The gamma extended Weibull family of distributions

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We introduce a new family of distributions called the gamma extended Weibull family. The proposed family includes several well-known models as special cases and defines at least seventeen new special models. Structural properties of this family are studied. Additionally, the maximum likelihood method for estimating the model parameters is discussed. An application to real data illustrates the usefulness of the new family. The results provide evidence that the proposed family outperforms other classes of lifetime models.

Keywords: Extended Weibull distributions; Hazard rate function; Maximum likelihood estimation; Moments; Stacy's generalized gamma distribution.

1. Introduction

Recently, Zografos and Balakrishnan [24] introduced and studied a broad family of univariate distributions through a particular case of Stacy's generalized gamma distribution, in the same way as Jones's family is defined following the beta distribution. This new family stems from the general class: if G denotes the baseline cumulative distribution function (cdf) of a random variable, then a generalized class of distributions can be defined by

$$F(x;\delta) = \gamma\{\delta, -\log[1-G(x)]\}, \tag{1.1}$$

where $x \in \mathscr{X} \subseteq \mathbb{R}$, $\delta > 0$, $\gamma(\delta,z) = \Gamma(\delta)^{-1}\int_0^z t^{\delta-1}\,\mathrm{e}^{-t}\mathrm{d}t$ denotes the incomplete gamma function and $\Gamma(\cdot)$ is the gamma function. This family of distributions has probability density function (pdf) given by

$$f(x;\delta) = \frac{1}{\Gamma(\delta)}\{-\log[1-G(x)]\}^{\delta-1}\,g(x). \tag{1.2}$$

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Moreover, the class of extended Weibull ($\mathscr{EW}$) distributions, as proposed by Gurvich *et al.* [8], has achieved a prominent position in new probability models. Its cdf is

$$G(x;\alpha,\boldsymbol{\xi}) = 1 - \exp[-\alpha H(x;\boldsymbol{\xi})], \tag{1.3}$$

where $x \in \mathscr{D} \subseteq \mathbb{R}_+$, $\alpha > 0$ and $H(x;\boldsymbol{\xi})$ is a non-negative monotonically increasing function which depends on the parameter vector $\boldsymbol{\xi}$. The corresponding pdf is given by

$$g(x;\alpha,\boldsymbol{\xi}) = \alpha \exp[-\alpha H(x;\boldsymbol{\xi})]\, h(x;\boldsymbol{\xi}), \tag{1.4}$$

where $h(x;\boldsymbol{\xi})$ is the derivative of $H(x;\boldsymbol{\xi})$.

Note that different functions $H(x;\boldsymbol{\xi})$ in equation (1.3) yield important statistical models such as: $H(x;\boldsymbol{\xi}) = x$ gives the exponential distribution; $H(x;\boldsymbol{\xi}) = x^2$ leads to the Rayleigh distribution; $H(x;\boldsymbol{\xi}) = \log(x/k)$ leads to the Pareto distribution and $H(x;\boldsymbol{\xi}) = \beta^{-1}[\exp(\beta x) - 1]$ gives the Gompertz distribution. Table 1 displays the functions $H(\cdot;\cdot)$ and $h(\cdot;\cdot)$ and the corresponding parameter vectors for special distributions.

In this paper, we derive a new family of distributions by compounding the classes of gamma and $\mathscr{EW}$ distributions. The compounding procedure follows by taking the $\mathscr{EW}$ family of distributions as the baseline distribution in (1.1). The gamma extended Weibull ($\mathscr{GEW}$) family of distributions contains as special models the modified Weibull, Pareto and Gompertz distributions, among those listed in Table 1.

The paper is organized as follows. In Section 2, we define the $\mathscr{GEW}$ class of distributions and obtain useful expansions for its cumulative and density functions. Some mathematical properties are derived and discussed in Sections 3-6: quantile function, order statistics, generating function, incomplete moments and mean deviations. Additionally, some information theory measures for the proposed family are derived. Formulas for the Rényi and Shannon entropies are presented in Section 7 and 8, respectively. In Section 9, we present expressions for the cross entropy and Kullback-Leibler divergence. The maximum likelihood method and the observed information matrix are investigated in Section 10. Some special cases are studied in some detail in Section 11. An application to a real data set is performed in Section 12 in order to illustrate the flexibility and potentiality of the new family. Finally, main conclusions are addressed in Section 13.

2. The $\mathscr{GEW}$ family of distributions

Taking the $\mathscr{EW}$ family of distributions as the baseline model in equation (1.1), we have

$$F(x;\delta,\alpha,\boldsymbol{\xi}) = \gamma[\delta, \alpha H(x;\boldsymbol{\xi})], \tag{2.1}$$

where $x \in \mathscr{D}$, $\alpha > 0$ and $\delta > 0$. The corresponding pdf has a very simple form

$$f(x;\delta,\alpha,\boldsymbol{\xi}) = \frac{\alpha^\delta}{\Gamma(\delta)} h(x;\boldsymbol{\xi}) H(x;\boldsymbol{\xi})^{\delta-1} \exp[-\alpha H(x;\boldsymbol{\xi})], \tag{2.2}$$

where $H(x;\boldsymbol{\xi})$ corresponds to a special distribution listed in Table 1 with cdf given in (1.3).

Table 1. Special distributions and corresponding $H(x;\boldsymbol{\xi})$ and $h(x;\boldsymbol{\xi})$ functions

Distribution	$H(x;\boldsymbol{\xi})$	$h(x;\boldsymbol{\xi})$	α	$\boldsymbol{\xi}$
Exponential ($x \geq 0$) [9]	x	1	α	$\emptyset$
Pareto ($x \geq k$) [9]	$\log(x/k)$	$1/x$	α	k
Rayleigh ($x \geq 0$) [18]	x^2	$2x$	α	$\emptyset$
Weibull ($x \geq 0$) [9]	x^γ	$\gamma x^{\gamma-1}$	α	γ
Modified Weibull ($x \geq 0$) [11]	$x^\gamma \exp(\lambda x)$	$x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)$	α	$[\gamma, \lambda]$
Weibull extension ($x \geq 0$) [23]	$\lambda[\exp(x/\lambda)^\beta - 1]$	$\beta\exp(x/\lambda)^\beta(x/\lambda)^{\beta-1}$	α	$[\gamma, \lambda, \beta]$
Log-Weibull ($-\infty < x < \infty$) [21]	$\exp[(x-\mu)/\sigma]$	$(1/\sigma)\exp[(x-\mu)/\sigma]$	1	$[\mu, \sigma]$
Phani ($0 < \mu < x < \sigma < \infty$) [17]	$[(x-\mu)/(\sigma-x)]^\beta$	$\beta[(x-\mu)/(\sigma-x)]^{\beta-1}$ $\times[(\sigma-\mu)/(\sigma-t)^2]$	α	$[\mu, \sigma, \beta]$
Weibull Kies ($0 < \mu < x < \sigma < \infty$) [10]	$(x-\mu)^{\beta_1}/(\sigma-x)^{\beta_2}$	$(x-\mu)^{\beta_1-1}(\sigma-x)^{-\beta_2-1}$ $\times[\beta_1(\sigma-x)+\beta_2(x-\mu)]$	α	$[\mu, \sigma, \beta_1, \beta_2]$
Additive Weibull ($x \geq 0$) [22]	$(x/\beta_1)^{\alpha_1}+(x/\beta_2)^{\alpha_2}$	$(\alpha_1/\beta_1)(x/\beta_1)^{\alpha_1-1}$ $+(\alpha_2/\beta_2)(x/\beta_2)^{\alpha_2-1}$	1	$[\alpha_1, \alpha_2, \beta_1, \beta_2]$
Traditional Weibull ($x \geq 0$) [14]	$x^b[\exp(cx^d-1)]$	$bx^{b-1}[\exp(cx^d)-1]$ $+cdx^{b+d-1}\exp(cx^d)$	α	$[b, c, d]$
Gen. power Weibull ($x \geq 0$) [15]	$[1+(x/\beta)^{\alpha_1}]^\theta - 1$	$(\theta\alpha_1/\beta)[1+(x/\beta)^{\alpha_1}]^{\theta-1}(x/\beta)^{\alpha_1}$	1	$[\alpha_1, \beta, \theta]$
Flexible Weibull extension($x \geq 0$) [1]	$\exp(\alpha_1 x - \beta/x)$	$\exp(\alpha_1 x-\beta/x)(\alpha_1+\beta/x^2)$	1	$[\alpha_1, \beta]$
Gompertz ($x \geq 0$) [6]	$\beta^{-1}[\exp(\beta x)-1]$	$\exp(\beta x)$	α	β
Exponential power ($x \geq 0$) [19]	$\exp[(\lambda x)^\beta]-1$	$\beta\lambda\exp[(\lambda x)^\beta](\lambda x)^{\beta-1}$	1	$[\lambda, \beta]$
Chen ($x \geq 0$) [4]	$\exp(x^b)-1$	$bx^{b-1}\exp(x^b)$	α	b
Pham ($x \geq 0$) [16]	$(a^x)^\beta - 1$	$\beta(a^x)^\beta\log(a)$	1	$[a, \beta]$

