International Journal of Algebra, Vol. 7, 2013, no. 19, 909 - 914
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2013.311111

The Galois Endomorphism Ring of a Galois Azumaya Extension

Xiaolong Jiang

Department of Mathematics
Sun Yat-Sen University,
Guangzhou, 510275 P.R. China
mcsjxl@mail.sysu.edu.cn

George Szeto

Department of Mathematics
Bradley University,
Peoria, Illinois 61625 USA
szeto@fsmail.bradley.edu

Abstract

Let B be a Galois Azumaya extension of B^G with Galois group G; that is, B is a Galois extension of B^G with Galois group G which is an Azumaya C^G-algebra where C is the center of B. Denote B^G by D and the endomorphism ring $\mathrm{Hom}({}_DB, {}_DB)$ of the left D-module endomorphisms of B by Ω. Then Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with the inner Galois group G' induced by G, but not a Galois Azumaya extension of $\Omega^{G'}$.

Mathematics Subject Classification: 13B05

Keywords: Azumaya algebras, Galois extensions, Hirata separable extensions, Endomorphism rings

1 Introduction

Let B be a ring with 1, G a finite automorphism group of B, C the center of B, and $B^G = \{b \in B|\ g(b) = b \text{ for each } g \in G\}$. In [1], we call B a Galois Azumaya extension of B^G with Galois group G if B is a Galois extension of B^G with Galois group G and B^G is an Azumaya C^G-algebra. Many properties of a Galois Azumaya extension were given in [1, 4, 7]. In [4], denote B^G by D and the endomorphism ring $\mathrm{Hom}({}_DB, {}_DB)$ of the left D-module endomorphisms of B by Ω. An expression of Ω is obtained in terms of a skew group ring and the opposite ring of B respectively. The purpose of the present paper is to show a Galois property of Ω. Observing that $G \subset \Omega$, we shall show that Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with the inner Galois group G' induced by the elements of G, but not a Galois Azumaya extension of $\Omega^{G'}$.

2 Preliminary

Throughout the paper, we assume that B is a ring with 1, C the center of B, D a subring of B with the same 1. As given in [1, 3, 8, 10], B is called a separable extension of D if the multiplication map: $B \otimes_D B \longrightarrow B$ splits as a B-bimodule homomorphism. In particular, if $D \subset C$, a separable extension B of D is called a separable D-algebra, and if $D = C$, a separable extension B of D is called an Azumaya C-algebra. If $B \otimes_D B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule, then B is called a Hirata separable extension of D. It is known that an Azumaya algebra is a Hirata separable extension. Let G be a finite automorphism group of B and $B^G = \{b \in B|\ g(b) = b \text{ for each } g \in G\}$. If there exist elements $\{a_i, b_i$ in B, $i = 1, 2, \cdots, m$ for some integer $m\}$ such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$, then B is called a Galois extension of B^G with Galois group G, and $\{a_i, b_i\}$ is called a G-Galois system for B. A Hirata separable Galois extension B of B^G is a Hirata separable extension and a Galois extension of B^G. If B is a Galois extension of B^G with Galois group G and B^G is an Azumaya C^G-algebra, then B is called a Galois Azumaya extension of B^G as studied in [1, 4, 7].

3 Galois Endomorphism Ring

In this section, let B be a Galois extension of B^G, denote B^G by D and $\mathrm{Hom}({}_DB, {}_DB)$ of the left D-module endomorphisms of B by Ω. Observing that $g(db) = dg(b)$ for all $d \in D$ and $b \in B$, we have $G \subset \Omega$. Denote the inner automorphism $f \longrightarrow gfg^{-1}$ by g' for each $g \in G$ and $f \in \Omega$, and $G' = \{g'|g \in G\}$. We shall show that Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with Galois group G'. We begin with embedding the opposite ring B^o of B in Ω and show that $G' \cong G$.

Lemma 3.1 *For any $b \in B$, let μ_b be the right multiplication map by b on B, $\mu_b(x) = xb$ for all $x \in B$, and $\mu_B = \{\mu_b | b \in B\}$. Then $B^o \cong \mu_B \subset \Omega$.*

Proof. It is straightforward to show that $b \longrightarrow \mu_b$ for $b \in B^o$ is an isomorphism from B^o to μ_B.

