Vol. 50, No. 3 DUKE MATHEMATICAL JOURNAL © September 1983

# THE GAUSS EQUATIONS AND RIGIDITY OF ISOMETRIC EMBEDDINGS

ERIC BERGER, ROBERT BRYANT, AND PHILLIP GRIFFITHS

**Table of Contents**



Received December 14, 1982. The first author's research was partially supported by the Canadian National Science and Engineering Research Council. The second author's research was partially supported by NSF Grant #MCS580-03237. The third author's research was partially supported by the Guggenheim Foundation and NSF Grant #MCS81-04249.

**0. Introduction.**

(a) Let $(\tilde{M}, ds^2)$ be a Riemannian manifold of dimension $n$. A classical problem in differential geometry is to study the existence and uniqueness of isometric embeddings

$$x : \tilde{M} \to \mathsf{E}^{n+r} \tag{1}$$

of $\tilde{M}$ in Euclidean space. In this paper we are primarily concerned with the local uniqueness question (the sequel [5] deals with local existence). Thus, we work in a neighborhood of a point $p \in \tilde{M}$, we assume given an isometric embedding (1) with image $x(\tilde{M}) = M$, and we ask how unique this embedding is. In this paper we shall prove one main general result, which we now state referring to the text and to [3] for explanation of the undefined terms.

MAIN THEOREM. *We consider local isometric embeddings* (1) *where the image is a general* submanifold* $M^n \subset \mathsf{E}^{n+r}$. *Then*

(i) *If* $r \leqq (n-1)(n-2)/2$ *the embedding depends only on constants.*

(ii) *If* $r = (n-1)(n-2)/2 + s$ *the embedding depends formally on functions of at most* $s$ *variables.*

(iii) *If the conditions*

$$r \leqq n \qquad n \geqq 8$$

*or*

$$\begin{cases} r \leqq 3 & n = 4 \\ r \leqq 4 & n = 5, 6 \\ r \leqq 6 & n = 7, 8 \end{cases}$$

*are satisfied, then the embedding* (1) *is unique up to rigid motion.*

We remark that this result was announced in [2], to which we refer for a general discussion of what was known classically concerning existence and uniqueness of local isometric embeddings.

We shall also give a detailed study of local isometric embeddings

$$x : \tilde{M}^3 \to \mathsf{E}^5 \tag{2}$$

in the first nonclassical case, and concerning these we find the following

THEOREM. *The local isometric embeddings* $x : \tilde{M}^3 \to \mathsf{E}^5$ *depend on at most six functions of one variable. Moreover, if the sectional curvatures of* $ds^2$ *are negative, this maximum deformability is achieved only for the four-parameter family of metrics described as follows: Let* $\mathsf{L}^4$ *be Lorentz four space, let* $H^3 \subset \mathsf{L}^4$ *be any convex hyperquadric, and let* $\tilde{M}^3 \subset H^3$ *be the open set of points where the induced metric* $ds^2$ *is positive definite. Then the analytic local isometric embeddings* $x : \tilde{M}^3 \to \mathsf{E}^5$ *depend on six functions of one variable.*

*"General" means that the 2nd fundamental form of $M$ at $p$ should lie in a dense Zariski open set (to be specified below) of all 2nd fundamental forms.

*Remarks.* The assumption of negativity on the sectional curvatures of $ds^2$ can be weakened to a suitable nondegeneracy assumption on the Ricci curvature. In this case, the class of metrics $ds^2$ must be enlarged to include the hyperquadrics in $\mathsf{E}^4$ as well as the spacelike regions of (not necessarily convex) hyperquadrics in $\mathsf{L}^4$.

Of course, one does not expect the "generic" $ds^2$ to be locally embeddable into $\mathsf{E}^5$. In this regard, one of our motivations for studying this over-determined problem was Cartan's remark in [6] that the generic $\tilde{M}^3 \subset \mathsf{E}^5$ is rigid. Our own calculations certainly make this seem plausible, but we were not able to prove this rigidity statement.

Finally, as is well known, the essential ingredient of isometric embeddings (1) is the Gauss equations, and in §5 we study these in detail using the theory of group representations. In particular, we find that even though a general $M^4 \subset \mathsf{E}^6$ is rigid, in contrast to previous local rigidity theorems this cannot be accounted for by the Gauss equations alone.

Our study is based on E. Cartan's theory of exterior differential systems, and in particular on their characteristic varieties. A general discussion of characteristic varieties of exterior differential systems is given in [4], and we have followed the notations and terminology from there (which also agrees with that in [3]).

In addition to the references cited below, there is a further bibliography in [2] giving sources for related work.

In the remainder of this introduction we shall discuss in more detail the contents of each of the sections of this paper.

(b) Sections 1 and 2 are preparation for the main part of the paper.

In §1 we review the structure equations of an abstract Riemannian manifold $(\tilde{M}, ds^2)$ and of a submanifold $M \subset \mathsf{E}^N$ in Euclidean space. In both cases we use the method of moving frames. At first glance this has the disadvantage of introducing a lot of extra variables; however, this is more than compensated for by keeping all of our computations intrinsic, thereby isolating the essential points.

Since the authors were unable to agree on whether or not to use indices, for important equations we have done both. In fact, each point of view has computational advantages.

The main ingredient of an isometric embedding

$$x : \tilde{M} \to \mathsf{E}^N$$

consists of the *Gauss equations* (cf. (1.37))

$$\gamma(H, H) = R \tag{3}$$

expressing the curvature $R$ as a quadratic polynomial in the 2nd fundamental form $H$. In §1(b) we derive these equations together with their 1st prolongation, the *Codazzi equations*

$$2\gamma(H,\nabla H)=\nabla R, \tag{4}$$

and in so doing lay the groundwork for discussing the isometric embedding system.

In §2(a) we set up the basic exterior differential system with independence condition $(I,\chi)$ that governs the local isometric embeddings (1). In doing this we modify Cartan's approach in several respects. First, we begin by setting up the naïve system $(I_0,\chi)$, which is the first thing one would think of doing in the problem. We then show that $(I_0,\chi)$ fails to be involutive, and therefore must be prolonged and the torsion equated to zero in order to obtain $(I,\chi)$. More importantly, we retain the spinning in both the tangent and normal variables as independent variables (i.e., we make no choice of frame), and this makes the prolongation theory of the isometric embedding go much more smoothly. The final system $(I,\chi)$ has the *Gauss equations as its symbol* and the *Codazzi equations as its torsion.*

To illustrate the isometric embedding system as set up in this paper, in §2(b) we give yet another proof of the classical theorem of Burstin–Cartan–Janet–Schaefly (BCJS-theorem). It seems to us that the present argument has the advantage of showing clearly that the proof consists of two parts: (i) the standard theory of differential systems (specifically, Cartan's test for involution); and (ii) a certain algebraic property of the Gauss equations that is forced by simply trying to verify Cartan's test.

(c) In section 3 we prove the Main Theorem stated above.

In outline the proof of parts (i) and (ii) is quite simple: According to the general theory of exterior differential systems, perhaps *the* fundamental invariant of such a system is furnished by the (complex) characteristic variety; accordingly, (i) and (ii) are simply consequences of general results about characteristic varieties applied to the isometric embedding system.

In more detail, if $(I,\chi)$ on $X$ is the isometric embedding system as set up in §2, then there is a vector bundle $V\to X$ (whose fibres may be thought of as the tangent spaces $T_x(\tilde{M})$) and the complex characteristic variety is constructed from the symbol of $(I,\chi)$ and is given by a family of projective algebraic varieties

$$\Xi_{\mathbf{C},x}\subset \mathbf{P}V^*_{\mathbf{C}}.$$

For an isometric embedding

$$x:\tilde{M}^n\to \mathbf{E}^{n+r}$$

whose image is a "general" submanifold $M^n\subset \mathbf{E}^{n+r}$, we will prove that (cf. Theorem A below)

$$\dim \Xi_{\mathbf{C},x}=\max(-1,r-1-(n-1)(n-2)/2), \tag{5}$$

where, by convention, $\dim \emptyset = -1$. When coupled with a general result from [4] concerning characteristic varieties, this gives (i) and (ii) in the Main Theorem.

We remark that the usual real characteristic variety $\Xi$ has the following geometric meaning: A point $(x,\xi) \in \mathsf{P}V^*$ (thus $x \in M$ and $\xi \in \mathsf{P}T_x^*(M)$) is in $\Xi$ if, and only if, there is a normal vector $\lambda \in N_x(M)$ such that the linear projection

$$M \to M_\lambda \subset \mathsf{E}^{n+1}$$

into the $\mathsf{E}^{n+1}$ spanned by $T_x(M)$ and $\lambda$ has the property that

$$\mathrm{II}_\lambda|_{\xi^\perp} \equiv 0 \tag{6}$$

where $\mathrm{II}_\lambda$ is the 2nd fundamental form of $M_\lambda$ at $x$ and $\xi^\perp$ is the hyperplane defined by $\xi$. Thus the linear section $M_\lambda \cap \xi^\perp$ is flat at $x$. Such $\xi$ are classically called the *asymptotic hyperplanes* of the embedded submanifold. It is interesting to note that if we consider $M$ as embedded in real projective space $\mathsf{R}P^{n+r} = \mathsf{E}^{n+r} \cup \{\text{hyperplane at infinity}\}$, then $\Xi$ is invariant under the projective transformations of $\mathsf{R}P^{n+r}$. Perhaps because of this, we shall find in [5] that $\Xi$ has an extraordinarily rich algebro-geometric structure.

The proof of the dimension statement (5) involves studying the Gauss equations of $M^n \subset \mathsf{E}^{n+r}$. It is well known that these equations, which are of a rather complicated quadratic nature, constitute the main feature of isometric embeddings, expressing as they do the link between the basic extrinsic invariant (the 2nd fundamental form) and the basic intrinsic invariant (the Riemann curvature tensor). The proof of (5) is made quite easy by the (to us miraculous) fact that when localized in the sense of algebra the Gauss equations become quite simple and are readily analyzed. This will be pursued further in [5] when we shall determine the degree, real dimension, and singularity structure of $\Xi$, culminating in a rather precise microlocal normal form for the isometric embedding system.

We would like to further comment on the proof of (5), as the method of localizing and applying results from commutative algebra may have further applications to problems in differential geometry in which derivatives of higher order (i.e., prolongations) are involved.

In §3(a) we give the basic localization of the Gauss equations. Very roughly speaking, in each complexified and projectivized cotangent space $\mathsf{P}(T_x^*)M(\otimes\mathsf{C}) \cong \mathsf{P}^{n-1}$ we interpret the symbol of the isometric embedding system as giving a mapping of coherent sheaves

$$\mathscr{K}^* \xrightarrow{\gamma} \mathscr{W}^*(2)$$

over $\mathsf{P}^{n-1}$ where $\mathscr{K}^*, \mathscr{W}^*$ both correspond to trivial vector bundles. (The quotient sheaf $\mathscr{M} = \mathscr{W}^*(2)/\gamma^*(\mathscr{K}^*)$ is the *characteristic sheaf*—cf. [4].) When localized at the point $dx^1 = \cdots = dx^{n-1} = 0$, $dx^n \neq 0$ the Gauss equations turn out to involve only the components $R_{\rho n \sigma n} = R_{\sigma n \rho n}$ of the curvature tensor, and therefore are expressed by simple equations involving one symmetric matrix. The dimension result (5) is a straightforward consequence of this fortuitous occurrence.