The survival function of the $\mathscr{G}\mathscr{E}\mathscr{W}$ family of distributions is given by

$$S(x;\delta,\alpha,\boldsymbol{\xi}) = 1 - \gamma[\delta, \alpha H(x;\boldsymbol{\xi})],$$

$x > 0$, and its hazard rate function becomes

$$\tau(x;\delta,\alpha,\boldsymbol{\xi}) = \frac{\alpha^\delta\, h(x;\boldsymbol{\xi})\, H(x;\boldsymbol{\xi})^{\delta-1}\,\exp[-\alpha H(x;\boldsymbol{\xi})]}{\Gamma(\delta)\, S(x;\delta,\alpha,\boldsymbol{\xi})},$$

$x > 0$.

2.1. *Expansions for the distribution and density functions*

Here, we derive useful expansions to obtain some important statistical quantities such as the noncentral moment, generating function and Rényi entropy. Raising the density function (2.2) to a positive power s gives

$$f(x;\delta,\alpha,\boldsymbol{\xi})^s = \frac{\alpha^{s\delta}}{\Gamma(\delta)^s}\, h(x;\boldsymbol{\xi})^s\, H(x;\boldsymbol{\xi})^{s(\delta-1)}\exp[-s\,\alpha H(x;\boldsymbol{\xi})].$$

For any real number $\delta > 0$, we have the following equality (see `http://functions.wolfram.com/ElementaryFunctions/Log/06/01/04/03/`)

$$\{-\log[1-G(x;\alpha,\boldsymbol{\xi})]\}^{\delta-1} = (\delta-1)\sum_{k=0}^{\infty}\binom{k+1-\delta}{k}\sum_{j=0}^{k}\frac{(-1)^{j+k}\binom{k}{j}p_{j,k}}{(\delta-1-j)}G(x;\alpha,\boldsymbol{\xi})^{\alpha+k-1}, \quad (2.3)$$

where the quantities $p_{j,k}$ can be obtained recursively, for $k = 1, 2, \ldots$, as

$$p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} [k - m(j+1)]\, c_m\, p_{j,k-m},$$

$p_{j,0} = 1$ and $c_k = (-1)^{k+1}/(k+1)$. Applying (2.3) in equation (1.2) and using the binomial expansion, we can express (2.2) as an infinite linear combination of $\mathscr{EW}$ densities. We have

$$f(x;\delta,\alpha,\boldsymbol{\xi}) = \sum_{r=0}^{\infty} v_r\, g(x;\alpha(r+1),\boldsymbol{\xi}), \tag{2.4}$$

where

$$v_r = \frac{(-1)^r\,(\delta-1)}{(r+1)\,\Gamma(\delta)} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-1)^{j+k}}{(\delta-1-j)} \binom{k}{j} \binom{k+1-\delta}{k}. \tag{2.5}$$

Equation (2.4) is the main result of this section.

3. Moments, generating function and log-moment

3.1. *Moments*

Let X be a random variable following the $\mathscr{GEW}$ distribution with parameters δ, α and $\boldsymbol{\xi}$, say $X \sim \mathscr{GEW}(\delta,\alpha,\boldsymbol{\xi})$. The nth noncentral moment is given by

$$\mathrm{E}(X^n) = \int_{\mathscr{D}} x^n \frac{\alpha^{\delta-1}}{\Gamma(\delta)} H(x;\boldsymbol{\xi})^{\delta-1}\, g(x;\alpha,\boldsymbol{\xi})\mathrm{d}x = \frac{\alpha^{\delta-1}}{\Gamma(\delta)}\, \mathrm{E}_Y[Y^n H(Y;\boldsymbol{\xi})^{\delta-1}]. \tag{3.1}$$

Here and henceforth Y, denotes a random variable following the $\mathscr{EW}$ distribution with pdf given by (1.4). We can also rewrite (3.1) as

$$\mathrm{E}\,(X^n) = \int_{\mathscr{D}} x^n f(x;\delta,\alpha,\boldsymbol{\xi})\mathrm{d}x = \frac{\alpha^{\delta}}{\Gamma(\delta)} \int_{\mathscr{D}} x^n H(x;\boldsymbol{\xi})^{\delta-1}\, h(x;\boldsymbol{\xi}) \exp\left[-\alpha H(x;\boldsymbol{\xi})\right] \mathrm{d}x.$$

Setting $u = H(x;\boldsymbol{\xi})$, we have $\mathrm{d}u = h(x;\boldsymbol{\xi})\mathrm{d}x$, $x = H^{-1}(u;\boldsymbol{\xi})$ and then

$$\mathrm{E}\,(X^n) = \frac{\alpha^{\delta}}{\Gamma(\delta)} \int_{\mathscr{A}} [H^{-1}(u;\boldsymbol{\xi})]^n\, u^{\delta-1} \exp(-\alpha\, u)\mathrm{d}u, \tag{3.2}$$

where $\mathscr{A} = \{u : H^{-1}(u;\boldsymbol{\xi}) \in \mathscr{D}\}$. The integral in (3.2) can be obtained in closed-form for some special models.

In Table 2, we list the $H^{-1}(x;\boldsymbol{\xi})$ function for some special cases. Table 3 provides $\mathrm{E}(X^n)$ for the exponential, Rayleigh, Weibull and Pareto distributions.