Lemma 3.2 *Let $G' = \{g' | g'(f) = gfg^{-1}$ for all $f \in \Omega\}$. Then $G' \cong G$ such that $g'(\mu_b) = \mu_{g(b)}$ for all $b \in B^o$ and $g \in G$.*

Proof. Let $\alpha : G \longrightarrow G'$ by $\alpha(g) = g'$ for all $g \in G$. Then, for $g, h \in G, \alpha(gh) = (gh)'$; and so for each $f \in \Omega$, $(\alpha(gh))(f) = (gh)'(f) = (gh)(f)(gh)^{-1} = gh(f)h^{-1}g^{-1} = g'(h'(f)) = (\alpha(g))(\alpha(h))(f)$. Also, let $\alpha(g) = 1'$, the identity of Ω. Then $\alpha(g)\mu_b = \mu_b$ for each $b \in B^o$. We have $g\mu_b g^{-1} = \mu_b$. But for all $x \in B$, $(g\mu_b g^{-1})(x) = g(g^{-1}(x)b) = x(g(b)) = \mu_{g(b)}(x)$, so $\mu_{g(b)}(x) = \mu_b(x)$. Thus $\mu_{g(b)} = \mu_b$; and so $g(b) = b$ by *Lemma* 3.1. This implies that $g = 1$, the identity of G. Therefore $G \cong G'$ such that for each $b \in B^o$, $g \in G$, $g'(\mu_b) = \mu_{g(b)}$.

By keeping the notations in *Lemma* 3.2, we next show a Galois property of Ω.

Theorem 3.3 *Let B be a Galois extension of B^G with Galois group G. Then Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with inner Galois group G' induced by and isomorphic with G.*

Proof. Since B is a Galois extension of B^G with Galois group G, there exists a Galois system $\{a_i, b_i \in B | i = 1, 2, \cdots, m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. Then by *Lemma* 3.2, $\sum_i g'(\mu_{b_i})\mu_{a_i} = \sum_i (\mu_{a_i g(b_i)}) = \mu_{(\sum_i a_i g(b_i))} = \mu_{\delta_{1,g}} = \delta_{1',g'}$ for each $g' \in G'$. Thus $\{\mu_{b_i}, \mu_{a_i} | i = 1, 2, \cdots, m\}$ is a Galois system for Ω with Galois group G' induced by and isomorphic with G by *Lemma* 3.2. Moreover, since G' is inner, Ω is a Hirata separable extension of $\Omega^{G'}$ ([8, Corollary 3]).

By *Theorem* 3.3, we derive an expression for Ω and $\Omega^{G'}$ respectively.

Corollary 3.4 *Let Z be the center of Ω, ZG the subalgebra generated by the elements of Z and G, and $V_\Omega(ZG)$ the commutator subalgebra of ZG in Ω. Then,*

(1) *$\Omega = \mu_B \Omega^{G'}$ generated by the elements of μ_B and $\Omega^{G'}$, and*

(2) *$\Omega^{G'} = V_\Omega(ZG)$, the subalgebra of all G-endomorphisms of Ω.*

Proof.

(1) By the proof of *Theorem* 3.3, Ω contains a Galois system $\{\mu_{b_i}, \mu_{a_i} | i = 1, 2, \cdots, m\}$ contained in μ_B, so $\mu_B \Omega^{G'}$ is a Galois extension of $\Omega^{G'}$ with Galois group induced by and isomorphic with G'. Thus $\Omega = \mu_B \Omega^{G'}$.

(2) For any $f \in \Omega^{G'}$, $g'(f) = f$ for all $g' \in G'$, so $gfg^{-1} = f$, $gf = fg$. Thus for all $x \in B$, $gf(x) = fg(x) = f(g(x))$; and so $\Omega^{G'} \subset V_\Omega(ZG)$. Conversely, for any $f \in V_\Omega(ZG)$ and $g \in G$, $fg(x) = gf(x)$, so $fg = gf$. Thus $g'(f) = gfg^{-1} = f$; and so $f \in \Omega^{G'}$. Therefore $\Omega^{G'} = V_\Omega(ZG)$.

