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The gamma beta ratio distribution

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Abstract. The important problem of the ratio of gamma and beta distributed random variables is considered. Six motivating applications (from efficiency modeling, income modeling, clinical trials, hydrology, reliability and modeling of infectious diseases) are discussed. Exact expressions are derived for the probability density function, cumulative distribution function, hazard rate function, shape characteristics, moments, factorial moments, variance, skewness, kurtosis, conditional moments, L moments, characteristic function, mean deviation about the mean, mean deviation about the median, Bonferroni curve, Lorenz curve, percentiles, order statistics and the asymptotic distribution of the extreme values. Estimation procedures by the methods of moments and maximum likelihood are provided and their performances compared by simulation. For maximum likelihood estimation, the Fisher information matrix is derived and the case of censoring is considered. Finally, an application is discussed for efficiency of warning-time systems.

1 Introduction

For given random variables X and Y, the distribution of the ratio X/Y is of interest in many areas of the sciences, engineering and medicine. In this paper, we study the distribution of $Z = X/Y$ when X and Y are independent random variables with X having the gamma distribution given by the probability density function (p.d.f.):

$$f_X(x) = \frac{\lambda^\beta x^{\beta-1} \exp(-\lambda x)}{\Gamma(\beta)} \tag{1.1}$$

(for $x > 0$, $\beta > 0$ and $\lambda > 0$) and Y having the beta distribution given by the p.d.f.:

$$f_Y(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a, b)} \tag{1.2}$$

(for $0 < y < 1$, $a > 0$ and $b > 0$), where $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ are the gamma and beta functions defined by

$$\Gamma(c) = \int_0^\infty t^{c-1} \exp(-t)\, dt$$

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and

$$B(c,d)=\int_0^1 t^{c-1}(1-t)^{d-1}\,dt,$$

respectively. We shall refer to the distribution of $Z=X/Y$ as the *gamma beta ratio distribution*. The study of the gamma beta ratio distribution is of importance in many applied areas. Six motivating examples are discussed in Section 2. A comprehensive treatment of the mathematical properties of the gamma beta ratio distribution including estimation issues is provided in Sections 3–10. An application is discussed in Section 11. Some of the results in Section 3 have appeared before in Nadarajah and Kotz (2005). They are reproduced here for completeness.

The calculations of this paper involve several more special functions, including the complementary incomplete gamma function defined by

$$\Gamma(a,x)=\int_x^\infty \exp(-t)t^{a-1}\,dt,$$

the ${}_1F_1$ hypergeometric function (also known as the confluent hypergeometric function) defined by

$${}_1F_1(a;b;x)=\sum_{k=0}^\infty \frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

the ${}_2F_1$ hypergeometric function (also known as the Gauss hypergeometric function) defined by

$${}_2F_1(a,b;c;x)=\sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},$$

the ${}_2F_2$ hypergeometric function defined by

$${}_2F_2(a,b;c,d;x)=\sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k(d)_k}\frac{x^k}{k!},$$

the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_n,a_{n+1},\dots,a_p\\ b_1,\dots,b_m,b_{m+1},\dots,b_q\end{matrix}\right)
=\frac{1}{2\pi\mathrm{i}}\int_L \frac{x^{-t}\Gamma(b_1+t)\cdots\Gamma(b_m+t)\Gamma(1-a_1-t)\cdots\Gamma(1-a_n-t)}{\Gamma(a_{n+1}+t)\cdots\Gamma(a_p+t)\Gamma(1-b_{m+1}-t)\cdots\Gamma(1-b_q-t)}\,dt$$

and, the generalized Kampé de Fériet function defined by

$$F_{C:D^{(1)};\dots;D^{(n)}}^{A:B^{(1)};\dots;B^{(n)}}\left((a):(b^{(1)});\dots;(b^{(n)});(c):(d^{(1)});\dots;(d^{(n)});x_1,\dots,x_n\right)
=\sum_{m_1=0}^\infty\cdots\sum_{m_n=0}^\infty \frac{((a))_{m_1+\cdots+m_n}((b^{(1)}))_{m_1}\cdots((b^{(n)}))_{m_n}}{((c))_{m_1+\cdots+m_n}((d^{(1)}))_{m_1}\cdots((d^{(n)}))_{m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!},$$

where $\mathrm{i} = \sqrt{-1}$, $a = (a_1, a_2, \ldots, a_A)$, $b^{(k)} = (b_{j,1}, b_{j,2}, \ldots, b_{j,B^{(k)}})$ for $j = 1, 2, \ldots, n$, $c = (c_1, c_2, \ldots, c_C)$, $d^{(k)} = (d_{j,1}, d_{j,2}, \ldots, d_{j,D^{(k)}})$ for $j = 1, 2, \ldots, n$ and $((f))_k = ((f_1, f_2, \ldots, f_p))_k = (f_1)_k (f_2)_k \cdots (f_p)_k$ denotes the product of ascending factorials with each ascending factorial defined as $(f_j)_k = f_j(f_j + 1) \cdots (f_j + k - 1)$ with the convention that $(f_j)_0 = 1$. For a description of the integration path, L, in the the Meijer G-function, see Section 9.3 in Gradshteyn and Ryzhik (2000). Detailed properties of these special functions can be found in Exton (1978), Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000).

2 Motivating applications

Here, we discuss six motivating examples from efficiency modeling, income modeling, clinical trials, hydrology, reliability and modeling of infectious diseases, where ratios of the form X/Y arise with X and Y being gamma and beta random variables. The assumption that X and Y are independent may not be realistic for some of the examples. However, the independence assumption could at least yield a first approximation for the distribution of the ratio. For large samples, it is known that the distribution assuming independence is consistent with that not assuming independence; see, for example, Cox and Hinkley (1974).

2.1 Over-reported income

In the economic literature, the over-reported income is commonly expressed by the multiplicative relationship $Z = X/Y$, where Y is a multiplicative error and X denotes the true income. It is well known that if Y has the power function distribution (a particular case of the beta distribution) then X is Pareto distributed if and only if Z is also; see Krishnaji (1970). In practice, the gamma distribution is often preferred as a model for income; see, for example, Grandmont (1987), Milevsky (1997), Sarabia et al. (2002) and Silver et al. (2002). This raises the important question: what is the distribution of the over-reported income $Z = X/Y$ if X is gamma distributed?

2.2 Hydrology

Let X and Y be independent random variables representing the areal precipitation and the annual stream flow, respectively. In hydrology, the interest is in the proportion of precipitation that ended up in stream flow, that is, $1/Z = Y/X$. It is known on physical grounds that Y is finite valued [see, e.g., Clarke (1979)]; therefore, it will be most reasonable to assume that X and Y are distributed according to (1.1) and (1.2), respectively, after suitable scaling.

2.3 Adaptive randomization

The purpose of outcome-adaptive randomization is to treat patients more effectively by weighting randomization probabilities in favor of better performing arms. [Berry and Eick (1995) and Berry (2004) discuss of the ethics and efficiency of adaptive randomization trials.] In most adaptively randomized clinical trials, the probability that a patient will be assigned to a given arm is proportional to $\exp(-\mu r) r^{\lambda}$ where r is the probability that the arm is in some sense best and $\mu > 0$ and $\lambda > 0$ are some trial design parameters. The value of r will be subject to some random error because it will depend on how many arms there are and on their respective strengths. Since r is a probability, the most reasonable model will be the beta distribution given by (1.2). The question is: what is the probability that a patient gets assigned an arm with a specific design? This is proportional to $\int_0^1 \exp(-\mu r) r^{\lambda} f(r)\, dr$, which entails computing the distribution of the ratio X/Y of gamma and beta random variables.

2.4 Expected efficiency

Suppose that a job can be performed in n possible ways with the resulting costs $c_1, c_2, \ldots, c_n$. Suppose too that the n ways are chosen with probabilities $p_1, p_2, \ldots, p_n$, where $p_1 + p_2 + \cdots + p_n = 1$. The expected efficiency of the job performed can be defined as $p_1/c_1 + p_2/c_2 + \cdots + p_n/c_n$, where p_i/c_i denotes the expected efficiency of choosing the ith possible way. In reality, both c_i and p_i will be subject to some random errors and so will the expected efficiency. Thus, in general, one can write the expected efficiency as Y/X, where X and Y are independent random variables representing the values of c_i and p_i, respectively. The most natural model for X will be the gamma distribution (it being the most popular model for skewed data) given by (1.1). The most natural model for Y will be the beta distribution (the only standard model for data on the unit interval) given by (1.2). Thus, inferences about the expected efficiency can be made by deriving the exact distribution of $Z = X/Y$ when X and Y are independent random variables with the p.d.f.s given by (1.1) and (1.2), respectively.

2.5 Modeling of infectious diseases

Importance of the Wells Riley equation to modeling of infectious diseases cannot be overlooked; see, for example, Fennelly et al. (2004), Fennelly and Nardell (1998), Liao et al. (2005), Nicas (1996, 2000) and Rudnick and Milton (2003). The original form of the Wells Riley equation [Nardell et al. (1991)] is given by

$$P = 1 - \exp\left(-\frac{ipqt}{Q}\right), \tag{2.1}$$

where $P =$ proportion of new disease cases among the susceptible persons; $D =$ number of new disease cases; $s =$ number of susceptible persons; $i =$ number of infectors; $p =$ breathing rate; $q =$ the rate at which an infector disseminates infectious particles; $t =$ time that infectors and susceptibles share a confined space or ventilation system; and $Q =$ rate of supply of outdoor air.