Table 2. The $H^{-1}(x;\boldsymbol{\xi})$ function

Distribution	$H^{-1}(x;\boldsymbol{\xi})$
Exponential power	$\frac{[\log(x+1)]^{1/\beta}}{\lambda}$
Chen	$[\log(x+1)]^{1/\beta}$
Weibull extension	$\lambda\left[\log\left(\frac{x}{\lambda}+1\right)\right]^{1/\beta}$
Log-Weibull	$\sigma\log(x)+\mu$
Kies	$\frac{x^{1/\beta}\sigma+\mu}{x^{1/\beta}+1}$
Gen. Power Weibull	$\beta\left[(x+1)^{1/\theta}-1\right]^{1/\alpha_1}$
Gompertz	$\frac{\log(\beta x+1)}{\beta}$
Pham	$\left[\frac{\log(x+1)}{\log(a)}\right]^{1/\beta}$

Table 3. Values of $\mathrm{E}(X^n)$

$\mathrm{E}(X^n)$			
Exponential	Rayleigh	Weibull	Pareto (for $k>\alpha$)
$\frac{\Gamma(n+\delta)}{\alpha^n\Gamma(\delta)}$	$\frac{\Gamma\left(\frac{n}{2}+\delta\right)}{\alpha^{\frac{n}{2}}\Gamma(\delta)}$	$\frac{\Gamma\left(\frac{n}{\gamma}+\delta\right)}{\alpha^{\frac{n}{\gamma}}\Gamma(\delta)}$	$\frac{\alpha^\delta k^n}{(k-\alpha)^\delta}$

3.2. *Moment generating function*

In a similar manner, the moment generating function (mgf) of the $\mathscr{GEW}$ family of distributions is given by

$$M(t)=\mathrm{E}\left(\mathrm{e}^{tX}\right)=\frac{\alpha^{\delta-1}}{\Gamma(\delta)}\,\mathrm{E}_Y\left[\mathrm{e}^{tX}H(Y;\boldsymbol{\xi})^{\delta-1}\right].$$

This equation can be expressed as

$$M(t)=\frac{\alpha^\delta}{\Gamma(\delta)}\int_{\mathscr{D}}h(x;\boldsymbol{\xi})H(x;\boldsymbol{\xi})^{\delta-1}\exp\left[-\alpha H(x;\boldsymbol{\xi})+tx\right]\mathrm{d}x.$$

Setting again $u=H(x;\boldsymbol{\xi})$, we obtain $M(t)=\alpha^\delta\Gamma^{-1}(\delta)\int_{\mathscr{A}}u^{\delta-1}\exp\left[-\alpha u+tH^{-1}(u;\boldsymbol{\xi})\right]\mathrm{d}u$.

3.3. *Log moment*

The kth log-moment of the $\mathscr{GEW}$ family of distributions reduces to

$$\begin{aligned}\mathrm{E}[\log^k(X)]&=\int_{\mathscr{D}}\log^k(x)f(x;\delta,\alpha,\boldsymbol{\xi})\mathrm{d}x=\frac{\alpha^{\delta-1}}{\Gamma(\delta)}\int_{\mathscr{D}}\log^k(x)H(x;\boldsymbol{\xi})^{\delta-1}g(x;\alpha,\boldsymbol{\xi})\mathrm{d}x\\&=\frac{\alpha^{\delta-1}}{\Gamma(\delta)}\mathrm{E}_Y[\log^r(Y)H(Y;\boldsymbol{\xi})^{\delta-1}].\end{aligned}$$

3.4. *Dependent-H moment*

Theorem 1. *Let Y and X be two random variables represented by the cdf's* (1.3) *and* (2.1)*, respectively. Thus, the following results hold:*

(1) $\mathrm{E}_X[H(X;\boldsymbol{\xi})^k] = \frac{\Gamma(\delta+k)}{\alpha^k\Gamma(\delta)}$;
(2) $\mathrm{E}_X\left\{\log[H(X;\boldsymbol{\xi})]^k H(X;\boldsymbol{\xi})^{r-\delta+1}\right\} = \frac{\alpha^{\delta-1}}{\Gamma(\delta)}\frac{\partial^k}{\partial r^k}\left[\frac{\Gamma(r+1)}{\alpha^r}\right]$;
(3) $\mathrm{E}_X\left\{\log^k[H(X;\boldsymbol{\xi})]\right\} = \frac{\alpha^{\delta-1}}{\Gamma(\delta)}\frac{\partial^k}{\partial \delta^k}\left[\frac{\Gamma(\delta)}{\alpha^{\delta-1}}\right]$.

Setting $\delta = 1$ in these equations, we obtain the corresponding expressions for the random variable Y. The proof of this theorem is given in the appendix B.

3.5. *Incomplete moments*

The kth incomplete moment of a random variable X following the $\mathscr{GEW}$ distribution is determined as

$$T_k(z) = E(X^k \mid X < z) = \int_{-\infty}^{z} x^k f(x;\delta,\alpha,\boldsymbol{\xi})\mathrm{d}x = \sum_{r=0}^{\infty} \nu_r T_k'(z)$$

and then

$$T_k(z) = \sum_{r=0}^{\infty} \nu_r T_k'(z), \tag{3.3}$$

where $T_k'(z) = \int_{-\infty}^{z} x^k g(x;\alpha(r+1),\boldsymbol{\xi})\mathrm{d}x$ is the kth incomplete moment of the $\mathscr{EW}$ distribution and the quantity ν_r is given in (2.5).

4. Quantile function and random number generator

The $\mathscr{GEW}$ quantile function can be expressed in terms of the quantile function of the gamma distribution and of the inverse function of H, which are denoted by $Q_\Gamma(\delta;u)$ and $H^{-1}(\cdot)$, respectively. From the $\mathscr{GEW}$ cumulative distribution $F(x;\delta,\alpha,\boldsymbol{\xi}) = \gamma[\delta,\alpha H(x;\boldsymbol{\xi})]$, we have $\gamma[\delta,\alpha H(x;\boldsymbol{\xi})] = u$ and, as a consequence, $\alpha H(x;\xi) = Q_\Gamma(\delta;u)$. Therefore, the $\mathscr{GEW}$ quantile function can be expressed as

$$Q(\delta,\alpha,\boldsymbol{\xi};u) = H^{-1}\left(\frac{Q_\Gamma(\delta;u)}{\alpha};\boldsymbol{\xi}\right). \tag{4.1}$$

For example, from Table 2, the quantile functions for the Chen and Pham distributions are given by
$[\log(Q_\Gamma(\delta,u)+1)]^{1/\beta}$ and $\left[\frac{\log(Q_\Gamma(\delta,u)+1)}{\log(a)}\right]^{1/\beta}$, respectively.

Hence, the generator for $X \sim \mathscr{GEW}(\delta,\alpha,\boldsymbol{\xi})$ can be given by the following algorithm:

1: Generate $U \sim U(0,1)$.
2: Specify a function $H(\cdot;\cdot)$ such as anyone in Table 1.
3: Obtain a outcome of X by $X = H^{-1}\left(\frac{Q_\Gamma(\delta;U)}{\alpha},\boldsymbol{\xi}\right)$.

