivitty ("ϵ- near zero", or "ENZ") to favourably modify electron-electron interaction, with the intent of engineering superconductor behaviour [110]. Even the mathematical framework of supersymmetry has been explored in a T-Optics context [111], being used as a way to redesign structures with identical external (e.g. scattering) properties, but preferred internal properties, such as having a lower index contrast.

The really important question to ask is this: To what extent might these exotic ideas for metamaterial applications be useful as anything except intriguing thought experiments? Could, for example, we ever expect to design a metamaterial system clean enough to really probe interesting theoretical ideas? Or will the often messy compromises needed in designing metamaterials always be too visible and so confuse the results? Whether or not it turns out that these exotic concepts have a substantial role to play in the future of T-Design remains to be seen, but at least they show that the unexpected is still very much a possibility.

7. Abstract futures

A future of particular interest to us is that of a progressive unification of different ray or wave models [1,35]. The central idea is to provide an over-arching theory applicable (or rather, reducible) to a range of specific physical models such as EM or acoustics. We could call this either a "transformation mechanics" or "transformation media", depending on whether you prefer to focus on the propagating wave or other quantity (e.g. heat), or on the material properties. In this article we have been using T-Design as our chosen label, but existing areas have names – e.g. T-Optics and T-Acoustics – which focus more on the wave or ray than on the material parameters which form the core behaviour of the selected T-device.

One early attempt is now published as part of an investigation into spatial carpet and spacetime carpet cloaks [3], but which has roots that go back three years to a conference presentation – at for Metamaterials in Barcelona 2011. This encodes two wave theories – both EM and a scalar acoustics ("p-Acoustics") – in a single formulation. What usually just appears to be EM, can in fact be used to encode a range of wave theories, including not only the expected EM, but also a simple linear acoustics. This is most easily seen in a version of the tensor equations relying on the dual tensor of F, which is

$$\partial_\alpha {*F}^{\alpha\beta} = 0, \qquad \partial_\delta G^{\gamma\delta} = 0, \qquad G^{\gamma\delta} = *\chi^{\gamma\delta}_{\alpha\beta} {*F}^{\alpha\beta}. \qquad (8)$$

How the space and time component parts of these tensors can interlink to allow for different types of propagating waves is summarized diagrammatically in Fig. 7.

This unified mathematics provides a backdrop upon which to deform a reference solution into a T-device,

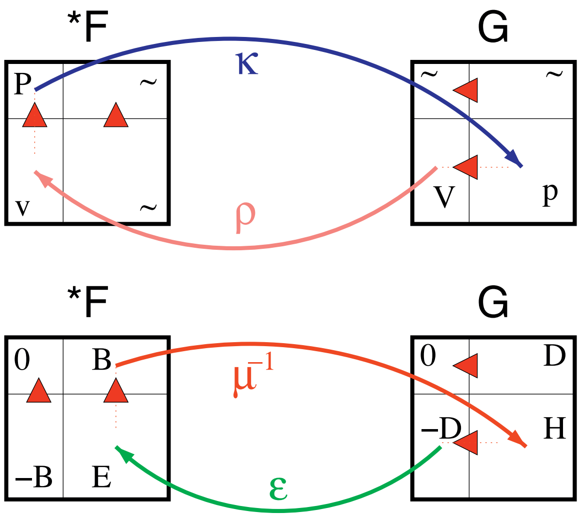

Fig. 7. Diagram showing how the wave fields pack into the field tensors $*F$ and G. Constitutive relations are indicated by arrows, and the space ↔ time connections of differential equations by red triangles. By following the arrows and triangles in the indicated loops we can see how use substitutions to construct second order wave equations for the fields. The top diagram shows for p-Acoustics how the occupation number P and velocity field $\vec{v}$ are related by $\partial_t P = \nabla \cdot \vec{V}$, and the momentum density $\vec{V}$ and pressure p are related by $\partial_t \vec{V} = \nabla \boldsymbol{p}$; with bulk modulus κ and density $\boldsymbol{\rho}$. The bottom diagram shows for EM how $\vec{B}$, $\vec{E}$ and $\vec{D}$, $\vec{H}$ are related by $\partial_t \vec{B} = \nabla \times \vec{E}$ and $-\partial_t \vec{D} = \nabla \times \vec{H}$; the curl operation being encoded in the antisymmetric 3×3 matrix re-representation of $\vec{E}$ and $\vec{H}$. See Ref. [3] for more details of this representation of acoustics and EM. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

after which the feasibility of implementing the device using a particular physical model (i.e. either EM or p-Acoustics) can be ascertained. A notable conclusion is that while T-devices have considerable design freedom in EM, that is *not* true of a p-Acoustics theory. However, the multi-tensor description used for this allows more waves than just those so far encoded, and extending this is something we are actively working on.

Of course, one might also eschew waves, and decide to take the simple path of dealing with only a ray transformation theory. This approach takes the T-device design as a mapping or morphism/diffeomorphism φ between one manifold $\mathcal{M}$ and another $\mathcal{N}$; the basic mathematics being outlined here earlier in Section 3. This retains the geodesics on the first manifold as geodesics on the second, despite the imposed T-device deformation that then induces a different metric on that second manifold. This metric then implies, for any chosen wave theory, a specific selection of physical parameters – perhaps permittivity ϵ_{ij} and permeability μ_{ij} (in EM), or bulk modulus κ and density ρ_{ij} in acoustics.