Probabilistic modeling based on (2.1) has gained much interest not just with respect to infectious diseases but also in other areas. Two popular models used with respect to (2.1) have been the gamma and beta distributions. For instance, Nicas (1996) stated the following: "... It was previously shown that the beta distribution on the interval [0, 1] is a good descriptor of respirator penetration values experienced by an individual worker from wearing to wearing, and of average respirator penetration values experienced by different workers. Based on the premise that the gamma distribution can reasonably describe the time-varying M. tb aerosol exposure levels experienced by health care workers...." The calculation with (2.1) clearly involves ratios of random variables.

2.6 Reliability

Let X and Y be independent random variables representing, respectively, the failure time of a component and the warning-time variable showing that the component will fail. In reliability engineering, $1/Z = Y/X$ will represent the efficiency of the warning-time system. Gamma distributions are popular models for failure time data and one would like the warning made within a fixed period of the time of operation; therefore, it will be most reasonable to assume that X and Y are distributed according to (1.1) and (1.2), respectively, after suitable scaling.

3 P.d.f. and c.d.f.

Theorem 1 expresses the p.d.f. and the c.d.f. of the gamma beta ratio distribution in terms of the confluent hypergeometric function and the ${}_2F_2$ hypergeometric function, respectively.

Theorem 1. *Suppose X and Y are distributed according to* (1.1) *and* (1.2), *respectively. The c.d.f. of $Z = X/Y$ can be expressed as*

$$F_Z(z) = \frac{B(b, a+\beta)(\lambda z)^\beta}{\Gamma(\beta+1)B(a,b)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \tag{3.1}$$

for $z > 0$. The corresponding p.d.f. of $Z = X/Y$ is

$$f_Z(z) = \frac{\lambda^\beta B(\beta+a, b)}{\Gamma(\beta)B(a,b)} z^{\beta-1} {}_1F_1(\beta+a; \beta+a+b; -\lambda z) \tag{3.2}$$

for $z > 0$.

Proof. The c.d.f. corresponding to (1.1) is $1 - \Gamma(\beta, \lambda x)/\Gamma(\beta)$. Thus, one can write the c.d.f. of X/Y as

$$
\begin{aligned}
\Pr(X/Y \leq z) &= \int_0^1 F_X(zy) f_Y(y)\, dy \\
&= 1 - \frac{1}{\Gamma(\beta) B(a,b)} \int_0^1 \Gamma(\beta, \lambda yz) y^{a-1} (1-y)^{b-1}\, dy \qquad (3.3) \\
&= 1 - \frac{1}{\Gamma(\beta) B(a,b)} I.
\end{aligned}
$$

Application of equation (2.10.2.2) in Prudnikov et al. (1986, Volume 2) shows that the integral I can be calculated as

$$
\begin{aligned}
I &= \Gamma(\beta) B(a,b) \\
&\quad - \frac{(\lambda z)^{\beta}}{\beta} B(b, a+\beta)\, {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z).
\end{aligned} \qquad (3.4)
$$

The result in (3.1) follows by substituting (3.4) into (3.3). The p.d.f. in (3.2) follows by differentiation and using properties of the hypergeometric function. □

Using special properties of the hypergeometric functions, one can derive several simpler forms for (3.1) and (3.2) when a, b and β take integer values. The following are worth noting:

- If $\beta = n \geq 1$ is an integer then

$$
F_Z(z) = 1 - \frac{1}{B(a,b)} \sum_{k=0}^{n-1} \frac{(\lambda z)^k}{k!} B(a+k, b)\, {}_1F_1(a+k; a+b+k; -\lambda z)
$$

  for $z > 0$.
- If $\beta = n \geq 1$ is an integer then

$$
f_Z(z) = \frac{\lambda^{-a} z^{-a-1}}{\Gamma(\beta) B(a,n)} \sum_{k=0}^{n} (-\lambda z)^{-k} \binom{n-1}{k} \{\Gamma(a+\beta+k) - \Gamma(a+\beta+k, \lambda z)\}
$$

  for $z > 0$.
- If $a + b + \beta = m \geq 1$ and $a + \beta = n \geq 1$ are integers then

$$
\begin{aligned}
f_Z(z) &= \frac{(-1)^{m-1} (1-m)_n (a)_{m-n} z^{-1} (\lambda z)^{\beta-m+1}}{(m-1)\Gamma(\beta)} \\
&\quad \times \left\{ \sum_{k=0}^{m-n-1} \frac{(n-m+1)_k (-\lambda z)^k}{k!(2-m)_k} - \exp(-\lambda z) \sum_{k=0}^{n-1} \frac{(1-n)_k (\lambda z)^k}{k!(2-m)_k} \right\}
\end{aligned}
$$

  for $z > 0$.

- If $a+\beta=n\geq 1$ is an integer then

$$f_Z(z)=\frac{(-1)^b z^{-1}(\lambda z)^{\beta-b}\exp(-\lambda z)}{\Gamma(\beta)B(a,b)} \times \sum_{k=0}^{n}(\lambda z)^k \binom{n-1}{k}\{\Gamma(b+k)-\Gamma(b+k,-\lambda z)\}$$

for $z>0$.

The formulas for $f_Z(z)$ and $F_Z(z)$ above can be used to save computational time since the computation of the hypergeometric functions in (3.1) and (3.2) can be more demanding. We note that the ${}_2F_2$ hypergeometric function in (3.1) has been reduced to the simpler confluent hypergeometric function. We also note that the confluent hypergeometric function in (3.2) has been reduced to the simpler complementary incomplete gamma function.

4 Hazard rate function

It follows from (3.1) and (3.2) that the hazard rate function (h.r.f.) of the gamma beta ratio distribution is

$$\lambda_Z(z)=\frac{A(z)}{B(z)} \tag{4.1}$$

for $z>0$,

$$A(z)=\beta\lambda^{\beta}B(a+\beta,b)z^{\beta-1}{}_1F_1(\beta+a;\beta+a+b;-\lambda z)$$

and

$$B(z)=\Gamma(\beta+1)B(a,b)-B(b,a+\beta)(\lambda z)^{\beta}{}_2F_2(\beta,a+\beta;\beta+1,a+b+\beta;-\lambda z).$$

5 Shape

Here, we derive shape characteristics of (3.2) and (4.1). Using the fact

$$\frac{\partial\,{}_1F_1(a;b;x)}{\partial x}=\frac{a}{b}\,{}_1F_1(a+1;b+1;x) \tag{5.1}$$

(see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric1F1/20/01/04/) one can see that the p.d.f., (3.2), is unimodal and the mode is the root of the equation

$$\frac{{}_1F_1(1+\beta+a;1+\beta+a+b;-\lambda z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda z)}=\frac{(\beta-1)(\beta+a+b)}{\lambda(\beta+a)z}.$$

Using the fact

$$ {}_1F_1(a;b;x) = \frac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a}[1+O(1/x)] + \frac{\Gamma(b)}{\Gamma(a)}\exp(x)x^{a-b}[1+O(1/x)] $$

as $x \to \infty$ (see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric1F1/06/02/), one can see that

$$ f_Z(z) \sim \frac{\Gamma(\beta+a)}{\lambda^a \Gamma(\beta) B(a,b)} z^{-1-a} \tag{5.2} $$

as $z \to \infty$. Using the fact ${}_1F_1(a;b;x) = 1 + O(x)$ as $x \to 0$ (see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric1F1/06/01/02/01/01/), one can see that

$$ f_Z(z) \sim \frac{\lambda^\beta B(\beta+a,b)}{\Gamma(\beta)B(a,b)} z^{\beta-1} \tag{5.3} $$

as $z \to 0$. Using the facts (5.1) and

$$ \frac{\partial\, {}_2F_2(a,b;c,d;x)}{\partial x} = \frac{ab}{cd}\, {}_2F_2(a+1,b+1;c+1,d+1;x) $$

(see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F2/20/01/06/), one can see that the hazard rate function, (4.1), is unimodal and the mode is the root of the equation

$$ \beta\lambda^\beta B(a+\beta,b) z^{\beta-2} B(z) C(z) = -\beta\lambda^\beta B(b,a+\beta) z^{\beta-1} A(z) D(z), $$

where

$$ \begin{aligned} C(z) &= (\beta-1)\, {}_1F_1(\beta+a;\beta+a+b;-\lambda z) \\ &\quad - \frac{(a+\beta)\lambda z}{a+b+\beta}\, {}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda z) \end{aligned} $$

and

$$ \begin{aligned} D(z) &= {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \\ &\quad - \frac{(a+\beta)\lambda z}{(\beta+1)(a+b+\beta)} \\ &\quad \times {}_2F_2(\beta+1, a+\beta+1; \beta+2, a+b+\beta+1; -\lambda z). \end{aligned} $$

Using the fact

$$ \begin{aligned} {}_2F_2(a,b;c,d;x) &= \frac{\Gamma(c)\Gamma(d)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)\Gamma(d-a)}(-x)^{-a}[1+O(1/x)] \\ &\quad + \frac{\Gamma(c)\Gamma(d)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)\Gamma(d-b)}(-x)^{-b}[1+O(1/x)] \\ &\quad + \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} x^{a+b-c-d} \exp(x)[1+O(1/x)] \end{aligned} $$

as $x \to \infty$ (see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F2/06/02/02/), one can see that

$$1 - F_Z(z) \sim \frac{\Gamma(\beta + a)}{a\lambda^a \Gamma(\beta) B(a, b)} z^{-a}$$

as $z \to \infty$ and so

$$\lambda_Z(z) \sim \frac{a\Gamma(b)}{\Gamma(\beta)} z^{-1}$$

as $z \to \infty$. It follows from (5.3) that

$$\lambda_Z(z) \sim \frac{\lambda^\beta B(\beta + a, b)}{\Gamma(\beta) B(a, b)} z^{\beta - 1}$$

as $z \to 0$. Clearly the tails of the p.d.f. and the hazard rate function are polynomial. It is also clear that the parameters a and β control, respectively, the upper and lower tails.