4.1. *Expansion for the quantile function*

The quantile function of the $\mathscr{GEW}$ distribution can be expressed in terms of a power series of a transformed variable v, which takes the form $v = p\,(u-t)^{\rho}$, for p, t and ρ known constants,

$$Q(u) = \sum_{i=0}^{\infty} m_i\, v^i, \tag{4.2}$$

where the coefficients m_i are suitably chosen real numbers. In Steinbrecher and Shaw [20], for the gamma distribution with shape parameter $\delta > 0$, equation (4.2) is defined by $v = [\Gamma(\delta+1)u]^{1/\delta}$ and

$$m_i = \begin{cases} 0, & \text{if } i = 0 \\ 1, & \text{if } i = 1 \\ a_{i+1}, & \text{if } i \geq 1, \end{cases}$$

where

$$a_{i+1} = \frac{1}{i(\delta+i)}\left\{\sum_{r=1}^{i}\sum_{s=1}^{i-s+1} a_r\, a_s\, a_{i-r-s+2}\, s\,(i-r-s+2) - \Delta(i)\sum_{r=2}^{i} a_r\, a_{i-r+2}\, r\,[r-\delta-(1-\delta)(i+2-r)]\right\},$$

$\Delta(i) = 0$ if $i < 2$ and $\Delta(i) = 1$ if $i \geq 2$. In this case, the first coefficients are $a_2 = 1/(\delta+1)$, $a_3 = (3\delta+5)/[2(\delta+1)^2(\delta+2)], \ldots$ Hence, the power series for the gamma quantile function can be expressed as

$$Q_{\Gamma}(\delta; u) = \sum_{i=0}^{\infty} m_i\, \Gamma(\delta+1)^{i/\delta}\, u^{i/\delta}. \tag{4.3}$$

Applying (4.3) to equation (4.1), it follows the $\mathscr{GEW}$ quantile function

$$Q(\delta, \alpha, \boldsymbol{\xi}; u) = H^{-1}\left(\frac{1}{\alpha}\sum_{i=0}^{\infty} m_i\, \Gamma(\delta+1)^{i/\delta}\, u^{i/\delta}; \boldsymbol{\xi}\right).$$

4.2. *Skewness and kurtosis*

There are several robust measures in the literature for location and dispersion. The median, for example, can be used for location and the interquartile range. Both the median and the interquartile range are based on quantiles. From this fact, Bowley [2] proposed a coefficient of skewness based on quantiles given by

$$SK = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)},$$

where $Q(\cdot)$ is the quantile function of a given distribution. It can be shown that Bowley's coefficient of skewness takes the value zero for symmetric distributions. Additionally, its largest value is one and the lowest is -1.

Moors [12] demonstrated that the conventional measure of kurtosis may be interpreted as a dispersion around the values $\mu + \sigma$ and $\mu - \sigma$. Thus, the probability mass focuses around μ or on the tails of the distribution. Therefore, based on this interpretation, Moors [12] proposed, as an alternative to the conventional coefficient of kurtosis, a robust measure based on octiles given by

$$KR = \frac{[Q(7/8) - Q(5/8)] + [Q(3/8) - Q(1/8)]}{Q(6/8) - Q(2/8)}.$$

5. Order statistics

In the following discussion, we derive the order statistics and their moments. The pdf of the ith order statistic $X_{i:n}$, for $i = 1,2,\ldots,n$, can be expressed as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)}F^{i-1}(x)\,[1-F(x)]^{n-i} = \frac{f(x)}{B(i,n-i+1)}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}F^{i+k-1}(x).$$

From equation (C.2) given in Appendix C, we obtain

$$f_{i:n}(x) = \frac{g(x)}{B(i,n-i+1)}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\sum_{m=0}^{\infty}\eta_{m,v}\,H(x;\boldsymbol{\xi})^{m+\delta(i+k)-1},$$

where $\eta_{m,v} = \alpha^{\delta-1}\,s_{m,v}/\Gamma(\delta)$ and $s_{m,v}$ is defined in this appendix. Additionally, from equation (C.2) given in Appendix C, the vth ordinary moment of $X_{i:n}$ becomes

$$\mathrm{E}(X_{i:n}^{v}) = \int_{\mathscr{D}} x^v f_{i:n}(x)\mathrm{d}x = \frac{1}{B(i,n-i+1)}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\mu_{v,i+k-1},$$

where the quantity $\mu_{v,i+k-1} = \mathrm{E}[X^v F^{i+k-1}(X)]$ is the probability weighted moment (pwm) of the $\mathscr{GEW}$ distribution.

6. Mean deviations

The mean deviations about the mean and the median for the $\mathscr{GEW}$ family of distributions can be expressed as

$$\delta_1(X) = \int_{\mathscr{D}} |x-\mu|\,f(x;\delta,\alpha,\boldsymbol{\xi})\,\mathrm{d}x \quad \text{and} \quad \delta_2(X) = \int_{\mathscr{D}} |x-M|\,f(x;\delta,\alpha,\boldsymbol{\xi})\,\mathrm{d}x,$$

respectively, where $\mu = \mathrm{E}(X)$ denotes the mean and $M = \mathrm{Median}(X)$ the median. The median follows from the nonlinear equation $F(M;\delta,\alpha,\boldsymbol{\xi}) = 1/2$. These quantities can be reduced to

$$\delta_1(X) = 2\,\mu\,F(\mu;\delta,\alpha,\boldsymbol{\xi}) - 2\,T_1(\mu) \quad \text{and} \quad \delta_2(X) = \mu - 2\,T_1(M),$$

where $T_1(z) = \int_{-\infty}^{z} x\,f(x;\delta,\alpha,\boldsymbol{\xi})\mathrm{d}x$ is the first incomplete moment.

From equation (3.3), the quantity $T_1(z)$ for the $\mathscr{GEW}$ distribution becomes $T_1(z) = \sum_{r=0}^{\infty} \nu_r\,T_1'(z)$, where $T_1'(z) = \int_{-\infty}^{z} x\,g(x;\alpha(r+1),\boldsymbol{\xi})\mathrm{d}x$ is the first incomplete moment of the $\mathscr{EW}$ distribution.

7. Rényi entropy

Let Y be a random variable with density $f(y;\theta)$ with support $y \in \mathscr{A} \subset \mathbb{R}$. The Rényi entropy is defined by

$$H_{\mathrm{R}}^{s}(Y) = \frac{1}{1-s}\log\left\{\mathrm{E}_Y[f(Y;\theta)^{s-1}]\right\} = \frac{1}{1-s}\log\left(\int_{\mathscr{A}} f(y;\theta)^s\mathrm{d}y\right),$$

where $s \in (0,\infty)\setminus\{1\}$.

