The Gap Lemma and Geometric Criteria for Instability of Viscous Shock Profiles

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Abstract

An obstacle in the use of Evans function theory for stability analysis of traveling waves occurs when the spectrum of the linearized operator about the wave accumulates at the imaginary axis, since the Evans function has in general been constructed only away from the essential spectrum. A notable case in which this difficulty occurs is in the stability analysis of viscous shock profiles. Here we prove a general theorem, the "gap lemma," concerning the analytic continuation of the Evans function associated with the point spectrum of a traveling wave into the essential spectrum of the wave. This allows geometric stability theory to be applied in many cases where it could not be applied previously.

We demonstrate the power of this method by analyzing the stability of certain undercompressive viscous shock waves. A necessary geometric condition for stability is determined in terms of the sign of a certain Melnikov integral of the associated viscous profile. This sign can easily be evaluated numerically. We also compute it analytically for solutions of several important classes of systems. In particular, we show for a wide class of systems that homoclinic (solitary) waves are linearly unstable, confirming these as the first known examples of unstable viscous shock waves. We also show that (strong) heteroclinic undercompressive waves are sometimes unstable. Similar stability conditions are also derived for Lax and overcompressive shocks and for $n \times n$ conservation laws, $n \geq 2$. © 1998 John Wiley & Sons, Inc.

1 Introduction

Traveling-wave solutions occur in many important systems modeling a variety of physical phenomena. The stability of these solutions is often an issue of key importance in understanding the types of physically observable phenomena that the system is capable of supporting, while instabilities often signal the onset of pattern formation. The stability of traveling waves is also an important issue in the construction of stable numerical approximation schemes. Our particular interest here is in *viscous shock waves*, which are traveling-wave solutions of systems of viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x \tag{1.1}$$

tending to asymptotic values $u_\pm$ as $x \to \pm\infty$.

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In general, traveling waves arise as stationary solutions $u = \bar{u}(x)$ of nonlinear systems of PDEs,

$$u_t = \mathcal{F}\left(\frac{\partial}{\partial x}, u\right). \tag{1.2}$$

The linearized equations about such a stationary solution are then

$$v_t = Mv\,; \qquad M := \frac{\partial}{\partial u}\mathcal{F}\bigg|_{u=\bar{u}(x)}. \tag{1.3}$$

The eigenvalue equation $Mw = \lambda w$ associated with M can be recast as a nonautonomous, linear system of ODEs of the form

$$W' = \mathbb{A}(x, \lambda)W\,. \tag{1.4}$$

In situations in which the underlying wave tends to limits at $\pm\infty$, the point spectrum of the wave is determined by the values of λ for which there is a nontrivial solution $W(x, \lambda)$ of (1.4) that decays to zero as $x \to \pm\infty$. This can be measured in terms of the vanishing of a certain Wronskian $D(\lambda)$ associated to this linear system called the *Evans function*. There is a substantial literature concerning the Evans function and its applications to the stability of traveling waves; see, e.g., [1, 9, 10, 11, 12, 16, 35]. However, a well-known obstacle to the use of these techniques is the presence of essential spectrum of M on or near the imaginary axis. This problem always occurs when certain physical effects such as conservation or dispersion are present, in particular for (1.1). It also occurs occasionally near certain types of singular limits of reaction-diffusion systems.

In this paper we prove a general result, the "gap lemma," on the analytic continuation of $D(\lambda)$ to a region inside the essential spectrum (in certain situations, this domain may be a Riemann surface). While it is no longer the case that the roots of the continuation of $D(\lambda)$ are necessarily eigenvalues of the differential operator, the analyticity of the continuation can play an important role in the search for eigenvalues in the right half-plane. In particular, it is then possible to use winding number calculations as well as other topological methods to count the number of roots of $D(\lambda)$ and so to obtain upper bounds for the number of eigenvalues of the associated differential operator.

We demonstrate the utility of this method by the stability analysis of *undercompressive* viscous shock waves. These have more sensitive stability properties than standard Lax shocks. Indeed, recent studies in [3, 4] suggest that many can be *unstable*, in sharp contrast to the usual case in conservation laws. However, no analytic results have previously been obtained in this direction. We provide a geometric, necessary condition for stability, which is easily checked numerically, and we calculate it analytically for several interesting example systems. Our results verify the observed numerical phenomenon of *instability*, the first analytic results of this kind for viscous shock waves. We emphasize that this instability is connected with point spectrum of M, that is, with the internal dynamics of the shock layer. This is a different type of instability from that observed for constant-state solutions in, e.g., [34], which is connected with the essential spectrum of M and far-field behavior.

This newly discovered instability has important consequences in regard to the large-time behavior of solutions of the underlying hyperbolic conservation laws. The standard picture that has emerged from the analysis of genuinely nonlinear, strictly hyperbolic systems is that the large-time dynamics are determined by the solution of the Riemann problem resolving the left and right states $u_\pm$ of the initial data. In certain situations, such as for small BV solutions of general systems, it has been possible to prove a rigorous theorem along these lines (see [29]). For more general systems that admit nonclassical waves, the existence of *unstable* undercompressive shock profiles presents a dramatically different set of possibilities for the large-time dynamics. In particular, numerical experiments for the hyperbolic conservation laws [3, 4] demonstrate that the solutions sometimes exhibit a threshold behavior in which the large-time dynamics are determined by the initial data $u_0(x)$ for finite x as well as $u_0(\pm\infty)$. This threshold can be understood heuristically in terms of the stable manifold $W^s(\bar{u}(x))$ in the infinite-dimensional state space for the initial data. In the case in which this manifold has codimension one, i.e., the Morse index of $\bar{u}(x)$ is one, $W^s(\bar{u}(x))$ defines a separatrix for different types of asymptotic behavior. This is also related to nonuniqueness of Riemann problem solutions of the hyperbolic equations, a phenomenon that was also observed in [3, 4].

The necessary condition for stability is obtained in Section 3 in terms of the sign of the quantity $D'(0)D(+\infty)$, where $D(\lambda)$ is the Evans function of the wave. It is here that the gap lemma continuation of the wave into the essential spectrum is needed.

It is not clear at this point under what conditions our necessary stability condition is also *sufficient*. In particular, the possibility of unstable complex eigenvalues must be addressed. Thus an extremely interesting open question is to obtain upper bounds for the Morse index of general shock profile solutions. In certain situations, it can determined through other methods that there are no unstable eigenvalues, and in this case, it has been possible to proceed from linearized stability to full nonlinear stability through careful estimates of the Green's function [31, 32, 33]. Most interestingly, it has recently be shown that these methods for proceeding from linearized to nonlinear stability are valid in a much more general setting and for other types of waves [41]. This imparts increased significance to the determination of sufficient criteria for linearized stability of shock profiles.

Finally, we point out that we have *not* made the standard assumption of a constant, scalar viscosity matrix B, since the form of B plays an important role in the phenomena we wish to study [22]. This generalization costs a surprising amount of additional effort, in particular, in Lemmas 2.7, 3.1, and 3.5, and Corollary 3.6. Lemma 3.5 in particular appears to be a simple case of a rather deep linear algebraic fact. The corresponding conjecture for the $n \times n$ case, made in Section 3.4, we regard as a key open problem for the theory.

Note. Following the completion of this manuscript, we have become aware that Kapitula and Sandstede have also proved a version of the gap lemma, in work simultaneous to ours [26].

2 The Gap Lemma

We begin by extending the Evans function framework of [1] to the more general setting required for (1.1). In particular, we show that the Evans function can be analytically continued into the region of essential spectrum.

2.1 Construction of the Evans Function: The Gap Lemma

The eigenvalue equation associated with the linearized equations (1.3) about a traveling-wave solution $\bar{u}(x)$ of (1.2) is a system of $N = 2n$ first-order, nonautonomous differential equations of the form (1.4), where the coefficient matrix $\mathbb{A}(x,\lambda)$ tends to limits $\mathbb{A}_{\pm}(\lambda)$ as $x \to \pm\infty$. All of the matrices $\mathbb{A}(x,\lambda), \mathbb{A}^{\pm}(\lambda)$ are analytic in λ. The linearized operator M about $\bar{u}(x)$ has an eigenvalue at λ when (1.4) admits a nontrivial solution W satisfying appropriate boundary conditions as $x \to \pm\infty$. The appropriate choice of boundary conditions for the study of nonlinear stability depends on the details of (1.2)–(1.3). For scalar equations and some reaction-diffusion systems, W can be prescribed to lie in a weighted L^∞ space [1, 24, 25, 36]. For standard systems of conservation laws, on the other hand, shock waves are *never* linearly stable under perturbations in any weighted L^∞ space, and the assignment of boundary conditions is more complicated [41]. These subtleties, however, concern linearly *stable* modes, $\mathrm{Re}\,\lambda = 0$. For a wide class of problems, it is possible to study linearly *unstable* modes, occurring as isolated eigenvalues $\mathrm{Re}\,\lambda > 0$, in a problem-independent way, requiring only that W be *bounded*.

In many important physical applications, it is the case that the linearized operator about a wave has no essential spectrum in the right half-plane $\mathrm{Re}\,\lambda > 0$.

In the case of viscous shock profiles, this condition defines the class of admissible viscosity matrices B (see (h3) below). In general, this will be the case when the following structural hypothesis is satisfied:

(h1) The asymptotic matrices $\mathbb{A}_\pm(\lambda)$ are both hyperbolic for all λ with positive real part, and the dimensions of the stable (respectively, unstable) subspaces, $S^\pm(\lambda)$ (respectively, $U^\pm(\lambda)$), of $\mathbb{A}_-(\lambda)$ and $\mathbb{A}_+(\lambda)$ is the *same* integer k (respectively, $N-k$), for all such λ for some k with $1 \le k < N$.

This structural feature is called *consistent splitting of the asymptotic systems* (see [1]). It follows from the resolvent formula together with the analyticity of $\mathbb{A}^\pm(\lambda)$ that this implies that the spectral projection operators associated to $S^\pm(\lambda)$ and $U^\pm(\lambda)$ are analytic in the right half-plane [27]. When (h1) holds, it is not difficult to show that the right half-plane consists of only normal points of the operator M in (1.3), so that the only spectrum in this region consists of isolated eigenvalues of finite multiplicity, and all bounded eigenfunctions in fact *decay exponentially* as $|x| \to \infty$ (see, e.g., [21, 1]). The determination of boundary conditions at $\pm\infty$ is therefore not an issue for $\operatorname{Re}\lambda > 0$, since all reasonable notions of spectrum are equivalent. Note, moreover, that the existence of such unstable eigenfunctions implies linearized instability of the wave under *most* perturbations, including those that decay rapidly at infinity.

Along with (h1), we make a second structural hypothesis that will be crucial in what follows:

(h2) $\mathbb{A}(x,\lambda) \to \mathbb{A}_\pm(\lambda)$ at exponential rate $O(e^{-\alpha|x|})$ as $x \to \pm\infty$ for some $\alpha > 0$, uniformly for λ in compact sets.

In the traveling-wave setting, this amounts to the assumption that $u_\pm$ are hyperbolic rest points of the associated traveling-wave ODE. It can thus be viewed as a nondegeneracy condition that is generically satisfied.

A useful tool for the location of point spectrum is the *Evans function*, introduced by Evans [9, 10, 11, 12] for special systems arising in neurophysiology. A quite general construction of the Evans function was given in [1] for all λ in the region of consistent splitting, i.e., *away from the essential spectrum of M*. Loosely speaking, the Evans function $D(\lambda)$ is a Wronskian of k solutions that decay as $x \to +\infty$ and $(N-k)$ solutions that decay as $x \to -\infty$. The precise definition is stated in terms of differential forms. It is shown that there exists a differential k-form $\eta(x,\lambda)$ which is associated to the unique k-dimensional subspace of solutions of (1.4) that decay as $x \to +\infty$, and an $(N-k)$–form $\zeta(x,\lambda)$ which is associated to the unique $(N-k)$–dimensional subspace of solutions that decay as $x \to -\infty$; that is, there exist forms associated to the stable and unstable manifolds of (1.4) at $\pm\infty$. The Evans function is then defined to be

$$D(\lambda) := e^{-\int_0^x \operatorname{tr} \mathbb{A}(s,\lambda)\, ds} \eta(x,\lambda) \wedge \zeta(x,\lambda) = \eta(0,\lambda) \wedge \zeta(0,\lambda)\,. \tag{2.1}$$

Clearly, D vanishes if and only if there is linear dependence between the subspaces of solutions decaying at $x \to \pm\infty$, or equivalently there is a solution decaying at both $\pm\infty$. Thus, the problem of locating unstable eigenvalues of M in $\{\operatorname{Re}\lambda > 0\}$ is equivalent to that of locating zeroes of D. The power of this approach comes from the observation that in this construction $D(\lambda)$ can be chosen *analytically* by analytic dependence of stable/unstable manifolds. In particular, it is proved in [1] that the algebraic multiplicity of an eigenvalue of M is equal to its order as a root of $D(\lambda)$. This makes possible the application of winding number and other topological arguments to the problem of counting eigenvalues. We remark that it is the requirement of analyticity that necessitates the use of differential forms, since it is not possible in general to make choices of *individual* solutions of (1.4) that are globally analytic in λ.

A frequently encountered problem with this program is that some portion of the imaginary axis may be contained in the essential spectrum. For shock profiles, this occurs at $\lambda = 0$ (see the remark at the end of Section 3.2). In such situations, it is necessary to analytically continue the Evans function through the essential spectrum in the left half-plane in order to use it as a tool in stability calculations, for the reasons mentioned above.

DEFINITION 2.1 Suppose that U and S are complementary $\mathbb{A}$-invariant subspaces for some $N \times N$ matrix $\mathbb{A}$. We define the *spectral gap* of U and S to be the difference β between the minimum real part of the eigenvalues of $\mathbb{A}$ restricted to U and the maximum real part of the eigenvalues of $\mathbb{A}$ restricted to S.

In certain situations [23, 24] there exist analytic continuations of the asymptotic subspaces $U^{\pm}(\lambda)$ and their complementary subspaces $S^{\pm}(\lambda)$ as λ moves across the boundary of the essential spectrum, such that a *positive spectral gap* $\beta^{\pm}(\lambda)$ is maintained between $U^{\pm}(\lambda)$ and $S^{\pm}(\lambda)$. In this case, the manifolds of solutions tangent to $U^{-}(\lambda)$ and $S^{+}(\lambda)$ at $\mp\infty$ remain uniquely determined, and it is then clear how to analytically extend the Evans function through the essential spectrum. Indeed, the argument of [1] still applies.

In the problem treated in this paper, as well as many other important physical systems, this is not the case, and the spectral gap becomes negative immediately upon entry into the essential spectrum. That is, there occurs a spectral *overlap* and the proper extensions of $\eta(x,\lambda)$ and $\zeta(x,\lambda)$ in (2.1) are no longer uniquely determined by the property that they are asymptotic to $S^+(\lambda)$ and $U^-(\lambda)$ (in a sense defined below). Indeed, it is not a priori clear that such analytic extensions exist. In such situations, the proof of analyticity in [1] breaks down. Below we give a different proof that permits negative spectral gaps as specified by a certain *gap condition*. The crucial idea is that, under certain circumstances, the appropriate manifolds of solutions can still be uniquely selected by the criterion of *maximal rate* of convergence to $S^+(\lambda)$ and $U^-(\lambda)$.

In the following, we shall construct the form $\eta(x,\lambda)$ that is associated to the space of solutions of (1.4) which decay to zero as $x \to +\infty$. For notational convenience, we shall delete the superscript $+$ for this part of the discussion. The same proof also applies to the continuation of the form $\zeta(\xi,\lambda)$ after a time reversal. As in [1], it will be most natural to work in projectivized, wedge product coordinates, the extension of $D(\lambda)$ then being obtained from invariant manifold methods. For completeness, we summarize the geometric setting developed in [1] that recasts the problem as the construction of a certain invariant manifold for a dynamical system. Consider a linear flow

$$W' = \mathbb{A}(x,\lambda)W \tag{2.2}$$

where $W \in \mathbb{C}^N$ and the coefficient matrix $\mathbb{A}(x,\lambda)$ tends to limits $\mathbb{A}_\pm(\lambda)$ as $x \to \pm\infty$ at an exponential rate $e^{-\alpha|x|}$ for some $\alpha > 0$.

In order to describe the evolution of k-dimensional subspaces of linear systems (1.4), it is natural to consider the flow induced by the equations on the Grassmannian $G_k(\mathbb{C}^N)$ obtained by forming the span of a set of k independent solutions $\Phi(x,\lambda) = \operatorname{span}\{W_i(x,\lambda) : 1 \le i \le k\}$ of the linear equations. Another, convenient way to characterize this (nonlinear) flow is through the Plücker embedding of $G_k(\mathbb{C}^N)$ into the space of projectivized k-forms $\mathbf{P}(\Lambda^k(\mathbb{C}^N))$. In particular, if Φ is a k-dimensional subspace with basis W_i, then the Plücker embedding is

$$\Phi \to \operatorname{span}\{W_1 \wedge \cdots \wedge W_k\}$$

i.e., we associate a k-dimensional subspace of $\mathbb{C}^N$ with a 1-dimensional subspace of $\Lambda^k(\mathbb{C}^N)$. This is an analytic embedding of the Grassmannian into a submanifold of the projectivized k-forms, $\mathbf{P}(\Lambda^k(\mathbb{C}^N))$, namely, the projectivized pure k-forms.

The linear equations (1.4) induce a flow on $\mathbf{P}(\Lambda^k(\mathbb{C}^N))$. In order to characterize this flow, note that the equations (1.4) induce a linear flow

$$\eta' = \mathbb{A}^{(k)}(x,\lambda)\eta \tag{2.3}$$

on the space of k-forms $\eta = W_1 \wedge \cdots \wedge W_k \in \Lambda^k(\mathbb{C}^N)$ via the Leibnitz rule

$$\begin{aligned} &\mathbb{A}^{(k)}(W_1 \wedge \cdots \wedge W_k) = \\ &\quad (\mathbb{A}W_1 \wedge \cdots \wedge W_k) + \cdots + (W_1 \wedge \cdots \wedge \mathbb{A}W_k)\,. \end{aligned} \tag{2.4}$$

The evolution of the k-plane of solutions of (3.1) is then determined by $\hat{\eta}(x,\lambda) = \mathrm{span}\{\eta(x,\lambda)\}$, which is an evolving point in the space of projectivized k-forms $\mathbf{P}(\Lambda^k(\mathbb{C}^N))$. We denote this nonlinear flow by

$$\hat{\eta}' = \hat{\mathbb{A}}^{(k)}(\hat{\eta}, x, \lambda)\,. \tag{2.5}$$

It is easily seen that for a given (constant) matrix $\mathbb{A}$, the eigenvectors of $\mathbb{A}^{(k)}$ are of form $V_1 \wedge \cdots \wedge V_k$, where $\mathrm{span}\{V_1, \cdots, V_k\}$ is an invariant subspace of $\mathbb{A}$, and that the corresponding eigenvalue is the trace of $\mathbb{A}$ on that subspace. It therefore follows that the span of such a k-form is a rest point of the associated asymptotic projectivized flow at $x = +\infty$, $\hat{\eta}' = \hat{\mathbb{A}}^{(k)}(\hat{\eta}, \lambda)$.

The equations for $\hat{\eta}$ can be computed in local (Plücker) coordinates. For example, in order to construct $\hat{\eta}(x,\lambda)$, select a basis

$$\{\eta_I(\lambda) = V_{i_1}(\lambda) \wedge \cdots \wedge V_{i_k}(\lambda) : I = (i_1, \ldots, i_k),\ \ i_1 < \cdots < i_k\}$$

for $\Lambda^k(\mathbb{C}^N)$, where $V_i(\lambda)$ is a basis of (generalized) eigenvectors of $\mathbb{A}(\lambda)$. A solution of (2.3) can then be expressed as a linear combination

$$\eta(x,\lambda) = \sum_I p_I(x,\lambda)\eta_I(\lambda)\,.$$

If $\eta(\lambda) = \eta_{I_0}(\lambda)$ for some I_0, then the local projectivized coordinates in a neighborhood of $\hat{\eta}_{I_0}(\lambda)$ are $q_I(x,\lambda) = p_I(x,\lambda)/p_{I_0}(x,\lambda)$ for all multi-indices $I \neq I_0$.