Figures 1 and 2 illustrate possible shapes of the p.d.f., (3.2), and the hazard rate function, (4.1), for selected values of a, b and β. The hypergeometric functions (3.2) and (4.1) were calculated using hypergeom([·],[·], ·) and hypergeom([·],[·], [·],[·], ·) functions in MAPLE.

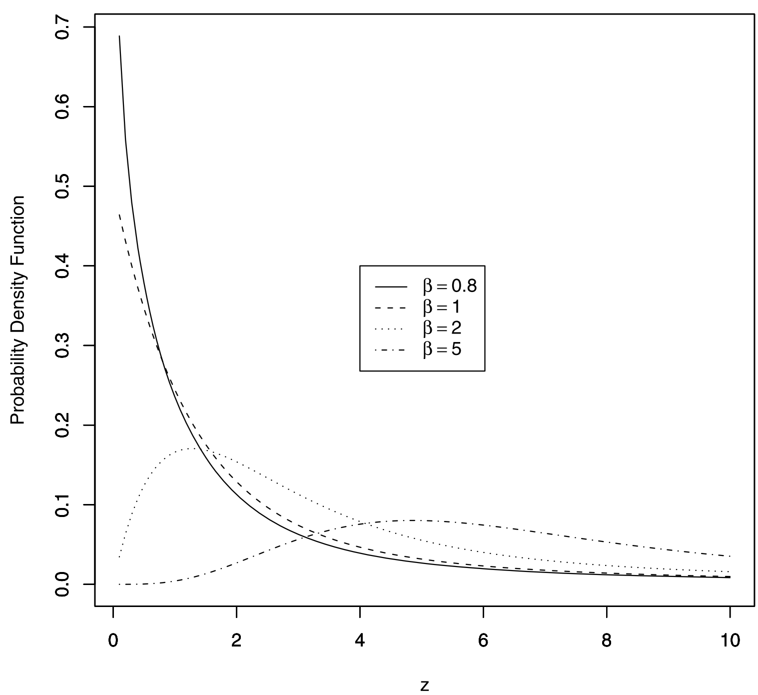

Figure 1 *Plots of the p.d.f., (3.2), for $a = b = 0.5$, $\lambda = 1$ and $\beta = 0.8, 1, 2, 5$.*

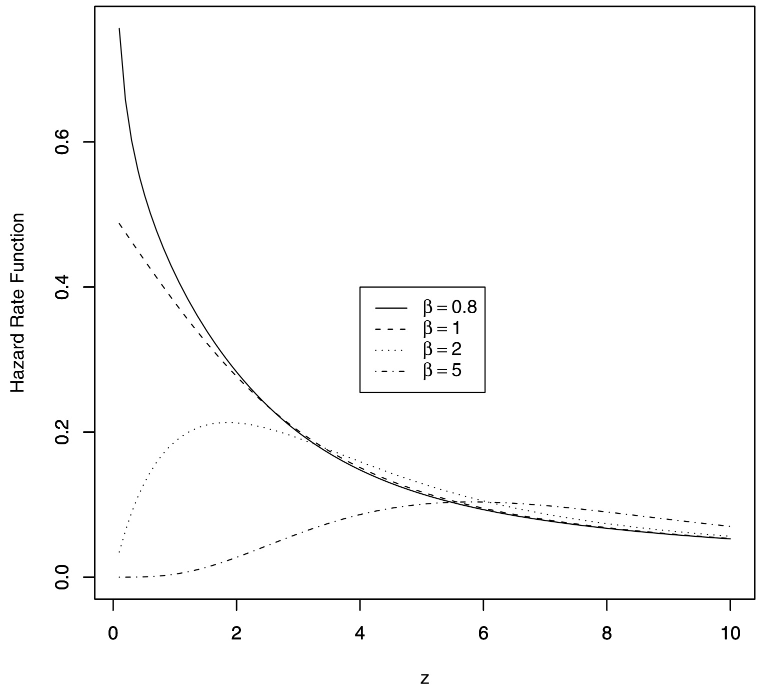

Figure 2 *Plots of the hazard rate function*, (4.1), *for* $a = b = 0.5$, $\lambda = 1$ *and* $\beta = 0.8, 1, 2, 5$.

6 Moment properties

The moments of the gamma beta ratio distribution can be derived by knowing the same for X and Y. It is well known [see, e.g., Johnson et al. (1994)] that

$$E(X^n) = \frac{\Gamma(\beta + n)}{\lambda^n \Gamma(\beta)}$$

and

$$E(Y^n) = \frac{B(a + n, b)}{B(a, b)}$$

for all real n such that $\beta + n \neq 0, -1, -2, \ldots, a + n \neq 0, -1, -2, \ldots$ and $a + b + n \neq 0, -1, -2, \ldots$. So, the nth moment of the gamma beta ratio distribution is

$$E(Z^n) = \frac{\Gamma(\beta + n) B(a - n, b)}{\lambda^n \Gamma(\beta) B(a, b)}$$

for all real n such that $\beta + n \neq 0, -1, -2, \ldots, a - n \neq 0, -1, -2, \ldots$ and $a + b - n \neq 0, -1, -2, \ldots$. The factorial moments, variance, skewness and the kurtosis can be calculated from the expression for $E(Z^n)$.

As mentioned in Section 2, the distribution of Z is useful as lifetime models. For such models, it is of interest to know what $E(Z^k \mid Z > z)$ is. Using Lemma 1 in the Appendix, it is easily seen that

$$E(Z^k \mid Z > z) = \frac{\lambda^\beta B(a+\beta, b)}{\{1 - F_Z(z)\}\Gamma(\beta) B(a,b)} J(k, z, a, b, \alpha, \lambda)$$

for all real k. The mean residual lifetime function is $E(Z \mid Z > z) - z$.

Some other important measures useful for lifetime models are the L moments due to Hoskings (1990). It can be shown using Lemma 2 in the Appendix that the kth L moment is

$$\lambda_k = \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} \binom{k-1+j}{j} \beta_j,$$

where

$$\beta_n = \frac{\Gamma(n\beta + \beta + 1) B^{n+1}(a+\beta, b)}{\lambda \Gamma(\beta) \Gamma^n(\beta+1) B^{n+1}(a,b)} I(k, n).$$

The L moments have several advantages over ordinary moments: for example, they apply for any distribution having finite mean; no higher-order moments need be finite.

Using the fact that the characteristic function (c.h.f.) of X is

$$E[\exp(\mathrm{i}tX)] = \left(\frac{\lambda}{\lambda - \mathrm{i}t}\right)^\beta,$$

the c.h.f. of the gamma beta ratio distribution can be expressed as

$$\begin{aligned}\phi(t) &= E\big(\exp(\mathrm{i}tX/Y)\big) \\ &= \int_0^1 \left(\frac{\lambda}{\lambda - \mathrm{i}t/y}\right)^\beta f_Y(y)\, dy = \frac{\lambda^\beta}{B(a,b)} \int_0^1 \frac{y^{a-1}(1-y)^{b-1}}{(\lambda - \mathrm{i}t/y)^\beta}\, dy \qquad (6.1) \\ &= \frac{1}{B(a,b)} \int_0^1 \frac{y^{a+\beta-1}(1-y)^{b-1}}{(y - \mathrm{i}t/\lambda)^\beta}\, dy = \frac{1}{B(a,b)} I.\end{aligned}$$

Application of equation (2.2.6.15) in Prudnikov et al. (1986, Volume 1) shows that the integral I can be calculated as

$$I = \left(-\frac{\mathrm{i}t}{\lambda}\right)^{-\beta} B(a+\beta, b)\, {}_2F_1\left(a+\beta, \beta; a+b+\beta; \frac{\lambda}{\mathrm{i}t}\right). \qquad (6.2)$$

Substituting (6.2) into (6.1), one obtains

$$\phi(t) = \frac{\lambda^\beta B(a+\beta, b)}{(-\mathrm{i}t)^\beta B(a,b)}\, {}_2F_1\left(a+\beta, \beta; a+b+\beta; \frac{\lambda}{\mathrm{i}t}\right). \qquad (6.3)$$