In §3(b) we prove that, when $r \leq (n-1)(n-2)/2$, the induced metric of a general $M^n \subset \mathsf{E}^{n+r}$ uniquely determines its 2nd fundamental form up to a general linear automorphism of the normal space. When, additionally, the conditions of (iii) in the Main Theorem are satisfied, the 2nd fundamental form is uniquely determined up to an orthogonal transformation, and our result follows easily from this.

We remark that our proof, which is a nonlinear commutative algebra argument, shows that the 2nd fundamental form is in fact determined by the sequence $(R, \nabla R, \dots, \nabla^{q_0} R)$ of covariant derivatives of the curvature for some (generally large) integer $q_0$. The example of a general $M^4 \subset \mathsf{E}^6$ shows that it is not determined by $R$ alone.

(d) In §4 we take up a set of examples of nongeneric behavior of the Gauss equations. These examples are based on Cartan's notion of exterior orthogonality of quadratic forms. A quadratic form $H$ on a vector space $V$ with values in an Euclidean vector space $W$ is said to be exteriorly orthogonal if

$$\gamma(H, H) = 0. \tag{7}$$

It turns out that the characteristic variety for an exteriorly orthogonal $H$ is much larger than the characteristic variety for a general $H$, at least in the case where $\dim W \leq \dim V$. We then give two examples to show how this pointwise phenomenon relates to geometric phenomena.

Our first example is classical, concerning the nondegenerate flat $M^n \subset \mathsf{E}^{2n}$. We show that the characteristic variety consists of the set of $\binom{n}{2}$ lines in $\mathsf{P}^{n-1}$ through pairs of $n$ points in general position. We then give a proof of Cartan's result that the analytic flat $M^n \subset \mathsf{E}^{2n}$ depend on $\binom{n}{2}$ functions of 2 variables $(n \geq 2)$.

Our second example starts with Cartan's observation about the problem of finding an isometric embedding of a hyperbolic space form $H^n$ into $\mathsf{E}^{n+r}$: There are no local solutions unless $r \geq n-1$. We then introduce a more general class of metrics, the *quasi-hyperbolic* metrics on $\tilde{M}^n$ characterized by the condition that there should exist a nondegenerate quadratic form $Q$ on $\tilde{M}^n$ satisfying

$$R = -\gamma(Q, Q). \tag{8}$$

(For the space form of sectional curvature $-1$, we may take $Q = ds^2$). Cartan's theorem for the space form immediately generalizes to quasi-hyperbolic metrics. It is important to remark that, when $n = 3$, quasi-hyperbolicity is an *open* condition on the metric $ds^2$.

In the case where $\dim W = \dim V - 1$ (the smallest value of $\dim W$ possible) we calculate the characteristic variety of an $H$ satisfying $\gamma(H, H) = -\gamma(Q, Q)$ and show that it consists of $n(n-1)$ (real) points. By the general theory of differential systems, it follows that the analytic isometric embeddings of an analytic quasi-hyperbolic metric $x : \tilde{M}^n \to \mathsf{E}^{2n-1}$ depend on at most $n(n-1)$ functions of one variable.

In [7], Cartan shows that, for the hyperbolic space form $H^n$, this maximum deformability is actually attained. We then go on to characterize the quasi-hyperbolic metrics satisfying the conditions that $Q$ be positive definite and that the maximum deformability be attained. These metrics turn out to have the simple characterization of being the metrics induced on convex space-like hyperquadrics in Lorentz $(n+1)$-space.

(e) In §5 we study the Gauss equations (3) using representation theory. Let $W$ be a vector space with inner product, $V$ a vector space (no inner product), and interpret the Gauss equations as a quadratic map

$$\gamma : W \otimes \mathrm{Sym}^2 V^* \to K \tag{9}$$

where $K \subset \mathrm{Sym}^2(\Lambda^2 V^*)$ is the space of curvature-like-tensors. The main observations are that $\gamma$ is $\mathrm{GL}(V^*)$ equivariant, and that there is a $\mathrm{GL}(V^*)$-*commutative* factorization

$$\begin{tikzcd} W \otimes \mathrm{Sym}^2 V^* \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & K \\ & \mathrm{Sym}^2(\mathrm{Sym}^2 V^*) \cong K \oplus \mathrm{Sym}^4 V^* \arrow[ur, "\lambda"'] & \end{tikzcd}$$

where for $w \in W$ and $\varphi \in \mathrm{Sym}^2 V^*$

$$\mu(w \otimes \varphi) = \langle w, w \rangle \varphi \circ \varphi$$

is the obvious quadratic mapping (Veronese mapping), and where $\lambda$ is the obvious projection. This allows us to analyze the Gauss equations using the map $\mu$, and from this prove Theorem H plus the following curious fact: Consider a general submanifold

$$M^4 \subset \mathsf{E}^6. \tag{10}$$

According to our Main Theorem this submanifold is rigid.

We recall that in classical rigidity theorems it is always shown that the equation $\gamma(H, H) = R$ has a unique solution $H$ up to the group $O(W)$ normal rotations. Now the Gauss equations for (10) correspond to (9) when $\dim W = 2$, $\dim V = 4$, and $\dim K = 20$. Then $\dim(W \otimes S^2 V^*) = 20$, and initially we thought that the general fibres of (9) were 1-dimensional corresponding to the invariance of $\gamma$ under $O(W)$. However, it turns out that for general $H$

$$\dim \gamma^{-1}(\gamma(H, H)) = 2,$$

so that rigidity in this case is *not* accounted for by uniquely (up to $O(W)$) solving the Gauss equations. This is one of the first examples to show the prolonged Gauss equations must also be considered in the study of submanifolds of low codimension.

## 1. Basic structure equations.

(a) *Structure equations of Riemannian manifolds.* In our work, it will be desirable to have a notation that distinguishes between an abstract Riemannian manifold and one embedded in Euclidean space. Accordingly, we denote by $(\tilde{M}, ds^2)$ an abstract Riemannian manifold $\tilde{M}$ with metric $ds^2$. In this section we shall review the structure equations of $(\tilde{M}, ds^2)$ (cf. [16]).

(i) We first do this using indices. By a *frame* $(p; \tilde{e}_1, \ldots, \tilde{e}_n)$ we mean a point $p \in \tilde{M}$ together with an orthonormal basis $\tilde{e}_i$ for $T_p(\tilde{M})$. The totality of all frames forms a manifold $\mathscr{F}(\tilde{M})$ fibered over $\tilde{M}$ with fibre isomorphic to the orthogonal group $O(n)$. If we denote this fibering by

$$\tilde{\pi} : \mathscr{F}(\tilde{M}) \to \tilde{M}$$

$$\tilde{\pi}^{-1}(p) = \{\text{all frames lying over } p\},$$

then it is well known that there are defined on $\mathscr{F}(\tilde{M})$ unique linear differential forms $\tilde{\omega}^i, \tilde{\psi}_j^i$ that satisfy the equations

$$\begin{cases} \tilde{\pi}^*(ds^2) = \sum_i (\tilde{\omega}^i)^2 \\ d\tilde{\omega}^i = -\tilde{\psi}_j^i \wedge \tilde{\omega}^j \\ \tilde{\psi}_j^i + \tilde{\psi}_i^j = 0. \end{cases} \tag{1.1}$$

*Note.* Recall that in any fibering

$$\pi : X \to Y$$

of manifolds the vertical tangent space

$$V_x = \ker\{\pi_* : T_x(X) \to T_{\pi(x)}(Y)\}$$

and horizontal cotangent space

$$H_x^* = \operatorname{image}\{\pi^* : T_{\pi(x)}^*(Y) \to T_x^*(X)\} = V_x^{\perp}$$

are well defined. At $x = (p; \tilde{e}_1, \ldots, \tilde{e}_n) \in \mathscr{F}(\tilde{M})$ the $\tilde{\omega}^i$ give a basis for $H_x^* \cong T_p^*(\tilde{M})$ dual to the basis $\tilde{e}_1, \ldots, \tilde{e}_n$ for $T_p(\tilde{M})$.

The uniqueness of the $\tilde{\psi}_j^i$ satisfying the second and third conditions results from the following well-known

(1.2) *Standard argument.* If $\phi_j^i$ is the difference of two solutions to the second and third equations in (1.1), then

$$\begin{cases} \phi_j^i \wedge \tilde{\omega}^j = 0 \\ \phi_j^i + \phi_i^j = 0. \end{cases}$$

By the Cartan lemma the first of these equations gives

$$\phi_j^i = C_{jk}^i \tilde{\omega}^k, \qquad C_{jk}^i = C_{kj}^i.$$

The second equation then gives

$$C_{jk}^i = -C_{ik}^j = -C_{ki}^j = C_{ji}^k = C_{ij}^k = -C_{kj}^i = -C_{jk}^i,$$

i.e.,

$$C_{jk}^i = 0.$$

The proof that $\phi_j^i$ satisfying the above pair of equations must be zero is the "standard argument". It will be used several times during this paper; it is for this reason that we have put it in a form that is easily referred to.

The matrix $\|\tilde{\psi}_j^i\|$ may be interpreted as defining the *Levi–Civita connection* associated to the $ds^2$. By the *Cartan structure equation* the curvature $\|\tilde{\Omega}_j^i\|$ is given by

$$\tilde{\Omega}_j^i = d\tilde{\psi}_j^i + \tilde{\psi}_k^i \wedge \tilde{\psi}_j^k. \tag{1.3}$$

It satisfies the properties

$$\begin{cases} \tilde{\Omega}_j^i \in \Lambda^2(\operatorname{span}\{\tilde{\omega}^i\}) & (\text{i.e., } \tilde{\Omega}_j^i \text{ is horizontal}) \\ \tilde{\Omega}_j^i + \tilde{\Omega}_i^j = 0. \end{cases}$$

Using the first of these we may define the components of the *Riemann curvature tensor* $R = \{R_{ijkl}\}$ by

$$\tilde{\Omega}_j^i = \tfrac{1}{2} R_{ijkl} \tilde{\omega}^k \wedge \tilde{\omega}^l. \tag{1.4}$$

The second property above together with (1.4) gives the symmetries

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}.$$

Exterior differentiation of (1.1) and (1.3) gives the first and second *Bianchi identities*

$$\begin{cases} \tilde{\Omega}_j^i \wedge \tilde{\omega}^j = 0 \\ d\tilde{\Omega}_j^i - \tilde{\Omega}_k^i \wedge \tilde{\psi}_j^k + \tilde{\psi}_k^i \wedge \tilde{\Omega}_j^k = 0. \end{cases} \tag{1.5}$$

To put these in more familiar form we consider any tensor $T = \{T_{i_1 \dots i_q}\}$ and set

$$T_{I\underbrace{ijk}} = T_{Iijk} + T_{Ikij} + T_{Ijki}$$

where $I$ is any index set containing $q - 3$ elements. If we denote by $\{R_{ijkl,m}\}$ the components of the covariant derivative of $R$, then (1.5) becomes

$$\begin{cases} R_{i\underbrace{jkl}} = 0 \\ R_{ij\underbrace{kl,m}} = 0. \end{cases} \tag{1.6}$$

(ii) In index-free notation we let $V$ be a fixed Euclidean vector space with inner product ( , ), and we define a frame to be an isometry

$$V \to T_p(\tilde{M}).$$

Once we have chosen an orthonormal basis for $V$ this is the same as our previous definition. There are now the following unique forms on $\mathscr{F}(\tilde{M})$

$$\begin{cases} \tilde{\omega} = V\text{-valued 1-form} \\ \tilde{\psi} = V \otimes V^*\text{-valued 1-form} \\ \tilde{\Omega} = V \otimes V^*\text{-valued 2-form} \end{cases}$$

satisfying the following index-free versions of (1.1) and (1.3).

$$\begin{cases} (\tilde{\omega}, \tilde{\omega}) = \tilde{\pi}^*(ds^2) \\ d\tilde{\omega} = -\tilde{\psi} \wedge \tilde{\omega} \\ \tilde{\psi} + {}^t\tilde{\psi} = 0 \\ d\tilde{\psi} + \tilde{\psi} \wedge \tilde{\psi} = \tilde{\Omega}. \end{cases} \tag{1.7}$$

The Bianchi identities (1.5) are

$$\begin{cases} \tilde{\Omega} \wedge \tilde{\omega} = 0 \\ D\tilde{\Omega} = 0 \end{cases} \tag{1.8}$$

where

$$D\tilde{\Omega} = d\tilde{\Omega} - \tilde{\Omega} \wedge \tilde{\psi} + \tilde{\psi} \wedge \tilde{\Omega}$$

is the covariant differential of $\tilde{\Omega}$.