In the above, it is not generally possible to choose the basis of eigenvectors $V_i(\lambda)$ analytically, or even continuously in λ. However, we have the following standard result:

LEMMA 2.2 *Suppose that the spectral projection operator $P_S(\lambda)$ associated to the invariant subspace $S(\lambda)$ of $\mathbb{A}(\lambda)$ defined in* (h1) *extends analytically to some simply connected domain Ω that contains $\{\mathrm{Re}\,\lambda > 0\}$. Then there is an analytic choice of basis $E_i(\lambda)$ for the continuation $S(\lambda) = P_S(\lambda)\mathbb{C}^N$ of $S(\lambda)$, and the k-form*

$$\eta(\lambda) = E_1(\lambda) \wedge \cdots \wedge E_k(\lambda) \tag{2.6}$$

gives an analytic section of $\hat{\eta}(\lambda)$, where $\hat{\eta}(\lambda)$ represents $S(\lambda)$ under the Plücker embedding.

If $\{V_i\}$ is a basis of $S(\lambda_0)$ for some $\lambda_0 \in \Omega$, then the basis $E_i(\lambda)$ can be initialized by setting $E_i(\lambda_0) = V_i$.

The proof is a standard but nontrivial result in matrix theory (see, e.g., [27], pp. 99–102). In order to conclude from Lemma 2.2 the existence of an analytic choice of solutions $\eta(x,\lambda)$ of (1.4) asymptotic to $\eta(\lambda)$ at $+\infty$, we impose two further hypotheses:

(h3) (*Geometric Separation*) The eigenvalues $\mu_i(\lambda)$ of $\mathbb{A}(\lambda)$ and the spectral projection operators $P_S(\lambda)$ and $P_U(\lambda)$ associated to $S(\lambda)$ and $U(\lambda)$, respectively, for $\operatorname{Re}\lambda > 0$ continue analytically to a simply connected domain Ω containing the right half-plane. Furthermore, the associated continuations $S(\lambda) = P_S(\lambda)\mathbb{C}^N$ and $U(\lambda) = P_U(\lambda)\mathbb{C}^N$ complement each other in $\mathbb{C}^N$ for $\lambda \in \Omega$.

(h4) (*Gap Condition*) $\beta(\lambda) > -\alpha$ for all $\lambda \in \Omega$, where $\beta(\lambda)$ is the spectral gap of the pair $S(\lambda), U(\lambda)$.

THEOREM 2.3 (Gap Lemma) *Let $\mathbb{A}(x,\lambda)$ be C^1 in x and analytic in λ, and suppose that* (h1)–(h4) *hold for some simply connected domain Ω containing $\operatorname{Re}\lambda > 0$. Then, for $\lambda \in \Omega$, there exists a solution $\hat{\eta}(x,\lambda)$ of* (2.5) *such that for any $\tilde{\alpha}$ satisfying $-\beta(\lambda) < \tilde{\alpha} < \alpha$, $\hat{\eta}(x,\lambda)$ converges at rate $e^{-\tilde{\alpha}|x|}$ to $\hat{\eta}(\lambda)$ as $x \to +\infty$, where $\hat{\eta}(\lambda)$ is associated to $S(\lambda)$ under the Plücker embedding. Moreover, this exponential decay rate determines $\hat{\eta}(x,\lambda)$* uniquely*, in that any other solution converging to $\hat{\eta}(\lambda)$ at $x = +\infty$ converges no faster than $e^{\beta x}$. Finally, $\hat{\eta}(x,\lambda)$ is analytic in λ for all $\lambda \in \Omega$.*

PROOF: Let $\hat{\eta}_{I_0}(\lambda) = \hat{\eta}(\lambda)$ be the analytic k-form associated to $S(\lambda)$ as in (2.6), where $I_0 = (1,\dots,k)$. We may (analytically) extend $E_i(\lambda)$, $1 \le i \le k$, to a full basis of $\mathbb{C}^N$ by applying Lemma 2.2 to $U(\lambda)$ to get another basis for the complementary space, which we denote by $E_i(\lambda),\ k+1 \le i \le N$. Let $\{q_I = p_I/p_{I_0}\}$ be the Plücker coordinates for the projectivized flow (2.4) relative to this basis of $\mathbb{C}^N$. The origin $q = 0$ of the local coordinate system corresponds to the rest point $\hat{\eta}_{I_0}(\lambda)$ of the vector field $\hat{\mathbb{A}}^{(k)}(\hat{\eta},\lambda)$. The equations for $q_I(x,\lambda)$ can then be calculated directly from (2.2) and the quotient rule. This yields a system of generalized Riccati equations,

$$q_I' = f_I(q,x,\lambda)\,,$$

where $q = (q_I)$ is in $\mathbb{C}^M$ with $M = \dim\Lambda^k(C^N) - 1$ and f_I is a quadratic polynomial in q.

As in [1], the above equation is augmented with an additional equation

$$\tau' = \kappa(1-\tau^2)\,, \tag{2.7}$$

where $-1 \leq \tau \leq 1$. The vector field $f_I(q, x, \lambda)$ can then be reparametrized by the change of variables $x = x(\tau) = (1/2\kappa) \log \frac{1+\tau}{1-\tau}$, yielding an autonomous system for $Y = (q, \tau) \in \mathbb{C}^M \times \mathbb{R}$,

$$Y' = F(Y, \lambda)\,, \tag{2.8}$$

where $F(Y, \lambda) = (f(q, \tau, \lambda), \kappa(1 - \tau^2))$. Similarly, as in [1, lemma 3.1], we find that the original matrix $\mathbb{A}$ reparametrized as $\mathbb{A}(\tau, \lambda)$ is $C^{1+\theta}$ in τ if $\mu = 2\kappa$ is chosen so that $\mu < \alpha$ and θ is chosen so that

$$0 < \theta < \frac{\alpha}{\mu} - 1\,, \tag{2.9}$$

since then the exponential blowup of $x(\tau)$ is offset by the exponential decay of $\mathbb{A}(x, \lambda)$.

For all λ, the flow (2.8) has a rest point $Y = (\mathbf{0}, -1)$ corresponding to the desired asymptotic subspace $S(\lambda)$; *we thus seek an analytic choice of solutions* $Y(\lambda, x)$ *approaching* $(\mathbf{0}, +1)$ *as* $x \to +\infty$. This rest point corresponds to the desired asymptotic subspace $S(\lambda)$. Linearizing about this rest state, we find that

$$\frac{\partial F}{\partial Y}(\mathbf{0}, +1, \lambda) = C(\lambda) = \begin{pmatrix} B(\lambda) & 0 \\ 0 & -\mu \end{pmatrix}, \tag{2.10}$$

where $B(\lambda) = d_q f(\mathbf{0}, +1, \lambda)$. The block diagonal form of $C(\lambda)$ follows because $\partial_\tau \mathbb{A}(+1, \lambda) = 0$ when $\mu = 2\kappa < \alpha$. Evidently, the eigenvalues of $C(\lambda)$ are $-\mu$ together with the eigenvalues of $B(\lambda)$.

The eigenvalues $\hat{\gamma}$ of $B(\lambda) = d_q f(\mathbf{0}, +1, \lambda) = d_{\hat{\eta}} \hat{\mathbb{A}}^{(k)}(\lambda)$ are the differences of eigenvalues of the original $\mathbb{A}^{(k)}(\lambda)$ matrix, $\hat{\gamma} = \mathrm{tr}(\hat{\eta}_I(\lambda)) - \mathrm{tr}(\hat{\eta}_{I_0}(\lambda))$, where $I \neq I_0$ and tr denotes the trace of $\mathbb{A}(\lambda)$ on the invariant subspace associated with $\hat{\eta}_I(\lambda)$. By the definition of $\beta(\lambda)$, we have $\mathrm{Re}\,\hat{\gamma} > \beta(\lambda)$ for every eigenvalue $\hat{\gamma}(\lambda)$ of $B(\lambda)$. The gap condition $\beta(\lambda) > -\alpha$ then ensures that all eigenvalues of $B(\lambda)$ have real part greater than $-\alpha$, so that $\kappa > 0$ can be chosen so that $\mathrm{Re}\,\hat{\gamma} > -2\kappa > -\alpha$ for all eigenvalues $\hat{\gamma}$ of $B(\lambda)$. It therefore follows that $-\mu = -2\kappa$ is the eigenvalue of smallest real part of $C(\lambda)$ and that $-\mu$ is simple. Thus, there is a unique solution/strongly stable manifold $Y(\lambda, x)$ approaching $(\mathbf{0}, +1)$ as $x \to +\infty$. Analytic dependence of Y on the parameter λ then follows from the general fact of analytic dependence of a one-dimensional, strongly stable manifold, proved for completeness in Lemma 3.8. Also by Lemma 3.8 we obtain convergence to the rest state $(\mathbf{0}, +1)$ as $x \to +\infty$ at rate $\tilde{\alpha} = \theta\mu$. By (2.9), θ can be taken to be arbitrarily close to $\frac{\alpha}{\mu} - 1$; hence $\tilde{\alpha}$ can be taken arbitrarily close to $(\frac{\alpha}{\mu} - 1)\mu = \alpha - \mu$. Since $\mu > 0$ was arbitrary, we obtain the claimed rate of decay. ∎

The following corollary is used to obtain a *local* factorization of the forms η and ζ obtained from the gap lemma in terms of wedges of individual solutions of the eigenvalue equations (1.4). This is used in Section 3 in the calculation of $D'(0)$. We remark that since each factor converges at the maximal rate to the corresponding eigenvector of $\mathbb{A}_\pm$, one can conclude by uniqueness that these products are the η and ζ given by the gap lemma. This local representation is crucial in our stability calculations. Note that *no gap condition is required for Corollary* 2.4; it is wholly a consequence of (h2). The gap condition is only required for global constructions.

COROLLARY 2.4 *Let $\mathbb{A}(x,\lambda)$ be C^1 in x and analytic in λ, and let it satisfy* (h2). *Let $C^+(\lambda)$ and $E^+(\lambda)$ be complementary $\mathbb{A}_+(\lambda)$-invariant subspaces, each analytic in λ in a neighborhood of λ_0, and let $\hat{\eta}^+(\lambda)$ denote the projectivized r- form associated to $C^+(\lambda)$ under the Plücker embedding, where r is the dimension of $C^+(\lambda)$. Then, in a neighborhood of λ_0, there exists a* (*not necessarily unique*) *$\hat{\eta}(x,\lambda)$ that is a solution of* (2.5) (*with $k = r$*) *such that for any $\tilde{\alpha}$ satisfying $\tilde{\alpha} < \alpha$, $\hat{\eta}(x,\lambda)$ converges at rate $e^{-\tilde{\alpha}|x|/2}$ to $\hat{\eta}^+(\lambda)$ as $x \to \infty$. Moreover, this solution is analytic in λ for λ near λ_0.*

PROOF: In the following, we suppress explicit reference to λ. By decomposing C^+ into component eigenspaces, we can reduce to the case that $\mathbb{A}_+$ restricted to C^+ has eigenvalues of real part μ_R for some unique μ_R. Define S^+ to be the direct sum of all eigenspaces of $\mathbb{A}_+$ restricted to E^+ having eigenvalues with real part smaller than $\mu_R - \alpha/2$, and U^+ to be the complement of S^+ in E^+. It follows that, in a neighborhood of λ_0, S^+ is separated from C^+ by a strictly positive spectral gap of at least $\alpha/2$, while $S^+ \oplus C^+$ is separated from U^+ by spectral gap $\beta > -\alpha/2$.

By the gap lemma applied to $S = S^+ \oplus C^+$ and $U = U^-$, there is an analytic, projectivized $(r+p)$–form $\hat{\eta}_{r+p}(x,\lambda)$ that converges to the $(r+p)$–form associated to $C^+ \oplus S^+$ at the claimed exponential rate $e^{-\alpha|x|/2}$; here p is the dimension of S^+. This then provides an analytic $(r+p)$–dimensional subspace $\Phi(x)$ of solutions of the original linear equations (1.4).

Since $\mathbb{A}_+$ restricted to S^+ has spectrum with real part less than the rest of the spectrum of $\mathbb{A}_+$ by at least $-\alpha/2$, it follows that there is a uniquely defined analytic subspace of solutions $\Phi_{S+}(x)$ of solutions of (1.4) which is asymptotic to S^+ as $x \to +\infty$ at the rate $e^{-\alpha x}$. This follows from the standard construction in [1]. This subspace contains every solution that tends to S^+, and so $\Phi_{S+}(x) \subset \Phi(x)$.

We now consider the projectivized flow (2.5) with $k = r$ on the space of projectivized r-forms, augmented with the τ-flow (2.7). It then follows that $(\hat{\Phi}(x), \tau(x))$ corresponds to a certain invariant submanifold of this projectivized, augmented flow, which tends to $((C^+ \oplus S^+)\hat{}, +1)$ as $x \to +\infty$.

The set $(\hat{\Phi}_{S^+}(x), \tau(x))$ is an invariant submanifold of $(\hat{\Phi}(x), \tau(x))$ that tends to $(\hat{S}^+, +1)$ as $x \to +\infty$. Let $(\hat{\eta}(x), \tau(x))$ be any solution in $(\hat{\Phi}(x), \tau(x))$ that is *not* in $(\hat{\Phi}_{S^+}(x), \tau(x))$. It follows that $\hat{\eta}(x)$ tends to the rest point $\hat{C}^+$ in the forward direction. This is because S^+ and C^+ have a positive spectral gap, and so $\hat{C}^+$ and $\hat{S}^+$ form an attractor/repeller pair for the asymptotic projectivized flow (2.5) (with $k = r$) at $\tau = +1$. The result then follows from standard theorems about ω-limit sets.

Thus $(\hat{\eta}(x), \tau(x))$ lies in the stable manifold of the rest point $(\hat{C}^+, +1)$ for equations (2.5) and (2.7) restricted to $(\hat{\Phi}(\tau), \tau), \tau \leq 1$. However, since C^+ and S^+ have a positive spectral gap of at least $\alpha/2$, it follows that the weakest stable eigenvalue of $(\hat{C}^+, +1)$ for the restricted flow has real part less than $-\alpha/2$, so that $\hat{\eta}(x)$ decays to its limit faster than $e^{-\alpha x/2}$. ∎

For completeness, we provide a lemma that describes the relation between a projectivized solution and the original linear system, and their behavior near $+\infty$.

LEMMA 2.5 *Suppose that $S_p(\lambda)$ and $U_p(\lambda)$ are complementary $\mathbb{A}_+(\lambda)$-invariant subspaces (where p is the dimension of $S_p(\lambda)$), and that the projection operators associated to these subspaces are analytic for $\lambda \in D$, where $D \subset \Omega$ is simply connected. Let $\eta_p(\lambda)$ be the analytic p-form* (2.6) *associated to $S_p(\lambda)$. Suppose also that there exists a solution $\hat{\eta}_p(x, \lambda)$ of* (2.5) *that is analytic for $\lambda \in D$ and that converges at rate $\mathbf{O}(e^{-\tilde{\alpha}x})$ to $\hat{\eta}_p(\lambda)$ as $x \to +\infty$ for some $\tilde{\alpha} < \alpha$. Then there is a solution $\eta_p(x, \lambda) \in \hat{\eta}_p(x, \lambda)$ of* (2.3) *such that*

$$\eta_p(x, \lambda) = e^{m_p(\lambda)x}(\eta_p(\lambda) + \mathbf{O}(e^{-\tilde{\alpha}x})) \tag{2.11}$$

as $x \to +\infty$, and $\eta_p(x, \lambda)$ is analytic for $\lambda \in D$.

If the gap condition $\beta_p(\lambda) > -\alpha$ holds in D, where $\beta_p(\lambda)$ is the spectral gap of the pair $S_p(\lambda), U_p(\lambda)$, then solution $\eta_p(x, \lambda)$ is uniquely determined by (2.11)*.*

PROOF: Let I_p be the p-multi-index associated to the subspace $S_p(\lambda)$ so that the projectivized form $\hat{\eta}_p(\lambda)$ corresponds to the origin of the local coordinate system $q_I = 0$, where $q_I = p_I/p_{I_p}, I \neq I_p$, are Plücker coordinates. The assumed decay rate on $\hat{\eta}_p(x, \lambda)$ then ensures that $|q_I(x, \lambda)| \leq Ke^{-\tilde{\alpha}x}$ as $x \to +\infty$. In particular, for x greater than some x_0, $|q_I(x, \lambda)| < \infty$, so that we can work in this single local coordinate system. From now on, we take $x \geq x_0$. It follows from the analyticity of $\hat{\eta}_p(x, \lambda)$ that each $q_I(x, \lambda)$ is analytic in λ for $x > x_0$.

Choose the solution $\eta_p(x,\lambda)$ of (2.3) that satisfies the initial conditions: $p_{I_p}(x_0,\lambda) \equiv 1$ and $p_I(x_0,\lambda) = q_I(x_0,\lambda)$ for all $I \neq I_p$. Then $\eta_p(x_0,\lambda)$ is an analytic section of $\hat{\eta}_p(x_0,\lambda)$ by virtue of the analyticity of $\hat{\eta}_p(x,\lambda)$.

Setting $\pi(x,\lambda) = e^{-m_p(x-x_0)} p_{I_p}(x)$, we find that $\pi' = \mathbf{O}(e^{-\tilde{\alpha}x})\pi$, where the coefficient of π in the $\mathbf{O}$-term is analytic in λ, since $q_I(x,\lambda)$ and $\mathbb{A} - \mathbb{A}_+$ are analytic in λ. Moreover, both of these quantities decay to zero as $x \to +\infty$ at the specified rate. Integration of this equation from x_0 to x gives $\pi(x) = e^{\varphi(x)}$, where $\varphi(x) = \int_{x_0}^{x} \mathbf{O}(e^{-\tilde{\alpha}s})\, ds$. Clearly, $\varphi(x)$ tends to a finite limit as $x \to +\infty$, so that $\pi(x,\lambda)$ tends to a finite, nonzero, analytic limit as well. The proof is completed by noting that

$$\eta_p(x,\lambda) = p_{I_p}(x,\lambda)(\eta_r(\lambda) + \sum_{I\neq I_r} q_I(x,\lambda)\eta_I(\lambda))\,,$$

so that the stated result follows upon replacing $\eta_p(\lambda)$ by $\eta_{I_p}(\lambda)/\pi(+\infty,\lambda)$. ∎

Lemma 2.5 applies in particular to solutions $\hat{\eta}_p(x)$ obtained by the gap lemma or the local Corollary 2.4. As an immediate consequence of Lemmas 2.3 and 2.5, we obtain the main result of Section 2.

THEOREM 2.6 *Let $\mathbb{A}(x,\lambda)$ satisfy* (h1)–(h4) *for $\lambda \in \Omega$ at $x = \pm\infty$. Then there is a unique analytic extension of the Evans $D(\lambda)$ function to Ω, with η and ζ as described in the gap lemma.*

We conclude with the standard observation that D is real-valued for real λ.

LEMMA 2.7 *Suppose that $\mathbb{A}(x,\lambda^*)^* = \mathbb{A}(x,\lambda)$, where * denotes complex conjugation.*

(i) *There exist bases $E_i^+(\lambda)$, $1 \leq i \leq k$, of $S^+(\lambda)$ and $E_i^-(\lambda), k+1 \leq k \leq N$, of $U^-(\lambda)$ depend analytically on λ for $\lambda \in \Omega$ and that are real-valued vectors for $\lambda \geq 0$.*

(ii) *The Evans function $D(\lambda)$ of Theorem* 2.6 *can be chosen to be real-valued* (*i.e., a real multiple of the standard N-form*) *for real $\lambda \geq 0$. Indeed, the same properties hold for $\eta(x,\lambda)$ and $\zeta(x,\lambda)$.*

PROOF: We first prove that the analytic forms $\eta^+(\lambda)$ and $\zeta^-(\lambda)$ can be chosen to be real-valued for real $\lambda \geq 0$. For such λ, the matrix $\mathbb{A}_+(\lambda)$ is a real matrix, and there is an analytic k-dimensional *real* $\mathbb{A}^+(\lambda)$-invariant subspace $S_r^+(\lambda)$ of $\mathbb{R}^N$ associated to the stable portion of the spectrum of $\mathbb{A}_+(\lambda)$ for positive λ. The subspace is formed in the usual way by taking real and imaginary parts of the associated eigenvectors whenever the latter are complex The original subspace $S^+(\lambda)$ is then the complexified span of a basis for $S_r^+(\lambda)$. Let $P_S(\lambda)$ be the projection operator on $\mathbb{R}^N$ for the subspace $S^+(\lambda)$ for real λ, and let $\{E_i^+(0),\ 1 \le i \le k\}$ be a (real) basis for $S_r^+(0)$. A construction of Kato [27], pp. 99–102] then provides an analytic basis $E_i^+(\lambda) = L(\lambda)E_i^+(0)$ for some analytic matrix $L(\lambda)$. The matrix $L(\lambda)$ is the solution of the differential equation $L' = Q(\lambda)L$, where Q is the commutator of P_S and P_S' and "prime" is $d/d\lambda$. It is clear from the construction in [27] that the solution $L(\lambda)$ is real-valued when the projection operator is restricted to $\mathbb{R}^N$ in the above manner and the initial conditions are chosen to be real. The wedge of these basis elements defines a real analytic form $\eta_r^+(\lambda)$ for $\lambda \ge 0$ that extends analytically to all $\lambda \in \Omega$. We may therefore drop the r and assume that $\eta^+(\lambda)$ is real for real λ.