4 Galois Azumaya Extensions

As defined in [2], let R be a commutative ring with 1, G a finite group and $f : G \times G \to \{\text{Units of } R\}$ a factor set. Then the R-algebra RG_f is called a projective group algebra with factor set f if $RG_f = \sum_{g \in G} RU_g$ which is an R-module with a free basis $\{U_g | g \in G\}$ such that $rU_g = U_g r$ for all $r \in R$ and $U_g U_h = U_{gh} f(g, h)$ for all $g, h \in G$. In this section, let B be a Galois Azumaya extension of B^G with Galois group G and C the center of B. We shall show that if G is inner, then $C = C^G$ and $B \cong B^G \otimes_C CG_f$ as Azumaya C-algebras. This will lead to the main result of this paper that Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$, but not a Galois Azumaya extension. An example is given to demonstrate this result.

Lemma 4.1 *If B is a Galois extension of B^G with an inner Galois group G such that $g(b) = U_g b U_g^{-1}$ for some invertible element $U_g \in B$, then $\{U_g \in G\}$ are free over C where C is the center of B.*

Proof. ([2, Theorem 6]).

Lemma 4.2 *If B is a Galois Azumaya extension of B^G with inner Galois group G such that $g(b) = U_g b U_g^{-1}$ for $g \in G$ and for some invertible element $U_g \in B$, then $C = C^G$ and $B \cong B^G \otimes_C CG_f$ as Azumaya C-algebras.*

Proof. Since B is a Galois extension of B^G with inner Galois group G, B is a Hirata separable extension of B^G ([8, Corllary 3]). Denote the commutator of B^G in B by $V_B(B^G)$. Then $V_B(V_B(B^G)) = B^G$ ([8, Proposition 4-(1)]). This implies that $C = C^G$. Moreover, B is a Galois Azumaya extension of B^G by hypothesis, so B^G is an Azumaya C^G-algebra; and so both B and B^G are Azumaya C-algebras. But then $B \cong B^G \otimes_C V_B(B^G)$ as Azumaya algebras (([3, theorem 4.3]). Since $g(b) = U_g b U_g^{-1}$ for each $b \in B$ and $g \in G$, $g(b) = U_g b U_g^{-1} = b$ for each $b \in B^G$ and $g \in G$. Thus $U_g b = b U_g$ for each $b \in B^G$. Therefore, $\sum_{g \in G} CU_g \subseteq V_B(B^G)$. Noting that $\sum_{g \in G} CU_g = CG_f$ by *Lemma* 4.1 and *Theorem* 6 in [2], we have $CG_f \subseteq V_B(B^G)$. Next we claim that $V_B(B^G) = CG_f$. Since B is a Galois and a Hirata separable extension of B^G, $V_B(B^G) = \oplus\sum_{g\in G} J_g$ where J_g is a rank one projective C-module for each g in G, $J_g = \{b \in B | bx = g(x)b\}$ for all $x \in B$ ([8, Proposition 4-(2)]). Since $U_g \in J_g$, $CG_f \subset \oplus\sum_{g\in G} J_g$. But the order of G is a unit in B ([8, Proposition 4-(3)]), so $\oplus\sum_{g\in G} J_g$ is a separable C-algebra and is equal to $V_B(B^G)$ ([6, theorem 1]). Since $V_B(B^G)$ is an Azumaya C-algebra, the separable subalgebra CG_f is a direct summand of $\oplus\sum_{g\in G} J_g$. Hence $\oplus\sum_{g\in G} J_g \cong CG_f \oplus M$ for some C-module M. Since $rank_C(J_g) = 1 = rank_C(CU_g)$ for each $g \in G$, $rank_C(M) = 0$. Thus $M = (0)$. Therefore $V_B(B^G) = CG_f$; and so $B \cong B^G \otimes_C CG_f$ as Azumaya C-algebras.

Next we show the main result.