We would like to remark further on the symmetries of the curvature tensor and its covariant derivative. As previously noted, from the third equation in (1.7) it follows that

$$\tilde{\Omega} + {}^t\tilde{\Omega} = 0. \tag{1.9}$$

Using the isomorphism ("lowering indices")

$$V \xrightarrow{\sim} V^*$$

given by the metric, we let $\tilde{\Omega}^*$ be the $V^* \otimes V^*$-valued 2-form corresponding to $\tilde{\Omega}$. By (1.9) it follows that $\tilde{\Omega}^*$ has values in $\Lambda^2 V^* \subset V^* \otimes V^*$. If we define $\tilde{\omega} \wedge \tilde{\omega}$ to be the unique $\Lambda^2 V$-valued 2-form satisfying

$$(\xi \wedge \eta)(\tilde{\omega} \wedge \tilde{\omega}) = \xi(\tilde{\omega}) \wedge \eta(\tilde{\omega})$$

for all $\xi, \eta \in V^*$, then it follows from the horizontality of $\tilde{\Omega}$ that

$$\tilde{\Omega}^* = \tfrac{1}{2} R \tilde{\omega} \wedge \tilde{\omega}$$

where the curvature tensor $R$ is a $\Lambda^2 V^* \otimes \Lambda^2 V^*$-valued function on $\mathscr{F}(\tilde{M})$.

*Definition.* We define the space of *curvature-like tensors*

$$K \subset \Lambda^2 V^* \otimes \Lambda^2 V^*$$

to be the kernel of the natural $\mathrm{GL}(V^*)$-equivariant mapping

$$\Lambda^2 V^* \otimes \Lambda^2 V^* \xrightarrow{\partial} V^* \otimes \Lambda^3 V^*$$

given by

$$\partial(\xi \wedge \eta \otimes \zeta \wedge \lambda) = \xi \otimes \eta \wedge \zeta \wedge \lambda - \eta \otimes \xi \wedge \zeta \wedge \lambda.$$

Choosing an orthonormal basis for $V$, $K$ is the space of tensors $T = \{T_{ijkl}\}$ satisfying

$$\begin{cases} T_{ijkl} = -T_{jikl} = -T_{ijlk} \\ T_{i\underbrace{jkl}} = 0. \end{cases} \tag{1.10}$$

It follows that the Riemann curvature tensor

$$R \in K.$$

For later use we remark that the Bianchi identity $T_{i\underbrace{jkl}} = 0$ implies that

$$K \subset \mathrm{Sym}^2(\Lambda^2 V^*) \subset \Lambda^2 V^* \otimes \Lambda^2 V^*. \tag{1.11}$$

Moreover, since $\partial$ is clearly surjective we easily compute that (cf. §5(g) below)

$$\dim K = \frac{n^2(n^2-1)}{12}. \tag{1.12}$$

*Remark.* When $n = 3$ the inclusion (1.11) is equivalent to the first Bianchi identity in (1.8). Put differently, the only time that the equation for $T \in \Lambda^2 V^* \otimes \Lambda^2 V^*$

$$T_{i\underbrace{jkl}} = 0$$

actually involves a sum of three nonzero terms is when all indices $i, j, k, l$ are distinct, and this is only possible when $n \geq 4$.

Next, we define the covariant differential $DR$ to be the unique $K$-valued 1-form that satisfies

$$\begin{aligned} DR(v_1, v_2, v_3, v_4) = d(R(v_1, v_2, v_3, v_4)) &- R(\tilde{\psi}(v_1), v_2, v_3, v_4) \\ &- R(v_1, \tilde{\psi}(v_2), v_3, v_4) - R(v_1, v_2, \tilde{\psi}(v_3), v_4) \\ &- R(v_1, v_2, v_3, \tilde{\psi}(v_4)) \end{aligned} \tag{1.13}$$

for all $v_1, v_2, v_3, v_4 \in V$. Then the second Bianchi identity asserts that there exists a unique $K \otimes V^*$-valued function $\nabla R$ satisfying

$$DR = \nabla R \omega \tag{1.14}$$

together with a symmetry that we now explain.

*Definition.* We define

$$K^{(1)} \subset K \otimes V^*$$

to be the kernel of the natural map

$$K \otimes V^* \subset \Lambda^2 V^* \otimes \Lambda^2 V^* \otimes V^* \to \Lambda^2 V^* \otimes \Lambda^3 V^*.$$

Then $\nabla R$ defined by (1.14) takes values in $K^{(1)}$.

Using indices, $K^{(1)}$ is given by tensors $\{T_{ijklm}\} \in \Lambda^2 V^* \otimes \Lambda^2 V^* \otimes V^*$ satisfying

$$\begin{cases} T_{i\underbrace{jkl}m} = 0 \\ T_{ij\underbrace{klm}} = 0. \end{cases}$$

We will discuss the higher derivatives $\nabla^k R$ where needed below.

For easy reference we collect the various structure equations on the frame bundle $\mathscr{F}(\tilde{M})$ as follows:

$$\begin{cases} (\tilde{\omega}, \tilde{\omega}) = \tilde{\pi}^*(ds^2) \\ d\tilde{\omega} = -\tilde{\psi} \wedge \tilde{\omega} \\ d\tilde{\psi} + \tilde{\psi} \wedge \tilde{\psi} = \tilde{\Omega} \\ \tilde{\psi} + {}^t\tilde{\psi} = 0 = \tilde{\Omega} + {}^t\tilde{\Omega} \\ \tilde{\Omega}^* = \frac{1}{2} R \tilde{\omega} \wedge \tilde{\omega} \\ R \in K = \ker\{\partial : \Lambda^2 V^* \otimes \Lambda^2 V^* \to V^* \otimes \Lambda^3 V^*\} \\ DR = \nabla R \omega \\ \nabla R \in K^{(1)} = (K \otimes V^*) \cap \left(\ker\{\Lambda^2 V^* \otimes \Lambda^2 V^* \otimes V^* \to \Lambda^2 V^* \otimes \Lambda^3 V^*\}\right). \end{cases} \tag{1.15}$$

(b) *Structure equations of submanifolds of Euclidean space.*

(i) By $\mathsf{E}^N$ we mean the coordinate space $\{x : x = (x^1, \ldots, x^N)\}$ having the usual flat metric

$$ds^2 = \sum_{a=1}^{N} (dx^a)^2.$$

Let $U$ be an $N$-dimensional Euclidean vector space (i.e., we fix an origin). By a *frame* for $\mathsf{E}^N$ we mean an isometry

$$F : U \to \mathsf{E}^N.$$

The image

$$F(O) = x(F)$$

will be called the *position vector* of the frame. Using the isomorphism

$$F_* : T_{(O)}(U) \xrightarrow{\sim} T_x(\mathsf{E}^N),$$

for each frame we make the identification

$$U = T_x(\mathsf{E}^N). \tag{1.16}$$

The manifold of all frames will be denoted by $\mathscr{F}$, and

$$x : \mathscr{F} \to \mathsf{E}^N \tag{1.17}$$

will denote the position vector map.

If we choose an orthonormal basis $u_1, \ldots, u_N$ for $U$ and set

$$\begin{cases} x = F(0) \\ e_a = F_*(u_a), \end{cases}$$

then the frame $F$ may be written as

$$F = (x; e_1, \ldots, e_N). \tag{1.18}$$

Equivalently, identifying $T_x(\mathsf{E}^N)$ with $\mathsf{E}^N$ there is a unique linear map (taking the origin to the origin)

$$e(F) : U \to \mathsf{E}^N$$

given by

$$e(F)(u) = F_*(u), \qquad u \in U.$$

Then

$$F(u) = x(F) + e(F)(u) \tag{1.19}$$

for all $u \in U$. We abbreviate (1.19) as

$$F = x + e, \tag{1.20}$$

and view $e$ as an $\mathsf{E}^N \otimes U^*$-valued function on $\mathscr{F}$.

With this notation, on the frame manifold $\mathscr{F}$ there are defined both a unique $U$-valued 1-form $\eta$ and a unique $U \otimes U^*$-valued 1-form $\phi$ that satisfy the *structure equations of a moving frame*

$$\begin{cases} dx = e \cdot \eta \\ de = e \cdot \phi \\ \phi + {}^t\phi = 0. \end{cases} \tag{1.21}$$

With $F$ given in coordinates by (1.18), equations (1.21) are (using the index range $1 \leq a, b \leq N$)

$$\begin{cases} dx = e_a \eta^a \\ de_a = e_b \phi_a^b \\ \phi_a^b + \phi_b^a = 0. \end{cases} \tag{1.22}$$

It is well known that, upon choice of a reference frame, we may identify $\mathscr{F}$ with the group of Euclidean motions of $\mathsf{E}^N$ (this is clear from (1.20)), and when this is done the components of $\eta$ and $\phi$ are the *Maurer–Cartan forms* on $\mathscr{F}$. The exterior derivatives of (1.21) give the *Maurer–Cartan equations*

$$\begin{cases} d\eta = -\phi \wedge \eta \\ d\phi + \phi \wedge \phi = 0. \end{cases} \tag{1.23}$$

We note that the use of vector-valued forms eliminates ambiguities concerning transformation rules. This will be even more true in the following discussion.