Using well-known transformation formulas for the Gauss hypergeometric function, one can obtain the following alternative forms of (6.3):

$$\phi(t)=\left(-\frac{\lambda}{\mathrm{i}t}\right)^{\lambda}\left(1-\frac{\lambda}{\mathrm{i}t}\right)^{-(a+\beta)}\frac{B(a+\beta,b)}{B(a,b)}{}_2F_1\left(a+\beta,a+b;a+b+\beta;\frac{\lambda}{\lambda-\mathrm{i}t}\right),$$

$$\phi(t)=\left(-\frac{\lambda}{\mathrm{i}t}\right)^{\lambda}\left(1-\frac{\lambda}{\mathrm{i}t}\right)^{-\beta}\frac{B(a+\beta,b)}{B(a,b)}{}_2F_1\left(\beta,b;a+b+\beta;\frac{\lambda}{\lambda-\mathrm{i}t}\right)$$

and

$$\phi(t)=\left(-\frac{\lambda}{\mathrm{i}t}\right)^{\lambda}\left(1-\frac{\lambda}{\mathrm{i}t}\right)^{b-\beta}\frac{B(a+\beta,b)}{B(a,b)}{}_2F_1\left(b,a+b;a+b+\beta;\frac{\lambda}{\mathrm{i}t}\right).$$

If a, b and β take integer values then, using special properties of the Gauss hypergeometric function, one can obtain the following elementary form of (6.3):

$$\phi(t)=\frac{t^a}{\lambda^a B(a,b)}\sum_{k=0}^{b-1}\sum_{l=0}^{\beta}\binom{b-1}{k}\binom{\beta}{l}(-1)^k(-\mathrm{i})^{\beta-l}(t/\lambda)^k P(a+\beta+k+l-1),$$

where $P(m)$ satisfies the recurrence relation

$$P(m)=\frac{1}{1+m-2\beta}\frac{(\lambda/t)^{m-1}}{(1+\lambda/t)^{\beta-1}}+\frac{m-1}{2\beta-m-1}P(m-2)$$

with the initial values

$$P(1)=\begin{cases}\dfrac{1}{2}\log\left(1+\dfrac{\lambda^2}{t^2}\right), & \text{if } \beta=1,\\[2ex] \dfrac{1}{2(1-\beta)}\left\{\left(1+\dfrac{\lambda^2}{t^2}\right)^{1-\beta}-1\right\}, & \text{if } \beta>1,\end{cases}$$

and

$$P(0)=\frac{\lambda}{(2\beta-1)t}\sum_{k=1}^{\beta-1}\frac{(2\beta-1)(2\beta-3)\cdots(2\beta-2k+1)}{2^k(\beta-1)(\beta-2)\cdots(\beta-k)}\left(1+\frac{\lambda^2}{t^2}\right)^{k-\beta}$$
$$+\frac{(2\beta-3)!!}{2^{\beta-1}(\beta-1)!}\arctan\left(\frac{\lambda}{t}\right).$$

7 Mean deviations and Bonferroni and Lorenz curves

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median—defined by

$$\delta_1(Z)=\int_0^{\infty}|z-\mu|f_Z(z)\,dz$$

and

$$\delta_2(Z) = \int_0^\infty |z - M| f_Z(z)\, dz,$$

respectively, where $\mu = E(Z)$ and $M = \text{Median}(Z)$ denotes the median. The measures $\delta_1(Z)$ and $\delta_2(Z)$ can be calculated using the relationships

$$\begin{aligned}
\delta_1(Z) &= \int_0^\mu (\mu - z) f_Z(z)\, dz + \int_\mu^\infty (z - \mu) f_Z(z)\, dz \\
&= \mu F_Z(\mu) - \int_0^\mu z f_Z(z)\, dz - \mu\{1 - F_Z(\mu)\} + \int_\mu^\infty z f_Z(z)\, dz \\
&= 2\mu F_Z(\mu) - 2\mu + 2\int_\mu^\infty z f_Z(z)\, dz
\end{aligned}$$

and

$$\begin{aligned}
\delta_2(Z) &= \int_0^M (M - z) f_Z(z)\, dz + \int_M^\infty (z - M) f_Z(z)\, dz \\
&= M F_Z(M) - \int_0^M z f_Z(z)\, dz - M\{1 - F_Z(M)\} + \int_M^\infty z f_Z(z)\, dz \\
&= 2\int_M^\infty z f_Z(z)\, dz - \mu.
\end{aligned}$$

By Lemma 1 in the Appendix,

$$\int_\mu^\infty z f_Z(z)\, dz = \frac{\lambda^\beta B(a+\beta, b)}{\Gamma(\beta) B(a, b)} J(1, \mu, a, b, \alpha, \lambda) \tag{7.1}$$

and

$$\int_M^\infty z f_Z(z)\, dz = \frac{\lambda^\beta B(a+\beta, b)}{\Gamma(\beta) B(a, b)} J(1, M, a, b, \alpha, \lambda), \tag{7.2}$$

so it follows that

$$\delta_1(Z) = 2\mu F_Z(\mu) - 2\mu + \frac{2\lambda^\beta B(a+\beta, b)}{\Gamma(\beta) B(a, b)} J(1, \mu, a, b, \alpha, \lambda)$$

and

$$\delta_2(Z) = \frac{2\lambda^\beta B(a+\beta, b)}{\Gamma(\beta) B(a, b)} J(1, M, a, b, \alpha, \lambda) - \mu.$$

Bonferroni and Lorenz curves [Bonferroni (1930)] have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are defined by

$$B(p) = \frac{1}{p\mu} \int_0^q t f_Z(t)\, dt \tag{7.3}$$

and

$$L(p) = \frac{1}{\mu} \int_0^q t f_Z(t)\, dt, \tag{7.4}$$

respectively, where $\mu = E(Z)$ and $q = F_Z^{-1}(p)$. Using (7.1) and (7.2), one can reduce (7.3) and (7.4) to

$$B(p) = \frac{1}{p} - \frac{\lambda^\beta B(a+\beta, b)}{p\mu\Gamma(\beta)B(a,b)} J(1, q, a, b, \alpha, \lambda)$$

and

$$L(p) = 1 - \frac{\lambda^\beta B(a+\beta, b)}{\mu\Gamma(\beta)B(a,b)} J(1, q, a, b, \alpha, \lambda),$$

respectively.

8 Percentiles

In this section, we provide a program for computing the percentage points z_p associated with the c.d.f. of the gamma beta ratio distribution. These values are obtained numerically by solving the equation

$$\frac{B(b, a+\beta)(\lambda z_p)^\beta}{\Gamma(\beta+1)B(a,b)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z_p) = p. \tag{8.1}$$

Evidently, this involves computation of the ${}_2F_2$ hypergeometric function and routines for this are widely available. We used the function hypergeom([·, ·],[·, ·], ·) in MAPLE. The following three-line program in MAPLE solves (8.1) for given p, λ, β, a and b:

```
cc:=Beta(b,a+beta)*(lambda*z)**beta/(GAMMA(beta+1)*Beta(a,b)):
ff:=cc*hypergeom([beta,a+beta],[beta+1,a+b+beta],-lambda*z):
fsolve(ff=p,z=0..10000):
```

We expect that this program could be useful for applications of the type described in Section 2. For instance, z_{1-p} will be the over reported income that will be exceeded with probability p; see Example 1 of Section 2. Similarly, in Example 2 of Section 2, the percentile points can be used to quantify the proportion of precipitation ended up in stream.

9 Order statistics

Suppose $Z_1, Z_2, \ldots, Z_n$ is a random sample from (3.2). Let $Z_{1:n} < Z_{2:n} < \cdots < Z_{n:n}$ denote the corresponding order statistics. It is well known that the p.d.f. and the c.d.f. of the kth order statistic, say $Y = Z_{k:n}$, are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F_Z^{k-1}(y)\{1 - F_Z(y)\}^{n-k} f_Z(y)$$
$$= \frac{n!}{(k-1)!(n-k)!} \sum_{m=0}^{n-k} \binom{n-k}{m} (-1)^m F_Z^{m+k-1}(y) f_Z(y)$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F_Z^j(y)\{1 - F_Z(y)\}^{n-j}$$
$$= \sum_{j=k}^{n} \sum_{m=0}^{n-j} \binom{n}{j} \binom{n-j}{m} (-1)^m F_Z^{j+m}(y),$$

respectively, for $k = 1, 2, \ldots, n$. Using Lemma 2 in the Appendix, the qth moment of Y can be expressed as

$$E(Y^q) = \frac{n!}{(k-1)!(n-k)!\lambda^q \Gamma(\beta)} \times \sum_{m=0}^{n-k} \binom{n-k}{m} (-1)^m \frac{\Gamma((m+k)\beta + q) B^{m+k}(a+\beta, b)}{\Gamma^{m+k-1}(\beta+1) B^{m+k}(a, b)} \times I(q, m+k-1)$$

for all real q such that $(m+k)\beta + q \neq 0, -1, -2, \ldots$ for all m.