This approach generalizes the concept of refractive index to one that goes beyond a simple isotropic scaling $g \rightarrow n^2 g$. A morphism can warp space so that at the dilation depends on direction as well as location. This means that the morphism defines a generalized refractive index $n(r, \theta, \phi)$ that depends on location (r), and on angular variables (θ, ϕ), but remains *independent* of polarization. We have recently shown [35] how such an index arises in standard T-Optics, and how it can be related to (equal) dielectric and magnetic material properties. It seems highly suggestive that the generalized index concept is the *key* parameter underlying all Transformation Media theories.

Remarkably, and in a way that points to a deeper connection between ray and wave transformations, the generalized *ray* radial cloak T-device induces the same permitivitty and permeability properties [35,66] as the original radial EM wave radial cloak [77,112]. One future direction might be to consider how much extra information needs to be transported along a ray with its location and direction before a ray theory becomes isomorphic to a wave theory... or can this ever be done? After all, what kind of ray theory might cope with the notion of impedance?

As hinted in the previous paragraph, there is an important distinction between the full-wave theory discussed first in this section, and a geodesic or ray-type theory as discussed second. This is the role of impedance: while index variation can be (or is) used to control wave or ray direction, it is changes in impedance that give rise to reflections and/or partial transmission. In a full-wave theory, containing both types of wave component, impedance is naturally included; however in a ray theory, that notion is neglected. This is why many types of p-Acoustics T-devices, which have less constitutive freedom, suffer greater theoretical constraints than their EM counterparts [3]. In this, the ray theories are most comparable to transformation approaches based on second order wave equations, notably the analogue gravity style methods (e.g. [73,113,116]). That said, neglecting the role of impedance is a very useful simplification that has long made device fabrication easier, particularly in the case of experimental demonstrations of acoustic cloaking [115], but it is worth being aware of the inherent technical limitations and approximations of ray methods.

Medium flow also might be addressed in a more elegant way – perhaps – using Lie derivatives[4]. Medium

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[4] Given two vector fields **U** and **V**, the Lie derivative of **U** along **V** is defined as $\mathcal{L}_{\mathrm{V}}\mathrm{U}$. The potential utility of Lie derivatives here is that the geometry is defined – even before introducing an affine

flow, and its consequences, are of course a common concern in fluid systems, and even have uses in T-Acoustics [73]. But it would be nice to just apply the right sort of derivative operator that "knows" about medium flow already, so as to simplify notation. However, from a practical perspective, the necessary (and rather abstract) mathematics will need to be re-expressed in a friendlier form if its uptake in the community is to be improved. Including the underlying medium velocity field as a parameter can extend previous work [117]. An object encased in, for example, a smooth spherical shell around which flows an inviscid fluid will (trivially) form a cloak since the surface of the shell is an equipotential surface. For supersonic flow, acoustic disturbances are constrained to travel along the flow lines.

8. Summary

In this discussion paper we have deliberately set out to provide an alternative 'futures agenda' to that promoted by the experimentalists and technologists whose focus is in translating Transformation Media concepts into practical systems and devices. Even the 'technological agenda' raises some issues over the use and impact of nonlinearity and dispersion on standard transformational schemes, as we saw in Section 2. The pursuit of foundational issues (Section 3) in transformation media starts, in our view, with a correct understanding of the mathematical framework; the realization that the abstract idea underlying transformational theories can be unnecessarily confused by the introduction of specific coordinate systems. How transformations induce curvature, for example, seems to us an important, and as yet largely unexplored concept. Our introduction of spacetime transformation optics in 2011, discussed in Section 4, was a significant conceptual advance that is only rather slowly being implemented experimentally. The future here is likely to be in the domain of signal processing where the spacetime cloak acts as a means of prioritizing data channels, rather than conceptually fruitless attempts to combine spacetime and spatial cloaks. The idea of conformal optical mapping (Section 5), so beautifully used in the early conception of a 'refractive-index-only' directional cloak, has yet to be exhausted, despite the constraints of Liouville's theorem: two dimensions is enough to explore interesting new horizons.

Transformation media that exploit hyperbolic characteristics have been a fruitful avenue for the analogue simulation of a variety of Minkowski-based relativistic scenarios, including spinning cosmic strings, multiverses, and universes containing tachyons. Whilst we feel it unlikely that the analogies commented on in Section 6 will achieve results that will impact the fields they mimic, they do, nevertheless, provide surprising and interesting links between apparently disparate areas of physics, and are worth pursuing. Our own futures agenda of achieving mathematical unity between different areas of transformation physics was discussed in Section 7. To date we have constructed an acoustic model that closely follows the tensor formulation of electromagnetism. In this way a certain unity between T-Optics and T-Acoustics has been achieved, at least at the level of waves. Rays appear to be an even more primitive concept, describable as geodesics on an affine manifold. Understanding ray transformations at this abstract level could lead to a transformation theory of wide cross-disciplinary application.

As a final comment, we acknowledge that predicting the future accurately is well known to be a tricky business. We therefore look forward, not only to the futures suggested here, but also to the unknown unknowns of the transformation physics programme.

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connection – with respect to only prior geometrical feature in the problem, namely the field **V** associated with the medium flow.

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