We suppose given an orthogonal direct sum decomposition (the motivation for this will appear shortly when we discuss submanifolds of $\mathsf{E}^{n+r}$)

$$U = V \oplus W, \tag{1.24}$$

and write accordingly

$$e(F) = e' + e''$$

where $e', e''$ are respectively $\mathsf{E}^N \otimes V^*, \mathsf{E}^N \otimes W^*$-valued functions on $U$. In terms of coordinates we set

$$N = n + r$$

and use the range of indices

$$\begin{cases} \quad 1 \le a, b, c \le N \\ \quad 1 \le i, j, k \le n \\ n+1 \le \mu, \nu \le n+r. \end{cases} \tag{1.25}$$

If we choose orthonormal bases $v_1, \dots, v_n$ for $V$ and $w_{n+1}, \dots, w_{n+r}$ for $W$, then by (1.18)

$$e' = (e_1, \dots, e_n)$$

$$e'' = (e_{n+1}, \dots, e_N).$$

With this notation there are uniquely defined on $\mathscr{F}$ the following: a

$$\begin{cases} V\text{-valued 1-form } \omega \\ W\text{-valued 1-form } \theta \\ V \otimes V^*\text{-valued 1-form } \psi \\ W \otimes V^*\text{-valued 1-form } \hbar \\ W \otimes W^*\text{-valued 1-form } \kappa \end{cases}$$

such that the structure equations (1.21) are

$$\begin{cases} dx = e'\omega + e''\theta \\ de' = e'\psi + e''\hbar \\ de'' = -e'{}^t\hbar + e''\kappa. \end{cases} \tag{1.26}$$

In terms of indices, (1.26) is

$$\begin{cases} dx = e_i\omega^i + e_\mu\theta^\mu \\ de_i = e_j\psi_i^j + e_\mu\hbar_i^\mu \\ de_\mu = -e_i\hbar_i^\mu + e_\nu\kappa_\mu^\nu. \end{cases}$$

To collect our notation, we write $\eta$, $\phi$, and the Maurer–Cartan equations (1.23) out in matrix blocks as follows:

$$\begin{cases} \eta = \begin{pmatrix} \omega \\ \theta \end{pmatrix} \\ \phi = \begin{pmatrix} \psi & -{}^t\hbar \\ \hbar & \kappa \end{pmatrix} \qquad \psi + {}^t\psi = 0 = \kappa + {}^t\kappa \\ d\omega = -\psi \wedge \omega + {}^t\hbar \wedge \theta \\ d\theta = -\hbar \wedge \omega - \kappa \wedge \theta \\ d\psi + \psi \wedge \psi = {}^t\hbar \wedge \hbar \\ d\hbar + \hbar \wedge \psi + \kappa \wedge \hbar = 0 \\ d\kappa + \kappa \wedge \kappa = \hbar \wedge {}^t\hbar. \end{cases} \tag{1.27}$$

(ii) We now assume given a smooth submanifold

$$M^n \subset \mathsf{E}^{n+r}$$

(we will always use $r$ for our codimension).

*Definition.* The manifold of *Darboux frames* (or *adapted frames*)

$$\mathscr{F}(M) \subset \mathscr{F}$$

is given by all frames (1.18) that satisfy

$$\begin{cases} x \in M \\ e_1, \dots, e_n \in T_x(M). \end{cases}$$

If we use our index range (1.25) and define the *normal spaces* to $M$ by

$$N_x(M) = T_x(M)^{\perp},$$

then a Darboux frame is given by $(x; e_i; e_\mu)$ where $x \in M$, the $e_i$ are a tangent frame at $x$, and the $e_\mu$ are a normal frame at $x$.

Equivalently, a Darboux frame is an isometry (cf. (1.24))

$$F : V \oplus W \to \mathsf{E}^{n+r}$$

satisfying

$$\begin{cases} F(O) = x \in M \\ F_*(V) = T_x(M) \\ F_*(W) = N_x(M). \end{cases}$$

Using the notation (1.26) the position vector mapping

$$x : \mathscr{F}(M) \to \mathsf{E}^{n+r}$$

has differential

$$dx \in T_x(M).$$

This is equivalent to

$$\theta|_{\mathscr{F}(M)} = 0.$$

We agree to omit the restriction signs (it being understood that we are working on $\mathscr{F}(M)$), and write this equation as

$$\theta = 0. \tag{1.28}$$

By (1.27) this implies that

$$0 = d\theta = \hbar \wedge \omega. \tag{1.29}$$

By the Cartan lemma, (1.29) gives

$$ℏ = H\omega \tag{1.30}$$

where $H$ is a $W \otimes S^2V^*$-valued function defined on $\mathscr{F}(M)$.

In terms of indices, (1.28)–(1.30) are respectively

$$\begin{cases} \theta^\mu = 0 \\ 0 = d\theta^\mu - ℏ_i^\mu \wedge \omega^i \\ ℏ_i^\mu = H_{ij}^\mu \omega^j, \qquad H_{ij}^\mu = H_{ji}^\mu . \end{cases} \tag{1.31}$$

*Definition.* The $W \otimes S^2V^*$-valued function $H$ on $\mathscr{F}(M)$ is the *2nd fundamental form* of $M^n \subset \mathsf{E}^{n+r}$.

It is clear that $H$ may be thought of as a section of $N(M) \otimes S^2T^*(M)$ over $M$, and for this reason we sometimes write

$$H \in \operatorname{Hom}\big(S^2T(M), N(M)\big).$$

Given $x \in M$ we may choose linear coordinates $(v^1, \ldots, v^n; w^{n+1}, \ldots, w^{n+r})$ centered at $x$ and with

$$T_x(M) = \operatorname{span}\{\partial/\partial v^1, \ldots, \partial/\partial v^n\}.$$

Then $M$ is given parametrically by

$$w^\mu = H_{ij}^\mu v^i v^j + (\text{higher order terms in } v^i)$$

where $H \equiv \|H_{ij}^\mu\|$ is the 2nd fundamental form of $M$ at $x$. It is well known that $H$ is the basic extrinsic invariant of $M$ in $\mathsf{E}^{n+r}$.

Again by (1.27), we have on $\mathscr{F}(M)$

$$d\omega = -\psi \wedge \omega. \tag{1.32}$$

This equation has the following meaning: given $M^n \subset \mathsf{E}^{n+r}$ there is an obvious abstract Riemannian manifold $(\tilde{M}, ds^2)$ together with an isometric embedding

$$x : \tilde{M} \to \mathsf{E}^{n+r}$$

with $x(\tilde{M}) = M$. There is also the obvious map

$$\mathscr{F}(M) \xrightarrow{\pi} \mathscr{F}(\tilde{M})$$

given by the tangential part of the Darboux frame (thus the fibres of $\pi$ correspond to spinning the normal frame). Since $x$ is an isometry, it is clear that

$$\pi^*\tilde{\omega} = \omega \tag{1.33}$$

and, by the "standard argument" (1.2), from the 2nd equation in (1.7) and (1.33) we have

$$\pi^* \tilde{\psi} = \psi. \tag{1.34}$$

Using the 4th equation in (1.7), (1.27), and (1.34) we deduce that

$$\pi^*(\tilde{\Omega}) = {}^t\!h \wedge h. \tag{1.35}$$

This is the main equation in the whole theory. Before putting it in more familiar form, we write (1.32)–(1.35) in terms of indices and omitting all pullback and restriction maps as

$$\begin{cases} d\omega^i + \psi^i_j \wedge \omega^j = 0 \\ \tilde{\omega}^i - \omega^i = 0 \\ \tilde{\psi}^i_j - \psi^i_j = 0 \\ \tilde{\Omega}^*_{ij} - \sum_\mu h^\mu_i \wedge h^\mu_j = 0. \end{cases}$$

*Definition.* We denote by

$$\gamma : (W \otimes S^2V^*) \times (W \otimes S^2V^*) \to K$$

the unique symmetric bilinear map that satisfies

$$\begin{aligned} \gamma(H,G)(v_1,v_2,v_3,v_4) = \frac{1}{2}\{&H(v_1,v_3)\cdot G(v_2,v_4) + H(v_2,v_4)\cdot G(v_1,v_3) \\ &- H(v_1,v_4)\cdot G(v_2,v_3) - H(v_2,v_3)\cdot G(v_1,v_4)\} \end{aligned} \tag{1.36}$$

for all $v_1, v_2, v_3, v_4 \in V$.

It is immediate that $\gamma(H,G)$ lies in $K$, the space of curvature-like tensors defined above (cf. (1.15)).

*Definition.* With the notation (1.36), (1.35) is equivalent to the *Gauss equations*, which by definition are the equations

$$\gamma(H,H) = R. \tag{1.37}$$

These provide the fundamental link between the intrinsic and extrinsic geometry of $M \subset \mathsf{E}^{n+r}$; they are ubiquitous in this work.

In terms of indices, (1.36) and (1.37) are

$$\gamma(H,G)_{ijkl} = \frac{1}{2}\sum_\mu \{H^\mu_{ik}G^\mu_{jl} + H^\mu_{jl}G^\mu_{ik} - H^\mu_{il}G^\mu_{jk} - H^\mu_{jk}G^\mu_{il}\} \tag{1.38}$$

$$R_{ijkl} = \sum_\mu \{H^\mu_{ik}H^\mu_{jl} - H^\mu_{il}H^\mu_{jk}\}. \tag{1.39}$$

We collect equations (1.28)–(1.30) and (1.32)–(1.35) as

$$\begin{cases} \theta = 0 \\ \hslash \wedge \omega = 0 \\ \hslash = H\omega \\ d\omega + \psi \wedge \omega = 0 \\ \tilde{\omega} - \omega = 0 \\ \tilde{\psi} - \psi = 0 \\ \tilde{\Omega}^* - {}^t\hslash \wedge \hslash = 0 \end{cases} \tag{1.40}$$

where it is understood that all these equations are taking place on $\mathscr{F}(M)$.

For later use we extend the map (1.36) to a bilinear map

$$\gamma : W \otimes S^2V^* \times W \otimes S^{q+2}V^* \to K \otimes S^qV^* \tag{1.41}$$

defined for $H \in W \otimes S^2V^*$ and $G \in W \otimes S^{q+2}V^*$ by the condition

$$\begin{aligned} \gamma(H,G)(w_1,w_2,w_3,w_4,v_1,\ldots,v_q) = \tfrac{1}{2}\{&H(w_1,w_3)\cdot G(w_2,w_4,v_1,\ldots,v_q) \\ &+ H(w_2,w_4)\cdot G(w_1,w_3,v_1,\ldots,v_q) \\ &- H(w_1,w_4)\cdot G(w_2,w_3,v_1,\ldots,v_q) \\ &- H(w_2,w_3)\cdot G(w_1,w_4,v_1,\ldots,v_q)\} \end{aligned} \tag{1.42}$$

where the $w$'s and $v$'s are arbitrary elements of $V$ (recall that $K \subset \bigotimes^4 V^*$). In terms of indices, we view elements in $W \otimes S^kV^*$ as $W$-valued polynomials of degree $k$ in the variables $x^1, \ldots, x^n$ and similarly for $K \otimes S^lV^*$, and then

$$\begin{aligned} &\gamma(H,G)_{ijklm_1\ldots m_q}x^{m_1}\ldots x^{m_q} \\ &= \frac{1}{2}\sum_\mu \{H^\mu_{ik}G^\mu_{jlm_1\ldots m_q}x^{m_1}\ldots x^{m_q} + H^\mu_{jl}G^\mu_{ikm_1\ldots m_q}x^{m_1}\ldots x^{m_q} \\ &\quad - H^\mu_{il}G^\mu_{jkm_1\ldots m_q}x^{m_1}\ldots x^{m_q} - H^\mu_{jk}G^\mu_{ilm_1\ldots m_q}x^{m_1}\ldots x^{m_q}\}. \end{aligned} \tag{1.43}$$

It is immediate that

$$\frac{\partial\gamma(H,G)}{\partial x^i} = \left(\frac{q+2}{q}\right)\gamma\left(H, \frac{\partial G}{\partial x^i}\right). \tag{1.44}$$

It is this relation that links the various prolongations of the isometric embedding system.