We can obtain the Rényi entropy for the $\mathscr{GEW}$ distribution as

$$
\begin{aligned}
H_{\mathrm{R}}^{s}(X) &= \frac{1}{1-s}\log\left(\frac{\alpha^{s\delta}}{\Gamma(\delta)^{s}}\int_{\mathscr{D}} H(x;\boldsymbol{\xi})^{s(\delta-1)}\,h(x;\boldsymbol{\xi})^{s}\exp\left[-s\,\alpha H(x;\boldsymbol{\xi})\right]\mathrm{d}x\right)\\
&= \frac{1}{1-s}\left\{s\delta\log(\alpha)-s\log[\Gamma(\delta)]+\log\left(\int_{\mathscr{D}} H(x;\boldsymbol{\xi})^{s(\delta-1)}\,h(x;\boldsymbol{\xi})^{s}\exp\left[-s\,\alpha H(x;\boldsymbol{\xi})\right]\mathrm{d}x\right)\right\}.
\end{aligned}
\tag{7.1}
$$

8. Shannon entropy

Let Y be defined as in Section 7. Here, we derive the Shannon entropy defined by

$$H_{\mathrm{S}}(Y)=\mathrm{E}_{Y}\{-\log[f(Y;\boldsymbol{\theta})]\}=-\int_{\mathscr{D}}\log[f(y;\boldsymbol{\theta})]f(y;\boldsymbol{\theta})\mathrm{d}y.$$

The log-likelihood function relative to one observation follows from (2.2) as

$$\log[f(x;\delta,\alpha,\boldsymbol{\xi})]=\log\left[\frac{\alpha^{\delta}}{\Gamma(\delta)}\right]+(\delta-1)\log[H(x;\boldsymbol{\xi})]-\alpha H(x;\boldsymbol{\xi})+\log[h(x;\boldsymbol{\xi})].$$

Thus, the Shannon entropy of X can be expressed as

$$H_{\mathrm{S}}(X)=-\log\left[\frac{\alpha^{\delta}}{\Gamma(\delta)}\right]-(\delta-1)\,\mathrm{E}_{X}\{\log[H(X;\boldsymbol{\xi})]\}+\alpha\,\mathrm{E}_{X}[H(X;\boldsymbol{\xi})]-\mathrm{E}_{X}\{\log[h(X;\boldsymbol{\xi})]\}.$$

Using Theorem 1, the following results hold: (i) $\mathrm{E}_{X}\{\log[H(X;\boldsymbol{\xi})]\}=\psi(\delta)-\log(\alpha)$, where $\psi(\cdot)$ is the digamma function and (ii) $\mathrm{E}_{X}[H(X;\boldsymbol{\xi})]=\delta/\alpha$. Finally, the Shannon entropy reduces to

$$H_{\mathrm{S}}(X)=\log\left[\frac{\Gamma(\delta)}{\alpha}\right]-(\delta-1)\,\psi(\delta)+\delta-\mathrm{E}_{X}\{\log[h(X;\boldsymbol{\xi})]\}. \tag{8.1}$$

9. Cross entropy and Kullback-Leibler Divergence

Let X and Y be two random variables with common support $\mathbb{R}_{+}$ whose densities are $f_X(x;\theta_1)$ and $f_Y(y;\theta_2)$, respectively. Cover and Thomas [5] defined the *cross entropy* as

$$C_X(Y)=\mathrm{E}_{X}\{-\log[f_Y(X;\theta_2)]\}=-\int_{0}^{\infty}f_X(z;\theta_1)\log[f_Y(z;\theta_2)]\mathrm{d}z.$$

Now, consider $X\sim\mathscr{GEW}(\delta_x,\alpha_x,\boldsymbol{\xi}_x)$ and $Y\sim\mathscr{GEW}(\delta_y,\alpha_y,\boldsymbol{\xi}_y)$. After some algebraic manipulations, we obtain

$$
\begin{aligned}
C_X(Y) &= -\int_{\mathscr{D}} f_X(z;\delta_x,\alpha_x,\boldsymbol{\xi}_x)\log[f_Y(z;\delta_y,\alpha_y,\boldsymbol{\xi}_y)]\mathrm{d}z\\
&= -\left\{\log\left[\frac{\alpha_y^{\delta_y}}{\Gamma(\delta_y)}\right]+\delta_y\,\mathrm{E}_X\left\{\log[H(X;\boldsymbol{\xi}_y)]\right\}-\alpha_y\,\mathrm{E}_X\left[H(X;\boldsymbol{\xi}_y)\right]+\mathrm{E}_X\{\log[h(X;\boldsymbol{\xi}_y)]\}\right\}.
\end{aligned}
\tag{9.1}
$$

An important measure in statistical information theory is the Kullback-Leibler divergence given by

$$D(X||Y) = C_X(Y) - H_S(X) = \mathrm{E}_X\left\{\log\left[\frac{f_X(X;\delta_x,\alpha_x,\boldsymbol{\xi}_x)}{f_Y(X;\delta_y,\alpha_y,\boldsymbol{\xi}_y)}\right]\right\}. \quad (9.2)$$

Applying (8.1) and (9.1) in equation (9.2) yields

$$\begin{aligned} D(X||Y) =& \mathrm{E}_X\left\{\log\left[\frac{h(X;\boldsymbol{\xi}_x)}{h(X;\boldsymbol{\xi}_y)}\right]\right\} + \log\left[\frac{\Gamma(\delta_y)\alpha_x}{\alpha_y^{\delta_y}\Gamma(\delta_x)}\right] - \delta_y\,\mathrm{E}_X[\log H(X;\boldsymbol{\xi}_y)] + (\delta_x-1)\psi(\delta_x) \\ & - \delta_x + \alpha_y\,\mathrm{E}_X[H(X;\boldsymbol{\xi}_y)]. \end{aligned}$$

10. Estimation and observed information matrix

The parameters of the $\mathscr{GEW}$ distribution can be estimated by the method of maximum likelihood. Let $x_1,\cdots,x_n$ be a random sample of size n from $X \sim \mathscr{GEW}(\delta,\alpha,\boldsymbol{\xi})$. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\delta,\alpha,\boldsymbol{\xi}^\top)^\top$ can be written as

$$l(\boldsymbol{\theta}) = n\delta\log(\alpha) - n\log[\Gamma(\delta)] + (\delta-1)\sum_{i=1}^{n}\log[H(x_i;\boldsymbol{\xi})] - \alpha\sum_{i=1}^{n}H(x_i;\boldsymbol{\xi}) + \sum_{i=1}^{n}\log[h(x_i;\boldsymbol{\xi})].$$

The components of the score vector $\mathbf{U}(\theta)$ are

$$U_\delta(\boldsymbol{\theta}) = \frac{\partial l(\theta)}{\partial\delta} = -n\psi(\delta) + \sum_{i=1}^{n}\log[H(x_i;\boldsymbol{\xi})] + n\log\alpha, \quad U_\alpha(\boldsymbol{\theta}) = \frac{\partial l(\theta)}{\partial\alpha} = \frac{n\delta}{\alpha} - \sum_{i=1}^{n}H(x_i;\boldsymbol{\xi})$$

and $U_{\boldsymbol{\xi}_k}(\boldsymbol{\theta}) = \frac{\partial l(\theta)}{\partial\boldsymbol{\xi}_k} = (\delta-1)\sum_{i=1}^{n}\frac{1}{H(x_i;\boldsymbol{\xi})}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} - \alpha\sum_{i=1}^{n}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k} + \sum_{i=1}^{n}\frac{1}{h(x_i;\boldsymbol{\xi})}\frac{\partial h(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k}$.