Sometimes one would be interested in the asymptotics of the extreme order statistics $M_n = \max(Z_1, \ldots, Z_n)$ and $m_n = \min(Z_1, \ldots, Z_n)$. Take the c.d.f. and the p.d.f. of the gamma beta ratio distribution as specified by (3.1) and (3.2), respectively. It can be seen from (5.2), (5.3) and an application of L'Hospital's rule that

$$\lim_{t\to\infty} \frac{1 - F_Z(tz)}{1 - F_Z(t)} = \lim_{t\to\infty} \frac{z f_Z(tz)}{f_Z(t)} = z^{-a}$$

and

$$\lim_{t\to\infty} \frac{F_Z(tz)}{F_Z(t)} = \lim_{t\to\infty} \frac{z f_Z(tz)}{f_Z(t)} = z^{\beta}.$$

So, it follows from Theorem 1.6.2 in Leadbetter et al. (1987) that there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$\Pr\{a_n(M_n - b_n) \leq t\} \to \exp(-t^{-a})$$

and

$$\Pr\{c_n(m_n - d_n) \leq t\} \to 1 - \exp(-t^{\beta})$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter et al. (1987), one can see that $1/a_n = F_Z^{-1}(1 - 1/n)$ and $b_n = 0$, where $F_Z^{-1}(\cdot)$ denotes the inverse function of $F_Z(\cdot)$.

10 Estimation issues

Here, we consider method of moments estimation and maximum likelihood estimation of the parameters in the gamma beta ratio distribution. We also provide the associated Fisher information matrices.

Suppose we have two independent random samples $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ from (1.1) and (1.2), respectively. Let $\bar{X}$, $\bar{Y}$, S_X^2 and S_Y^2 denote the sample means and sample variances. By equating the theoretical and empirical moments

$$E(X) = \bar{X}, \qquad E(Y) = \bar{Y}, \qquad \mathrm{Var}(X) = S_X^2, \qquad \mathrm{Var}(Y) = S_Y^2,$$

we obtain the method of moments estimators (MMEs) as

$$\widehat{\beta} = \frac{\bar{X}^2}{S_X^2}, \qquad \widehat{\lambda} = \frac{\bar{X}}{S_X^2}, \qquad \widehat{a} = \bar{Y}\left[\frac{\bar{Y}(1-\bar{Y})}{S_Y^2} - 1\right],$$

$$\widehat{b} = (1-\bar{Y})\left[\frac{\bar{Y}(1-\bar{Y})}{S_Y^2} - 1\right].$$

The maximum likelihood estimator (MLE) of λ is the root of the equation

$$\psi(\bar{X}\lambda) - \log \lambda = \frac{1}{n}\sum_{i=1}^{n} \log X_i,$$

where $\psi(x) = d \log \Gamma(x)/dx$ is the digamma function. The MLE $\widehat{\beta} = \bar{X}\widehat{\lambda}$. The corresponding Fisher information matrix is given by

$$E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) = \frac{n\beta}{\lambda^2}, \qquad E\left(-\frac{\partial^2 \log L}{\partial \beta^2}\right) = n\psi'(\beta),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \lambda\, \partial \beta}\right) = -\frac{n}{\lambda}.$$

The MLEs of a and b are the simultaneous solutions of the equations

$$\psi(a) - \psi(a+b) = \frac{1}{n}\sum_{i=1}^{n} \log Y_i$$

and

$$\psi(b) - \psi(a+b) = \frac{1}{n}\sum_{i=1}^{n} \log(1 - Y_i).$$

Some rearrangement shows that a is root of the equation

$$\psi(a) - \psi\left(a + \psi^{-1}\left(\psi(a) - \frac{1}{n}\sum_{i=1}^{n}\log Y_i + \frac{1}{n}\sum_{i=1}^{n}\log(1-Y_i)\right)\right) = \frac{1}{n}\sum_{i=1}^{n}\log Y_i.$$

Similarly, b is root of the equation

$$\begin{aligned}\psi(b) - \psi\left(b + \psi^{-1}\left(\psi(b) + \frac{1}{n}\sum_{i=1}^{n}\log Y_i - \frac{1}{n}\sum_{i=1}^{n}\log(1-Y_i)\right)\right)\\ = \frac{1}{n}\sum_{i=1}^{n}\log(1-Y_i).\end{aligned}$$

The Fisher information matrix for the estimators of (a, b) is given by

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = n\psi'(a) - n\psi'(a+b),$$

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = n\psi'(b) - n\psi'(a+b)$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial a\,\partial b}\right) = -n\psi'(a+b).$$

Sometimes the observations are on $Z = X/Y$, and not on the original variables, X and Y. Suppose $Z_1, Z_2, \ldots, Z_n$ is a random sample on Z. The MMEs of the four parameters can be obtained as the simultaneous solutions of the equations

$$E(Z^k) = \frac{1}{n}\sum_{i=1}^{n} Z_i^k \tag{10.1}$$

for $k = 1, 2, 3, 4$, where the theoretical moments are given in Section 6. The MLEs are the simultaneous solutions of the equations

$$\begin{aligned}n\psi(\beta) + n\psi(\beta+a+b) - n\psi(\beta+a) - n\log\lambda\\ = \sum_{i=1}^{n}\log Z_i + \sum_{i=1}^{n}\frac{\partial\, {}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)/\partial\beta}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)},\end{aligned} \tag{10.2}$$

$$\frac{n\beta}{\lambda} = \frac{\beta+a}{\beta+a+b}\sum_{i=1}^{n} Z_i \frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z_i)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)}, \tag{10.3}$$

$$\begin{aligned}n\psi(a) + n\psi(\beta+a+b) - n\psi(\beta+a) - n\psi(a+b)\\ = \sum_{i=1}^{n}\frac{\partial\, {}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)/\partial a}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)}\end{aligned} \tag{10.4}$$

and

$$n\psi(\beta+a+b)-n\psi(a+b)=\sum_{i=1}^{n}\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)/\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)}. \qquad (10.5)$$

The Fisher information matrix for the estimators of (β, λ, a, b) is given by

$$\begin{aligned}
E\left(-\frac{\partial^2 \log L}{\partial\beta^2}\right)
&= n\psi'(\beta)+n\psi'(\beta+a+b)-n\psi'(\beta+a)\\
&\quad -nE\left[\frac{\partial^2\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial\beta^2}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\quad +nE\left[\left\{\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial\beta}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right\}^2\right],
\end{aligned}$$

$$\begin{aligned}
E\left(-\frac{\partial^2 \log L}{\partial\beta\,\partial\lambda}\right)
&= -\frac{n}{\lambda}+\frac{n(\beta+a)}{\beta+a+b}E\left[Z\frac{\partial\,{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)/\partial\beta}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\quad +\frac{nb}{(\beta+a+b)^2}E\left[Z\frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\quad -\frac{n(\beta+a)}{\beta+a+b}\\
&\qquad \times E\left[Z\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial\beta_1}\right.\\
&\qquad\quad \times F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)\\
&\qquad\quad \left./\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right],
\end{aligned}$$

$$\begin{aligned}
E\left(-\frac{\partial^2 \log L}{\partial\beta\,\partial a}\right)
&= n\psi'(\beta+a+b)-n\psi'(\beta+a)\\
&\quad -nE\left[\frac{\partial^2\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial\beta\,\partial a}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\quad +nE\left[\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial\beta}\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial a}\right.\\
&\qquad \left./\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right],
\end{aligned}$$

$$
\begin{aligned}
&E\left(-\frac{\partial^2 \log L}{\partial\beta\,\partial b}\right)\\
&\quad = n\psi'(\beta+a+b)\\
&\qquad - nE\left[\frac{\partial^2{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial\beta\,\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\qquad + nE\left[\frac{\partial{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial\beta}\frac{\partial{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial b}\right.\\
&\qquad\qquad \left./\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right],
\end{aligned}
$$

$$
\begin{aligned}
&E\left(-\frac{\partial^2 \log L}{\partial\lambda^2}\right)\\
&\quad = \frac{n\beta}{\lambda^2} - \frac{n(\beta+a)(\beta+a+1)}{(\beta+a+b)(\beta+a+b+1)}\\
&\qquad\qquad \times E\left[Z^2\frac{{}_1F_1(\beta+a+2;\beta+a+b+2;-\lambda Z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\qquad + \frac{n(\beta+a)^2}{(\beta+a+b)^2}E\left[\left\{Z\frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right\}^2\right],
\end{aligned}
$$

$$
\begin{aligned}
&E\left(-\frac{\partial^2 \log L}{\partial\lambda\,\partial a}\right)\\
&\quad = \frac{nb}{(\beta+a+b)^2}E\left[Z\frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\qquad + \frac{n(\beta+a)}{\beta+a+b}E\left[Z\frac{\partial{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)/\partial a}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]\\
&\qquad - \frac{n(\beta+a)}{\beta+a+b}E\left[Z{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)\right.\\
&\qquad\qquad\qquad \times\frac{\partial{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial a}\\
&\qquad\qquad\qquad \left./\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right],
\end{aligned}
$$

$$
\begin{aligned}
&E\left(-\frac{\partial^2 \log L}{\partial\lambda\,\partial b}\right)\\
&\quad = -\frac{n(\beta+a)}{(\beta+a+b)^2}E\left[Z\frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]
\end{aligned}
$$

$$+\frac{n(\beta+a)}{\beta+a+b}E\left[Z\frac{\partial\,{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)/\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]$$

$$-\frac{n(\beta+a)}{\beta+a+b}E\left[Z\,{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z)\times\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial b}\Big/\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right],$$

$$E\left(-\frac{\partial^2\log L}{\partial a^2}\right)=n\psi'(a)+n\psi'(\beta+a+b)-n\psi'(\beta+a)-n\psi'(a+b)-nE\left[\frac{\partial^2\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial a^2}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]+nE\left[\left\{\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial a}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right\}^2\right],$$

$$E\left(-\frac{\partial^2\log L}{\partial a\,\partial b}\right)=n\psi'(\beta+a+b)-n\psi'(a+b)-nE\left[\frac{\partial^2\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial a\,\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]+nE\left[\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial a}\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}{\partial b}\Big/\{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)\}^2\right]$$

and

$$E\left(-\frac{\partial^2\log L}{\partial b^2}\right)=n\psi'(\beta+a+b)-n\psi'(a+b)-nE\left[\frac{\partial^2\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial b^2}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right]+nE\left[\left\{\frac{\partial\,{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)/\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z)}\right\}^2\right].$$