By analogy with $DR$ and $\nabla R$ we now define $DH$ and $\nabla H$. Thus $DH$ is the unique $W \otimes S^2V^*$-valued 1-form on $\mathscr{F}(M)$ that satisfies

$$DH(w^*; v_1, v_2) = d(H(w^*; v_1, v_2)) + H(\kappa(w^*); v_1, v_2) - H(w^*; \psi(v_1), v_2) - H(w^*; v_1, \psi(v_2)) \qquad (1.45)$$

for every $w^* \in W^*$ and $v_1, v_2 \in V$ (and where $\kappa, \psi$ are as in (1.27)). Using (1.13), exterior differentiation of the Gauss equations (1.37) gives the *Codazzi equations*

$$2\gamma(H, DH) = DR. \qquad (1.46)$$

Using

$$\not{h} = H\omega$$

and (1.27) (where we write the next to last equation there as $D\not{h} = 0$), we obtain by exterior differentiation that

$$0 = DH \wedge \omega.$$

By the Cartan lemma this implies that

$$DH = \nabla H \cdot \omega$$

where

$$\nabla H = H^{\mu}_{ijk} e_{\mu} \otimes x^i x^j x^k$$

is a $W \otimes S^3V^*$-valued function on $\mathscr{F}(M)$ (it is the symmetry of $\nabla H$ in its lower indices that is important). Using this equation together with (1.14) we may write (1.45) as

$$2\gamma(H, \nabla H) = \nabla R. \qquad (1.47)$$

At this point, we insert a few remarks about the relations among the higher co-variant derivatives of the tensors $R$ and $H$. These relationships will become a key point in our study of the overdetermined isometric imbedding problem.

Just as we can construct new vector spaces from $W$ and $V$ by taking tensor products, duals and invariant subspaces, the associated bundle construction shows that for each such vector space $\hat{P}$ we can construct a corresponding vector bundle $P$ over $M$ (using $\mathscr{F}(M)$ as the principal $O(W) \otimes (V)$-bundle). The connection on $\mathscr{F}(M)$ naturally induces a connection $\nabla_P: C^\infty(P) \to C^\infty(P \otimes T^*)$. This family of connections commutes with all bundle maps $P \to Q$ induced from corresponding invariant vector space maps $\hat{P} \to \hat{Q}$. For this reason, we often omit the subscript on $\nabla$.

For each section of $C^\infty(P)$, we have an $O(W) \times O(V)$-equivariant function $\mathscr{F}(M) \to \hat{P}$. Conversely, given an $O(W) \times O(V)$-equivariant $\hat{P}$-valued function on $\mathscr{F}(M)$, we may construct a section of $C^\infty(P)$. For this reason, we need not make a distinction between the two concepts.

The relation with covariant differentiation is given as follows: If $\sigma \in C^\infty(P)$

and $\hat{\sigma} : \mathscr{F}(M) \to \hat{P}$ is the corresponding function, we have

$$d\hat{\sigma} = -(\psi \oplus \kappa) \cdot \hat{\sigma} + \nabla \hat{\sigma} \omega. \tag{1.48}$$

The notation $x \cdot p \in \hat{P}$ for $x \in so(V) \oplus so(W)$ and $p \in \hat{P}$ denotes the natural Lie algebra action induced by the representation of $O(V) \times O(W)$ on $\hat{P}$. When there is no danger of confusion, we omit the circumflexes.

Iteration of $\nabla$ then gives rise to a series of differential operators $\nabla^q$ on $\mathscr{F}(M)$ taking equivariant functions $\mathscr{F}(M) \to \hat{P}$ to equivariant functions $\mathscr{F}(M) \to \hat{P} \otimes \bigotimes^q V^*$. There is also a symmetrized operator $\nabla^{(q)}$ with values in $\hat{P} \otimes S^q V^*$ obtained by using the canonical projections $\bigotimes^q V^* \to S^q V^*$. Clearly, both $\nabla^q$ and $\nabla^{(q)}$ are linear over the constants.

In particular, the Gauss equations yield a $K$-valued equivariant function $\sigma = R - \gamma(H,H)$ which is identically zero, hence we have

$$\nabla^{(q)}(R - \gamma(H,H)) \equiv 0. \tag{1.49}$$

Applying the Leibnitz rule, we see that

$$\nabla^{(q)}R = 2\gamma(H, \nabla^{(q)}H) + \{\text{terms involving } \nabla^{(\kappa)}H \text{ with } \kappa < q\}. \tag{1.50}$$

At this point, it is important to remark on the ranges of these operators. For example, we have already seen that $\nabla^{(1)}H = \nabla H$ takes values in $W \otimes S^3V^*$, a proper subspace of $(W \otimes S^2V^*) \otimes V^*$. It follows that $\nabla^{(q)}H$ takes values in $(W \otimes S^3V^*) \otimes S^{q-2}V^* \cap (W \otimes S^2V^*) \otimes S^q(V^*)$, i.e., that $\nabla^{(q)}H$ takes values in $W \otimes S^{q+2}(V^*)$.

In addition, $\nabla R$ takes values in the proper subspace $K^{(1)}$ of $K \otimes V^*$; hence $\nabla^{(q)}R$ takes values in the subspace

$$K^{(q)} = (K \otimes S^qV^*) \cap (K^{(1)} \otimes S^{q-1}V^*).$$

In §5, it will be shown that $K^{(q)}$ may also be characterized as the image of the map $\gamma$ defined in (1.41). Equation (1.50) will turn out to be a key point in our theory.

## 2. The isometric embedding system.

(a) *Setting up the system.* In this section we shall use terminology from the theory of exterior differential systems, and for this we have followed the notations and definitions used in [3] and [4].

(i) Let $(\tilde{M}, ds^2)$ be an abstract Riemannian manifold. We want to set up a differential system with independence condition whose admissible integral manifolds are in one-to-one correspondence with the (local) isometric embeddings

$$x : \tilde{M} \to \mathsf{E}^{n+r}. \tag{2.1}$$

In simplest terms, the PDE system for isometric embeddings is just

$$(dx, dx) = ds^2 \tag{2.2}$$

where the left hand side is the symmetric inner product of the differential of the map (2.1). This is indeed a 1st order PDE whose solutions correspond to isometric embeddings (2.1), and we could take as exterior differential system that given by considering (2.2) as such a system. However, we are certainly going to have to differentiate (2.2) since neither the curvature nor 2nd fundamental form appear explicitly in the equation. This will in turn necessitate introducing either local coordinates or an arbitrary choice of frame field. For our purposes it is more convenient to keep things entirely intrinsic by working on appropriate frame bundles.

(ii) Working backwards we first let

$$M^n \subset \mathsf{E}^{n+r}$$

be a submanifold and consider it as the image of an isometric embedding (2.1).

*Definition.* We define

$$\mathscr{F}(\tilde{M}, x) \subset \mathscr{F}(M) \times \mathscr{F}$$

to be the set of pairs of frames

$$\{(p; \tilde{e}_i); (x; e_i; e_\mu)\}$$

satisfying the conditions

$$\begin{cases} x(p) = x \\ x_*(\tilde{e}_i) = e_i . \end{cases} \tag{2.3}$$

We thus have a commutative diagram of maps

$$\begin{tikzcd} & \mathscr{F}(\tilde{M}, x) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathscr{F}(\tilde{M}) \arrow[d] & & \mathscr{F}(M) \arrow[ll, dashed, "\pi"'] \arrow[d] \\ \tilde{M} \arrow[rr, "x"] & & M \subset \mathsf{E}^{n+r} \end{tikzcd} \tag{2.4}$$

where

$$\begin{cases} \pi_1\{(p; \tilde{e}_i), (x; e_i; e_\mu)\} = (p; \tilde{e}_i) \\ \pi_2\{(p; \tilde{e}_i), (x; e_i; e_\mu)\} = (x; e_i; e_\mu) \\ \pi \cdot (x; e_i; e_\mu) = (p, e_i) \qquad \text{where} \quad x(p) = x \quad \text{and} \quad x_*(\tilde{e}_i) = e_i . \end{cases} \tag{2.5}$$

For $\eta$ a differential form on $\mathscr{F}(\tilde{M})$, we shall denote again by $\eta$ the form

$$\pi_1^*(\eta),$$

and similarly for forms on $\mathscr{F}$. We thus have

$$\begin{cases} \omega^i - \tilde{\omega}^i = 0 \\ \theta^\mu = 0 \\ \chi = \bigwedge_i \omega^i \bigwedge_{j<k} \psi_k^j \bigwedge_{\mu<\nu} \kappa_\nu^\mu \neq 0. \end{cases} \tag{2.6}$$

Without indices we abbreviate (2.6) as

$$\begin{cases} \omega - \tilde{\omega} = 0 \\ \theta = 0 \\ \chi = \omega \wedge \psi \wedge \kappa \neq 0. \end{cases} \tag{2.7}$$

We may consider (2.7) as a differential system with independence condition $(I_0, \chi)$ on the manifold

$$X_0 = \mathscr{F}(\tilde{M}) \times \mathscr{F}, \tag{2.8}$$

and it is essentially clear that its admissible integral manifolds are locally of the form $\mathscr{F}(\tilde{M}, x)$ for an isometric embedding (2.50); this will be made precise below following our discussion of Cauchy characteristics (cf. (2.26)).

Using (1.7) and (1.27) the exterior derivatives of the equations (2.7) are

$$\begin{cases} d(\omega - \tilde{\omega}) \equiv -(\psi - \tilde{\psi}) \wedge \omega & \operatorname{mod}\{\omega - \tilde{\omega}, \theta\} \\ d\theta \equiv \hbar \wedge \omega & \operatorname{mod}\{\omega - \tilde{\omega}, \theta\} \end{cases} \tag{2.9}$$

and the symbol relations are

$$(\psi - \tilde{\psi}) + {}^t(\psi - \tilde{\psi}) = 0. \tag{2.10}$$

Using these equations we shall show that $(I_0, \chi)$ fails to be involutive (this reflects the fact that neither the curvature nor 2nd fundamental form has yet appeared). For this we remark that even though the independence conditions contain $N = n + n(n-1)/2 + r(r-1)/2$ linear differential forms, only the forms $\omega^1, \ldots, \omega^n$ are relevant in checking for involution. This is clear from (2.9) and the reason for it will be explained below.

In the following, then, we shall simply test for involutivity of a differential system with structure equation (2.9), (2.10) and independence condition $\omega \neq 0$. Setting $\sigma = (\psi - \tilde{\psi})$ the matrix used in Cartan's test for involution is

$$\begin{Vmatrix} \sigma_1^1 & \cdots & \sigma_n^1 \\ \vdots & & \vdots \\ \sigma_1^n & & \sigma_n^n \\ \hbar_1^1 & & \hbar_n^1 \\ \vdots & & \vdots \\ \hbar_1^r & & \hbar_n^r \end{Vmatrix}$$

Since the symbol relations (2.10) are just $\sigma_j^i + \sigma_i^j = 0$ we have for the Cartan characters

$$\begin{cases} s_1' = n-1+r \\ s_2' = n-2+r \\ \vdots \\ s_n' = r. \end{cases}$$

Thus

$$s_1' + 2a_2' + \cdots + ns_n' = (n+3r-1)n(n+1)/6. \tag{2.11}$$

On the other hand, admissible integral elements are given by linear equations

$$\begin{cases} \psi_j^i - \tilde{\psi}_j^i - C_{jk}^i\omega^k = 0 \\ \quad \hbar_i^\mu - H_{ij}^\mu\omega^j = 0 \end{cases} \tag{2.12}$$

where

$$C_{jk}^i = C_{kj}^k, \quad H_{ij}^\mu = H_{ji}^\mu, \quad \text{and by (2.10)} \quad C_{jk}^i + C_{ik}^j = 0.$$

It follows from the standard reasoning (1.2) that

$$C_{jk}^i = 0 \tag{2.13}$$

while $H = (H_{ij}^\mu e_\mu \otimes \omega^i\omega^j)$ is an arbitrary element of $W \otimes S^2V^*$. Hence the space of admissible integral elements lying over any point has dimension $rn(n+1)/2$, and by (2.11) the systems fails to be involutive when $n \geqq 2$.