The partitioned observed information matrix for the $\mathscr{GEW}$ distribution is

$$J(\boldsymbol{\theta}) = -\begin{pmatrix} U_{\delta\delta} & U_{\delta\alpha} & | & U_{\delta\xi}^\top \\ U_{\alpha\delta} & U_{\alpha\alpha} & | & U_{\alpha\xi}^\top \\ -- & -- & -- & -- \\ U_{\delta\xi} & U_{\alpha\xi} & | & U_{\xi\xi} \end{pmatrix},$$

whose elements are $U_{\delta\delta}(\boldsymbol{\theta}) = -\psi^{(1)}(\delta)$, $U_{\delta\alpha}(\boldsymbol{\theta}) = n\alpha^{-1}$, $U_{\alpha\alpha}(\boldsymbol{\theta}) = -n\delta\alpha^{-2}$,

$$\begin{aligned} U_{\delta\boldsymbol{\xi}_k}(\boldsymbol{\theta}) =& \sum_{i=1}^{n}\frac{1}{H(x_i;\boldsymbol{\xi})}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k}, \qquad U_{\alpha\boldsymbol{\xi}_k}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k}, \quad \text{and} \\ U_{\boldsymbol{\xi}_k\boldsymbol{\xi}_j}(\boldsymbol{\theta}) =& (\delta-1)\sum_{i=1}^{n}\frac{1}{H(x_i;\boldsymbol{\xi})}\left[\frac{\partial^2 H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k\boldsymbol{\xi}_j} - \frac{1}{H(x_i;\boldsymbol{\xi})}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k}\frac{\partial H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_j}\right] \\ & - \alpha\sum_{i=1}^{n}\frac{\partial^2 H(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k\boldsymbol{\xi}_j} + \sum_{i=1}^{n}\frac{1}{h(x_i;\boldsymbol{\xi})}\left[\frac{\partial^2 h(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k\boldsymbol{\xi}_j} - \frac{1}{h(x_i;\boldsymbol{\xi})}\frac{\partial h(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_k}\frac{\partial h(x_i;\boldsymbol{\xi})}{\partial\boldsymbol{\xi}_j}\right]. \end{aligned}$$

11. Two special models

11.1. *The gamma modified Weibull distribution*

For $H(x;\gamma) = x^{\gamma}\exp(\lambda x)$ and $h(x;\gamma) = x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)$, we obtain the gamma modified Weibull ($\mathscr{GMW}$) density $f(x;\,\delta,\alpha,\gamma,\lambda) = \alpha^{\delta}\Gamma^{-1}(\delta)x^{\gamma\delta-1}(\gamma+\lambda x)\exp[\delta\lambda x-\alpha x^{\gamma}\exp(\lambda x)]$, where $x>0$ and $\lambda,\gamma \geq 0$. If $\delta=1$, it gives as special case the modified Weibull ($\mathscr{MW}$) distribution proposed by Lai *et al.* [11]. In addition, when $\lambda = 0$, it gives the Weibull distribution.

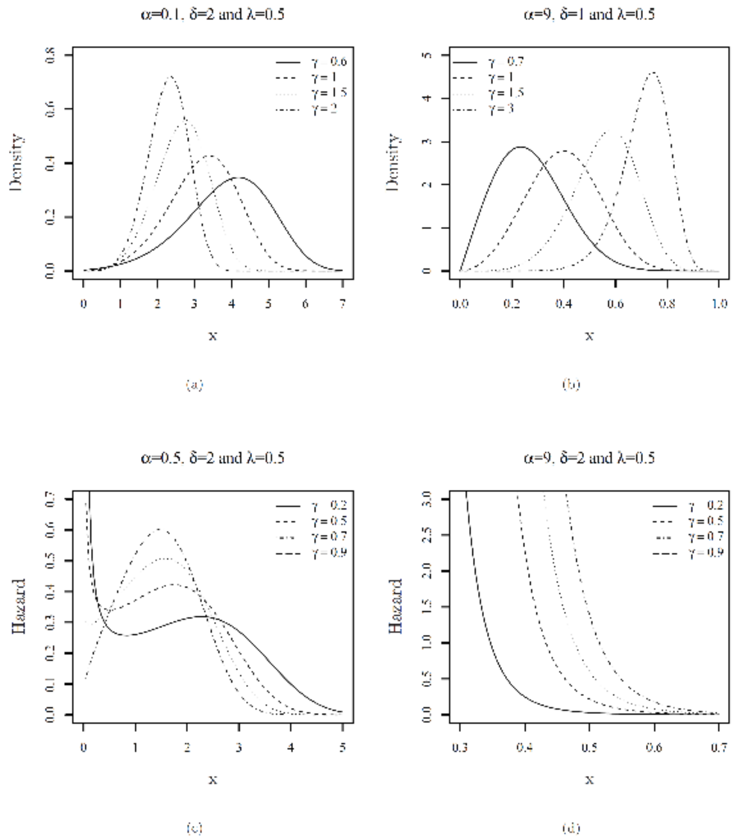


Fig. 1. Plots of the $\mathscr{GMW}$ density and Hazard rate function for some parameter values.

From equation (2.4), we can obtain $f(x;\,\delta,\alpha,\gamma,\lambda) = \sum_{r=1}^{\infty}\nu_r\, g(x;\,\alpha(r+1),\gamma,\lambda)$, where $g(x;\,\alpha(r+1),\gamma,\lambda)$ is the $\mathscr{MW}$ density function with parameters $\alpha(r+1),\gamma$ and λ. The $\mathscr{GMW}$ hazard function is $\tau(x;\,\delta,\alpha,\gamma,\lambda) = [\Gamma(\delta)\,S(x;\delta,\alpha,\gamma,\lambda)]^{-1}\ \alpha^{\delta}(\gamma+\lambda x)\,x^{\gamma\delta-1}\exp(\delta\lambda x-\alpha x^{\gamma}\mathrm{e}^{\lambda x})$, where $x>0$.

The raw moment of a random variable X following the $\mathscr{GMW}$ distribution has closed-form computed from (2.4) as

$$\mathrm{E}(X^k) = \sum_{r=0}^{\infty}\nu_r\,\mu_k'(r), \tag{11.1}$$

where $\mu_k'(r) = \int_0^{\infty} x^k g(x;\,\alpha(r+1),\gamma,\lambda)\mathrm{d}x$ denotes the raw moment of the $\mathscr{MW}$ distribution with parameters $\alpha(r+1),\gamma$ and λ. Carrasco *et al.* (2008) [3] obtained an infinite representation for this moment as

$$\mu_k'(r) = \sum_{i_1,\ldots,i_k=1}^{\infty} \frac{A_{i_1,\ldots,i_k}\,\Gamma(s_k/\gamma+1)}{[\alpha\,(r+1)]^{s_k/\gamma}}, \qquad (11.2)$$

where $A_{i_1,\ldots,i_k} = a_{i_1},\ldots,a_{i_k}$ and $s_k = i_1,\ldots,i_k$, and $a_i = \frac{(-1)^{i+1} i^{i-2}}{(i-1)!}\left(\frac{\lambda}{\gamma}\right)^{i-1}$.

Hence, the moments of the $\mathscr{GMW}$ distribution can be computed directly from equations (11.1) and (11.2).