The partial derivatives of the confluent hypergeometric function can be calculated by using the facts that

$$\frac{\partial {}_1F_1(a;b;z)}{\partial a} = \sum_{i=0}^{\infty} \frac{(a)_i \psi(a+i) z^i}{(b)_i i!} - \psi(a) {}_1F_1(a;b;z)$$

and

$$\frac{\partial {}_1F_1(a;b;z)}{\partial b} = \psi(b) {}_1F_1(a;b;z) - \sum_{i=0}^{\infty} \frac{(a)_i \psi(b+i) z^i}{(b)_i i!};$$

see http://functions.wolfram.com/07.20.20.0001.01 and http://functions.wolfram.com/07.20.20.0003.01.

Often with lifetime data, one encounters censoring. There are different forms of censoring: Type I censoring, Type II censoring, etc. Here, we consider the general case of multicensored data: there are n subjects of which:

- n_0 are known to have the values $t_1, \ldots, t_{n_0}$.
- n_1 are known to belong to the interval $[s_{i-1}, s_i]$, $i = 1, \ldots, n_1$.
- n_2 are known to have exceeded r_i, $i = 1, \ldots, n_2$, but not observed any longer.

Note that $n = n_0 + n_1 + n_2$. Note too that Type I censoring and Type II censoring are contained as particular cases of multicensoring. In this case, the maximum likelihood equations, (10.2) to (10.5), generalize to

$$\begin{aligned}
& n_0\psi(\beta) + n_0\psi(\beta+a+b) - n_0\psi(\beta+a) - n_0 \log\lambda \\
& \quad = \sum_{i=1}^{n_0} \log t_i + \sum_{i=1}^{n_0} \frac{\partial {}_1F_1(\beta+a;\beta+a+b;-\lambda t_i)/\partial\beta}{{}_1F_1(\beta+a;\beta+a+b;-\lambda t_i)} \\
& \qquad + \sum_{i=1}^{n_1} \frac{D_1(s_i) - D_1(s_{i-1})}{F_Z(s_i) - F_Z(s_{i-1})} - \sum_{i=1}^{n_2} \frac{D_1(r_i)}{1 - F_Z(r_i)},
\end{aligned}$$

$$\begin{aligned}
\frac{n_0\beta}{\lambda} = & \frac{\beta+a}{\beta+a+b} \sum_{i=1}^{n_0} Z_i \frac{{}_1F_1(\beta+a+1;\beta+a+b+1;-\lambda Z_i)}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)} \\
& + \sum_{i=1}^{n_1} \frac{D_2(s_i) - D_2(s_{i-1})}{F_Z(s_i) - F_Z(s_{i-1})} - \sum_{i=1}^{n_2} \frac{D_2(r_i)}{1 - F_Z(r_i)},
\end{aligned}$$

$$\begin{aligned}
& n_0\psi(a) + n_0\psi(\beta+a+b) - n_0\psi(\beta+a) - n_0\psi(a+b) \\
& \quad = \sum_{i=1}^{n_0} \frac{\partial {}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)/\partial a}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)} \\
& \qquad + \sum_{i=1}^{n_1} \frac{D_3(s_i) - D_3(s_{i-1})}{F_Z(s_i) - F_Z(s_{i-1})} - \sum_{i=1}^{n_2} \frac{D_3(r_i)}{1 - F_Z(r_i)}
\end{aligned}$$

and

$$n_0\psi(\beta+a+b) - n_0\psi(a+b) = \sum_{i=1}^{n_0} \frac{\partial {}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)/\partial b}{{}_1F_1(\beta+a;\beta+a+b;-\lambda Z_i)} + \sum_{i=1}^{n_1} \frac{D_4(s_i)-D_4(s_{i-1})}{F_Z(s_i)-F_Z(s_{i-1})} - \sum_{i=1}^{n_2} \frac{D_4(r_i)}{1-F_Z(r_i)},$$

where $F_Z(\cdot)$ is given by (3.1),

$$\begin{aligned}
D_1(z) = \frac{(\lambda z)^\beta \Gamma(a+b)}{\Gamma(a)} & \times \left\{ \frac{\Gamma'(a+\beta)+\log(\lambda z)\Gamma(a+\beta)}{\Gamma(a+b+\beta)\Gamma(\beta+1)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \right. \\
& - \frac{\Gamma(a+\beta)\Gamma'(a+b+\beta)}{\Gamma^2(a+b+\beta)\Gamma(\beta+1)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \\
& - \frac{\Gamma(a+\beta)\Gamma'(\beta+1)}{\Gamma(a+b+\beta)\Gamma^2(\beta+1)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \\
& \left. + \frac{\Gamma(a+\beta)}{\Gamma(a+b+\beta)\Gamma(\beta+1)} \frac{\partial {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z)}{\partial \beta} \right\},
\end{aligned}$$

$$\begin{aligned}
D_2(z) = \frac{\Gamma(a+\beta)\Gamma(a+b)(\lambda z)^\beta}{\Gamma(a)\Gamma(a+b+\beta)\Gamma(\beta+1)} & \times \left\{ \frac{\beta}{\lambda} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \right. \\
& - \frac{\beta(a+\beta)z}{(a+b+\beta)(\beta+1)} \\
& \left. \times {}_2F_2(\beta+1, a+\beta+1; \beta+2, a+b+\beta+1; -\lambda z) \right\},
\end{aligned}$$

$$\begin{aligned}
D_3(z) = \frac{(\lambda z)^\beta}{\Gamma(\beta+1)} & \times \left\{ \frac{\Gamma'(a+\beta)\Gamma(a+b)+\Gamma(a+\beta)\Gamma'(a+b)}{\Gamma(a+b+\beta)\Gamma(a)} \right. \\
& \times {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \\
& - \frac{\Gamma'(a)\Gamma(a+\beta)\Gamma(a+b)}{\Gamma(a+b+\beta)\Gamma^2(a)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \\
& - \frac{\Gamma'(a+b+\beta)\Gamma(a+\beta)\Gamma(a+b)}{\Gamma^2(a+b+\beta)\Gamma(a)}
\end{aligned}$$

$$\times {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z)$$
$$+ \frac{\Gamma(a+\beta)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+\beta)\Gamma(\beta+1)}$$
$$\left. \times \frac{\partial {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z)}{\partial a} \right\}$$

and

$$D_4(z) = \frac{(\lambda z)^\beta \Gamma(a+\beta)}{\Gamma(a)\Gamma(\beta+1)}$$
$$\times \left\{ \frac{\Gamma'(a+b)}{\Gamma(a+b+\beta)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \right.$$
$$- \frac{\Gamma'(a+b+\beta)\Gamma(a+b)}{\Gamma^2(a+b+\beta)} {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z)$$
$$\left. + \frac{\Gamma(a+b)}{\Gamma(a+b+\beta)} \frac{\partial {}_2F_2(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z)}{\partial b} \right\}.$$

The partial derivatives of the ${}_2F_2$ hypergeometric function can be calculated by using the facts that

$$\frac{\partial {}_2F_2(a,b;c,d;z)}{\partial a} = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i \psi(a+i) z^i}{(c)_i (d)_i i!} - \psi(a) {}_2F_2(a,b;c,d;z),$$

$$\frac{\partial {}_2F_2(a,b;c,d;z)}{\partial b} = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i \psi(b+i) z^i}{(c)_i (d)_i i!} - \psi(b) {}_2F_2(a,b;c,d;z),$$

$$\frac{\partial {}_2F_2(a,b;c,d;z)}{\partial c} = \psi(c) {}_2F_2(a,b;c,d;z) - \sum_{i=0}^{\infty} \frac{(a)_i (b)_i \psi(c+i) z^i}{(c)_i (d)_i i!}$$

and

$$\frac{\partial {}_2F_2(a,b;c,d;z)}{\partial d} = \psi(d) {}_2F_2(a,b;c,d;z) - \sum_{i=0}^{\infty} \frac{(a)_i (b)_i \psi(d+i) z^i}{(c)_i (d)_i i!};$$

see http://functions.wolfram.com/07.25.20.0001.01, http://functions.wolfram.com/07.25.20.0004.01, http://functions.wolfram.com/07.25.20.0007.01 and http://functions.wolfram.com/07.25.20.0010.01. The Fisher information matrix for the estimators of (β, λ, a, b) for the case of censoring is too complicated to be presented here.