(iii) According to the general theory we must prolong $(I_0, \chi)$ to obtain a new Pfaffian system $(I_0^{(1)}, \chi)$ on the manifold $X_0^{(1)}$ of admissible integral elements of $(I_0, \chi)$. According to (2.8) and (2.12), (2.13) we see that

$$X_0^{(1)} = \mathscr{F}(\tilde{M}) \times \mathscr{F} \times (W \otimes S^2V^*).$$

The Pfaffian system $(I_0^{(1)}, \chi)$ is generated by the Pfaffian equations

$$\begin{cases} \omega - \tilde{\omega} = 0 \\ \quad \theta = 0 \\ \psi - \tilde{\psi} = 0 \\ \hbar - H\omega = 0 \end{cases} \tag{2.14}$$

with the same independence condition $\chi = \omega \wedge \psi \wedge \kappa \neq 0$. Using (1.15) and

(1.27) the exterior derivatives of (2.14) are

$$\begin{cases} d(\omega - \tilde{\omega}) \equiv 0 \\ d\theta \equiv 0 \\ d(\psi - \tilde{\psi}) \equiv \frac{1}{2}\{\gamma(H,H) - R\}\omega \wedge \omega \\ d(\not{h} - H\omega) \equiv -DH \wedge \omega \end{cases} \tag{2.15}$$

where all congruences are modulo the exterior ideal generated by $\{\omega - \tilde{\omega}, \theta, \psi - \tilde{\psi}, \not{h} - H\omega\}$. We remark that in the 3rd equation we are using $\not{h} \equiv H\omega$ and the notation (1.36) (cf. also (1.38)), and in the last equation we are using the notation (1.45). The 1-form $DH$ is defined on $X_0^{(1)} = \mathscr{F}(\tilde{M}) \times \mathscr{F} \times (W \otimes S^2V^*)$ and has values in $W \otimes S^2V^*$ (i.e., $(DH)_{ij}^{\mu} = (DH)_{ji}^{\mu}$); aside from this symmetry there are no other symbol relations in (2.15). Using this observation it is easy to verify that (2.14) is a Pfaffian system in dual good form whose tableau is always involutive.

However, the system $(I_0^{(1)}, \chi)$ itself is not involutive because, due to the 3rd equation, the torsion is nonzero. Annihilating the torsion exactly forces the Gauss-equations (1.37) to hold, and it is at this point that the geometry at last appears.

Namely, on $X_0^{(1)}$ we consider the locus

$$\gamma(H,H) = R \tag{2.16}$$

(recall that both sides are $K$-valued functions on $X_0^{(1)}$), and assuming that the $C^\infty$ equations (2.16) contain solutions that are smooth as a submanifold of $X_0^{(1)}$ we give the

*Definitions.* (i) We let $X \subset X_0^{(1)}$ be any locally closed submanifold that is an open subset of the solutions to (2.16) and on which the independence condition $\chi = \omega \wedge \psi \wedge \kappa \neq 0$ is valid;

(ii) The *isometric embedding system* $(I, \chi)$ is the restriction to $X$ of the system $(I_0^{(1)}, \chi)$.

By (2.14) and (2.15) the isometric embedding system may be written as

$$\begin{cases} \omega - \tilde{\omega} = 0 \\ \theta = 0 \\ \psi - \tilde{\psi} = 0 \\ \not{h} - H\omega = 0 \\ \left.\begin{array}{l} d(\omega - \tilde{\omega}) \equiv 0 \\ d\theta \equiv 0 \\ d(\psi - \tilde{\psi}) \equiv 0 \\ d(\not{h} - H\omega) \equiv -\pi \wedge \omega \end{array}\right\} \quad \operatorname{mod}\{\omega - \tilde{\omega}, \theta, \psi - \tilde{\psi}, \not{h} - H\omega\} \end{cases} \tag{2.17}$$

where the relations (2.16) hold and where $\pi = DH|_X$ is a $W \otimes S^2V^*$-valued 1-form. The symbol relations of (2.17) are obtained by exterior differentiation of (2.16), and by (1.46), (1.47) they are

$$2\gamma(H,\pi) \equiv \nabla R\omega \qquad \operatorname{mod}\{\omega - \tilde{\omega}, \theta, \psi - \tilde{\psi}, h - H\omega\}. \tag{2.18}$$

In summary:

*The isometric embedding system* $(I,\omega)$ *is defined on a smooth open subset of the solutions to the Gauss equations* (2.16), *and its symbol is given by the Codazzi equations* (2.18).

Since (2.17) is our main object of interest it may be useful to write it out in indices. Setting $I = \{\omega - \tilde{\omega}, \theta, \psi - \tilde{\psi}, h - H\omega\}$ this is

$$\begin{cases} \omega^i - \tilde{\omega}^i = \theta^\mu = \psi^i_j - \tilde{\psi}^i_j = 0 & \\ h^\mu_i - H^\mu_{ij}\omega^j = 0, \qquad H^\mu_{ij} = H^\mu_{ji} & \\ d(\omega^i - \tilde{\omega}^i) \equiv d\theta^\mu \equiv d(\psi^i_j - \tilde{\psi}^i_j) \equiv 0 & \operatorname{mod} I \\ d(h^\mu_i - H^\mu_{ij}\omega^j) \equiv -\pi^\mu_{ij} \wedge \omega^j & \operatorname{mod} I, \quad \pi^\mu_{ij} = \pi^\mu_{ji} \\ \gamma(H,\pi)_{ijkl} \equiv R_{ijkl,m}\omega^m & \operatorname{mod} I, \end{cases} \tag{2.19}$$

where the last equation constitutes the symbol relations (2.18) (cf. (1.43) for the definition of the left hand side).

(iv) We shall now give some properties of the isometric embedding system. The terminology we use is taken from [3].

$$(I,\chi) \textit{ is a Pfaffian system in dual good form.} \tag{2.20}$$

In fact, this is true of the 1st prolongation of any differential system. In our case, however, more is true. Namely, if using (2.19) we make the correspondence in notation

$$\begin{cases} \omega^i - \tilde{\omega}^i, \psi^i_j - \tilde{\psi}^i_j \leftrightarrow \theta^\alpha \\ \quad h^\mu_i - H^\mu_{ij}\omega^j \leftrightarrow \theta^\alpha_i \end{cases} \tag{2.21}$$

then $(I,\chi)$ is a Pfaffian system of the special form given in §4 of [4] (this means that it formally looks like the differential system arising from a 2nd order PDE system).

> $(I,\chi)$ *is embeddable, so that it is locally equivalent to a* 1*st order P.D.E. system. However, it is* not *locally equivalent to a* 2*nd order P.D.E. system.* (2.22)

The reason for the first assertion is that *any* 1st prolongation is locally embeddable and is therefore locally equivalent to a 1st order PDE system.

However, since the integrability condition $d\omega^i \equiv 0 \bmod\{\theta^\alpha, \omega^i\}$ (and not just $d\omega^i \equiv 0 \bmod\{\theta^\alpha, \theta_i^\alpha, \omega^i\}$) *fails* to hold, we may infer the second assertion in (2.22).

(2.23) The *Cauchy characteristic system* ([3])

$$A(I) \subset T(X)$$

*of* $(I, \omega)$ *is given by*

$$A(I) = \operatorname{span}\{\omega - \tilde{\omega}, \theta, \psi - \tilde{\psi}, \hbar - H\omega, \pi, \omega\}^{\perp}.$$

It follows that $A(I)$ is a sub-bundle of $T(X)$ of fibre dimension $n(n-1)/2 + r(r-1)/2$ that is coframed by the 1-forms

$$\tfrac{1}{2}(\psi_j^i + \tilde{\psi}_j^i), \kappa_\mu^\nu$$

(intuitively, $A(I)$ is generated by the vector fields

$$\tfrac{1}{2}(\partial/\partial\psi_j^i + \partial/\partial\tilde{\psi}_j^i), \partial/\partial\kappa_\mu^\nu). \tag{2.24}$$

It is well known (loc. cit.) that the sub-bundle $A(I)$ is completely integrable, and since our independence condition is $\chi = \Lambda_i \omega^i \Lambda_{i<j} \psi_j^i \Lambda_{\nu<\mu} \kappa_\nu^\mu \neq 0$ it follows that the $N = n + n(n-1)/2 + r(r-1)/2$-dimensional admissible integral manifolds of $(I, \chi)$ are foliated by the $n(n-1)/2 + r(r-1)/2$-dimensional leaves of the Cauchy characteristic system.

In fact, this has a simple geometric interpretation. By (2.24) the leaves of the Cauchy characteristic foliation are isomorphic to the product $O(n) \times O(r)$ of orthogonal groups and correspond to spinning the normal frame $\{e_\mu\}$ and to spinning the tangent frames $\{\tilde{e}_i\}, \{e_i\}$ *at the same rate* (i.e., by the same element of $O(n)$). If $N \subset X$ is any admissible integral manifold of $(I, \omega)$ then we may enlarge $N$ by adding on all Cauchy characteristic leaves passing through points of $N$ (i.e., we spin the tangential and normal frames as above). Assuming this has been done we have a diagram

$$\begin{array}{ccc} N & \hookrightarrow & X \subset \mathscr{F}(\tilde{M}) \times \mathscr{F} \times (W \otimes S^2 V^*) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \Gamma & \longrightarrow & \tilde{M} \times \mathsf{E}^{n+r} \end{array} \tag{2.25}$$

where $\pi : N \to \Gamma$ is the fibering by Cauchy characteristics. From the geometric picture it is clear that there is an induced map $\Gamma \to M \times \mathsf{E}^{n+r}$ such that the image is locally the graph of an isometric embedding $x : \tilde{M} \to \mathsf{E}^{n+r}$. Then $N = \mathscr{F}(\tilde{M}, x)$ for this embedding, and we have shown that

*The admissible integral manifolds of* $(I, \chi)$ *give local isometric embeddings* (2.1). (2.26)

Moreover, the converse of (2.26) is valid in the sense that, given an isometric embedding $x : \tilde{M}^n \to \mathsf{E}^{n+r}$, the adapted frame bundle $\mathscr{F}(\tilde{M}, x) \subset \mathscr{F}(\tilde{M}) \times \mathscr{F}$ is an admissible integral manifold of $(I, \chi)$.