11.2. *The gamma Pareto distribution*

For $H(x;k) = \log(x/k)$ and $h(x;k) = 1/x$, we obtain the gamma Pareto ($\mathscr{GP}$) density

$$f(x;\delta,\alpha,k) = \frac{\alpha^{\delta}\,k^{\alpha}}{\Gamma(\delta)\,x^{\alpha+1}}\left[\log\left(\frac{x}{k}\right)\right]^{\delta-1}, x \geq k. \qquad (11.3)$$

The hazard rate function is

$$\tau(x;\delta,\alpha,k) = \frac{\alpha^{\delta} k^{\alpha}\left[\log(x/k)\right]^{\delta-1}}{\Gamma(\delta)\,x^{\alpha+1}\,S(x;\delta,\alpha,k)}.$$

From equations (2.4) and (11.3), we obtain the sth ordinary moment of X

$$\mathrm{E}(X^s) = \alpha k^s \sum_{r=0}^{\infty} \frac{v_r(r+1)}{\left[\alpha(r+1)-s\right]}, \quad \text{for} \quad \alpha > s,$$

where the coefficients v_r are given by (2.5).

From equation (7.1) we obtain the Rényi entropy of the $\mathscr{GP}$ distribution, which is valid for $s > 1$, as

$$H_{\mathrm{R}}^{s}(X) = \frac{1}{1-s}\left\{s\,\delta\log(\alpha) - s\log\left[k^{s\alpha}\int_k^{\infty} x^{1-s\delta}\,\log^{s(\delta-1)}(x/k)\mathrm{d}x\right]\right\}.$$

Similarly, we obtain from equation (8.1) the Shannon entropy given by $H_{\mathrm{S}}(X) = \log[\frac{\Gamma(\delta)}{\alpha}] - (\delta-1)\,\psi(\delta) + \delta + \alpha\,\delta\,k^{\alpha-\delta}$.

12. Application

We assess the efficiency of the proposed model in an analysis of real data. We compare the fits of some $\mathscr{GEW}$ distributions and those of some sub-models such as the $\mathscr{GMW}$, gamma Weibull ($\mathscr{GW}$), gamma Rayleigh ($\mathscr{GR}$), $\mathscr{GP}$, $\mathscr{MW}$ and Weibull distributions. In order to estimate the parameters of these submodels in the class of the $\mathscr{GEW}$ distributions, we adopt the maximum likelihood method (as discussed in Section 10) using the subroutine `NLMixed` of the software SAS. The data for this application, consisting of the failure times of 20 mechanical components given in Murthy *et al*. [13] are listed in Table 4.

Table 6 displays the MLEs of the parameters and the values of the statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC). From the values of these statistics, we verify that the $\mathscr{GP}$ model provides a better fit to

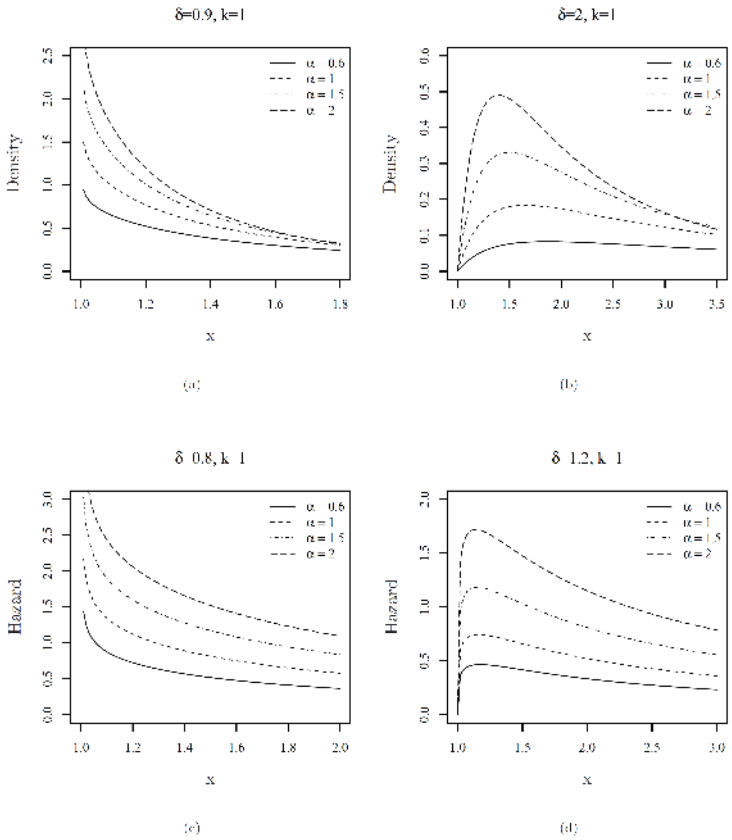


Fig. 2. Plots of the $\mathscr{G}\mathscr{P}$ density and hazard rate function for some parameter values.

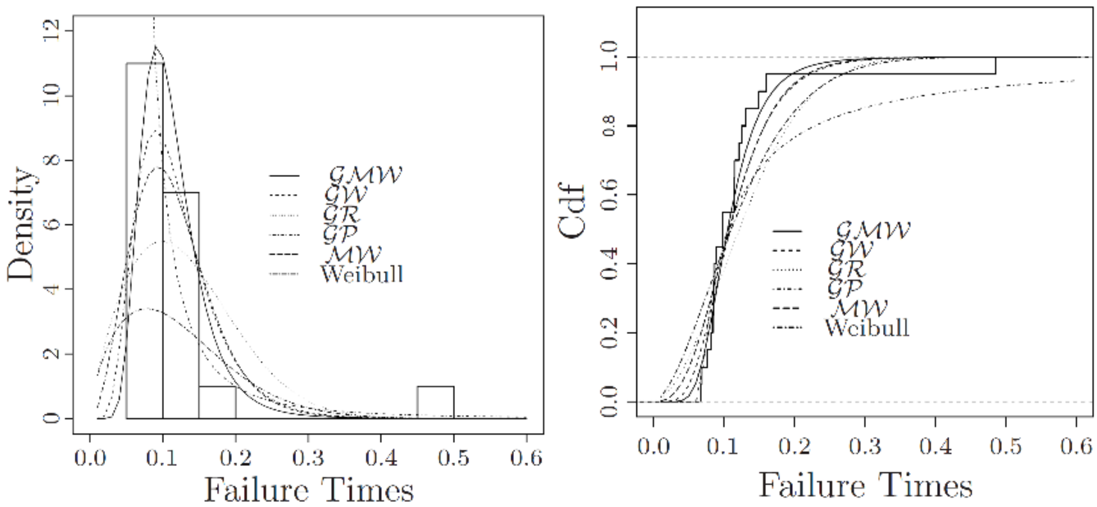
Table 4. The failure times of 20 mechanical components

0.067	0.068	0.076	0.081	0.084	0.085	0.085	0.086	0.089	0.098
0.098	0.114	0.114	0.115	0.121	0.125	0.131	0.149	0.160	0.485

Table 5. The K-S statistics and $-2\ell(\widehat{\theta})$ for some fitted models

Model	K–S	$-2\ell(\widehat{\theta})$
$\mathscr{G}\mathscr{M}\mathscr{W}$	0.8761	-71.2
$\mathscr{G}\mathscr{W}$	0.1855	-65.4
$\mathscr{G}\mathscr{R}$	0.3001	-50.7
$\mathscr{G}\mathscr{P}$	0.2518	-80.4
$\mathscr{M}\mathscr{W}$	0.8007	-61.7
Weibull	0.2641	-52.8

these data. Additionally, the $\mathscr{G}\mathscr{E}\mathscr{W}$ and $\mathscr{G}\mathscr{W}$ models are much better than the $\mathscr{M}\mathscr{W}$ and Weibull models.