The gap lemma can then be proved in both *real* projective space (for $\lambda \ge 0$) and in *complex* projective space (for $\lambda \in \Omega$), where the asymptotic limit in each case is $\hat{\eta}^+(\lambda)$. We then apply Lemma 2.5 to each of these two forms to obtain solutions of the original linear equations. However, when we restrict the complex form to $\lambda \ge 0$, it must coincide with the real form by uniqueness. Hence, the real-valued form extends analytically to all $\lambda \in \Omega$. ∎

Remarks

1. It is precisely the exponential decay of $\mathbb{A}$ that was not fully exploited in previous analyses. If we set $\alpha = 0$, note that condition (h4) reduces to the usual positive gap condition, $\beta > 0$.

2. In the above discussion, we avoided specifying a functional space by the observation that any reasonable choice would give the same spectral theory for $\operatorname{Re}\lambda > 0$. This has important consequences also for $\operatorname{Re}\lambda = 0$, and the study of linear *stability*. In some cases, in particular for systems of conservation laws, there may be *no* functional space appropriate for the study of stability. Yet one can still define a useful, "effective spectrum" using the gap lemma by analytic continuation from $\operatorname{Re}\lambda > 0$ [41].

3. In the problem at hand, the domain Ω can be taken to be a neighborhood containing the origin and the right half-plane in the complex plane. However, we mention another interesting possibility that does not immediately present itself but that also falls under the theory developed above. The eigenvalues $\mu_j^\pm(\lambda)$ of $\mathbb{A}_\pm(\lambda)$, being algebraic functions of the entries of $\mathbb{A}_\pm(\lambda)$, may in general be expected to have branch points in the left half-plane.

   In certain situations, in particular in the *weak shock strength limit*, these branch points fall within the maximal region prescribed by the gap condition. In this case, it may be advantageous to take R to be some portion of a Riemann surface *including* these branch points. The analytic extension of $D(\lambda)$ to such a Riemann surface follows directly from the gap lemma with no change necessary. In the weak limit of shock profiles, it is not difficult to show that any unstable eigenvalue necessarily lies in a small neighborhood of the origin. Since the branch points of the Evans function also coalesce towards $\lambda = 0$ in the weak limit, it is essential to lift the Evans function to a portion of such a Riemann surface in order to perform a winding-number calculation for the Evans function that counts all eigenvalues near the origin. Indeed, this is *necessary* if one hopes to conclude stability properties of weak waves by carrying out the limit to a constant-state solution.

The weak limit in even the simplest case of a scalar viscous shock wave is a heat equation, for which the eigenvalues $\mu_1^\pm(\lambda) = -\sqrt{\lambda}$ and $\mu_2^\pm(\lambda) = +\sqrt{\lambda}$ have branch points occurring directly at the origin. In this case, the limiting Riemann surface R for D is at least two-sheeted; in fact, one must consider a four-sheeted surface if it is to be topologically stable under perturbations of the branch points. Thus, any uniform limit must likewise take place on some portion of a four-sheeted surface. The topology of this surface should play an interesting role in calculating the winding number of the continued Evans function D on $\mathbb{R}$. The weak limit of the undercompressive and overcompressive profiles necessarily occurs in the context of a two-dimensional system, and the Riemann surface required in this case is even more complicated.

These issues arise in trying to establish *sufficiency* of our stability conditions in the weak shock limit for general systems of conservation laws, a current topic of our investigation. In this context, we mention also a previously proposed program [15] to analyze stability of weak *Lax* shocks by a rescaling argument, using the singular perturbation techniques of [1] to reduce to a scalar calculation in the single slow mode. This technique would avoid the introduction of a Riemann surface, since branch points are scaled out to infinity. The gap lemma appears to be the main technical ingredient needed to make this argument rigorous.

Related issues arise in the analysis of traveling waves near certain singular limits of reaction-diffusion systems, for example,

$$\begin{aligned} \varepsilon^p u_t &= \varepsilon^2 u_{xx} + f(u,v)\,, \\ v_t &= D v_{xx} + g(u,v)\,, \end{aligned}$$

where ε is a small parameter and $p = 1$. In the case where $p = 0$, treated in [17], the essential spectrum of traveling waves remains uniformly bounded away from the origin in the left half-plane. In this case, the methods in [1] and in [17] based upon the Evans function and an attendant topological invariant called the *stability index* provide an effective framework for stability calculations. However, when $p = 1$, the essential spectrum approaches the imaginary axis from the left as $\varepsilon \to 0$, along with a branch point of the Evans function. In previous studies, it has not been possible to implement the methods in [1, 17] in this sort of limiting regime. The program outlined above involving continuation of the Evans function to a portion of a Riemann surface should also make available the methods in [1] and [17] in the analysis of this type of singular limit.

2.2 Large $|\lambda|$ Behavior

We next prove another result of a somewhat general nature that provides an estimate for the global behavior for all $x \in \mathbb{R}$ of solutions $\hat{\eta}(x, \lambda)$ of (2.5) as $\lambda \to +\infty$ along rays within the resolvent set (in particular, along the positive real axis; see Corollary 4.3). A result of this nature was proved in [1, prop. 2.2] for traveling waves of parabolic systems under the assumption that the viscosity matrix B is a constant. That is not the case here, and we show that the geometric argument in [1] actually applies in greater generality.

In the limit $\lambda \to +\infty$, the eigenvalue equations (3.1) can typically be rescaled to resolve the system into an $\mathbf{O}(1)$ limit system modulo a small error,

$$W' = \mathbb{B}(x)W + \delta\Theta(x)W\,. \tag{2.12}$$

The rescaling typically occurs both in the dependent variables W and in the independent variable x. The nature of the rescaling is highly dependent upon the structure of the systems under consideration. For example, the rescalings required for parabolic systems and for higher-order equations are quite different. It is therefore useful to formulate a general result by abstracting some of the common structural features that are present in such rescaled eigenvalue equations.

In the rescaled coordinates, there is an asymptotic $\mathbf{O}(1)$-limit $\mathbb{B}(x)$ and an error term, which is measured in terms of a small parameter $\delta = \lambda^{-p}$ for some $p > 0$. The rescaled system system typically has the following structure:

(h5) The matrix $\mathbb{B}(x)$ tends to limits $\mathbb{B}_{\pm}$ as $x \to \pm\infty$ and has a complementary pair of $\mathbb{B}(x)$-invariant subspaces $U(x)$ and $S(x)$ of dimensions k and $N-$

k, respectively, which admit a uniformly positive spectral gap $\beta(x)$ for all x:

$$\beta(x) > \beta > 0\,.$$

(h6) The matrix $\mathbb{B}(x)$ is bounded and slowly varying in the (rescaled) variable x, and $\Theta(x)$ is bounded, i.e., for some $c > 0$,

$$\|\mathbb{B}\|_\infty, \|\Theta\|_\infty \le c\,; \qquad \|\mathbb{B}'\|_\infty \le c\delta\,.$$

In [1], the matrix $\mathbb{B}(x)$ is constant.

PROPOSITION 2.8 *Suppose that* (h3) *and* (h4) *hold. Let $\hat{\zeta}_{\mathbb{B}(x)}$ be the projectivized k-form associated to $U(x)$, and let $\hat{\zeta}(x)$ be the unique solution of* (2.12) *satisfying $\hat{\zeta}(x) \to \hat{\zeta}_-$ where $\hat{\zeta}_-$ is the projectivized k-form associated to the subspace U^- of $\mathbb{B}_-$. If ρ is some globally defined metric on $\mathbf{P}(\Lambda^{(N-k)}(\mathbb{C}^N))$, then*

$$\rho(\hat{\zeta}(x), \hat{\eta}_{\mathbb{B}(x)}) \le C\delta$$

for some $C > 0$ that must be chosen sufficiently large relative to β.

PROOF: As before, it will be convenient to consider the induced flow on the space of projectivized k-forms $\hat{\zeta} \in \mathbf{P}(\Lambda^{(N-k)}(\mathbb{C}^N))$ of the form

$$\hat{\zeta}' = \hat{\mathbb{B}}_\delta^{(N-k)}(x, \hat{\zeta})\,,$$

where $\mathbb{B}_\delta(x) = \mathbb{B}(x) + \delta\Theta(x)$. First, consider the "frozen" system determined by setting $x \equiv x_0$, $\delta = 0$, in the arguments of the right-hand side:

$$\hat{\zeta}' = \hat{\mathbb{B}}^{(N-k)}(x_0, \hat{\zeta})\,. \tag{2.13}$$

By the spectral gap assumption, the vector field $\hat{\mathbb{B}}^{(N-k)}(x_0, \hat{\zeta})$ has an attracting rest point $\hat{\zeta}_{\mathbb{B}(x_0)}$. As x_0 is varied, this determines a curve of rest points in the corresponding frozen systems. Around each $\hat{\zeta}_{\mathbb{B}(x_0)}$, there exists an attracting neighborhood $\Omega(x_0)$ varying smoothly in terms of the matrix $\mathbb{B}(x_0)$. Moreover, the other eigenspaces of $\mathbb{B}(x_0)$ remain uniformly separated (in $\mathbb{C}\mathbf{P}^{N-1}$) from the eigenvectors in $U(x_0)$, since the spectral gap $\beta(x) > \beta$ is uniformly positive for all x. It therefore follows that the attracting neighborhood $\Omega(x_0)$ contains a uniform r-ball about $\hat{\zeta}_{\mathbb{B}(x_0)}$ in the ρ metric for all $x \in \mathbb{R}$.

A more precise description of this neighborhood can be obtained in the Plücker coordinates $q = (q_I) \in \mathbb{C}^{M-1}$ defined above with I_0 equal to the multi-index for the k-form $\eta_{\mathbb{B}(x_0)}$. Let $f(q, x, \delta)$ be the vector field induced by the matrix $\mathbb{B}(x) + \delta\Theta(x)$, and let $f(q, x_0)$ be the vector field induced by $\mathbb{B}(x_0)$. It follows from (h4) that for x near x_0, we have the estimate

$$f(q, x, \delta) = f(q, x_0) + \mathbf{O}(\delta)$$

for q in compact sets. Since the Jacobian matrix $B(x_0) = d_q f(0, x_0)$ of the unperturbed vector field is uniformly negative definite for all $x_0 \in \mathbb{R}$, there is an inner product $(\cdot, \cdot)_{x_0}$ defined by the eigenvectors of $B(x_0)$ that induces a norm $\| \cdot \|_{x_0}$ on $\mathbb{C}^{M-1}$ such that $\mathrm{Re}(B(x_0)q, q)_{x_0} < -\beta \|q\|^2_{x_0}$. We therefore take $\Omega(x_0) = q^{-1}(\{||q||_{x_0} < r\})$. The norm and the inner product vary smoothly with x_0 since the coefficient matrices do, and by (ii) their x-derivatives are of order δ.

As x_0 varies, the neighborhoods $\Omega(x_0)$ sweep out a tube Ω in $\mathbf{P}(\Lambda^k(\mathbb{C}^N)) \times \mathbb{R}$. This tube is positively invariant for small δ. For example, if $(\hat{\zeta}, x_0) \in \partial\Omega = \partial\Omega(x_0) \times \mathbb{R}$, then in the Plücker coordinates defined at $x = x_0$ we have that the perturbed flow (2.12) can be expressed for x near x_0 as $q' = f(q, x_0) + \mathbf{O}(\delta)$. For $\|q\|_{x_0} = r$ we have that

$$\frac{d}{dx}\|q\|^2_x = \mathrm{Re}(B(x_0)q + \mathbf{O}(r^2), q)_{x_0} + \mathbf{O}(r\delta) < -\beta r^2 + \mathbf{O}(r^3) + \mathbf{O}(r\delta)$$

at $x = x_0$. Now let $r = C\delta$ for some $C > 0$. The condition for the right-hand side of the above inequality to be negative is that $(-\beta C + K_1 \delta C^2 + K_2) < 0$, where $K_i > 0$ are the constants in the two $\mathbf{O}$-terms. This quadratic expression in C has two positive roots $0 < C_1 < C_2$ if $\delta < \beta^2/(4K_1K_2)$, where C_1 is approximately K_2/β. For $C \in (C_1, C_2)$ it then follows that for small $x - x_0 > 0$, we have that $\|q(x)\|_x < r^2$. Thus, the tube Ω is positively invariant. ∎

Remark. A similar estimate holds for the solution $\hat{\eta}(x)$ that tends to the k-form $\hat{\eta}^+$ in the forward direction.

3 Stability of Viscous Shock Waves

Using the technical framework derived in the previous section, we now derive a necessary condition for the stability of viscous shock solutions of (1.2), using an Evans function argument developed in [23]. The basic idea is to compute the sign of $D'(0)D(+\infty)$, where $+\infty$ here represents the limiting sign as $\lambda \to +\infty$ along the real axis and D denotes the Evans function. Since D restricted to the real axis is real (Lemma 2.7), this gives an index for the parity of the number of positive real zeroes of D, each of which corresponds to an unstable mode of (1.3). The utility of the method comes from the observation [23] that the sign of this index can be related to the dynamics of the underlying traveling-wave

ODE. We determine this relation in the shock-wave case for waves of undercompressive, Lax, and overcompressive types. In the second case, the index reduces further to a purely algebraic quantity. In the first, it is determined by the sign of what is essentially a Melnikov integral Γ for the separation function of the invariant manifolds of $u_\pm$ in the ODE describing traveling waves, suitably normalized by proper choice of basis. In each case, we obtain a readily (numerically) computable condition for instability.

3.1 Mathematical Preliminaries

Equations and Assumptions

Consider a viscous conservation law

$$u_t + f(u)_x = (B(u)u_x)_x\,, \quad u, f \in \mathbb{R}^n\,, \tag{3.1}$$

Such equations arise in a variety of physical contexts, where B typically has real, nonnegative eigenvalues (for example, in fluid dynamics, magnetohydrodynamics, and multiphase flow). Here we make the assumption of *strict parabolicity*,

(H1) $\mathrm{Re}(b_j) > 0$ for each eigenvalue $b_1, \dots, b_n$ of B.

A distinctive feature of viscous conservation laws is the appearance of *viscous shock wave* solutions, rapidly varying traveling waves

$$u(x,t) = \bar{u}(x - st) \qquad \lim_{\xi \to \infty} \bar{u}(\xi) = u_\pm \tag{3.2}$$

connecting asymptotically constant states $u_\pm$. The associated *viscous profile* $\bar{u}(\xi)$ thus satisfies the traveling wave ODE

$$u' = B(u)^{-1}[f(u) - f(u_-) - s(u - u_-)] \tag{3.3}$$

By appropriate choice of coordinate frame, we will always take shock speed $s = 0$, so that we consider *stationary viscous shocks* $u = \bar{u}(x)$. However, it is important to recall the dependence of (3.3) on all $n+1$ parameters, (u_-, s).

We make the further assumption that $u_\pm$ are *strictly hyperbolic* both in the PDE and ODE sense. That is, denoting $A_\pm = f'(u_\pm)$, $B_\pm = B(u_\pm)$, we assume that

(H2) $A_\pm$ has distinct, real eigenvalues $a_j^\pm$, while $B_\pm^{-1} A_\pm$ has eigenvalues $\gamma_j^\pm$ with nonzero real part.

From (3.3), this has the important consequence that $\bar{u}$ decays exponentially in all derivatives,

$$|D^k(\bar{u}(x) - u_\pm)| = O(e^{-\alpha|x|}) \quad \text{as } x \to \pm\infty\,,\ \alpha \geq \min|\gamma_j^\pm| > 0\,. \tag{3.4}$$

Our third and final assumption is that both $u_\pm$ satisfy a weak version of the *stable viscosity matrix* criterion of [34], i.e.,

(H3) The constant solutions $u \equiv u_\pm$ of (3.1) are linearly stable with respect to L^2.

The stable viscosity matrix criterion has been shown to hold in many physical systems of interest [28]. As we will show below, (H3) is equivalent to $\sigma_{\text{ess}}(M) \subset \{\operatorname{Re}\lambda \leq 0\}$ or, alternatively, the consistent splitting hypothesis (h1) of Section 2. Since here we are interested in instabilities arising from *point spectrum*, this is an appropriate hypothesis to make. In any case, it can be checked by Laplace or Fourier transform methods, as in [28, 34]. For our analysis, we require only a weakened version (H3′), to be described below.

Linearized Equations

Linearizing (3.1) about $\overline{u}(x)$, we obtain

$$v_t = Mv = -(Av)_x + (Bv_x)_x$$

as the approximate equations governing the evolution of a small perturbation $v(x,t) = u(x,t) - \overline{u}(x)$, where $u(x,t)$ is a nearby solution of (3.1). Here, $B(x) = B(\overline{u}(x))$, while $A(x)$ is the matrix defined by the relation

$$Av = f'(\overline{u})v - B'(\overline{u})v\overline{u}_x\,. \tag{3.5}$$

The eigenvalue equations $Mw = \lambda w$ associated with M are

$$(Bw')' = (Aw)' + \lambda w\,; \quad w(\pm\infty) = 0\,. \tag{3.6}$$

We express these equivalently as a first-order system

$$W' = \mathbb{A}(x,\lambda)W\,; \quad W(\pm\infty) = 0\,, \tag{3.7}$$

with $W = (w, w')^\top$ and

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ \lambda B^{-1} + B^{-1}A' & B^{-1}A - B^{-1}B' \end{pmatrix} \tag{3.8}$$

The asymptotic systems at $x = \pm\infty$ for (3.8)–(3.7) are the linear, constant-coefficient systems

$$W' = \mathbb{A}_\pm(\lambda)W\,, \tag{3.9}$$

where

$$\mathbb{A}_\pm = \begin{pmatrix} 0 & I \\ \lambda B_\pm^{-1} & B_\pm^{-1}A_\pm \end{pmatrix}$$

and $B_\pm = B(u_\pm)$, $A_\pm = f'(u_\pm)$. Dropping subscripts, the eigenvectors of

$$\mathbb{A} = \begin{pmatrix} 0 & I \\ \lambda B^{-1} & B^{-1}A \end{pmatrix}$$

are of the form $V = (v, \mu v)^\top$, where μ is the associated eigenvalue and

$$\left(\lambda B^{-1} + \mu B^{-1}A - \mu^2 I\right) v = 0\,. \tag{3.10}$$

We now state an algebraic condition that is equivalent to (H3).

(H3*) The eigenvalues $\mu^\pm(\lambda)$ of $\mathbb{A}_\pm(\lambda)$ have nonvanishing real part for all $\operatorname{Re}\lambda > 0$.

By using (H3*), it is easy to verify that there are exactly n stable and n unstable eigenvalues for $\operatorname{Re}\lambda > 0$ by checking their signs as $\lambda \to +\infty$ along the real axis. Let the eigenvalues of $\mathbb{A}_\pm(\lambda)$ then be denoted by $\mu_j^\pm(\lambda)$, indexed in order of increasing real part for all $\operatorname{Re}\lambda > 0$. Thus for such λ, $\operatorname{Re}\mu_j^\pm(\lambda) < 0$ for $j \leq n$ and > 0 for $j > n$. By (H3) and standard considerations (cf. [8, 21]),

$$\begin{aligned}\sigma_{\mathrm{ess}}(M) &\subset \{\lambda : \operatorname{Re}\mu_i^\pm(\lambda) \geq 0, 1 \leq i \leq n\} \\ &\quad \cup \{\lambda : \operatorname{Re}\mu_i^\pm(\lambda) \leq 0, n+1 \leq i \leq 2n\} \\ &\subset \{\operatorname{Re}\lambda \leq 0\}\,.\end{aligned}$$

The boundary of this set is part of the essential spectrum of M. This is exactly the union of the spectra of the constant-coefficient operators $M_\pm$ associated with the asymptotic systems

$$v_t = M_\pm v := -A_\pm v_x + B_\pm v_{xx}\,. \tag{3.11}$$

It is easy to see that these operators have *only* essential spectrum with respect to L^2. Since (3.11) are precisely the linearized equations around the constant solutions $u \equiv u_\pm$, it follows that the two forms of (H3) are indeed equivalent.