(b) *Proof of the Burstin–Cartan–Janet–Schaefly (BCJS) theorem.* We will give a proof of the following classical

BCJS THEOREM. *Let $(\tilde{M}, ds^2)$ be a real analytic Riemannian manifold. Then given any point $p \in \tilde{M}$ there exists a neighborhood (still denoted by $\tilde{M}$) and real analytic isometric embedding*

$$x : \tilde{M} \to \mathsf{E}^{n(n+1)/2}.$$

*Moreover, $x$ depends on $n$ functions of $n-1$ variables.*

Our proof will consist in showing that the isometric embedding system $(I, \omega)$ is involutive (according to the definition using Cartan's test) and then applying the Cartan–Kähler theorem. In principle, our argument is similar to the original one of Cartan [6] (cf. also [16]), and of course is also roughly the same as the recent proofs [9] and [13]. However, by using the exterior differential system $(I, \chi)$ the solution of the Gauss equations and computation of the Cartan characters may be simpler.

We remark that, for the obvious reason, we shall call $n(n+1)/2$ the *embedding dimension* and $n(n-1)/2 = (n(n+1)/2) - n = n(n-1)/2$ the *embedding codimension*. Our proof will show that the isometric embedding system for $x : \tilde{M} \to \mathsf{E}^{n+r}$ is involutive when $r \geq n(n-1)/2$ but cannot be involutive for general $\tilde{M}$ when $r < n(n-1)/2$. (The meaning of "general" will be clarified below.) Moreover, the argument may be trivially modified to cover the case where $\mathsf{E}^{n+r}$ is replaced by any real analytic Riemannian manifold of the same dimension.

(i) We recall the definitions of the spaces $K \subset S^2(\Lambda^2 V^*)$ of curvature-like tensors and $W \otimes S^2 V^*$ of algebraic 2nd fundamental forms, and of the Gauss map

$$\gamma : W \otimes S^2 V^* \to K \tag{2.27}$$

(cf. §1(a) and (1.36)–(1.38)). The differential of (2.27) at a point $H \in W \otimes S^2 V^*$ is the map

$$d\gamma(H) : W \otimes S^2 V^* \to K \tag{2.28}$$

given for $G \in W \otimes S^2 V^*$ by

$$d\gamma(H)(G) = 2\gamma(H, G)$$

(cf. (1.36) and (1.38)). We recall that the mapping $\gamma$ is *submersive* at $H$ if (2.28) is surjective.

*Definitions.* (i) $H \in W \otimes S^2V^*$ is *ordinary* if there exists a basis $v_1, \ldots, v_n$ for $V$ such that the vectors

$$H_{\sigma\rho} = H(v_\rho, v_\sigma) \in W, \qquad 1 \le \rho \leqq \sigma \le n-1$$

are linearly independent. (In particular, this requires that $\dim W = r \ge n(n-1)/2$, which we assume to be the case.)

(ii) We denote by $\mathscr{H} \subset W \otimes S^2V^*$ the dense Zariski open set of ordinary algebraic 2nd fundamental forms.

The main computational step in the proof of the BCJS theorem is the following

(2.29) PROPOSITION. (i) $\gamma : \mathscr{H} \to K$ *is everywhere submersive, and* (ii) $\gamma : \mathscr{H} \to K$ *is surjective.*

*Remark.* This result is the main step in the classical proofs of the BCJS theorem using exterior differential systems; our proof is somewhat different. This result will be considerably sharpened in §5(h) below.

*Proof of* (i). Letting $\dim W = r \geqq n(n-1)/2$ we have by (1.12)

$$\dim K = n^2(n^2-1)/12$$

$$\dim W \otimes S^2V^* = rn(n+1)/2.$$

Using the notation

$$\gamma_H(G) = 2\gamma(H, G),$$

by (2.28) it will suffice to show that

$$\dim \ker \gamma_H \leqq rn(n+1)/2 - n^2(n^2-1)/12 \tag{2.30}$$

for $H \in \mathscr{H} \subset W \otimes S^2V^*$. Let $v_1, \ldots, v_n \in V$ be the basis in the definition of ordinary for $H$ and consider the mappings

$$\begin{cases} \mu_1 : \ker \gamma_H \to \underbrace{W \oplus \cdots \oplus W}_{n} \\ \qquad \vdots \\ \mu_p : \ker \gamma_H \to \underbrace{W \oplus \cdots \oplus W}_{n + (n-1) + \cdots + (n-p+1)} \\ \qquad \vdots \end{cases}$$

defined for $2 \leq p \leq n$ by

$$\begin{cases} \mu_1(G) = (G_{11}, G_{12}, \ldots, G_{1n}) \\ \quad \vdots \\ \mu_p(G) = \mu_{p-1}(G) \oplus (G_{pp}, G_{p,p+1}, \ldots, G_{pn}) \\ \quad \vdots \end{cases}$$

Since $G$ is symmetric it is clear that $\mu_n$ is injective; i.e.,

$$\dim \ker \gamma_H = \dim \operatorname{im} \mu_n . \tag{2.31}$$

We define $t_1, t_2, \ldots, t_n$ by

$$t_1 + \cdots + t_p = \dim \operatorname{im} \mu_p .$$

By (2.30) and (2.31) we must show that

$$t_1 + \cdots + t_n \leqq rn(n+1)/2 - n^2(n^2-1)/12. \tag{2.32}$$

By (2.36) (or (2.38)) the condition $G \in \ker \gamma_H$ is

$$H_{ik} \cdot G_{jl} + H_{jl} \cdot G_{ik} - H_{il} \cdot G_{jk} - H_{jk} \cdot G_{il} = 0$$

for all $i, j, k, l$. Among these relations are the following two sets that express the components of $G_{pl}$, $p \leqq l \leqq n$, in terms of the $G_{ij}$, $1 \leqq i, j \leqq p-1$

$$H_{ik} \cdot G_{pl} = H_{il} \cdot G_{pk} + H_{pk} \cdot G_{il} - H_{pl} \cdot G_{ik} \tag{2.33}$$

where $1 \leqq i \leqq k \leqq p-1$, $l \geqq p$, and

$$H_{ik} \cdot G_{pl} - H_{il} \cdot G_{pk} = -H_{pl} \cdot G_{ik} + H_{pk} \cdot G_{il} \tag{2.34}$$

where $i < p \leqq k < l$.

Since there are $p(p-1)/2$ pairs $(i,k)$ with $1 \leqq i \leqq k \leqq p-1$ and $n-p+1$ choices of $l$ with $p \leqq l \leqq n$, and since by the assumption that $H$ be ordinary the vectors $H_{ik} \in W$ are independent, there are

$$(p(p-1)/2)(n-p+1)$$

independent relations (2.33).

Since there are $p-1$ choices for $i < p$ and $(n-p+1)(n-p)/2$ choices for pairs $(k,l)$ with $p \leqq k < l \leqq n$, it again follows from the assumption that $H$ be ordinary that there are

$$(p-1)((n-p+1)(n-p)/2)$$

independent relations (2.34).

Having fixed $\mu_{p-1}(G)$ (i.e., the $G_{ij}$ for $1 \leqq i, j \leqq p-1$), it follows that there are at most

$$r(n-p+1)-\left[(p(p-1)/2)(n-p+1)+(p-1)((n-p+1)(n-p)/2)\right] \tag{2.35}$$

choices for the component of the vectors $G_{pl}$, $p \leqq l \leqq n$. (The "at most" is because there may be additional equations to (2.33) and (2.34).) Noting that the two numbers in the brackets in (2.35) add up to

$$n(n-p+1)(p-1)/2$$

it follows immediately that (2.35) is equal to $(n-p+1)(r-n(p-1)/2)$. Thus

$$t_p \leqq (n-p+1)(r-n(p-1)/2). \tag{2.36}$$

Summing we find by an elementary calculation that

$$t_1+\cdots+t_n \leqq rn(n+1)/2-n^2(n^2-1)/12,$$

which is (2.32).

*Proof of* (ii). To prove that $\gamma(\mathscr{H})=K$ it suffices by homogeneity to show that there is an $H \in \mathscr{H}$ with

$$\gamma(H,H)=(0). \tag{2.37}$$

In fact, by part (i) the image $\gamma(\mathscr{H})$ will contain a neighborhood of $(0) \in K$, and then since $\gamma(\lambda H, \lambda H)=\lambda^2\gamma(H,H)$ we must have $\gamma(\mathscr{H})=K$.

We then choose elements $H_{ij}=H_{ji} \in W$, $1 \leqq i, j \leqq n-1$, so that

$$\begin{cases} H_{ii} \cdot H_{ii} = \alpha & \\ H_{ii} \cdot H_{jj} = H_{ij} \cdot H_{ij} = 1, & i \neq j \\ H_{ij} \cdot H_{kl} = 0 & \text{otherwise,} \end{cases} \tag{2.38}$$

and we set $H_{in}=0$ for $1 \leqq i \leqq n$. So long as $\dim W = r \geqq n(n-1)/2$ and $\alpha > 1$, it is possible to choose vectors so that (2.38) is satisfied. It is clear that $H$ is ordinary, and by (1.36) we have (2.37). Q.E.D.

(ii) Referring to the notation and discussion in §1(a) we set

$$Y=\mathscr{F}(\tilde{M}) \times \mathscr{F} \times \mathscr{H},$$

and in $Y$ we consider the locus (cf. (2.16))

$$X=\{\gamma(H,H)=R\}.$$

A point of $X$ thus consists of a tangent frame $(p;\tilde{e}_i)$ to $\tilde{M}$, a frame $(x;e_i,e_\mu)$ in $\mathsf{E}^{n(n+1)/2}$, and an *ordinary* algebraic 2nd fundamental form $H \in W \otimes S^2V^*$ such that the Gauss equations $\gamma(H,H)=R(p)$ are satisfied. It follows from proposition (2.29) that $X$ is a smooth submanifold of codimension $n^2(n^2-1)/12$ in $Y$ and that the projection

$$X \to \mathscr{F}(\tilde{M}) \times \mathscr{F}$$

is submersive. We therefore may consider the isometric embedding system $(I,\chi)$ (cf. (2.17) or, in indices, (2.19)) on $X$.