(a) (b)

Fig. 3. Estimated densities, cumulative and empirical distributions from the fitted $\mathscr{GMW}$,$\mathscr{GM}$,$\mathscr{GR}$,$\mathscr{GP}$,$\mathscr{MW}$ and Weibull models for the failure times.

Table 6. MLEs of the model parameters and the statistics AIC, BIC and AICC

Model	Estimates				AIC	BIC	AICC
$\mathscr{GMW}$ $(\delta,\alpha,\gamma,\lambda)$	93.9631	223.2331	0.3563	−0.6258	−63.2	−59.2	−60.5
$\mathscr{GW}$ (δ,α,γ)	72.2393	128.7642	0.2622		−59.4	−57.9	−56.4
$\mathscr{GR}$ (δ,α)	0.8884	39.7374			−46.7	−44.8	−46.0
$\mathscr{GP}$ (δ,α,k)	0.7490	1.5826	0.0670		−76.4	−74.4	−75.6
$\mathscr{MW}$ (α,γ,λ)	828.48	2.9129	−5.4296		−55.7	−52.7	−54.2
Weibull (α,γ)	25.9723	1.6422			−48.8	−46.9	−48.1

More information is provided by a visual comparison of the fitted density functions and the histogram of the data. The plots of the fitted $\mathscr{GMW}$,$\mathscr{GW}$,$\mathscr{GR}$,$\mathscr{GP}$,$\mathscr{MW}$ and Weibull density functions and estimated cumulative functions are given in Figure 3. Based on these plots, we conclude that the new distributions provide adequate fits. Table 5 lists the values of the Kolmogorov-Smirnov (K-S) statistic and of $-2\ell(\widehat{\theta})$.

13. Conclusion

We propose and study the gamma extended Weibull ($\mathscr{GEW}$) family of distributions. The new density function can be expressed as a mixture of extended Weibull density functions. This result is important to derive some mathematical properties of the new family including moments, generating function, mean deviations, Shannon entropy, Rényi entropy, Cross entropy and Kullback-Leibler Divergence. We also derive the density function of the order statistics and their moments. Two special distributions are investigated in some detail. The model parameters are estimated by maximum likelihood. An example to real data illustrates the importance and potentiality of the new family.

Appendix A. Theoretical background

For a positive integer s, we have that (see Gradshteyn and Ryzhik [7])

$$\left(\sum_{m=0}^{\infty} a_m x^m\right)^s = \sum_{m=0}^{\infty} t_{s,m} x^m, \tag{A.1}$$

where the coefficients $t_{s,m}$, for $m = 1,2,\ldots$, are obtained by the recurrence equation $t_{s,0} = a_0^s$ and $t_{s,m} = (ma_0)^{-1} \sum_{j=1}^{m} [j(s+1)-m]\; a_j t_{s,m-j}$.

Appendix B. Proof of Theorem 1

From equation (2.2), we have

$$\int_{\mathscr{D}} H(x;\boldsymbol{\xi})^r g(x;\alpha,\boldsymbol{\xi})\mathrm{d}x = \frac{\Gamma(r+1)}{\alpha^r}. \tag{B.1}$$

The kth derivative with respect to r at both sides of equation (B.1) yields

$$\int_{\mathscr{D}} \log^k[H(x;\boldsymbol{\xi})] H(x;\boldsymbol{\xi})^r g(x;\alpha,\boldsymbol{\xi})\mathrm{d}x = \frac{\partial^k}{\partial r^k}\left[\frac{\Gamma(r+1)}{\alpha^r}\right]$$

and then $\mathrm{E}\{\log^k[H(X;\boldsymbol{\xi})] H(X;\boldsymbol{\xi})^{r-\delta+1}\} = \frac{\alpha^{\delta-1}}{\Gamma(\delta)}\frac{\partial^k}{\partial r^k}\left[\frac{\Gamma(r+1)}{\alpha^r}\right]$.

Using equation (B.1) with $r = \delta - 1$ and differentiating k times with respect to δ,

$$\int_{\mathscr{D}} \log^k[H(x;\boldsymbol{\xi})] H(x;\boldsymbol{\xi})^{\delta-1} g(x;\alpha,\boldsymbol{\xi})\mathrm{d}x = \frac{\partial^k}{\partial \delta^k}\left[\frac{\Gamma(\delta)}{\alpha^{\delta-1}}\right].$$

After multiplying both sides of this equation by $\frac{\alpha^{\delta-1}}{\Gamma(\delta)}$, we can write $\mathrm{E}\{\log^k[H(X;\boldsymbol{\xi})]\} = \frac{\alpha^{\delta-1}}{\Gamma(\delta)}\frac{\partial^k}{\partial \delta^k}\left[\frac{\Gamma(\delta)}{\alpha^{\delta-1}}\right]$.

Appendix C. A linear combination for the quantity $f(x;\delta,\alpha,\xi)F(x;\delta,\alpha,\xi)^v$

First, we derive a power series expansion for $F(x;\delta,\alpha,\boldsymbol{\xi})^v$. From equation (2.1), we have

$$F(x;\delta,\alpha,\boldsymbol{\xi})^v = \left(\frac{[\alpha H(x;\boldsymbol{\xi})]^\delta}{\Gamma(\delta)}\right)^v \left(\sum_{m=0}^{\infty} \frac{(-\alpha)^m}{(\delta+m)m!} H(x;\boldsymbol{\xi})^m\right)^v = \left(\frac{[\alpha H(x;\boldsymbol{\xi})]^\delta}{\Gamma(\delta)}\right)^v \left(\sum_{m=0}^{\infty} w_m H(x;\boldsymbol{\xi})^m\right)^v,$$

where $w_m = (-\alpha)^m/[(\delta+m)m!]$. We assume that v is a positive integer, and then the Eq. A.1 implies that

$$F(x;\delta,\alpha,\boldsymbol{\xi})^v = \left(\frac{[\alpha H(x;\boldsymbol{\xi})]^\delta}{\Gamma(\delta)}\right)^v \sum_{m=0}^{\infty} w_{m,v} H(x;\boldsymbol{\xi})^m = \sum_{m=0}^{\infty} w_{m,v}\left(\frac{\alpha^\delta}{\Gamma(\delta)}\right)^v H(x;\boldsymbol{\xi})^{m+v\delta} = \sum_{m=0}^{\infty} s_{m,v} H(x;\boldsymbol{\xi})^{m+v\delta}, \tag{C.1}$$

where $s_{m,v} = w_{m,v}\left(\frac{\alpha^\delta}{\Gamma(\delta)}\right)^v$ and the coefficients $w_{m,v}$ for $m = 1,2,\ldots$ are obtained from the recurrence relation in Eq. A.1. Combining this result and the expansion (C.1), we have

$$f(x;\delta,\alpha,\xi)F(x;\delta,\alpha,\xi)^v \sum_{m=0}^{\infty} s_{m,v} H(x;\boldsymbol{\xi})^{m+v\delta} = \frac{\alpha^{\delta-1}}{\Gamma(\delta)} g(x;\alpha,\boldsymbol{\xi}) \sum_{m=0}^{\infty} s_{m,v} H(x;\boldsymbol{\xi})^{m+\delta(v+1)-1}, \tag{C.2}$$

where the last equation holds because of (1.4).

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