We remark that our instability analysis requires only the weaker condition

(H3′) $\mathrm{Re}(\mu_j^\pm) \neq 0$ for real $\lambda > 0$,

since all our calculations are confined to the real axis. We make the stronger assumption (H3) only to simplify certain ancillary discussion. There are situations in which this observation may be important; for example, a class of viscous shock waves is pointed out in [40] that are L^2-stable under suitably localized perturbations even though their endstates do not satisfy the stable viscosity matrix criterion, i.e., $\sigma_{\text{ess}}(M)$ is not confined to the stable half-plane.

3.2 Undercompressive Shocks, $n = 2$

The specifics of our stability analysis depend on the *type* of the shock being considered. We now specialize to our case of main interest, of undercompressive type shocks in 2×2 systems. We will carry out this case in some detail; the other cases are analogous.

The type of a viscous shock wave is determined by the signs of $a_j^\pm$. Let i denote the number of positive a_j^- plus the number of negative a_j^+. A shock is of *Lax* type if $i = n + 1$, *undercompressive* if $i = n$, and *overcompressive* if $i = n + 2$. Note that this is consistent with the usual classification of inviscid shocks by relative propagation speeds of characteristics with respect to the shock, which, in the stationary case considered here, are the same $a_j\pm$. Evidently, i is the number of hyperbolic characteristics incoming to the shock, hence the description in terms of "compressivity." There are other possibilities for the values of i corresponding to higher degrees of under- or overcompressivity [33]. However, these cases are more degenerate in the weak shock strength limit and hence rarely occur [4]. For dimension $n = 2$, they do not occur.

Specializing to dimension $n = 2$, we find that undercompressive shock waves are precisely saddle-saddle connections, and the condition for undercompressivity becomes simply

$$\det B(u_\pm)^{-1}(f'(u_\pm) - sI) < 0,$$

or in the stationary case, $\det B_\pm^{-1} A_\pm < 0$. With (H1), this gives

(H2′) $A_\pm$ and $B_\pm^{-1} A_\pm$ have real eigenvalues $a_1^\pm < 0 < a_2^\pm$, $\gamma_1^\pm < 0 < \gamma_2^\pm$.

Having fixed the type of the rest points, we can now explicitly describe the behavior of the asymptotic systems.

LEMMA 3.1

(i) *Let* (H1), (H2′)*, and* (H3) *hold,* $n = 2$*. Then, for* $\mathrm{Re}\,\lambda > 0$*, the matrix* $\mathbb{A}_\pm(\lambda)$ *in* (3.9) *has eigenvalues* $\mu_1^\pm(\lambda), \mu_2^\pm(\lambda) < 0 < \mu_3^\pm(\lambda), \mu_4^\pm(\lambda)$ (*with* *ordering referring to real parts) such that the (generalized) eigenspace* $S^\pm(\lambda)$ *(respectively,* $U^\pm(\lambda)$*) associated to* $\mu_1^\pm(\lambda), \mu_2^\pm(\lambda)$ *(respectively,* $\mu_3^\pm(\lambda), \mu_4^\pm(\lambda)$*) depends analytically on* λ*.*

(ii) *For each* j*, there is an analytic extension of* $\mu_j^\pm(\lambda)$ *to some neighborhood* N *of* $\lambda = 0$*. For* $\lambda \in N$ *there also exists an analytic choice of an individual eigenvector* $V_j^\pm(\lambda)$ *corresponding to each eigenvalue* $\mu_j^\pm(\lambda)$*. The eigenvalues* $\mu_j^\pm(0)$ *are*

$$\mu_1^\pm(0) = \gamma_1^\pm\,, \qquad \mu_2^\pm(0) = 0\,, \qquad \mu_3^\pm(0) = 0\,, \qquad \mu_4^\pm(0) = \gamma_2^\pm\,,$$

and the associated eigenvectors $V_j^\pm = (v_j^\pm, \mu_j^\pm v_j^\pm)$ *are*

$$\begin{aligned} V_1^\pm(0) &= \begin{pmatrix} s_1^\pm \\ \gamma_1^\pm s_1^\pm \end{pmatrix}, \qquad & V_2^\pm(0) &= \begin{pmatrix} r_2^\pm \\ 0 \end{pmatrix}, \\ V_3^\pm(0) &= \begin{pmatrix} r_1^\pm \\ 0 \end{pmatrix}, \qquad & V_4^\pm(0) &= \begin{pmatrix} s_2^\pm \\ \gamma_2^\pm s_2^\pm \end{pmatrix}. \end{aligned} \tag{3.12}$$

Here $\gamma_1^\pm < 0 < \gamma_2^\pm$ *are the eigenvalues of* $B_\pm^{-1}A_\pm$ *and* $s_1^\pm, s_2^\pm$ *are the associated eigenvectors, while* $a_1^\pm < 0 < a_2^\pm$ *are the eigenvalues of* $A_\pm$ *and* $r_1^\pm, r_2^\pm$ *are the associated eigenvectors. The spectral projection operators* $P_{S^\pm(\lambda)}$ *and* $P_{U^\pm(\lambda)}$ *associated to the subspaces* $S^\pm(\lambda)$ *and* $U^\pm(\lambda)$ *have analytic extensions to the neighborhood* $\Omega = \{\operatorname{Re}\lambda > 0\} \cup N$*.*

(iii) *There exist choices of 2-forms* $\eta^\pm(\lambda)$ *and* $\zeta^\pm(\lambda)$ *associated to* $S^\pm(\lambda)$ *and* $U^\pm(\lambda)$*, respectively, which are analytic in the domain* Ω*, which are real-valued for real* λ*, and which, for* $\lambda \in N$*, satisfy*

$$\eta^\pm(\lambda) = k(\lambda) V_1^\pm(\lambda) \wedge V_2^\pm(\lambda)\,, \qquad \zeta^\pm(\lambda) = l(\lambda) V_3^\pm(\lambda) \wedge V_4^\pm(\lambda)\,,$$

for some scalar functions $k(\lambda)$ *and* $l(\lambda)$ *satisfying* $k(0) = l(0) = 1$*.*

PROOF: By assumption (H3), the number of eigenvalues μ with $\pm$ real parts is constant for $\operatorname{Re}(\lambda) > 0$. Taking $\lambda \to +\infty$, we find from (3.10) that the μ_j approximately satisfy $\det(B^{-1} - (\mu^2/\lambda)I) = 0$, which has roots

$$\mu = \pm\sqrt{\lambda/b_j}\,,$$

where $b_j > 0$ are the eigenvalues of $B_\pm$. This confirms the 2-2 splitting of roots for $\operatorname{Re}(\lambda) > 0$ with *strict* inequality and consequent separation of the eigenspaces $S^{\pm}(\lambda)$ and $U^{\pm}(\lambda)$. The analytic dependence of these subspaces then follows from standard matrix perturbation theory and (H3), since (H3) implies that these subspaces have a positive spectral gap for $\operatorname{Re}\lambda > 0$ (e.g., [27]).

On the other hand, the analytic expansion of the spectrum of $\mathbb{A}_{\pm}(\lambda)$ about $\lambda = 0$ follows by standard bifurcation theory [20]. Substituting $\lambda = 0$ into (3.10) to obtain

$$\mu_j(B_{\pm}^{-1}A_{\pm} - \mu_j I)v_j = 0\,,$$

we find that there is a root $\mu = 0$ of multiplicity 2 and two distinct roots $\mu = \gamma_j^{\pm}$, $v = s_j^{\pm}$. Corresponding to each distinct root (for $j = 1, 4$), there is an analytic $\mu_j(\lambda)$ that trivially satisfies

$$\mu_j = \gamma_j + \mathbf{O}(\lambda) \quad \text{and} \quad V_j = \begin{pmatrix} s_j \\ \gamma_j s_j \end{pmatrix} + \mathbf{O}(\lambda)\,.$$

The remaining two roots ($j = 2, 3$) bifurcate from $(\lambda, \mu) = (0, 0)$. Writing (3.10) as

$$\left(\lambda_{j\pm} I + \mu_j A_{\pm} - \mu_j^2 B_{\pm}\right) v_j = 0$$

and linearizing about $(\lambda, \mu) = (0, 0)$, we obtain $(\lambda_{\pm} + \mu A_{\pm})v = 0$. Since by assumption $A_{\pm}$ has a full set of eigenvalue-eigenvector pairs $(a_j^{\pm}, r_j^{\pm})$, this is a bifurcation from a simple root, and we obtain two analytic solutions of form $\mu_j^{\pm}(\lambda) = -\lambda/a_j^{\pm} + \mathbf{O}(\lambda^2)$, $v_j^{\pm} = r_j^{\pm} + \mathbf{O}(\lambda)$. The existence of analytic choices of eigenvectors $V_j^{\pm}(\lambda)$ in a neighborhood of $\lambda = 0$ then follows from standard linear algebra and the analyticity of the eigenvalues.

The analyticity of the eigenvalues and the four eigenvectors in a neighborhood N of $\lambda = 0$ then implies that the two spectral projection operators $P_{S^{\pm}(\lambda)}$ and $P_{U^{\pm}(\lambda)}$ continue analytically to the domain Ω. The existence of analytic choices of 2-forms $\eta^{\pm}(\lambda)$ and $\zeta^{\pm}(\lambda)$ then follows from Lemma 2.2. They can be chosen to be real for real λ by Lemma 2.7. They are related to the wedges of the $V_i(\lambda)$ in the indicated manner by our initialization of $\eta^{\pm}(0)$ and $\zeta^{\pm}(0)$. ∎

We remark that it is not obvious that the local forms consisting of the wedges of individual eigenvectors extend globally to analytic objects on all of Ω.

PROPOSITION 3.2 *On the neighborhood* $\Omega = \{\operatorname{Re}\lambda > 0\} \cup N$, *there exist solutions* $\eta(x, \lambda), \zeta(x, \lambda)$ *of* (2.3) *(with* $k = 2$*) such that*

$$\begin{aligned} \eta(x, \lambda) &= e^{(\mu_1^+ + \mu_2^+)x}(\eta^+(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to +\infty\,, \\ \zeta(x, \lambda) &= e^{(\mu_3^- + \mu_4^-)x}(\zeta^-(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to -\infty\,, \end{aligned} \tag{3.13}$$

where α is as in (3.4) *and $\eta^+(\lambda), \zeta^-(\lambda)$ are the analytic forms defined in Lemma* 3.1. *The solutions $\eta(x,\lambda), \zeta(x,\lambda)$ depend analytically on λ.*

On a neighborhood of $\lambda = 0$, there are individual solutions $\varphi_j(x,\lambda)$ of (3.7) *that are analytic in λ and satisfy*

$$\begin{aligned} \varphi_j(x,\lambda) &= e^{\mu_j^+ x}(V_j^+(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to +\infty\,,\ j = 1,2,\\ \varphi_j(x,\lambda) &= e^{\mu_j^\pm x}(V_j^-(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to -\infty\,,\ j = 3,4, \end{aligned} \tag{3.14}$$

where $V_j^\pm(\lambda)$ are as in Lemma 3.1. *Furthermore, for $\lambda \in N$*

(3.15)

$$\begin{aligned} \eta(x,\lambda) &= k(\lambda)\varphi_1(x,\lambda) \wedge \varphi_2(x,\lambda)\\ &= k(\lambda)e^{(\mu_1^\pm + \mu_2^+)x}(V_1^+(\lambda) \wedge V_2^+(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to +\infty\,,\\ \zeta(x,\lambda) &= l(\lambda)\varphi_3(x,\lambda) \wedge \varphi_4(x,\lambda)\\ &= l(\lambda)e^{(\mu_3^- + \mu_4^-)x}(V_3^-(\lambda) \wedge V_4^-(\lambda) + \mathbf{O}(e^{-\alpha|x|/2}))\,, \quad x \to -\infty\,. \end{aligned}$$

PROOF: By Lemma 3.1, the gap conditions (H3) and (H4) hold uniformly on $\operatorname{Re}\lambda \geq 0$ for the subspaces associated to $\eta^+(\lambda)$ and $\eta^-(\lambda)$. It then follows from Theorem 2.3 and Lemma 2.5 that there exist unique 2-form solutions $\eta(x,\lambda)$ and $\zeta(x,\lambda)$ satisfying (3.13). On the other hand, the local result Corollary 2.4 together with Lemma 2.5 implies the existence of analytic choices of $\varphi_j(x,\lambda)$ in a neighborhood N of $\lambda = 0$ satisfying (3.14). It follows that their wedge products converge to $V_1^+(\lambda) \wedge V_2^+(\lambda)$ and $V_3^-(\lambda) \wedge V_4^-(\lambda)$ at the correct asymptotic rate; hence they must be equal to $\eta(x,\lambda)/k(\lambda)$ and $\zeta(x,\lambda)/l(\lambda)$ by uniqueness. ∎

COROLLARY 3.3 *The Evans function*

$$D(\lambda) = e^{-\int_0^x \operatorname{tr}\mathbb{A}(s,\lambda)ds}\eta(x,\lambda) \wedge \zeta(x,\lambda)$$

is analytic in λ on the domain Ω. In the neighborhood N of $\lambda = 0$,

$$D(\lambda) = k(\lambda)l(\lambda)e^{-\int_0^x \operatorname{tr}\mathbb{A}(s,\lambda)ds}\varphi_1(x,\lambda) \wedge \varphi_2(x,\lambda) \wedge \varphi_3(x,\lambda) \wedge \varphi_4(x,\lambda)$$

with $\varphi_i(x,\lambda)$ as in Proposition 3.2.

As described in Section 2, away from the essential spectrum of M, the Wronskian $D(\lambda)$ vanishes precisely at eigenvalues of M. Thus, we can search for unstable eigenvalues in $\operatorname{Re}\lambda > 0$ by looking for zeroes of D.

Remarks

1. The factors $k(\lambda)$ and $l(\lambda)$ clearly play no role in the calculation of the sign of $D'(0)$, since $k(0) = l(0) = 1$ and $D(0) = 0$. We can therefore replace the local expression for $D(\lambda)$ by

   (3.16)
   $$D(\lambda) = e^{-\int_0^x \mathrm{tr}\mathbb{A}(s,\lambda)\, ds}\varphi_1(x,\lambda) \wedge \varphi_2(x,\lambda) \wedge \varphi_3(x,\lambda) \wedge \varphi_4(x,\lambda)$$

   during this part of the calcuation.

2. If (H3) is replaced by the weaker assumption (H3$'$), we obtain by the same arguments the result of Corollary 3.3 for λ on a neighborhood of the nonnegative real axis. This is enough to carry out all our later analysis.

3. Further expansion of $\mu_2^{\pm}$ and $\mu_3^{\pm}$ in the argument of Lemma 3.1 shows that $\mathrm{Re}\,\mu_2^{\pm}$ and $\mathrm{Re}\,\mu_3^{\pm}$ exchange signs as λ crosses zero along the real axis, i.e., the spectral gap becomes negative. Thus, our application of the gap lemma was truly necessary to treat this case.

3.3 The Stability Condition

We are now ready to carry out our main calculations. As in Lemma 3.1, let $r_j^{\pm}$ and $s_j^{\pm}$ denote the eigenvectors of $A_{\pm}$ and $B_{\pm}^{-1}A_{\pm}$, respectively. Choose an orientation of the $V_j^{\pm}$ of Lemma 3.1 so that, at $\lambda = 0$, $v_1^+ = s_1^+$ and $v_4^- = s_2^-$ point in the asymptotic direction of $\bar{u}_x$ at $\pm\infty$, respectively.

Note. We shall, with a slight abuse of notation, adopt the following convention. Given a top differential form $\eta \in \bigwedge^n(\mathbb{C}^n)$, $\mathrm{sgn}\,\eta$ will denote the sign of the coefficient of the form relative to the *positive* orientation on $\mathbb{C}^n$ for any n.

LEMMA 3.4 *Let* (H1)*,* (H2$'$)*, and* (H3) *hold,* $n = 2$*, and suppose that* r_1^-*,* r_2^+ *are linearly independent,* $r_j^{\pm}$ *as specified in Lemma* 3.1*. Then,*

(i) $D(0) = 0$ *and*

(ii)

(3.17)
$$\mathrm{sgn}\, D'(0) = \mathrm{sgn}\left[\Gamma(r_1^- \wedge r_2^+)\right]$$

where

(3.18)
$$\Gamma = \int_{-\infty}^{\infty} e^{\int_0^x \mathrm{tr}(B^{-1}A)dx}\left[\overline{u}_x \wedge B^{-1}(\overline{u} - u_*)\right]dx$$

and u_* is the unique point such that

$$u_* = \begin{cases} u_- + \alpha_- r_1^- \\ u_+ + \alpha_+ r_2^+ . \end{cases} \tag{3.19}$$

PROOF: We can compute $\operatorname{sgn} D(0)$ and $\operatorname{sgn} D'(0)$ using (3.16) by the accompanying remark.

(i) At $\lambda = 0$, φ_1 is the unique solution (up to constant multiples) of (3.7) decaying at $x = +\infty$, and φ_4 is the unique solution decaying at $x = -\infty$. Since $W = (\overline{u}_x, \overline{u}_{xx})^\top$ satisfies (3.7) for $\lambda = 0$ with $W(\pm\infty) = 0$, we find that both φ_1 and φ_4 must be multiples of W; hence their wedge product is zero, and $D(0) = 0$.

(ii) Without loss of generality, fix $\varphi_1 = \varphi_4 = (\overline{u}_x, \overline{u}_{xx})$. Using the Leibniz rule and the dependence of φ_1 and φ_4, we find that

$$\begin{aligned} D'(0) &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \left[\left(\frac{\partial \varphi_1}{\partial \lambda} \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \right) \right. \\ &\qquad\qquad \left. + \left(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \frac{\partial \varphi_4}{\partial \lambda} \right) \right] \\ &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \left[\varphi_2 \wedge \varphi_3 \wedge \varphi_1 \wedge \left(\frac{\partial \varphi_4}{\partial \lambda} - \frac{\partial \varphi_1}{\partial \lambda} \right) \right] . \end{aligned} \tag{3.20}$$

Denoting each φ_j as $(w_j, w_j')^\top$, we have that w_j satisfies equation (3.6), or

$$(Bw_j')' = (Aw_j)' + \lambda w_j . \tag{3.21}$$

Differentiating with respect to λ at $\lambda = 0$, for $j = 1, 4$, we find that $z_j = \partial w_j / \partial \lambda$ satisfies

$$\begin{aligned} (Bz_j')' &= (Az_j)' + \lambda z_j + w_j \\ &= (Az_j)' + \overline{u}_x . \end{aligned} \tag{3.22}$$

From the boundary condition $\varphi_1(+\infty) \equiv 0$, we further have $z_1(+\infty)$, $z_1'(+\infty) = 0$, while $z_4(-\infty), z_4'(-\infty) = 0$. Thus, integrating (3.22) from x to $+\infty$ and $-\infty$ to x, respectively, we find that z_1 and z_4 satisfy equations

$$z_1{}' = B^{-1}Az_1 + B^{-1}(\overline{u} - u_+) \tag{3.23}$$

and

$$z_4{}' = B^{-1}Az_4 + B^{-1}(\overline{u} - u_-)\,. \tag{3.24}$$

Functions w_2, w_3, and w_1, on the other hand, by (3.21) satisfy

$$(Bw_j{}')' = (Aw_j)' \tag{3.25}$$

at $\lambda = 0$. Using the boundary conditions $w_2(+\infty) = r_2^+,\ w_2{}'(+\infty) = 0$ and $w_3(-\infty) = r_1^-,\ w_3{}'(-\infty) = 0$ from Lemma 3.1 and Proposition 3.2, and integrating (3.25) from x to $+\infty$ and $-\infty$ to x, respectively, we thus obtain

$$\begin{aligned} w_2{}' &= B^{-1}Aw_2 - B^{-1}A_+r_2^+ \\ &= B^{-1}Aw_2 - B^{-1}a_2^+r_2^+\,, \end{aligned} \tag{3.26}$$

and

$$w_3{}' = B^{-1}Aw_3 - B^{-1}a_1^-r_1^-\,, \tag{3.27}$$

where $a_j^\pm$ as before denotes the eigenvalue of $A_\pm$ associated with $r_j^\pm$. Finally, the boundary conditions $w_1(\pm\infty), w_1'(\pm\infty) = 0$ give

$$w_1{}' = B^{-1}Aw_1\,. \tag{3.28}$$

Combining (3.20) with (3.23) through (3.24) and (3.26) through (3.28) and performing elementary matrix manipulations, we obtain

(3.29)

$$\begin{aligned} D'(0) = e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} &\begin{pmatrix} w_2 \\ B^{-1}(Aw_2 - a_2^+r_2^+) \end{pmatrix} \\ &\wedge \begin{pmatrix} w_3 \\ B^{-1}(Aw_3 - a_1^-r_1^-) \end{pmatrix} \\ &\wedge \begin{pmatrix} w_1 \\ B^{-1}Aw_1 \end{pmatrix} \wedge \begin{pmatrix} z_4 - z_1 \\ B^{-1}[A(z_4 - z_1) + (u_+ - u_-)] \end{pmatrix} \\ = e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} &\begin{pmatrix} w_2 \\ Aw_2 - a_2^+r_2^+ \end{pmatrix} \\ &\wedge \begin{pmatrix} w_3 \\ Aw_3 - a_1^-r_1^- \end{pmatrix} \wedge \begin{pmatrix} w_1 \\ Aw_1 \end{pmatrix} \wedge \begin{pmatrix} z_4 - z_1 \\ A(z_4 - z_1) + (u_+ - u_-) \end{pmatrix} \end{aligned}$$

$$= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} \begin{pmatrix} w_2 \\ a_2^+ r_2^+ \end{pmatrix} \wedge \begin{pmatrix} w_3 \\ a_1^- r_1^- \end{pmatrix} \wedge \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} z_4 - z_1 \\ -(u_+ - u_-) \end{pmatrix}.$$

Now, setting $\alpha_2 = \alpha_+/a_2^+$ and $\alpha_3 = \alpha_-/a_1^-$ so that $\alpha_2 a_2^+ r_2^+ + \alpha_3 a_1^- r_1^- = (u_+ - u_-)$, we obtain from (3.29) that

(3.30)

$$\begin{aligned} D'(0) &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} \begin{pmatrix} w_2 \\ a_2^+ r_2^+ \end{pmatrix} \\ &\quad \wedge \begin{pmatrix} w_3 \\ a_1^- r_1^- \end{pmatrix} \wedge \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} z_4 - z_1 + \alpha_2 w_2 + \alpha_3 w_3 \\ 0 \end{pmatrix} \\ &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \\ &\quad \times \det B^{-1}[w_1 \wedge (z_4 - z_1 + \alpha_2 w_2 + \alpha_3 w_3)](a_2^+ a_1^-)[r_2^+ \wedge r_1^-]. \end{aligned}$$

Defining $\Gamma = e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} w_1 \wedge (z_4 - z_1 + \alpha_2 w_2 + \alpha_3 w_3)$ and noting that $a_1^- < 0 < a_2^+$, $\det B^{-1} > 0$, we obtain (3.17).