Theorem 4.3 *If B is a Galois Azumaya extension of B^G with Galois group G, then Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with inner Galois group G' induced by and isomorphic with G, but not a Galois Azumaya extension.*

Proof. Since B is a Galois extension of B^G with Galois group G, Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with inner Galois group G' induced by and isomorphic with G by *Theorem* 3.3 where $G' = \{g' | g'(x) = gxg^{-1}\}$ for all $x \in \Omega$ and $g \in G$. Next we claim that Ω is not a Galois Azumaya extension of $\Omega^{G'}$ with Galois group G'. Assume to the contrary that Ω is a Galois Azumaya extension of $\Omega^{G'}$. Since $g'(x) = gxg^{-1}$ for all $x \in \Omega$ and $g \in G$, the factor set $f : G' \times G' \to \{\text{units of the center of } \Omega\}$ is trivial. That is, $f(g', h') = gh(gh)^{-1} = 1$ for all $g', h' \in G'$. On the other hand, by *Theorem* 5 in [8] or *Theorem* 3.4 in [4], the center of Ω is C^G which is the center of B^G, so $\Omega = \Omega^{G'} \otimes_{C^G} C^G G_f$ as Azumaya algebras by *Lemma* 4.2. Since f is trivial, $C^G G'_f = C^G G$ which is a group algebra of G over C^G. Thus $\Omega = \Omega^{G'} \otimes_{C^G} C^G G'_f = \Omega^{G'} \otimes_{C^G} C^G G$ as Azumaya C^G-algebras. But the center of a group algebra $C^G G$ is not C^G, so $C^G G$ is not an Azumaya C^G-algebra. Therefore we have a contradiction; and so Ω is not a Galois Azumaya extension.

We conclude the present paper with an example to demonstrate the main results as given by *Theorem* 3.3 and *Theorem* 4.3.

Example. Let $M_2(Z)$ be the matrix ring of order 2 over the ring of integers Z, $B = M_2(Z) \oplus M_2(Z)$ and $G = \{1, g | g(x, y) = (y, x)\}$ for $(x, y) \in B$. Then,

(1) B is a Galois Azumaya extension of B^G where $B^G \cong M_2(Z)$;

(2) B^G is an Azumaya C^G-algebra where $C = Z \oplus Z$ which is the center of B, and $C^G \cong Z$;

(3) Denote B^G by D and $\text{Hom}({}_D B, {}_D B)$ by Ω. Then Ω is a Galois and a Hirata separable extension of $\Omega^{G'}$ with inner Galois group G' induced by and isomorphic with G by *Theorem* 3.3; and

(4) Since the group algebra ZG is not an Azumaya Z-algebra, Ω is not a Galois Azumaya extension of $\Omega^{G'}$ by $Theorem$ $4, 3$.

References

[1] R. Alfaro and G. Szeto, Skew Group Rings Which Are Azumaya, *Comm. in Algebra,* **22**(6) 1995, 2255-2261.

[2] F.R. DeMeyer, Some Notes on the General Galois Theory of Rings, *Osaka J. Math.,* **2** 1965, 117-127.

[3] F.R. DeMeyer and E. Ingraham, Separable Algebras over Commutative Rings, **181**, Springer Verlag, Berlin, Heidelberg, New York, 1971.

[4] X.L. Jiang and G. Szeto, The Endomorphism Ring of a Galois Azumaya Extension, *International J. Math.,* **7**(11), 2013, 527-532.

[5] X.L. Jiang and G. Szeto, On a Composition Series of a General Azumaya Galois Extension, *South Asian J. Math.,* **2**(1), 2012, 11-15.

[6] T. Kanzaki, On Galois Algebra over a Commutative Ring, *Osaka J. Math.,* **2** 1965, 309-317.

[7] A. Paques, V. Rodriques, A. Sant'Ana, Galois Correspondences for Partial Galois Azumaya Extensions, *J. Algebra and Its Applications,* **10** 2011, 835-847.

[8] K. Sugano, On a Special Type of Galois Extensions, *Hokkaido J. Math.,* **9** 1980, 123-128.

[9] K. Sugano, Note on Separability of Endomorphism Rings, *Hokkaido J. Math.,* **XXI** 1971, 196-208.

[10] G. Szeto, On Azumaya Crossed Products, *Bull. Malays. Math. Sci. Soc.,* **2** 2012, 529-535.

Received: November 1, 2013