We now compare the performances of the two estimation methods. For this purpose, we generated samples of size $n = 20$ from (3.2) for $\alpha = 1, 2, \ldots, 5$, $a = 1, 2, \ldots, 5$ and $b = 1, 2, \ldots, 5$, and λ fixed as $\lambda = 1$. For each sample, we computed the MMEs and the MLEs, by solving the equations (10.1) and (10.2)–(10.5). We repeated this process 100 times and computed the average of the estimates (AE) and the mean squared error (MSE). The computer package R was used for the calculations. The results for selected α, a and b are reported in Table 1.

Note that for many cases MME does not exist because $a \le 4$. For other cases, it is clear that the MLE performs consistently better than the MME for all values of α, a and b and with respect to the AE and MSE. This is expected of course.

11 Application

The results in Sections 3–10 can be applied to the practical problems discussed in Section 2 in several different ways. For example, consider the problem discussed in Section 2.6. Suppose we have two different warning-time systems, say A and B, and that we wish to compare their performances. According to Section 2.6, the efficiencies of the two systems can be represented by the random variables $1/Z_1$ and $1/Z_2$, where Z_1 and Z_2 are distributed according to (3.1)–(3.2). So, the probability that system A is more efficient than system B can be expressed as

$$\begin{aligned} R &= \Pr(1/Z_1 > 1/Z_2) = \Pr(Z_2 > Z_1) = 1 - \Pr(Z_2 < Z_1) \\ &= 1 - \int_0^\infty F_{Z_2}(z) f_{Z_1}(z)\, dz. \end{aligned} \tag{11.1}$$

If Z_i, $i = 1, 2$, has the parameters $(\beta_i, \lambda_i, a_i, b_i)$, $i = 1, 2$, then we can express (11.1) as

$$R = 1 - \frac{\lambda_1^{\beta_1} \lambda_2^{\beta_2} B(\beta_1 + a_1, b_1) B(b_2, a_2 + \beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2 + 1) B(a_1, b_1) B(a_2, b_2)} L,$$

where

$$\begin{aligned} L = \int_0^\infty & z^{\beta_1 + \beta_2 - 1} {}_1F_1(\beta_1 + a_1; \beta_1 + a_1 + b_1; -\lambda_1 z) \\ & \times {}_2F_2(\beta_2, a_2 + \beta_2; \beta_2 + 1, a_2 + b_2 + \beta_2; -\lambda_2 z)\, dz. \end{aligned}$$

Using equation (2.21.1.1) in Prudnikov et al. (1986, Volume 3), the integral, L, can be calculated to give

$$R = 1 - \frac{\lambda_1^{2\beta_1 + \beta_2} \lambda_2^{\beta_2} \Gamma(b_1) \Gamma(b_2)}{\Gamma^2(\beta_1) B(a_1, b_1) B(a_2, b_2)} L^*, \tag{11.2}$$

where

$$L^* = G^{2,3}_{4,4}\left(\frac{\lambda_2}{\lambda_1} \middle| \begin{matrix} 1 - \beta_1 - \beta_2, 1 - \beta_2, 1 - a_2 - \beta_2, a_1 + b_1 - \beta_2 \\ 0, a_1 - \beta_2, -\beta_2, 1 - a_2 - b_2 - \beta_2 \end{matrix} \right).$$

If estimates on the parameters are available (either from prior knowledge or by applying the procedures in Section 10 to some data) then (11.2) can provide a useful measure of the relative performance of the two systems.

Table 1 *Comparison of MLE versus MME*

			MLE						MME					
α	a	b	AE($\widehat{\alpha}$)	AE($\widehat{a}$)	AE($\widehat{b}$)	MSE($\widehat{\alpha}$)	MSE($\widehat{a}$)	MSE($\widehat{b}$)	AE($\widehat{\alpha}$)	AE($\widehat{a}$)	AE($\widehat{b}$)	MSE($\widehat{\alpha}$)	MSE($\widehat{a}$)	MSE($\widehat{b}$)
1	2	3	1.032	2.255	3.458	0.027	0.414	0.894	NA	NA	NA	NA	NA	NA
1	2	4	1.035	2.390	4.832	0.033	0.623	2.389	NA	NA	NA	NA	NA	NA
1	2	5	1.032	2.366	6.042	0.031	0.531	3.802	NA	NA	NA	NA	NA	NA
1	3	2	1.008	3.478	2.358	0.022	3.145	1.364	NA	NA	NA	NA	NA	NA
1	3	4	1.027	3.739	4.895	0.032	1.712	3.754	NA	NA	NA	NA	NA	NA
1	3	5	1.024	3.440	5.759	0.031	1.217	3.424	NA	NA	NA	NA	NA	NA
1	4	2	1.001	4.751	2.344	0.030	3.316	0.635	NA	NA	NA	NA	NA	NA
1	4	3	1.086	4.407	3.227	0.030	2.235	0.862	NA	NA	NA	NA	NA	NA
1	4	5	1.015	4.471	5.800	0.037	1.902	3.797	NA	NA	NA	NA	NA	NA
1	5	2	1.039	5.709	2.188	0.035	4.065	0.407	1.120	6.582	2.600	0.039	4.177	0.413
1	5	3	1.008	6.004	3.583	0.031	4.869	1.546	1.045	7.251	3.964	0.032	5.524	1.763
1	5	4	1.010	5.644	4.545	0.034	3.206	2.406	1.132	5.818	4.807	0.040	3.274	2.503
2	1	3	2.064	1.142	3.545	0.066	0.120	1.893	NA	NA	NA	NA	NA	NA
2	1	4	2.028	1.144	4.588	0.071	0.253	3.156	NA	NA	NA	NA	NA	NA
2	1	5	2.023	1.179	6.249	0.086	0.185	9.054	NA	NA	NA	NA	NA	NA
2	3	1	2.014	3.273	1.080	0.081	1.334	0.112	NA	NA	NA	NA	NA	NA
2	3	4	2.038	3.403	4.517	0.069	1.593	2.398	NA	NA	NA	NA	NA	NA
2	3	5	2.024	3.456	5.663	0.070	1.183	3.204	NA	NA	NA	NA	NA	NA
2	4	1	2.038	4.703	1.115	0.076	3.729	0.195	NA	NA	NA	NA	NA	NA
2	4	3	2.058	4.926	3.636	0.103	3.172	1.597	NA	NA	NA	NA	NA	NA
2	4	5	2.008	4.803	5.905	0.071	2.784	4.890	NA	NA	NA	NA	NA	NA
2	5	1	1.995	5.913	1.126	0.059	3.815	0.114	2.455	6.489	1.128	0.065	4.067	0.137
2	5	3	2.059	5.718	3.420	0.081	2.805	1.241	2.233	6.607	3.550	0.088	3.410	1.390
2	5	4	2.062	5.844	4.672	0.073	6.047	3.883	2.072	6.877	5.558	0.074	6.161	4.452
3	1	2	3.029	1.109	2.162	0.105	0.094	0.474	NA	NA	NA	NA	NA	NA
3	1	4	3.020	1.185	4.828	0.118	0.155	3.748	NA	NA	NA	NA	NA	NA
3	1	5	3.034	1.183	6.118	0.142	0.160	4.994	NA	NA	NA	NA	NA	NA
3	2	1	2.974	2.374	1.183	0.123	0.845	0.160	NA	NA	NA	NA	NA	NA
3	2	4	3.027	2.320	4.671	0.118	0.586	2.800	NA	NA	NA	NA	NA	NA
3	2	5	3.034	2.328	6.012	0.119	0.647	5.885	NA	NA	NA	NA	NA	NA
3	4	1	3.037	4.664	1.077	0.150	3.039	0.091	NA	NA	NA	NA	NA	NA
3	4	2	2.971	4.682	2.268	0.115	3.348	0.617	NA	NA	NA	NA	NA	NA
3	4	5	3.018	4.637	5.823	0.111	2.904	3.952	NA	NA	NA	NA	NA	NA
3	5	1	3.078	5.864	1.135	0.144	5.461	0.135	3.228	5.899	1.353	0.153	5.931	0.161
3	5	2	3.044	5.656	2.190	0.107	4.121	0.471	3.623	5.864	2.559	0.125	4.188	0.495
3	5	4	3.004	6.006	4.935	0.099	6.135	4.563	3.120	6.966	5.595	0.102	6.623	4.940
4	1	2	4.012	1.255	2.455	0.140	0.235	0.989	NA	NA	NA	NA	NA	NA
4	1	3	4.084	1.139	3.555	0.194	0.153	2.441	NA	NA	NA	NA	NA	NA
4	1	5	4.025	1.162	5.761	0.203	0.119	4.656	NA	NA	NA	NA	NA	NA
4	2	1	4.078	2.267	1.108	0.198	0.894	0.146	NA	NA	NA	NA	NA	NA
4	2	3	4.028	2.373	3.541	0.152	0.760	1.543	NA	NA	NA	NA	NA	NA
4	2	5	4.017	2.335	5.944	0.201	0.501	4.283	NA	NA	NA	NA	NA	NA
4	3	1	4.036	3.582	1.192	0.183	2.454	0.208	NA	NA	NA	NA	NA	NA
4	3	2	4.007	3.731	2.501	0.165	2.305	1.019	NA	NA	NA	NA	NA	NA
4	3	5	4.089	3.738	6.147	0.176	2.211	6.682	NA	NA	NA	NA	NA	NA
4	5	1	4.041	5.809	1.135	0.195	5.407	0.129	4.309	6.462	1.337	0.230	5.631	0.156
4	5	2	4.039	6.039	2.426	0.189	7.561	1.113	4.485	6.599	2.600	0.191	8.781	1.206
4	5	3	4.044	6.073	3.679	0.145	7.698	3.103	4.050	6.144	4.036	0.146	8.815	3.331