To evaluate Γ, set $\tilde{z}_4 = z_4 + \alpha_3 w_3$ and $\tilde{z}_1 = z_1 - \alpha_2 w_2$, so that $\tilde{z}_4 - \tilde{z}_1 = z_4 - z_1 + \alpha_2 w_2 + \alpha_3 w_3$. By (3.23) through (3.24) and (3.26) through (3.27) combined with (3.30), both $\tilde{z}_1$ and $\tilde{z}_4$ satisfy the ODE

$$\tilde{z}' = B^{-1}A\tilde{z} + B^{-1}(\overline{u} - u_*),$$

where u_* is defined by (3.19). Further, $\tilde{z}_4(-\infty)$ and $\tilde{z}_1(+\infty)$ are bounded, since $z_4(-\infty) = z_1(+\infty) = 0$ and both $w_3(-\infty)$ and $w_2(+\infty)$ are bounded. Since w_1 satisfies $w_1' = B^{-1}Aw_1$, we thus have that the Wronskian $(w_1 \wedge \tilde{z}_4)$ satisfies

$$\begin{aligned} (w_1 \wedge \tilde{z}_4)' &= \Big[\Big(B^{-1}Aw_1 \wedge \tilde{z}_4\Big) + \Big(w_1 \wedge B^{-1}A\tilde{z}_4\Big)\Big] \\ &\quad + \Big(w_1 \wedge B^{-1}(\overline{u} - u_*)\Big) \\ &= \operatorname{tr}\Big(B^{-1}A\Big)(w_1 \wedge \tilde{z}_4) + \Big[w_1 \wedge B^{-1}(\overline{u} - u_*)\Big]. \end{aligned}$$

It follows from Duhamel's principle that

(3.31)
$$(w_1 \wedge \tilde{z}_4)\Big|_{x=0} = \lim_{y\to-\infty}\Big[(\overline{u}_x(y)\wedge\tilde{z}_4(y))e^{\int_y^0 \mathrm{tr}(B^{-1}A)dx} + \int_y^0 e^{\int_x^0 \mathrm{tr}(B^{-1}A)dx}(\overline{u}_x \wedge B^{-1}(\overline{u}-u_*))dx\Big].$$

Since $\tilde{z}_4(y)\to$ const and $\overline{u}_x(y)\sim e^{\gamma_2 y}$ as $y\to-\infty$, while

$$e^{\int_y^0 \mathrm{tr}(B^{-1}A)} \sim e^{-(\gamma_1+\gamma_2)y},$$

the first term on the right-hand side of (3.31) goes to zero as $e^{-\gamma_1 y}$, where, as before, $\gamma_1 < 0 < \gamma_2$ denote the eigenvalues of $B^{-1}A$. Thus, we obtain

$$(w_1\wedge\tilde{z}_4)\Big|_{x=0} = \int_{-\infty}^0 e^{-\int_0^x \mathrm{tr}(B^{-1}A)}\overline{u}_x \wedge B^{-1}(\overline{u}-u_*)dx$$

and, by a symmetric calculation at $+\infty$,

$$(w_1\wedge\tilde{z}_1)\Big|_{x=0} = -\int_0^\infty e^{-\int_0^x \mathrm{tr}(B^{-1}A)}\Big(\overline{u}_x\wedge B^{-1}(\overline{u}-u_*)\Big)dx\,.$$

Combining and using $\Gamma(x)\equiv\Gamma(0) = w_1\wedge(\tilde{z}_4-\tilde{z}_1)\big|_{x=0}$ gives (3.18), completing the proof.

∎

LEMMA 3.5

(i) *Let* $V_j^\pm = (v_j^\pm, \tilde{v}_j^\pm)^\top$ *be eigenvectors or generalized eigenvectors of* $\mathbb{A}_\pm(\lambda)$ *associated to* $\mu_i^\pm(\lambda)$ *for (fixed)* $\lambda\in\Omega$. *Then* $v_1^+\wedge v_2^+\neq 0$ *and* $v_3^-\wedge v_4^-\neq 0$.

(ii) *For real* $\lambda\geq 0$, *let* $E_i^\pm(\lambda) = (e_i^\pm(\lambda), f_i^\pm(\lambda))$ *be the* (*real*) *analytic bases of* $S^+(\lambda)$ (*for* $i=1,2$) *and* $U^\pm(\lambda)$ (*for* $i=3,4$) *of Lemma* 2.7, *initialized so that* $E_i^\pm(0)=V_i^\pm(0)$. *Then*

(3.32)
$$\mathrm{sgn}\, e_1^+(\lambda)\wedge e_2^+(\lambda) = \mathrm{sgn}\, s_1^+\wedge r_2^+\,,$$
$$\mathrm{sgn}\, e_3^-(\lambda)\wedge e_4^-(\lambda) = \mathrm{sgn}\, r_1^-\wedge s_2^-\,.$$

for all real $\lambda\geq 0$.

PROOF: We will prove the lemma for $V_1^+ \wedge V_2^+$; the claim for $V_3^- \wedge V_4^-$ follows by a symmetric argument. For notational convenience, we drop the $+$ superscript. Suppose that $V_1 = (v_1, \mu_1 v_1)^\top$, $V_2 = (v_2, \mu_2 v_2)^\top$, are both genuine eigenvectors, but $v_1 \wedge v_2 = 0$, or without loss of generality $v_1 = v_2$. Then, by (3.10),

$$0 = \left(\lambda B^{-1} + \mu_1 B^{-1}A - \mu_1^2 I\right) v_1 = \left(\lambda B^{-1} + \mu_2 B^{-1}A - \mu_2^2 I\right) v_1 \tag{3.33}$$

Subtracting and simplifying, we have $[B^{-1}A - (\mu_1+\mu_2)I]v_1 = 0$, implying that v_1 is an eigenvector s of $B^{-1}A$ with eigenvalue γ. From (3.33), it then follows that v_1 must be an eigenvector of B^{-1} and thus A as well, with eigenvalues b and a, respectively. Taking $v_1 \equiv s$, we thus find that equation (3.33) is satisfied whenever $\lambda b + \mu_j \gamma - \mu_j^2 = 0$; hence the μ_j are *identically equal to* $\gamma/2 \pm \sqrt{\gamma^2/4 + \lambda b}$ for all λ on which they are defined. But, $\operatorname{Re} b > 0$, (H1), implies that $\gamma/2 \pm \sqrt{\gamma^2/4 + \lambda b}$ have real parts of opposite sign for large real $\lambda > 0$, contradicting the definition of μ_1 and μ_2 in Lemma 3.1.

A similar calculation shows that $v_1 \wedge v_2 = 0$ is also impossible in the case that $\mu_1 = \mu_2$ and V_2 is a generalized eigenvector, i.e., $V_1 = (v_1, \mu_1 v_1)^\top$ as before, while $V_2 = (v_1, w)^\top$. For then $\tilde{V}_2 = V_2 - V_1 = (0, \tilde{w})^\top$ is a generalized eigenvector as well, and thus

$$\begin{pmatrix} -\mu_1 I & I \\ \lambda B^{-1} & B^{-1}A - \mu_1 I \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{w} \end{pmatrix} = \alpha \begin{pmatrix} v_1 \\ \mu_1 v_1 \end{pmatrix},$$

giving $\tilde{w} = \alpha v_1$ and therefore $B^{-1}Av_1 = 2\mu_1 v_1$. Since we already have

$$\left(\lambda B^{-1} + \mu_1 B^{-1}A - \mu_1^2 I\right) v_1 = 0$$

by the fact that V_1 is a genuine eigenvector, this implies that $\lambda B^{-1} v_1 = -\mu_1^2 v_1$. As in the previous case, we can conclude that v_1 is an eigenvector of A as well; hence v_1 is real by (H2). Since B and A are real, it follows from $B^{-1}Av_1 = 2\mu_1 v_1$ that μ_1 is real Thus, $-\mu_1^2/\lambda$ is an eigenvalue of B^{-1} with negative real part, contradicting (H1).

Now let $\eta(\lambda) = E_1(\lambda) \wedge E_2(\lambda)$ where the $E_i(\lambda)$ are as in Lemma 2.7. In particular, the $E_i(\lambda)$ are analytic in Ω, and they are real for $\lambda \geq 0$. Let $E_i(\lambda) = (e_i(\lambda), f_i(\lambda))^\top$, $i = 1, 2$, and suppose that $\eta(\lambda) = \bar{V}_1 \wedge \bar{V}_2$ where the $\bar{V}_i = (\bar{v}_i, \bar{w}_i)$ are (generalized) eigenvectors of $\mathbb{A}(\lambda)$ that also span $S(\lambda)$. It then follows that $E_i(\lambda) = \sum_{j=1}^{2} \alpha_{i,j} \bar{V}_j$ for some matrix α with $\det \alpha = 1$. However, we then have that $e_1(\lambda) \wedge e_2(\lambda) = \bar{v}_1 \wedge \bar{v}_2$. By the above, the former and hence the latter wedge products are nonvanishing. However, the former wedge is continuous in λ, which verifies (3.32). ∎

Lemma 3.5 appears to be rather fundamental for conservation laws. For example, when $\lambda = 0$, it reduces to the statement that the unstable/stable manifold of A is transverse to the stable/unstable manifold of $B^{-1}A$, a fact that is important in the study of initial boundary problems [38, 39].

COROLLARY 3.6 *If* (H1), (H2′), *and* (H3) *hold,* $n = 2$, *then*

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn}(s_1^+ \wedge r_2^+)(r_1^- \wedge s_2^-) \neq 0$$

as $\lambda \to +\infty$ *along the real axis.*

PROOF: We restrict our attention to real $\lambda > 0$. Rescaling (3.8) by the change of variables $\tilde{x} = |\lambda^{1/2}|x$, we obtain, after dropping tildes,

$$(B(x)w')' = \delta(A(x)w)' + w \tag{3.34}$$

where $\delta = \lambda^{-1/2}$, or

$$W' = \mathbb{B}(x)W + \Theta(x,\delta)W \tag{3.35}$$

where

$$\begin{gathered} \mathbb{B}(x) = \begin{pmatrix} 0 & I \\ B^{-1}(\bar{u}(x)) & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} w \\ w' \end{pmatrix}, \\ |\Theta(x,\delta)|,\ |\mathbb{B}'(x)| = \mathbf{O}(\delta)\,. \end{gathered} \tag{3.36}$$

Recall that in the course of defining $D(\lambda)$ we have specified an analytic choice of the 2-forms $\eta^+(\lambda)$ and $\zeta^-(\lambda)$ associated with the unstable/stable subspaces of the matrix $\mathbb{A}_\pm(\lambda)$ for the unscaled system. Through the scaling transformation, this induces an analytic choice of 2-forms $\eta^+(\delta)$ and $\zeta^-(\delta)$ associated with the stable and unstable subspaces of the matrix $\mathbb{B}_\delta^\pm = \mathbb{B}(\pm\infty) + \Theta(\pm\infty,\delta)$. Let $E_j^\pm(\delta) = (e_j^\pm(\delta), f_j^\pm(\delta))$ be the bases for the stable ($j = 1,2$) and unstable ($j = 3,4$) subspaces of $\mathbb{B}_\delta^\pm$ of Lemma 2.7 under the rescaling; then (3.32) also holds for the rescaled quantities, with λ replaced by δ.

Now, consider the matrix $\mathbb{B}(x)$ which, for large x, is $\mathbf{O}(\delta)$ close to $\mathbb{B}_\delta^\pm$. Furthermore, by (H1) and (3.36), the eigenvalues of $\mathbb{B}(x))$ are of the form $\pm b_i(x)^{-1/2},\ i = 1,2$, where $b_1(x)$ and $b_2(x)$ are the eigenvalues of $B(\bar{u}(x))$. It therefore follows that $\mathbb{B}(x)$ has two-dimensional stable and unstable subspaces, which are $\mathbf{O}(\delta)$ close to those of $\mathbb{B}_\delta^\pm$ for sufficiently large $|x|$ in the sense that

$$\begin{aligned} |\eta_{\mathbb{B}(x)} - \eta^+(\delta)| &= \mathbf{O}(\delta)\,, \quad x \to +\infty\,, \\ |\zeta_{\mathbb{B}(x)} - \zeta^-(\delta)| &= \mathbf{O}(\delta)\,, \quad x \to -\infty\,, \end{aligned} \tag{3.37}$$

for some appropriate 2-forms $\eta_{\mathbb{B}(x)}$ and $\zeta_{\mathbb{B}(x)}$, which represent the stable and unstable subspaces of $\mathbb{B}(x)$.

By (H1) the stable and unstable subspaces of $\mathbb{B}(x)$ are uniformly separated with a positive spectral gap $\beta > 2 \max \operatorname{Re} b_i(x)^{-1/2}$; it follows that there exist smooth bases $\bar{E}_1(x), \bar{E}_2(x)$ and $\bar{E}_3(x), \bar{E}_4(x)$ of the stable and unstable subspaces of $\mathbb{B}(x)$ such that

$$\eta_{\mathbb{B}(x)} = \bar{E}_1(x) \wedge \bar{E}_2(x)\,, \qquad \zeta_{\mathbb{B}(x)} = \bar{E}_3(x) \wedge \bar{E}_4(x)\,.$$

In the event that $\mathbb{B}(x)$ is not diagonalizable for some x, the basis elements $\bar{E}_i(x)$ may consist of combinations of the eigenvectors (or generalized eigenvectors) V_i of $\mathbb{B}(x)$ in order to ensure continuity with respect to x. However, for each *fixed* x there exist eigenvectors (or generalized eigenvectors) V_i of $\mathbb{B}(x)$ such that $\eta_{\mathbb{B}(x)} = V_1 \wedge V_2$ and $\zeta_{\mathbb{B}(x)} = V_3 \wedge V_4$. Let $\bar{E}_i(x) = (\bar{e}_i(x), \bar{f}_i(x))$ and $V_i = (v_i, w_i)$; it then follows as in the proof of Lemma 3.5 that

$$\begin{aligned} \operatorname{sgn} \bar{e}_1(x) \wedge \bar{e}_2(x) &= \operatorname{sgn} v_1 \wedge v_2\,, \\ \operatorname{sgn} \bar{e}_3(x) \wedge \bar{e}_4(x) &= \operatorname{sgn} v_3 \wedge v_4\,. \end{aligned} \tag{3.38}$$

Suppose first that $B(\bar{u}(x))$ and consequently $\mathbb{B}(x)$ are diagonalizable. It then follows that there are bases $\{v_1, v_2\}$ and $\{v_3, v_4\}$ of eigenvectors of $B(\bar{u}(x))$ and that

$$\begin{aligned} V_1 &= \begin{pmatrix} v_1 \\ -b_1^{-1/2} v_1 \end{pmatrix}, & V_2 &= \begin{pmatrix} v_2 \\ -b_2^{-1/2} v_2 \end{pmatrix}, \\ V_3 &= \begin{pmatrix} v_3 \\ +b_1^{-1/2} v_3 \end{pmatrix}, & V_4 &= \begin{pmatrix} v_4 \\ +b_2^{-1/2} v_4 \end{pmatrix}, \end{aligned}$$

where $b_i = b_i(x)$. Thus, by the above we have that

$$\zeta_{\mathbb{B}(x)} = \begin{pmatrix} v_3 \\ +b_1^{-1/2} v_3 \end{pmatrix} \wedge \begin{pmatrix} v_4 \\ +b_2^{-1/2} v_4 \end{pmatrix}$$

and

$$\tilde{\eta}_{\mathbb{B}(x)} = \begin{pmatrix} v_1 \\ -b_1^{-1/2} v_1 \end{pmatrix} \wedge \begin{pmatrix} v_2 \\ -b_2^{-1/2} v_2 \end{pmatrix}.$$

Performing elementary column manipulations, we now find that

(3.39)

$$\begin{aligned}\operatorname{sgn}\eta_{\mathbb{B}(x)}\wedge\zeta_{\mathbb{B}(x)} &= \operatorname{sgn}\begin{pmatrix} v_1 \\ -b_1^{-1/2}v_1\end{pmatrix}\wedge\begin{pmatrix} v_2 \\ -b_2^{-1/2}v_2\end{pmatrix}\wedge\begin{pmatrix} v_1 \\ +b_1^{-1/2}v_1\end{pmatrix}\\ &\quad\wedge\begin{pmatrix} v_2 \\ +b_2^{-1/2}v_2\end{pmatrix}\\ &= \operatorname{sgn}\begin{pmatrix} v_1 \\ -b_1^{-1/2}v_1\end{pmatrix}\wedge\begin{pmatrix} v_2 \\ -b_2^{-1/2}v_2\end{pmatrix}\wedge\begin{pmatrix} 2v_1 \\ 0\end{pmatrix}\wedge\begin{pmatrix} 2v_2 \\ 0\end{pmatrix}\\ &= \operatorname{sgn} 4\det B^{-1/2}(v_1\wedge v_2)(v_3\wedge v_4)\\ &= \operatorname{sgn}[\bar{e}_1(x)\wedge\bar{e}_2(x)][\bar{e}_3(x)\wedge\bar{e}_4(x)],\end{aligned}$$

where the final equality follows from (H1) and (3.38). Though we have derived (3.39) under the assumption that B is diagonalizable, it holds in general by continuity of $\eta_{\mathbb{B}}$ and $\zeta_{\mathbb{B}}$ with respect to $\mathbb{B}$, a consequence of their spectral separation.

Let $\hat{\eta}(x,\delta)$ and $\hat{\zeta}(x,\delta)$ be the stable and unstable manifolds of the projectivized flow (2.13) induced by (3.35) on the appropriate (projectivized) exterior power space, so that $\hat{\eta}(x,\delta)\to\hat{\eta}^+(\delta)$ and $\hat{\zeta}(x,\delta)\to\hat{\zeta}^-(\delta)$ for $x\to+\infty$ and $x\to-\infty$, respectively. It then follows from Proposition 2.8 that for sufficiently small δ,

$$\hat{\eta}(x,\delta)=\hat{\eta}_{\mathbb{B}(x)}+\mathbf{O}(\delta)\,,\qquad \hat{\zeta}(x,\delta)=\hat{\zeta}_{\mathbb{B}(x)}+\mathbf{O}(\delta)\,,$$

where the approximation is in terms of some appropriate local coordinate on the projective space. It then follows from (3.37) and Lemma 2.5 that there are solutions $\eta(x,\delta),\zeta(x,\delta)$ of the original linear equations that satisfy

$$\eta(x,\delta)=k(x)(\eta_{\mathbb{B}(x)}+\mathbf{O}(\delta))\,,\qquad \zeta(x,\delta)=l(x)(\zeta_{\mathbb{B}(x)}+\mathbf{O}(\delta))\,, \tag{3.40}$$

where $k(x)$ and $l(x)$ are real and positive scalar functions. This is because both $\eta(x,\delta)$ and $\eta_{\mathbb{B}(x)}$ are real for real $\lambda\geq 0$, so that k and l can be chosen real, and they are nonvanishing by uniqueness of solutions of linear ODEs. Finally, they are positive by (3.37) and the specified behavior of η and ζ as $x\to+\infty$ and $-\infty$, respectively.

It follows from (3.40) that the Evans function $D(\delta)$ of the rescaled system (3.34) satisfies

$$D(\delta)\overset{\text{def}}{=}\eta(0,\delta)\wedge\zeta(0,\delta)$$

$$= K\eta_{\mathbb{B}(0)} \wedge \zeta_{\mathbb{B}(0)} + \mathbf{O}(\delta)$$

where $K > 0$. By (3.38)–(3.39) it then follows that

$$\begin{aligned}\operatorname{sgn} D(\delta) &= \operatorname{sgn}[\bar{e}_1(0) \wedge \bar{e}_2(0)][\bar{e}_3(0) \wedge \bar{e}_4(0)] \\ &= \operatorname{sgn}[\bar{e}_1(x) \wedge \bar{e}_2(x)][\bar{e}_3(-x) \wedge \bar{e}_4(-x)]\end{aligned}$$

for all $x \geq 0$, since each of the two above wedge products is continuous in x and nonvanishing and continuous for all x. By translating back to the original, the unscaled system (3.8) takes the limit of the above as $x \to +\infty$ for all real $\lambda > 0$ sufficiently large. It then follows from (3.37) and the above for large$\lambda > 0$ that

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn}\left[e_1^+(\lambda) \wedge e_2^+(\lambda)\right]\left[e_3^-(\lambda) \wedge e_4^-(\lambda)\right],$$

where the $e_i^{\pm}(\lambda)$'s are as in Lemma 3.5. Since each of the two wedge products is nonvanishing and continuous for all $\lambda \geq 0$, the sign of each wedge on the right-hand side of the above is maintained for all $\lambda \geq 0$. Evaluating the right side at $\lambda = 0$, by Lemma 3.5 we finally obtain

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn}\left[r_1^+ \wedge s_2^+\right]\left[r_1^- \wedge s_2^-\right]$$

for all sufficiently large $\lambda > 0$. ∎

THEOREM 3.7 (Main Theorem) *A necessary condition for bounded linearized stability of an undercompressive viscous shock wave $u = \overline{u}(x)$, given* (H1)–(H3)*, $n = 2$, and $r_1^- \wedge r_2^+ \neq 0$ is*

$$\text{(S)} \qquad (\Gamma)(s_1^+ \wedge r_2^+)(r_1^- \wedge s_2^-)(r_1^- \wedge r_2^+) \geq 0\,,$$

where s_2^- and s_1^+ are chosen with the orientation of $\overline{u}_x$ at $\mp\infty$, respectively.[1] *More precisely, the number of unstable modes is odd if the sign in* (S) *is < 0 and even if the sign is > 0.*

PROOF: If (S) fails, then by Lemma 3.4 and Corollary 3.6, we have $D(0) = 0$ and $D(\lambda)$ of opposite sign from $D'(0)$ for large real λ. Thus, $D(\lambda)$ must have a real zero, $\lambda > 0$, contradicting stability. More precisely, there are an odd number of real, unstable zeroes and an even number of unstable zeroes occurring in complex conjugate pairs. ∎

[1] Recall that we have forced this normalization by choosing $\varphi_1 = \varphi_4 = \overline{u}_x$ and by the conventions (3.12) and (3.15).

Remarks

1. The quantity Γ has a geometrical interpretation as a *Melnikov integral* associated with the traveling-wave ODE (3.3). Precisely, fix a line L transverse to the orbit $\overline{u}$. Varying the parameters (u^-, s), define $u^+ = u^+(u^-, s)$ to be the unique rest point guaranteed by the implicit function theorem to lie near the original point $u_+ = \overline{u}(+\infty)$, and define the separation function $d(u^-, s)$ to be the signed distance between the intersections with L of the unstable manifold from u^- and the stable manifold from u^+ (Figure 3.1). Then Γ is exactly the derivative of the separation function at $(u^-, 0)$ in the special direction $(\alpha_- r_1^-, 1)$, i.e., $\Gamma = \langle \partial d/\partial(u^-, s)\big|_{(u_-,0)}, (\alpha_- r_1^-, 1)\rangle$ (cf., e.g., [20]).

   This is the unique direction in which both $u_\pm$ vary along the outgoing characteristic directions r_2^+ and r_1^-, that is, in the directions that can be reached by concatenating Lax one and two waves with the original undercompressive shock. The condition $\Gamma \neq 0$ thus corresponds to the *transversality condition* of [42], expressing local well-posedness near data (u_-, u_+) of the Riemann problem for the associated inviscid system, because the persistence of a connection in these directions would imply nonuniqueness of Riemann solutions at the linearized level. It is worth noting, in the degenerate case $r_1^+ \wedge r_2^- = 0$, that (3.29) leads by analogous matrix manipulations to the stability condition

   $$(\tilde{\Gamma})(s_1^+ \wedge r_2^+)(r_1^- \wedge s_2^-)(r_1^- \wedge (u_+ - u_-)) \geq 0\,,$$

   where $\tilde{\Gamma}$ is the derivative of the separation function in the direction $(r_1^-, 0)$. The condition $\tilde{\Gamma} \neq 0$ is again the condition for linearized well-posedness of the associated Riemann problem.

   In the case of a homoclinic profile, $u^+ = u^-$, we have $\alpha_- = \alpha_+ = 0$, and the computations in the proof of Lemma 3.4 simplify considerably. In particular, Γ becomes simply $\partial d/\partial s$, the derivative of separation with respect to change in propagation speed. Thus, as s is increased, breaking the homoclinic connection, $\mathrm{sgn}(\Gamma)$ measures whether the unstable manifold at u_- spirals inward ($\Gamma < 0$) toward the node-type rest point in the interior of the original homoclinic orbit or possibly a surrounding limit cycle or outward ($\Gamma > 0$) toward infinity or a rest point in the exterior of the original orbit. Alternatively, $\mathrm{sgn}(\Gamma)$ determines the orientation with which the unstable manifold at $x = -\infty$ of (3.3) augmented with $\dot{s} = 0$ intersects the stable manifold at $x = +\infty$. The relation between stability with respect to (3.1) and geometry of connections in (3.3) was first observed in [23].

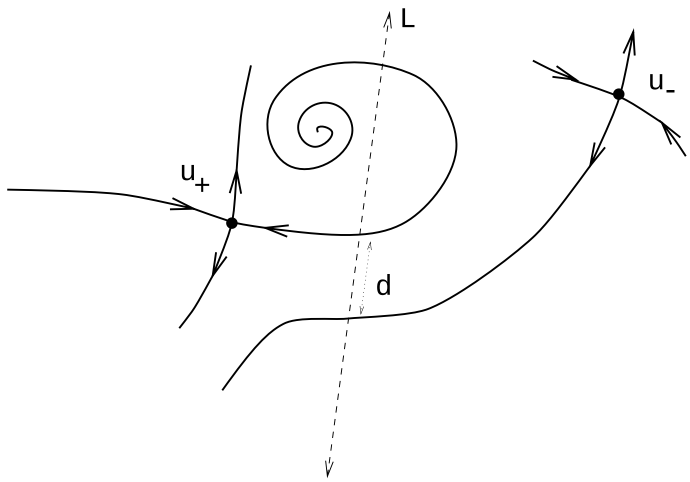

Figure 3.1. The separation function.

2. Note that the arguments of Lemma 3.5 and Corollary 3.6 require only hypothesis (H1); hence they apply in the case $n = 2$ to shocks of any type. It is only the behavior of D near $\lambda = 0$ that varies. Note also that for the argument of Corollary 3.6, the result of Corollary 3.6 is required only on the real, nonnegative axis.

3.4 Lax and Overcompressive Shocks, $n \geq 2$

The analysis of Section 3.2 can also be applied to Lax and overcompressive-type waves, in which cases (S) reduces to a simple algebraic condition. There is likewise an extension to the case $n > 2$ of more than two conservation laws; however, the resulting stability condition appears to require numerical computation for its evaluation. We briefly sketch these results below. First, however, we note the following useful fact, a slight generalization of theorem 2.4 in [34].

LEMMA 3.8 *Assuming* (H1) *through* (H3)*, the unstable/stable manifolds of A and $B^{-1}A$ have equal dimension.*

PROOF: By Fourier transform, (H3) is also equivalent to

(H3†) $\operatorname{Re}\sigma(-ikA_\pm - k^2 B_\pm) \leq 0$ for all real k; alternatively,

$$\operatorname{Re}\sigma(-ikA_\pm - k^2 B_\pm^\varepsilon) \leq -\varepsilon k^2 \quad \text{for all real } k, \tag{3.41}$$

where $B_\pm^\varepsilon := B_\pm + \varepsilon I$, $\varepsilon > 0$.

Condition (3.41) is the strict version of the Majda-Pego stable-viscosity-matrix criterion applied to $B^\varepsilon_\pm$. By continuity, it is sufficient to prove the lemma for $B^\varepsilon_\pm$, all $\varepsilon > 0$. This follows as in theorem 2.4 of [34] by the observation that (3.41) (indeed, (H3†) as well) is preserved along the homotopy $B^\varepsilon_\pm(\theta) = (1-\theta)B^\varepsilon_\pm + \theta I$, $0 \le \theta \le 1$. For this implies that $B^\varepsilon_\pm(\theta)^{-1}A_\pm$ (necessarily nonsingular, by (H2)) has no pure imaginary eigenvalues for all $0 \le \theta \le 1$; hence the number of stable/unstable eigenvalues is independent of θ. ∎

Lax Shocks

By Lemma 3.8 a viscous Lax shock can be defined as a shock for which $A_\pm$ have real eigenvalues $a_1^\pm$ and $a_2^\pm$, and, if we denote the eigenvalues of $B_\pm^{-1}A_\pm$ as usual by $\gamma_1^\pm$ and $\gamma_1^\pm$, there holds in place of (H2′) either:

1-shock: $0 < a_1^- < a_2^-$, $\operatorname{Re}\gamma_1^-, \operatorname{Re}\gamma_2^- > 0$, and $a_1^+ < 0 < a_2^+$, or

2-shock: $a_1^- < 0 < a_2^-$ and $a_1^+ < a_2^+ < 0$, $\operatorname{Re}\gamma_1^+, \operatorname{Re}\gamma_2^+ < 0$.

Without loss of generality, consider the case of a 2-shock. The behavior of φ_3 and φ_4 is similar to the undercompressive case. However, now φ_1 and φ_2 both decay at $+\infty$ at $\lambda = 0$. Redefining φ_1 and φ_2 if necessary as linear combinations of the original versions, we can arrange as in the undercompressive case that $\varphi_1 = \varphi_4 = \bar{u}_x$ at $\lambda = 0$ and carry out the calculations of Lemma 3.4 exactly as before.

We first note that the eigenvalues of $B_+^{-1}A_+$ can now be complex. In this case, it is easily seen that there is a basis of eigenvectors s_1^+, s_2^+ such that the wedge $s_1^+ \wedge s_2^+$ is real. Thus, if $E_i^\pm(\lambda) = (e_i^\pm(\lambda), f_i^\pm(\lambda))$ are the basis elements of Lemma 3.5, we now obtain the formula

$$\operatorname{sgn} e_1^+(0) \wedge e_2^+(0) = \operatorname{sgn} s_1^+ \wedge s_2^+ ;$$

by our choice of basis, the latter quantity is well-defined.

As before, we find that $D(0) = 0$. However, (3.29) now becomes

$$\begin{aligned} D'(0) &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} \begin{pmatrix} w_2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} w_3 \\ a_1^- r_1^- \end{pmatrix} \wedge \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \\ &\qquad \wedge \begin{pmatrix} z_4 - z_1 \\ -(u_+ - u_-) \end{pmatrix} \\ &= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} (w_1 \wedge w_2)(-a_1^- r_1^- \wedge (u_+ - u_-)). \end{aligned}$$

Since $(w_1 \wedge w_2)$ is a Wronskian for the ODE $w' = B^{-1}Aw$, we have that its sign is that of $e_1^+(0) \wedge e_2^+(0)$. Recalling that $a_1^- < 0$, then

$$\operatorname{sgn} D'(0) = \operatorname{sgn}(e_1^+(0) \wedge e_2^+(0))(r_1^- \wedge (u_+ - u_-)) .$$

Likewise, Corollary 3.6 goes through essentially unchanged (see the remark above) to give

$$\operatorname{sgn} D(\lambda) = \operatorname{sgn}(e_1^+(0) \wedge e_2^+(0))(r_1^- \wedge s_2^-) \neq 0$$

as $\lambda \to +\infty$ along the real axis. Combining, we obtain the stability condition

$$(r_1^- \wedge (u_+ - u_-))(r_1^- \wedge s_2^-) \geq 0 ,$$

where s_2^- is chosen with the orientation of $\overline{u}_x$ at $-\infty$. More precisely, the number of unstable modes is odd if the sign of the left hand side is < 0 and even if the sign is > 0. Note that this is an *algebraic* condition that is much simpler than in the undercompressive case.

For a weak ($|u_+ - u_-|$ small) Lax shock, s_2^-, which points in the direction of $\bar{u}_x$ at $-\infty$, lies approximately parallel to $(u_+ - u_-)$. Thus, *the stability condition is always satisfied for sufficiently weak Lax shocks*. Further, as argued in the undercompressive case, $(r_1^- \wedge s_2^-) \neq 0$. Thus, if we start with a weak shock and move u_+ along the Hugoniot curve of u_-, i.e., the curve of rest points of (3.3) traced out as s is varied, we find that the stability condition will remain satisfied so long as

$$(r_1^- \wedge (u_+ - u_-)) \neq 0 .$$

This transversality condition is again the condition for local well-posedness of the (hyperbolic) Riemann problem (u_-, u_+) and is also the condition that the L^1 asymptotic state of a perturbed viscous shock be determinable by conservation of mass alone (see related discussion in [33, 42], respectively).

Overcompressive Shocks

Similarly, a viscous overcompressive can be defined as a shock for which $A_\pm$ have real eigenvalues $a_1^\pm$ and $a_2^\pm$, and again, if we denote the eigenvalues of $B_\pm^{-1}A_\pm$ by $\gamma_1^\pm$ and $\gamma_1^\pm$, there holds in place of (H2′):

(H2″) $0 < a_1^- < a_2^-$, with $\operatorname{Re}\gamma_1^-, \operatorname{Re}\gamma_2^- > 0$ and $a_1^+ < 0 < a_2^+ < 0$, with $\operatorname{Re}\gamma_1^-, \operatorname{Re}\gamma_2^- < 0$.

In this case, φ_1 and φ_2 decay at $+\infty$ and φ_3 and φ_4 decay at $-\infty$; hence there are two zero eigenfunctions $\bar{u}_x$ and $\bar{u}_\alpha$, where $\bar{u}^\alpha(x)$ denotes the one-parameter family of possible orbits from u_- to u_+ [13, 33]. Again, by change of coordinates we can arrange that $\varphi_1 = \varphi_4 = \bar{u}_x$ and $\varphi_2 = \varphi_3 = \bar{u}_\alpha$ at $\lambda = 0$ and then carry out the calculations as before to find that $D'(0) = D(0) = 0$, while

$$D''(0) = e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1} \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} w_2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} z_3 - z_2 \\ -\int_{-\infty}^{+\infty} \bar{u}_\alpha dx \end{pmatrix} \wedge \begin{pmatrix} z_4 - z_1 \\ -(u_+ - u_-) \end{pmatrix}$$
$$= e^{-\int_0^x \operatorname{tr}(B^{-1}A)dx} \det B^{-1}(w_1 \wedge w_2) \left(\int_{-\infty}^{+\infty} \bar{u}_\alpha dx \wedge (u_+ - u_-) \right),$$

and the stability condition becomes

$$\left(\int_{-\infty}^{+\infty} \bar{u}_\alpha dx \wedge (u_+ - u_-) \right) (s_1^- \wedge s_2^-) \geq 0\,,$$

where s_1^- and s_2^- are chosen with the directions of $\bar{u}_x$ and $\bar{u}_\alpha$, respectively, as $x \to -\infty$.

This is no longer a purely algebraic condition but, like the undercompressive condition, involves the variational equations about the traveling wave. Note that the sign of $(\bar{u}_\alpha \wedge \bar{u}_x)$ cannot change along the profile $\bar{u}$ or else $\bar{u}$ would intersect nearby profiles $\bar{u}^\alpha$, a contradiction of uniqueness of solutions to the traveling-wave ODE. Moreover, it can be arranged to be nonvanishing near either $x = -\infty$ or $x = +\infty$ by suitable definition of the family $\bar{u}^\alpha$. For shocks lying approximately along a line, $\bar{u}_x$ lies approximately in the direction $(u_+ - u_-)$; hence the stability condition is satisfied. Indeed, nonlinear stability of such profiles has been established in certain cases [13]. For sufficiently weak shock strength, there generically exist connecting profiles lying approximately on a line [4], but not all orbits in the one-parameter family of profiles have this property. The stability of general overcompressive profiles would be an interesting topic for further investigation. We mention that the condition $\int_{-\infty}^{+\infty} \bar{u}_\alpha \, dx \wedge (u_+ - u_-) \neq 0$ again is the condition that the L^1 asymptotic state of a perturbed shock be determined by conservation of mass. However, in the overcompressive case there is no clear connection to well-posedness of the Riemann problem, just as in the undercompressive case the connection to asymptotic distribution of mass is lost.

$n \times n$ Systems

Similar analyses can be carried out in the $n \times n$ case, provided we assume the n-dimensional version of Lemma 3.5, at least for λ real. We conjecture that this holds whenever A and B are simultaneously symmetrizable (cf. [38] for the special case $\lambda = 0$) but have not been able to prove it so far. This would be a very interesting issue to resolve.

The resulting stability conditions again relate in a natural way to properties of the associated inviscid system. As the calculations are identical to what has come before, we will indicate here only the nature of the results.

For example, a Lax 3-shock solution of a 3×3 system satisfying

$$a_1^- < a_2^- < 0 < a_3^- \quad \text{and} \quad a_1^+ < a_2^+ < a_3^+ < 0\,,$$

and similarly for $\operatorname{Re} \gamma_j^\pm$, gives the stability condition

$$(r_1^- \wedge r_2^- \wedge (u_+ - u_-))(r_1^- \wedge r_2^- \wedge s_3^-) \geq 0\,.$$

Similarly, a Lax 2-shock satisfying

$$a_1^- < 0 < a_2^- < a_3^- \quad \text{and} \quad a_1^+ < a_2^+ < 0 < a_3^+\,,$$

and similarly for $\operatorname{Re} \gamma_j^\pm$, gives the condition

$$\begin{aligned}\operatorname{sgn}(r_1^- \wedge r_3^+ \wedge (u_+ - u_-))(\bar{u}_x(-\infty) \wedge w_2^+(-\infty) \wedge s_2^-)&\\ ((\bar{u}_x(+\infty) + \wedge s_2^+ \wedge r_3^+)(r_1^- \wedge s_2^- \wedge (\bar{u}_x(-\infty)) \geq 0\,,&\end{aligned}$$

where, as in the case $n = 2$, we have normalized $\phi_1 = \phi_6 = \bar{u}_x$ at $\lambda = 0$, and $s_2^\pm$ are independent of $\bar{u}_x(\pm\infty)$ (in an abuse of notation, denoting limiting direction).