We can also obtain measures of the gain in efficiency, say, by how much system A is more efficient than system B. For example,

$$\frac{\beta_1(a_1+b_1-1)}{\lambda_1(a_1-1)}-\frac{\beta_2(a_2+b_2-1)}{\lambda_2(a_2-1)}$$

gives a measure of gain in terms of the mean,

$$\frac{\beta_2(\beta_2+1)(a_2+b_2-1)(a_2+b_2-2)}{\lambda_2^2(a_2-1)(a_2-2)}-\frac{\beta_1(\beta_1+1)(a_1+b_1-1)(a_1+b_1-2)}{\lambda_1^2(a_1-1)(a_1-2)}$$
$$-\frac{\beta_2^2(a_2+b_2-1)^2}{\lambda_2^2(a_2-1)^2}+\frac{\beta_1^2(a_1+b_1-1)^2}{\lambda_1^2(a_1-1)^2}$$

gives a measure of gain in terms of the variance,

$$\begin{aligned}&2\mu_2 F_{Z_2}(\mu_2)-2\mu_1 F_{Z_1}(\mu_1)-2\mu_2+2\mu_1\\&\quad+\frac{2\lambda_2^{\beta_2}B(a_2+\beta_2,b_2)}{\Gamma(\beta_2)B(a_2,b_2)}J(1,\mu_2,a_2,b_2,\alpha_2,\lambda_2)\\&\quad-\frac{2\lambda_1^{\beta_1}B(a_1+\beta_1,b_1)}{\Gamma(\beta_1)B(a_1,b_1)}J(1,\mu_1,a_1,b_1,\alpha_1,\lambda_1)\end{aligned}$$

give a measure of gain in terms of the mean deviation about the mean [where $\mu_1 = E(Z_1)$ and $\mu_2 = E(Z_2)$], and so on.

12 Conclusions

Motivated by practical problems ranging from efficiency modeling to modeling of infectious diseases, we have studied mathematical properties of the ratio of gamma and beta random variables assumed to be independent. We have derived exact and explicit expressions for many characteristics of the ratio, including its p.d.f., c.d.f., h.r.f., moments, mean deviation about the mean, mean deviation about the median, percentiles, order statistics and the asymptotic distribution of the extreme values. We have also derived estimation procedures by the methods of moments and maximum likelihood. Finally, an illustration of applicability of the mathematical results is given in the context of efficiency of warning-time systems.

Appendix

We need the following lemmas.

Lemma 1. *Let Z be a random variable with its p.d.f. specified by* (3.2). *We have*

$$\int_x^\infty z^k f_Z(z)\,dz=\frac{\lambda^\beta B(a+\beta,b)}{\Gamma(\beta)B(a,b)}J(k,x,a,b,\alpha,\lambda)$$

for all real k, *where*

$$J(k,x,a,b,\alpha,\lambda) = -\frac{x^{k+\beta}}{k+\beta} {}_2F_2(\beta+a, k+\beta; \beta+a+b, k+\beta+1; -\lambda x).$$

Proof. Using (3.2), we can write

$$\int_x^\infty z^k f_Z(z)\,dz = \frac{\lambda^\beta B(a+\beta,b)}{\Gamma(\beta)B(a,b)} J(k,x,a,b,\alpha,\lambda),$$

where

$$J(k,x,a,b,\alpha,\lambda) = \int_x^\infty z^{k+\beta-1} {}_1F_1(\beta+a; \beta+a+b; -\lambda z)\,dz.$$

The result follows by applying http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric1F1/21/01/02/01/0001/ to calculate this integral. □

Lemma 2. *Let* Z *be a random variable with its c.d.f. and p.d.f. specified by* (3.1) *and* (3.2), *respectively. We have*

$$\int_0^\infty z^k F_Z^n(z) f_Z(z)\,dz = \frac{\Gamma(n\beta+\beta+k)B^{n+1}(a+\beta,b)}{\lambda^k\Gamma(\beta)\Gamma^n(\beta+1)B^{n+1}(a,b)} I(k,n)$$

for all real k *such that* $n\beta+\beta+k \neq 0,-1,-2,\ldots$, *where*

$$\begin{aligned} I(k,n) = F^{1:2;\ldots;2;1}_{0:2;\ldots;2;1}\big((n\beta+\beta+k) &: (\beta,a+\beta);\ldots;(\beta,a+\beta);(b); \\ &- : (\beta+1,a+b+\beta);\ldots;(\beta+1,a+b+\beta);(a+b+\beta); \\ &-1,\ldots,-1,1\big). \end{aligned}$$

Proof. Using (3.1) and (3.2), we can write

$$\int_0^\infty z^k F_Z^n(z) f_Z(z)\,dz = \frac{\lambda^{(n+1)\beta} B^{n+1}(a+\beta,b)}{\Gamma(\beta)\Gamma^n(\beta+1)B^{n+1}(a,b)} J(k,n),$$

where

$$\begin{aligned} J(k,n) = \int_0^\infty & z^{k+n\beta+\beta-1}\{{}_2F_2(\beta,a+\beta;\beta+1,a+b+\beta;-\lambda z)\}^n \\ & \times {}_1F_1(\beta+a;\beta+a+b;-\lambda z)\,dz. \end{aligned}$$

Using the fact ${}_1F_1(a;b;x) = \exp(x)\,{}_1F_1(b-a;b;-x)$ (see http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric1F1/17/02/02/) and the series expansions for hypergeometric functions, we can calculate $J(k,n)$ as

$$J(k,n) = \int_0^\infty z^{k+n\beta+\beta-1}\{{}_2F_2(\beta,a+\beta;\beta+1,a+b+\beta;-\lambda z)\}^n$$

$$
\begin{aligned}
&\times {}_1F_1(b; \beta + a + b; \lambda z) \exp(-\lambda z)\, dz \\
&= \int_0^{\infty} \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{i=0}^{\infty} (\beta)_{i_1} (a+\beta)_{i_1} \cdots (\beta)_{i_n} (a+\beta)_{i_n} (b)_i \\
&\qquad \Big/ \big((\beta+1)_{i_1} (a+b+\beta)_{i_1} \cdots (\beta+1)_{i_n} (a+b+\beta)_{i_n} \\
&\qquad \times (a+b+\beta)_i\big) \\
&\qquad \times \frac{(-1)^{i_1+\cdots+i_n} \lambda^{i_1+\cdots+i_n+i}}{i_1! \cdots i_n! i!} z^{i_1+\cdots+i_n+i+k+n\beta+\beta-1} \\
&\qquad \times \exp(-\lambda z)\, dz \\
&= \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{i=0}^{\infty} (\beta)_{i_1} (a+\beta)_{i_1} \cdots (\beta)_{i_n} (a+\beta)_{i_n} (b)_i \\
&\qquad \Big/ \big((\beta+1)_{i_1} (a+b+\beta)_{i_1} \cdots (\beta+1)_{i_n} (a+b+\beta)_{i_n} \\
&\qquad \times (a+b+\beta)_i\big) \\
&\qquad \times \frac{(-1)^{i_1+\cdots+i_n} \lambda^{i_1+\cdots+i_n+i}}{i_1! \cdots i_n! i!} \\
&\qquad \times \int_0^{\infty} z^{i_1+\cdots+i_n+i+k+n\beta+\beta-1} \exp(-\lambda z)\, dz \\
&= \frac{1}{\lambda^{k+n\beta+\beta}} \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{i=0}^{\infty} (\beta)_{i_1} (a+\beta)_{i_1} \cdots (\beta)_{i_n} (a+\beta)_{i_n} (b)_i \\
&\qquad \Big/ \big((\beta+1)_{i_1} (a+b+\beta)_{i_1} \cdots (\beta+1)_{i_n} \\
&\qquad \times (a+b+\beta)_{i_n} (a+b+\beta)_i\big) \\
&\qquad \times \frac{(-1)^{i_1+\cdots+i_n} \lambda^{i_1+\cdots+i_n+i}}{i_1! \cdots i_n! i!} \\
&\qquad \times \Gamma(i_1 + \cdots + i_n + i + k + n\beta + \beta) \\
&= \frac{\Gamma(n\beta+\beta+k)}{\lambda^{k+n\beta+\beta}} \sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{i=0}^{\infty} (\beta)_{i_1} (a+\beta)_{i_1} \cdots (\beta)_{i_n} (a+\beta)_{i_n} \\
&\qquad \Big/ \big((\beta+1)_{i_1} (a+b+\beta)_{i_1} \cdots (\beta+1)_{i_n} \\
&\qquad \times (a+b+\beta)_{i_n}\big) \\
&\qquad \times \frac{(b)_i (n\beta+\beta+k)_{i_1+\cdots+i_n+i}}{(a+b+\beta)_i} \\
&\qquad \times \frac{(-1)^{i_1+\cdots+i_n} 1^i}{i_1! \cdots i_n! i!}.
\end{aligned} \tag{A.1}
$$

The result of the lemma follows by using the definition of the generalized Kampé de Fériet function to calculate the multiple sum in (A.1). □

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