In both cases, vanishing of the first factor corresponds, as in the 2×2 case, with loss of well-posedness in the associated Riemann problem, as well as indeterminacy of the L^1 asymptotic state. Likewise, in the weak shock limit, the two terms coincide, so that the stability condition is always satisfied for weak shocks. The new feature is that, in the 2-shock case, the second term can now also vanish, which in the 2-shock case detects the occurrence of a second L^2 zero eigenfunction and in the 3-shock case detects an L^∞ zero eigenfunction like that discussed in the second remark at the end of Section 3.3. Note also that in the 2-shock case, the sign of the second term for strong shocks is not easy to evaluate, requiring a knowledge of the geometry of solutions of the traveling-wave ODE.

Finally, we mention the interesting case of a homoclinic shock solution of a 3×3 system, for example: $a_1^- < 0 < a_2^- < a_3^-$ and $a_1^+ < 0 < a_2^+ < a_3^+$, and similarly for $\operatorname{Re} \gamma_j^{\pm}$. This leads to the stability condition

$$(\Gamma)(s_1^+ \wedge r_2^+ \wedge r_3^+)(r_1^- \wedge s_2^- \wedge s_3^-)(r_1^- \wedge r_2^+ \wedge r_3^+) \geq 0\,,$$

where (Γ) is the derivative with respect to speed s of the separation function, defined, similarly as in the remark following Section 3.2, to be the distance between the two-dimensional unstable manifold at u_- and the one-dimensional stable manifold at u_+, as measured in a plane orthogonal to $\bar{u}$ passing through $\bar{u}(0)$. Similar formulae arise for the other two possible configurations of the characteristic speeds $a_j^{\pm}$.

Remarks

1. When the stability conditions are satisfied *strictly*, our analysis is suggestive of linearized stability; however, we lack a second estimate as in [23, 35], establishing that there is at most one eigenvalue in the unstable half-plane $\mathbb{C}^+ = \{\lambda : \operatorname{Re}(\lambda) > 0\}$. This would be a very interesting issue to resolve. When the stability condition evaluates to zero (i.e., the appropriate derivative of the Evans function vanishes at $\lambda = 0$), then the wave is necessarily linearly *unstable* with respect to L^p, $p < \infty$. For either there is a generalized eigenvector, giving linear growth, or else there is an extra genuine eigenvector, which is only in the space L^∞ (recall that φ_j may be bounded only at $-\infty$, $j = 1, 2$, and $+\infty$, $j = 3, 4$, for $\lambda = 0$). Thus, an L^2 perturbation can excite an L^∞ asymptotic mode, changing the endstates of the shock. This latter situation occurs, for example, in the case of a curved shock of system (4.1) below, with $s = 0$ and $B_{21} = 0$. Numerical experiments [5] verify that the shock is neutrally stable in L^∞ and unstable in L^2. Indeed, any perturbation with nonzero mass in the v-direction results in wave splitting similar to that described in [30] for overcompressive waves. For a rigorous, functional analytic treatment of this topic, we refer the reader to [41].

2. We conjecture that for *weak shocks*, the only zeroes of D in a neighborhood of $\operatorname{Re} \lambda \geq 0$ are those bifurcating from the zeroes at the origin of the Evans function for the limiting constant-state solution, and that their number should be proportional to its multiplicity, namely, one in the Lax case and two in the under- and overcompressive cases. Together with the results of [41], this would imply that (S) is both necessary and sufficient for stability. However, as touched upon in Section 2, the fact that the

Evans function for the constant solution has a branch point at the origin makes this a rather subtle issue. For strong shocks, it is perhaps not to be expected that there be an upper bound on the number of unstable modes. Such a result would appear to require further structural information.

4 Explicit Calculations for Model Systems

The stability conditions obtained in the previous section are in each case readily evaluated numerically and hence provide a useful tool for determining the stability of viscous shock waves. Nevertheless, it is preferable to have an analytical method for their evaluation. In this final section, we explicitly evaluate the stability condition in the most interesting, undercompressive case for several interesting classes of equations.

4.1 Example Systems

In [3, 4], it was pointed out that behavior of weak undercompressive shocks is generically much more complicated than that of weak Lax shocks. In particular, the existence was demonstrated of homoclinic and double heteroclinic cycles of undercompressive shocks, which seem to be associated with "threshold" behavior of the type described in the introduction. The arguments in [4], being of general nature, involved the unfolding of a rather complicated three-parameter bifurcation. However, the main features can be illustrated quite simply in a reduced setting.

Consider the class of 2×2 systems with f quadratic and B constant for which additionally the traveling-wave ODE (3.2) is *Hamiltonian*, i.e.,

$$\operatorname{tr}(B^{-1}(f'(u) - sI)) \equiv 0$$

for some choice of s. This is a codimension-2 subclass of the quadratic flux models, as parametrized by the coefficients of B and f. Generically, such systems can be reduced by an affine change of variables to the canonical form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + B \begin{pmatrix} \frac{\varepsilon}{2}v^2 - \frac{1}{2}u^2 + v \\ uv \end{pmatrix}_x = B \begin{pmatrix} u \\ v \end{pmatrix}_{xx} \tag{4.1}$$

where $\varepsilon = \pm 1$; this follows by an argument similar to that of [37]. For $\varepsilon = 1$, $B = I$, this reduces to Holden's model, a variant of the complex Burgers equation.

We will always work with this normal form. For the choice $s = 0$, (3.2) then becomes a Hamiltonian system

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \frac{\varepsilon}{2}v^2 - \frac{1}{2}u^2 + v \\ uv \end{pmatrix} - \begin{pmatrix} \frac{\varepsilon}{2}v_-^2 - \frac{1}{2}u_-^2 + v_- \\ u_- v_- \end{pmatrix}, \tag{4.2}$$

with $H(u,v) = \frac{1}{2}(\frac{\varepsilon}{3}v^3 - u^2 v + v^2) - (\frac{\varepsilon}{2}v_-^2 v - \frac{1}{2}u_-^2 v + v_- v - u_- v_- u)$ preserved along orbits.

For such systems, we can easily recover the observations of [4]. For $v_- = 0$, we find that the orbits through (u_-, v_-) lie on the level set $H(u,v) = \frac{v}{2}(\frac{\varepsilon}{3}v^2 - u^2 + v + u_-^2) = 0$, hence along the line $v = 0$ and the conic section $\frac{\varepsilon}{3}v^2 + v = u^2 - u_-^2$. For $|u_-|$ sufficiently small, these form a 2-cycle, or heteroclinic loop, connecting rest states (u_-, v_-) and (u_+, v_+) of (4.2) (Figure 4.1). This loop contains a nonlinear center (u_c, v_c), which is a local minimum of H. By dimensionality, we can expect that curved and straight-line connections each persist separately under small changes in (u_-, v_-) for an appropriate choice of s; in fact, this is the case.

For $s = 0$ and $v_- \neq 0$, on the other hand, it is easily checked that $H(u_-, v_-) \neq H(u_+, v_+)$, so that the connections from (u_-, v_-) to (u_+, v_+) and vice versa are broken. However, the local minimum (u_c, v_c) persists, together with its surrounding periodic orbits/bounded level sets. These must be bounded by a homoclinic loop (Figure 4.2) from either (u_-, v_-) or (u_+, v_+), according to the sign of v_-, since a heteroclinic loop is impossible unless $v_- = 0$.

To summarize:

For (u_-, v_-, s) near $(0,0,0)$, the local phase portrait of (4.2) consists of three rest points, two saddles $u_\pm$ and a spiral node or center u_c. Moreover, for (u_-, v_-) on an open wedge $0 < v < k(u_-)u_-$, there are three (necessarily different) choices of s leading to $u_- \to u_+$, $u_+ \to u_-$, and $u_- \to u_-$ connections. That is, there are *three different types* of undercompressive shock waves involving each hyperbolic rest point u_-: two heteroclinic profiles and one homoclinic, or solitary wave, profile. All these features hold generically for weak undercompressive shocks [4]. This is quite different from the simple local phase portrait of a weak Lax shock, which involves only two rest points and admits connections in only one direction [18, 34].

Remark. Note that weak undercompressive shocks bifurcate from the special parameters $(u_-, v_-, s) = (0,0,0)$ for which the three rest points coincide. That is, they always lie near the isolated state $(u_*, v_*) = (0,0)$ by contrast with weak gas dynamic shocks, which can be found near any state. Further, the

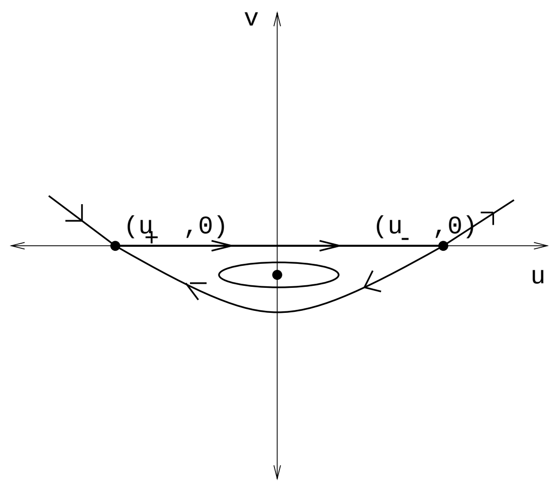

Figure 4.1. Double cycle.

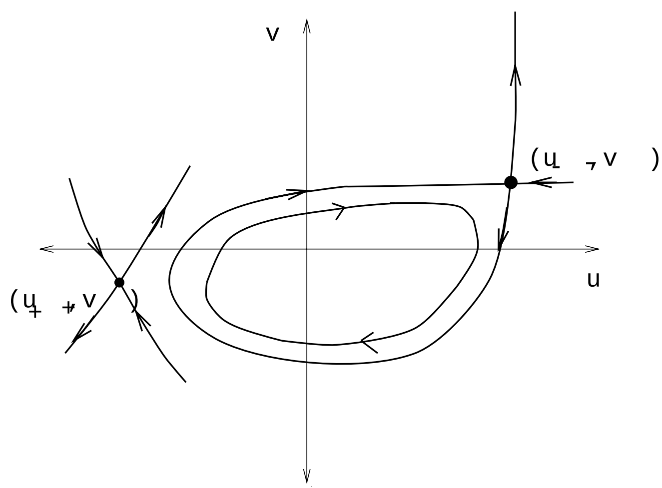

Figure 4.2. Homoclinic wave.

bifurcation point $(0,0)$ is generically a point of strict hyperbolicity, since

$$f'(0) = B\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_{11} \\ 0 & B_{21} \end{pmatrix}.$$

Thus, for weak shocks, the orbits described above *lie entirely in the hyperbolic region* of the corresponding inviscid equation, $u_t + f(u)_x = 0$. What makes possible the nonclassical shocks is not failure of hyperbolicity but rather neutral satisfaction of the stable viscosity matrix condition (H3) at the point $(0,0)$ [4]. We emphasize that the endstates $u_\pm$ *do* satisfy (H3) however.

4.2 Calculations

We now explicitly evaluate the stability condition (S) for examples of each of the three types of undercompressive wave illustrated in the previous section: the curved and straight heteroclinic waves and the homoclinic (solitary) wave. In particular, we show:

1. Generic, straight-line, heteroclinic profiles of any system satisfy the stability condition.

2. Homoclinic profiles of example systems (4.1) are *linearly unstable*. Likewise, *convex* homoclinic profiles are linearly unstable for arbitrary systems such that B is a multiple of the identity.

3. Curved heteroclinic profiles of example systems (4.1) satisfy the stability condition for a sufficiently small propagation speed s in the case $\varepsilon = 1$. In the case $\varepsilon = -1$, there are two types of curved profile, one stable and one unstable; however, the unstable type are "strong" shocks in the sense that they feature large excursions.

These analytic results confirm numerical observations of [3] for the homoclinic wave and are consistent with those for the heteroclinic waves. The homoclinic and large-excursion heteroclinic waves are particularly significant as the first rigorous examples of an unstable viscous shock wave.

Homoclinic Waves

LEMMA 4.1 *Let* (3.3) *be Hamiltonian for* $s = 0$. *If* $\overline{u}(x)$ *is a* convex *homoclinic orbit, then* $\operatorname{sgn}(\Gamma) = \operatorname{sgn}(s_2 \wedge s_1)$.

PROOF: By (3.5), (3.3) is Hamiltonian if and only if $\mathrm{tr}(B^{-1}A) \equiv 0$. Thus, (S) reduces to

$$\Gamma = \int_{-\infty}^{+\infty} \overline{u}_x \wedge B^{-1}(\overline{u} - u_*)dx = \int_{u\in C} du \wedge B^{-1}(u - u_*)\,,$$

where C, the image of $\overline{u}(x)$, is traversed in the direction of $\overline{u}_x$. This is equal to the boundary integral

$$\pm \int_C [B^{-1}(u - u_*)] \cdot n_C \, dC\,,$$

where the sign is $\pm$ according as $\overline{u}_x \wedge n_C \lesseqgtr 0$, n_C denoting outward normal to C. The sign of $\overline{u}_x \wedge n_C$ can be conveniently evaluated at the point u^-, where s_2 points in the direction of $\overline{u}_x$, while s_1, by convexity, points out of C (Figure 4.3), giving $\mathrm{sgn}(\bar{u}_x \wedge n_C) = \mathrm{sgn}(s_2 \wedge s_1)$. Thus,

$$\mathrm{sgn}(\Gamma) = \mathrm{sgn}(s_2 \wedge s_1) \int_{u\in C} B^{-1}(u - u_*) \cdot n_C \, dc$$

and, by Gauss-Green,

$$\begin{aligned}\mathrm{sgn}(\Gamma) &= \mathrm{sgn}(s_2 \wedge s_1) \int_{u\in C^{\text{interior}}} \mathrm{div}(B^{-1}(u - u_*))du \\ &= \mathrm{sgn}(s_2 \wedge s_1)|C^{\text{interior}}| \ \mathrm{tr}\left(B^{-1}\right) \\ &= \mathrm{sgn}(s_2 \wedge s_1)\end{aligned}$$

as claimed. ∎

LEMMA 4.2 *If* $B = \text{const}$ *and* $f(u)$ *is quadratic, then homoclinic orbits of* (3.3) *are convex.*

PROOF: This is a corollary of Lemma B.1 in Appendix B. ∎

PROPOSITION 4.3 *For the class of example systems* (4.1) *with* $B > 0$, $\det B > 0$, *zero-speed homoclinic shocks are linearly unstable.*

PROOF: From Lemmas 4.1 and 4.2, we find that $\overline{u}(x)$ is unstable if

$$(s_1 \wedge s_2)(s_1 \wedge r_2)(r_1 \wedge s_2)(r_1 \wedge r_2) > 0\,. \tag{4.3}$$

For $B = I$, $r_j = s_j$, and we see that (4.3) becomes $(r_1 \wedge r_2)^4 > 0$, so that in this case homoclinics are unstable. Now, vary B among the class of positive,

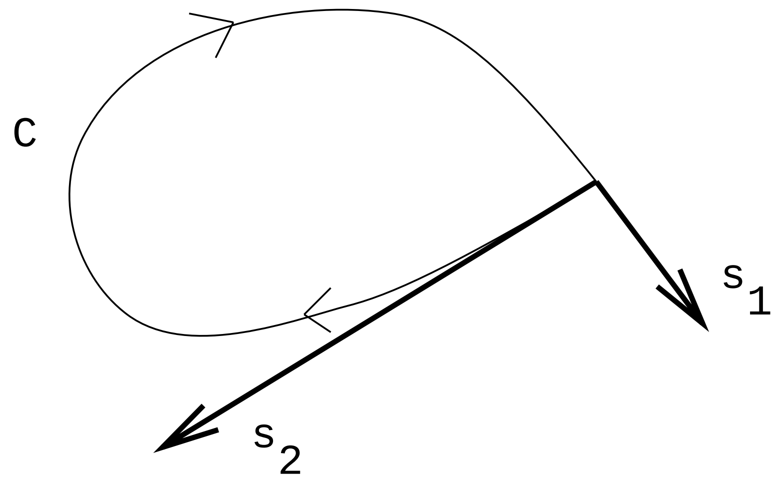

Figure 4.3.

nonsingular matrices. $(s_1 \wedge s_2)$ and $(r_1 \wedge r_2)$ cannot vanish, since A and $B^{-1}A$ are hyperbolic. But neither can $s_1 \wedge r_2$ or $r_1 \wedge s_2$ vanish (Corollary 3.6). Thus, the sign of each factor remains constant under perturbation of B, and the product is always positive. ∎

Remark. From the relation $\Gamma = \partial d/\partial s > 0$ noted in the remark ending Section 3.2, we find that *all* homoclinics (in some bounded neighborhood of the origin) with sufficiently small speed in fact have zero speed.

PROPOSITION 4.4 *If $\overline{u}(x)$ is convex and $B(u)$ is (or is sufficiently near) a multiple of the identity matrix, then $\overline{u}(x)$ is unstable. In particular, homoclinic shock waves are linearly unstable for all quadratic flux models with $B = I$.*

PROOF: For homoclinic waves, $u_* = u_-$. Thus, if C, the image of $\overline{u}(x)$, is *convex*, then $(u - u_*)$ points out of C^{interior} for all $u \in C$, and $\bar{u}_x \wedge (u - u_*)$ has the sign of $\bar{u}_x \wedge n_C$, which was shown in the proof of Lemma 4.1 to be $\text{sgn}(s_2 \wedge s_1)$. Thus, $\text{sgn}(\Gamma) = \text{sgn}(s_2 \wedge s_1)$. Recalling that $s_j = r_j$ for $B = I$, we therefore have $\text{sgn}\,\Gamma(s_1 \wedge r_2)(r_1 \wedge s_2)(r_1 \wedge r_2) = -\,\text{sgn}(r_1 \wedge r_2)^4 < 0$, giving instability. ∎

Remark. In the Hamiltonian case, the system admits nested families of periodic viscous profiles inside the homoclinic profiles. It follows from a general result in [14] that the periodic profiles of sufficiently large wavelength are unstable when the bounding homoclinic profile has an isolated eigenvalue of finite multiplicity λ_0 in the right half-plane. More precisely, it is shown in [14] that a periodic profile of sufficiently large wavelength has a loop of spectrum in a neighborhood of λ_0. Presumably all the periodic waves are unstable; however, this question remains open. It would require a different calculation than the one given here for homoclinic waves.

Straight-Line-Profile Heteroclinic Waves

This case is particularly straightforward.

PROPOSITION 4.5 *For $B = \text{const}$ (any model), all straight-line profiles satisfy the stability condition* (S).

PROOF: It is sufficient to consider the case that $r_1^- \wedge r_2^+ \neq 0$. If $\overline{u}(x)$ lies on a straight line between u^- and u^+, then $\overline{u}_x$ is always parallel to $\Delta u = (u^+ - u^-)$. Thus, we can choose $s_2^- = s_1^+ = \Delta u$. Recalling that $s_2^- \wedge r_1^-$ and $s_1^+ \wedge r_2^+$ never vanish (Corollary 3.6), we have that $\alpha_\pm \neq 0$ in (3.3). Thus, we can normalize r_1^- and r_2^+ so that (3.19) becomes simply

$$u_* = \begin{cases} u_- + r_1^- \\ u_+ + r_2^+ \end{cases}, \tag{4.4}$$

or equivalently

$$\Delta u = r_1^- - r_2^+ . \tag{4.5}$$

Taking wedge products with r_1^- and r_2^+, we thus have

$$r_2^+ \wedge \Delta u = r_2^+ \wedge r_1^- = r_1^- \wedge \Delta u . \tag{4.6}$$

Further, note that $\Delta u \wedge B^{-1}(u - u_*)$ in Γ is for each $u \in [u_-, u_+]$ a convex combination of $\Delta u \wedge B^{-1}(u_+ - u_*) = -s_1^+ \wedge B^{-1} r_2^+$ and $\Delta u \wedge B^{-1}(u_- - u_*) = -s_2^- \wedge B^{-1} r_1^-$. Using $B^{-1} A s_1^+ = \gamma_1 s_1^+$, $A r_2^+ = a_2^+$, with $\gamma_1^+ < 0$, $a_2^+ > 0$, we have

$$\begin{aligned} \operatorname{sgn}\big(- s_1^+ \wedge B^{-1} r_2^+ \big) &= \operatorname{sgn}\big(B^{-1} A s_1^+ \wedge B^{-1} A r_2^+\big) \\ &= \operatorname{sgn}\big[\det\big(B^{-1} A\big)\big(s_1^+ \wedge r_2^+\big)\big] \\ &= -\operatorname{sgn}\big(s_1^+ \wedge r_2^+\big) \\ &= -\operatorname{sgn}\big(\Delta u \wedge r_2^+\big) , \end{aligned}$$

and a symmetric argument shows that $\operatorname{sgn}(-s_2^- \wedge B^{-1} r_1^-) = -\operatorname{sgn}(\Delta u \wedge r_1^-)$. By (4.6), we conclude that

$$\operatorname{sgn}(\Delta u \wedge B^{-1}(u - u_*)) = -\operatorname{sgn}(\Delta u \wedge r_2^+)$$

for $u \in [u_-, u_+]$, and consequently

$$\Gamma = \int_0^1 e^{-\int_0^{x(t)} \operatorname{tr}(B^{-1}A)} \Delta u \wedge B^{-1}(u_- + t\Delta u - u_*)\, dt$$

has the sign of $-\Delta u \wedge r_2^+$. Combining, we find that (S) reduces to

$$-(\Delta u \wedge r_2^+)(\Delta u \wedge r_2^+)(r_1^- \wedge \Delta u)(r_1^- \wedge r_2^+) \geq 0\,,$$

which by (4.6) is equivalent to $(\Delta u \wedge r_2^+)^2 (r_1^- \wedge r_2^+)^2 \geq 0$ and is always satisfied. ∎

Curved Heteroclinic Waves

The case of curved heteroclinic profiles is subtler than the previous two. However, for the example systems (4.1), it can be treated by a combination of the arguments used in the homoclinic and straight-line cases, at least for zero-speed (Hamiltonian) waves. As described above, the traveling-wave ODE (4.2), for system (4.1) with $s = 0$ has a curved heteroclinic connection if and only if $(u^-, v^-) = (\alpha, 0), \alpha > 0$, in which case there is also a straight-line connection $(u^+, v^+) \to (u^-, v^-)$ (Figure 4.1). Further, we have by direct computation that

$$A_- = B \begin{pmatrix} -\alpha & 1 \\ 0 & \alpha \end{pmatrix} ; A_+ = B \begin{pmatrix} \alpha & 1 \\ 0 & -\alpha \end{pmatrix} . \tag{4.7}$$

LEMMA 4.6 *Let* $\det B > 0$ *and* $A_\pm$ *as in* (4.7)*. Then* $r_2^+ \wedge r_1^- = 0$ *if and only if* $B_{21} = 0$*, in which case* r_2^+ *and* r_1^- *are multiples of* $(1, 0)^\top$*. Otherwise, normalizing so that* (4.4) *holds, we have* $r_1^- \wedge r_2^+ > 0$*.*

PROOF: Note first that r_1^- and r_2^+ are well-defined so long as $A_\pm$ have simple eigenvalues, in particular for $\det B > 0$, since then $\det(A_\pm) < 0$ and both $A_\pm$ are hyperbolic. Further, $r_1^- \wedge r_2^+ = 0$ if and only if, without loss of generality, $r_2^+ = r_1^- = r$, or $(A_+/a_2^+ - A_-/a_1^-)r = 0$. Equivalently, $0 = \det(a_1^- A_+ - a_2^+ A_-) = \alpha^2 \det(B)(a_1^- + a_2^+)^2$ by (4.7), so that $a_1^- = -a_2^+$. Noting that $\det A_- = -\alpha^2 \det B = \det A_+$, we find so long as $\det B \neq 0$, that $a_2^- = -a_1^+$ as well, and thus $\operatorname{tr}(A_-) = -\operatorname{tr}(A_+)$, or

$$0 = \operatorname{tr}(A_- + A_+) = \operatorname{tr} B \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2B_{21}\,.$$

This proves the first assertion.

It follows that as B is varied continuously, with $B_{21} \neq 0, \det B \neq 0$, there is a continuous choice of r_1^- and r_2^+ satisfying (4.4), and therefore $\operatorname{sgn}(r_1^- \wedge r_2^+)$

is independent of the choice of B so long as $B_{21} \neq 0$. Thus, to prove the second assertion it is sufficient to verify that $r_1^- \wedge r_2^+ > 0$ for two choices of B, with $\det B > 0$ and $B_{21} >$, $B_{21} < 0$.

Choosing $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, we have

$$
\begin{aligned}
A_- &= \begin{pmatrix} 0 & 0 \\ -\alpha & 1 \end{pmatrix}, \qquad & A_+ &= \begin{pmatrix} 0 & 0 \\ -\alpha & -1 \end{pmatrix}, \\
r_1^- &= \begin{pmatrix} -\frac{\alpha}{2} \\ -\frac{\alpha^2}{2} \end{pmatrix}, \qquad & r_2^+ &= \begin{pmatrix} \frac{\alpha}{2} \\ -\frac{\alpha^2}{2} \end{pmatrix},
\end{aligned}
\tag{4.8}
$$

so that

$$
r_1^- \wedge r_2^+ = \det \begin{pmatrix} -\frac{\alpha}{2} & \frac{\alpha}{2} \\ -\frac{\alpha^2}{2} & -\frac{\alpha^2}{2} \end{pmatrix} > 0\,.
$$

Though $\det B = 0$, $A_\pm$ have full sets of eigenvectors and $r_1^- \wedge r_2^+ \neq 0$, so that r_1^- and r_2^+ are still continuously defined for small perturbations of B. In particular, for

$$
B = \begin{pmatrix} 0 & \theta \\ -1 & 0 \end{pmatrix}, \quad \theta > 0 \text{ small},
$$

it is easily checked that $r_1^- \wedge r_2^+ > 0$.

Similarly, choosing $B = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$, we obtain

$$
\begin{aligned}
A_- &= \begin{pmatrix} -\alpha & 1 \\ -\alpha^2 & 2\alpha \end{pmatrix}, \qquad & A_+ &= \begin{pmatrix} \alpha & 1 \\ \alpha^2 & 0 \end{pmatrix}, \\
r_1^- &= c \begin{pmatrix} 1 \\ \frac{\alpha}{2}(3-\sqrt{5}) \end{pmatrix}, \qquad & r_2^+ &= d \begin{pmatrix} 1 \\ \frac{\alpha}{2}(\sqrt{5}-1) \end{pmatrix},
\end{aligned}
\tag{4.9}
$$

where $c, d < 0$, since $\frac{\alpha}{2}(\sqrt{5}-1) > \frac{\alpha}{2}(3-\sqrt{5}) > 0$. Thus,

$$
\begin{aligned}
r_1^- \wedge r_2^+ &= cd \det \begin{pmatrix} 1 & 1 \\ \frac{\alpha}{2}(3-\sqrt{5}) & \frac{\alpha}{2}(\sqrt{5}-1) \end{pmatrix} \\
&= \left(\frac{cd\alpha}{2}\right) \left[\left(\sqrt{5}-1\right) - \left(3-\sqrt{5}\right)\right] > 0\,,
\end{aligned}
$$

as claimed. This completes the proof. ∎

PROPOSITION 4.7 *For models of type* (4.1) *with* $\det B > 0$, $\varepsilon = 1$, *all zero-speed, curved, undercompressive waves satisfy the stability condition* (S).

PROOF: Since there must exist a straight-line profile in the opposite sense $u^+ \to u^-$ (Figure 4.4), we are free to choose

$$s_2^+ = s_1^- = \Delta u = (u_+ - u_-)\,.$$

It is sufficient to consider the case that $r_1^- \wedge r_2^+ \neq 0$, for which we may choose r_1^- and r_2^+ so that (3.19) becomes (4.5) and $\Delta u \wedge r_2^+ = r_1^- \wedge r_2^+$ as in the proof of Proposition 4.5. Since

$$\begin{aligned}\Delta u \wedge B^{-1} r_2^+ &= s_2^+ \wedge B^{-1} r_2^+ \\ &= (1/\gamma_2^+ a_2^+)(B^{-1} A s_2 \wedge B^{-1} A r_2^+) \\ &= (\det(B^{-1}A)/\gamma_2^+ a_2^+)(\Delta u \wedge r_2^+)\end{aligned}$$

and similarly for $\Delta u \wedge B^{-1} r_1^-$, we thus have

$$\operatorname{sgn}(\Delta u \wedge B^{-1} r_2^+) = -\operatorname{sgn}(r_1^- \wedge r_2^+) = \operatorname{sgn}(\Delta u \wedge B^{-1} r_1^-). \tag{4.10}$$

Denoting the curved orbit $u^- \to u^+$ by C_1, the straight line orbit $u^+ \to u^-$ by C_2, and their concatenation by $C_1 \cup C_2$, we have that

$$\begin{aligned}\Gamma &= \int_{C_1} du \wedge B^{-1}(u - u_*) \\ &= \int_{C_1 \cup C_2} du \wedge B^{-1}(u - u_*) - \int_{C_2} du \wedge B^{-1}(u - u_*)\,.\end{aligned}$$

As in the proof of Lemma 4.1, the first term has sign of $s_2^- \wedge -s_1^-$ (note that we have here chosen orientation of s_1^- opposite from that of the entering profile, giving the change in sign), which by direct computation is positive. As in the proof of Proposition 4.5, (4.10) implies that the second term has sign of $(r_1^- \wedge r_2^+)$, which by Lemma 4.6 is also positive. Hence, $\Gamma > 0$, and (S) reduces to

$$(r_1^- \wedge r_2^+)(s_1^+ \wedge r_2^+)(r_1^- \wedge s_2^-) \geq 0\,.$$

The first factor, again, is positive. As observed before in the proof of Proposition 4.3, $\operatorname{sgn}(s_1^+ \wedge r_2^+)(r_1^- \wedge s_2^-)$ does not change under perturbations of B, and for the choices (4.8)–(4.9) can be checked to have positive sign. Thus, (S) is satisfied, verifying the claim. ∎

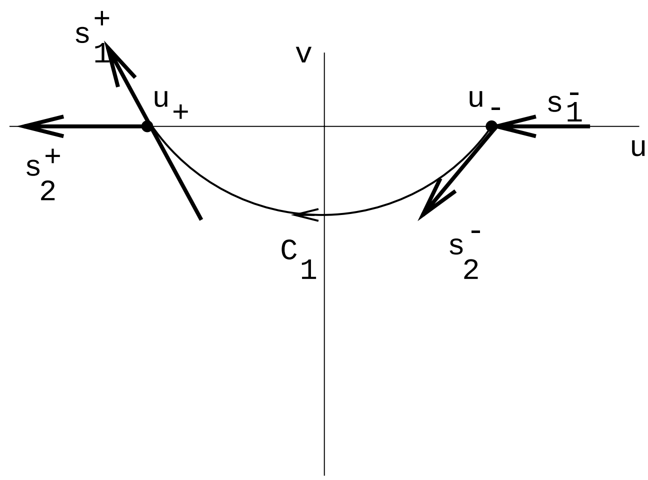

Figure 4.4.

The Case $\varepsilon = -1$

In the case $\varepsilon = -1$, $s = 0$, $v_- = 0$, not treated in Proposition 4.7, the conic section describing the curved shock becomes an ellipse and there appear *two*, lower and upper, curved connections in the clockwise and counterclockwise sense, respectively (Figure 4.5). In the limit as $|u_-| \to 0$, the profile of the lower shock shrinks to a neighborhood of the origin as in the $\varepsilon = 1$ case. However, the upper shock approaches a fixed ellipse tangent to the origin. In this sense, it is an extraneous solution in the weak shock strength limit. Virtually the same calculation as in the proof of Proposition 4.7 shows that the lower shock satisfies (S), while the upper shock does not, the difference in $\operatorname{sgn}(\Gamma)$ being due to its reverse sense. Thus, in this case we find an example of an *unstable heteroclinic profile*, albeit a somewhat degenerate one.

We remark in passing that curved connections persist for arbitrarily large shock strength when $\varepsilon = -1$, in contrast to the case $\varepsilon = 1$; this is related to the fact that the elliptic region in the $\varepsilon = -1$ case is unbounded. Note that the upper profile is always a "strong" shock in the sense that its orbit makes an order one excursion, even when $|u_+ - u_-|$ is taken arbitrarily small. Thus, truly weak curved shocks are only of the lower, stable variety.

Remarks. Though it is not obvious how to conveniently evaluate condition (S) for curved shocks in the non-Hamiltonian case $s \neq 0$, we do obtain from Proposition 4.7 that (S) holds for small s by continuity.

It would be interesting to try to evaluate the stability condition for homo-

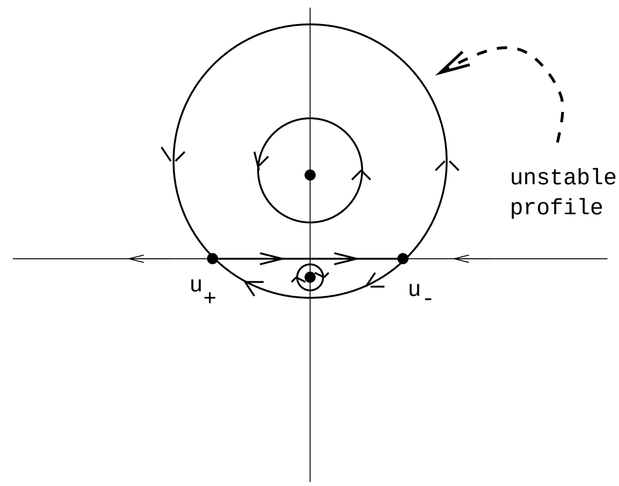

Figure 4.5. The case $\varepsilon = -1$.

clinic waves in a wider class of 2×2 systems to determine whether they are indeed unstable in general. Another interesting direction for future study would be to consider homoclinic waves in the 3×3 case. A key step in our proof of homoclinic instability is the Poincare-Bendixson argument of Lemma 4.1, which depends on planarity of the orbit. Thus, it is conceivable that a stable *nonplanar orbit* might exist, even if there are no stable planar orbits.

Appendix A: Analytic Dependence of the Unstable Manifold

PROPOSITION A.1 *Let the parametrized flow $dY/dx = F(Y,\lambda)$ have rest point $Y_0(\lambda)$ for each λ, and suppose that $C(\lambda) = \partial F/\partial Y(Y_0,\lambda)$ has a single, strongly unstable eigenvector $V(\lambda)$ with eigenvalue $\mu(\lambda)$, where F is $C^{1+\theta}$ in Y and F, Y_0, V, and μ are analytic in λ. Then there is a unique solution/unstable manifold $\phi(\cdot,\lambda)$ depending analytically on λ such that*

$$\phi(\lambda, x) = Ve^{\mu x}\big(1 + \mathbf{O}(e^{\theta\mu x})\big) .$$

PROOF: By the analytic change of coordinates $Y \to Y - Y_0$, we may assume without loss of generality that $Y_0 \equiv 0$ and

$$Y' = C(\lambda)Y + \mathbf{O}(|Y|^{1+\theta}) . \tag{A.1}$$

Setting $Y = ze^{\mu x}$, we obtain

$$z' = (C(\lambda) - \mu)z + e^{\theta\mu x}|z|^{1+\theta}R(z,x) ,$$

$R = \mathbf{O}(1)$. We must find a solution $z(x) \to V$ as $x \to -\infty$, or, by Duhamel's principle, $z = \mathcal{T}z$, where

$$\mathcal{T}z(x) = V + \int_{-\infty}^{x} e^{(C(\lambda)-\mu)(x-y)} e^{\theta\mu y} |z(y)|^{1+\theta} |R(z,y)| dy\,.$$

Because the top eigenvalue μ of C is simple, we have $|e^{(C-\mu)(x-y)}| = \mathbf{O}(1)$ for $y < x$; hence $\mathcal{T}$ is a contraction on $Z_{M,V} = \{z : z \in L^\infty(-\infty, -M)$ and $|z|_{L^\infty(-\infty,-m)} \leq 2|V|\}$ for M sufficiently large, since then

$$\begin{aligned}\int_{-\infty}^{x} \left|e^{(C(\lambda)-\mu)(x-y)}\right| \left|e^{\theta\mu y}\right| |R(y)| dy &= \mathbf{O}(1) \int_{-\infty}^{x} e^{\theta\mu y}\, dy \\ &= \mathbf{O}(1) \frac{e^{\theta\mu x}}{\theta\mu} < \frac{1}{2}\end{aligned} \tag{A.2}$$

for $x \leq -M$.

It follows that $\mathcal{T} : Z_{M,V} \to Z_{M,V}$ and there is a unique solution $|z| \leq |2V|$ of $z = \mathcal{T}z$. Since $\mathcal{T}$ clearly preserves analyticity in λ, z is analytic as well, as the uniform limit of analytic iterates. Finally, applying (A.2) a second time, we obtain

$$z = V + \mathbf{O}(|V|) e^{\theta\mu x} = V(1 + \mathbf{O}(e^{\theta\mu x}))\,,$$

proving the theorem. ∎

Remark. The argument of Lemma 3.8 also gives analyticity in λ for F merely C^1, with $\mathbf{O}(|z|^{1+\theta})$ replaced by $\mathbf{o}(|z|)$ in (A.1) and (3.1) replaced by $\phi e^{-\mu x} \to V$.

Appendix B: Convexity of Homoclinic Orbits

We define a cycle of a dynamical system to be a simple closed curve formed by a succession of orbits in the positive sense, with all rest points hyperbolic; examples are periodic and homoclinic orbits, or the 2-cycles described in the introduction. In this appendix, we prove the basic proposition given below about quadratic dynamical systems in the plane. Our analysis is based on a property used frequently in [6, 7], namely, that rest points p and q lying on a line L divide L into three intervals across which the flow is strictly transverse.

PROPOSITION B.1 *Any cycle of a planar, quadratic dynamical system is convex.*

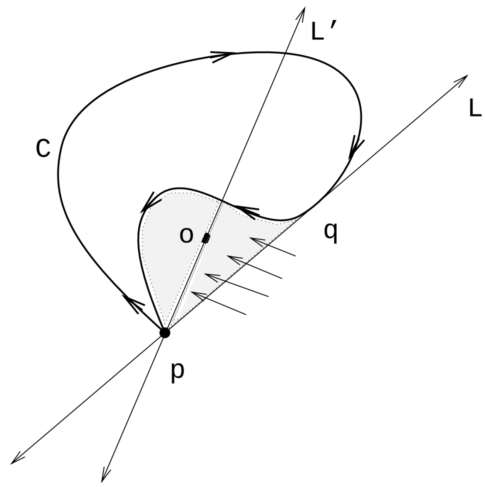

Figure B.1. Nonconvex cycle (shaded region is Re).

PROOF: Let C be a cycle that is not convex. Then there exists a support line L tangent to C at two points p and q but not on the whole interval pq. Restricted to L, the component of the flow orthogonal to L is quadratic in arc length, with zeroes at p and q. Since C does not coincide with pq, this component is not identically zero and therefore is of one sign between p and q. Without loss of generality, suppose the flow crosses pq everywhere in the C direction, and C flows from q to p (Figure 4.5). Then p must be a rest point or else all orbits entering the invariant region Re (shaded region in the figure) between pq and C sufficiently near to p would pass through p, violating uniqueness. In fact, p must be a saddle, as a rest point lying on a cycle, with C entering along the stable manifold. Of the infinitely many orbits entering Re across pq, none can terminate at p by uniqueness of the stable manifold, and at most one can terminate at q (in which case q would be a rest point with a unique stable manifold as well). We conclude that there must be a rest point o strictly interior to Re.

Now, consider the line L' passing through the two rest points p and o. Since Re is clearly outside C, L' enters and then leaves C^{interior} as it is traversed from o in the direction away from p. Put another way, C crosses L' twice on the side of o away from p, each with opposite sense. But, as before, the component of the flow orthogonal to L' is quadratic in arc length, with zeroes at p and o, so it must be of one sign on the side of o away from p. From this contradiction the claim follows. ∎

Proposition B.1 has the following interesting consequence.

COROLLARY B.2 *In any* 2*-cycle of a planar quadratic vector field, one connection lies along a straight line.*

PROOF: Label the two saddles as p and q and the line between them as L. As above, the component of the vector field orthogonal to L must be either identically zero, in which case there exists a straight-line connection as claimed, or else of constant sign on each of the three intervals of L subdivided by p and q. By way of contradiction, assume that the latter case holds.

By Proposition B.1, the two saddle-saddle connections are confined to opposite half-spaces divided by the line pq and the stable and unstable manifolds at p and q that are not involved in the 2-cycle must lie outside the the cycle. It follows that, near p and q, the flow inside the cycle is *circulatory* in the direction of the cycle; hence the flow crosses the segment pq in opposite directions near p as it does near q. But, as observed above, the flow must cross everywhere in the same direction. Thus, we have a contradiction, proving the claim. ∎

Corollary B.2 confirms a conjecture made in [2]. Related investigations may be found in [19].

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