(2.39) PROPOSITION. *The system $(I,\chi)$ is involutive.*

*Proof.* We shall apply Cartan's test as given in [3]. We let $v_1,\dots,v_n \in V$ be a basis such that the vectors $H_{ij}=H(v_i,v_j)$, $1 \leqq i,j \leqq n-1$ are linearly independent in $W$. In computing the reduced Cartan characters $s_1',s_2',\dots,s_n'$ only the last equation in (2.17) and symbol relation (2.18) are relevant (in terms of indices, these are the last two equations in (2.19)). Setting $L=\operatorname{span}\{\omega-\tilde{\omega},\theta,\psi-\tilde{\psi},\mathscr{h}-H\omega\}$ we write these as

$$d(\mathscr{h}-H\omega)_i \equiv \pi_{ij}\wedge\omega^j \qquad \operatorname{mod} L \tag{2.40}$$

$$\gamma(H,\pi) \equiv \tfrac{1}{2}\nabla R\omega \qquad \operatorname{mod} L \tag{2.41}$$

where $\pi_{ij}=\pi_{ji}$ is a $W$-valued 1-form. The matrix used to compute the $s_p'$ is

$$\begin{Vmatrix} \pi_{11} & \cdots & \pi_{1n} \\ \vdots & & \vdots \\ \pi_{n1} & \cdots & \pi_{nn} \end{Vmatrix}$$

and $s_1'+\cdots+s_p'$ is the number of independent 1-forms in the first $p$ columns. If we use the symmetry $\pi_{ij}=\pi_{ji}$ and set $\dim W=r$, then there are at most

$$r(n+(n-1)+\cdots+(n-p+1))$$

independent 1-forms $\pi_{ij}$, $1 \leqq i \leqq n$ and $1 \leqq j \leqq p$. However, we must take into account the symbol relations (2.41). The rank here is the same as the rank of the corresponding homogeneous equations

$$\gamma(H,\pi) \equiv 0 \qquad \operatorname{mod} L, \tag{2.42}$$

and referring to proof of (i) in proposition (2.29) it follows that

$$s_1'+\cdots+s_p' = t_1+\cdots+t_p.$$

Thus $s_p'=t_p$ for all $p$, and by (2.36) we obtain by an elementary calculation that

$$s_1'+2s_2'+\cdots+ns_n' \leqq r\left(\frac{n(n+1)(n+2)}{6}\right)-\frac{n^2(n^2-1)(n+2)}{24}. \tag{2.43}$$

On the other hand, by Cartan's test the space $V_x(I,\chi)$ of admissible integral elements lying over $x \in X$ satisfies

$$\dim V_x(I,\chi) \leqq s_1' + 2s_1' + \cdots + ns_n', \tag{2.44}$$

with equality holding (for all $x \in X$) if, and only if, the differential system $(I,\chi)$ is involutive. Since the admissible integral elements are given by linear equations

$$\pi_{ij} - H_{ijk}^{(1)}\omega^k = 0$$

where $H^{(1)} \in W \otimes S^3V^*$ satisfies

$$\gamma(H, H^{(1)}) = \nabla R,$$

it is clear that

$$\dim W \otimes S^3V^* - \dim K^{(1)} \leqq \dim V_x(I,\chi). \tag{2.45}$$

If we show that

$$\begin{aligned} \dim W \otimes S^3V^* &= r\left(\frac{n(n+1)(n+2)}{6}\right) \\ \dim K^{(1)} &= \frac{n^2(n^2-1)(n+2)}{24}, \end{aligned} \tag{2.46}$$

then the inequalities (2.43)–(2.45) must all be equalities, from which we conclude first that $V_x(I,\chi)$ is nonempty (i.e., the torsion of the system $(I,\omega)$ is zero) and that the tableau of $(I,\chi)$ is involutive. This will complete the proof of (2.39).

Now the first equation in (2.46) is clear. The second follows from the exactness of the sequence

$$0 \to K^1 \to K \otimes V^* \to \Lambda^2V^* \otimes \Lambda^3V^* \to V^* \otimes \Lambda^4V^* \to 0 \tag{2.47}$$

and the formula (1.12).

*Remark.* The best way to compute the dimension of all the spaces $K^{(q)}$ is by representation theory; this will be done in §5 below.

## 3. Localization of the Gauss equations.

(a) *Proof of* (i) *and* (ii) *in the Main Theorem.* In this section we will complete the proofs of parts (i) and (ii) in our Main Theorem. The idea is to apply Theorem IV from [4] to the isometric embedding system discussed above. What makes this feasible is the following remarkable fact:

(3.1) *Even though the Gauss equations* (*cf.* (1.36)–(1.39)) *are of a complicated quadratic character, their algebraic localizations are quite simple.*

In fact, we may go even further and say that very many of the nonobvious properties of the Gauss equations and Riemann curvature tensor (e.g., the two Bianchi identities and the fact that these generate all symmetries on the curvature and its covariant derivatives) became quite transparent when localized in the sense of algebra.

In this discussion we will use the following notations: $V$ is an $n$-dimensional real vector space (it is important to note that we will *not* use a metric on $V$);[1] $W$ is an $r$-dimensional Euclidean vector space with inner product $w \cdot w'$; with the natural identification

$$W \otimes S^2V^* = \operatorname{Hom}(S^2V, W) \tag{3.2}$$

we let $H \in W \otimes S^2V^*$ be a fixed element; $K \subset S^2(\Lambda^2V^*)$ is the space of curvature-like tensors (cf. (1.11) and the discussion just above this equation, and recall that the definition of $K$ also does not require a metric on $V$);

$$\gamma_H : W \otimes S^2V^* \to K \tag{3.3}$$

is the mapping defined by (cf. (1.36))

$$\gamma_H(G) = \gamma(H, G)$$

(this requires a metric on $W$ but not on $V$); $V_{\mathbf{C}}$, $W_{\mathbf{C}}$, and $K_{\mathbf{C}}$ are the complexified vector spaces, and the Euclidean inner product on $W$ is uniquely extended to a complex symmetric bilinear form; for a point $\xi \in \mathbf{P}V_{\mathbf{C}}^*$, we denote by

$$L_\xi \subset V_{\mathbf{C}}^*$$

the corresponding line and define

$$\gamma_{H,\xi} : W_{\mathbf{C}} \otimes S^2L_\xi \to K_{\mathbf{C}}$$

by

$$\gamma_{H,\xi}(w \otimes \eta^2) = \gamma_H(w \otimes \eta^2)$$

where $w \in W_{\mathbf{C}}$ and $\eta \in L_\xi$; and finally $\Sigma_{\mathbf{C}} \subset \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$ will be a *proper* subvariety to be defined below. It will turn out that $\Sigma_{\mathbf{C}}$ is defined by *real*

[1] Once we have lowered indices so that the curvature tensor

$$R \in K \subset S^2(\Lambda^2V^*)$$

(cf. (1.10)), the Gauss equations use a metric in the normal space $W$ but do *not* use one in the cotangent space $V$. This turns out to be extremely important when we pass to the study of the complexified characteristic variety in $\mathbf{P}V_{\mathbf{C}}^*$ (otherwise the quadric in $\mathbf{P}V_{\mathbf{C}}^*$ of vectors of length zero would enter into our considerations).

*homogeneous* polynomials on $\mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}}) = \mathrm{Hom}(S^2V, W)_{\mathbf{C}}$. This implies that

$$\Sigma = \Sigma_{\mathbf{C}} \cap \mathrm{Hom}(S^2V, W)$$

is a proper subvariety.

*Definitions.* (i) The (complex) characteristic variety of $H$ is the subvariety $\Xi_{H,\mathbf{C}}$ of $\mathbf{P}V_{\mathbf{C}}^*$ defined by

$$\Xi_{H,\mathbf{C}} = \{\xi \in \mathbf{P}V_{\mathbf{C}}^* : \gamma_{H,\xi} \text{ is not injective}\}.$$

(ii) We say that $H \in \mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$ is *general* in case it does not lie in the proper subvariety $\Sigma_{\mathbf{C}}$.

*Remarks.* (i) For any differential system in dual good form the symbol mapping and characteristic variety are defined (cf. [4]). The above is just the definition of the characteristic variety for the isometric embedding system restricted to lie over the point $H$.

(ii) Strictly speaking the definition of general doesn't make sense since we have not said what the special subvariety $\Sigma_{\mathbf{C}}$ is; but we prefer to define $\Sigma_{\mathbf{C}}$ when it arises naturally during the proof of the following result.

THEOREM A. *If $H \in \mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$ is general and $r \leqq n(n-1)/2$, then*

$$\dim \Xi_{H,\mathbf{C}} = \max\{-1, r-1-(n-1)(n-2)/2\}.$$

*Remarks.* In particular,

$$\Xi_{H,\mathbf{C}} = \emptyset \qquad \text{if} \quad r \leqq (n-1)(n-2)/2$$

$$\Xi_{H,\mathbf{C}} = \text{isolated points} \qquad \text{if} \quad r = (n-1)(n-2)/2 + 1.$$

It is clear that (i) and (ii) in the Main Theorem follow from Theorem IV in [4] and the discussion in §2 above (cf. (2.17) and (2.18)).

*Proof of Theorem A.*[2] We shall first define a natural inclusion

$$j_\xi : S^2(V_{\mathbf{C}}^*/L_\xi) \otimes S^2(L_\xi) \hookrightarrow K_{\mathbf{C}}. \tag{3.4}$$

[2]Intuitively, the reason that the localized Gauss equations are so simple is the following. If we choose a basis $\omega^1, \ldots, \omega^n$ for $V_{\mathbf{C}}^*$ such that $\xi = [0, \ldots, 0, 1]$, then the localized Gauss mapping

$$\gamma_{H,\xi} : W_{\mathbf{C}} \otimes S^2L_\xi \to K_{\mathbf{C}}$$

is given, for

$$G = G_{nn}^\alpha w_\alpha \otimes (\omega^n)^2 \in W_{\mathbf{C}} \otimes S^2L_\xi,$$

*Definition of $j_\xi$.* There is a natural map

$$S^2(V_{\mathbf{C}}^*/L_\xi) \otimes S^2(L_\xi) \to S^2(\Lambda^2 V_{\mathbf{C}}^*) \tag{3.5}$$

given, for $\alpha, \beta \in V_{\mathbf{C}}^*/L_\xi$ and $\eta \in L_\xi$, by

$$(\alpha \circ \beta) \otimes \eta^2 \to (\alpha \wedge \eta) \circ (\beta \wedge \eta). \tag{3.6}$$

It is clear that this map is well defined. To give it in coordinates we let $\omega^1, \ldots, \omega^n$ be a basis for $V_{\mathbf{C}}^*$ such that $\xi = [0, \ldots, 0, 1]$, i.e., $\xi = \omega^n$ and use the additional index-range

$$1 \leqq \rho, \qquad \sigma \leqq n-1.$$

We may assume that $\eta = \omega^n$ and then, by (3.6), (3.5) is given in coordinates by

$$\sum_{\rho,\sigma} q_{\rho\sigma}\omega^\rho\omega^\sigma \otimes (\omega^n)^2 \to \sum_{\rho,\sigma} q_{\rho\sigma}\omega^\rho \wedge \omega^n \otimes \omega^\sigma \wedge \omega^n \tag{3.7}$$

where $q_{\rho\sigma} = q_{\sigma\rho}$ is a symmetric $(n-1) \times (n-1)$ matrix. To see that the image of (3.5) lies in the subspace $K_{\mathbf{C}} \subset S^2(\Lambda^2 V_{\mathbf{C}}^*)$, we have, using the formula for $\partial$ in §1 (cf. also (1.15))

$$\begin{aligned}\partial\Big(\sum_{\rho,\sigma} q_{\rho\sigma}\omega^\rho \wedge \omega^n \otimes \omega^\sigma \wedge \omega^n\Big) &= 2\sum_{\rho,\sigma} q_{\rho\sigma}\omega^\rho \otimes \omega^n \wedge \omega^\sigma \wedge \omega^n \\ &\quad - 2\sum q_{\rho\sigma}\omega^n \otimes \omega^\rho \wedge \omega^\sigma \wedge \omega^n \\ &= 0.\end{aligned}$$

(What is going on here is that the Bianchi identity $T_{ijkl} = 0$ involves a 3-term sum only for components of the curvature having *4* distinct indices.)

Next, for each $\xi \in \mathbf{P}V_{\mathbf{C}}^*$ we denote by

$$\xi^\perp \subset V_{\mathbf{C}}$$

---

by

$$\begin{aligned}\gamma_{H,\xi}(G) &= \gamma(H, G) \\ &= \sum_\alpha H_{ik}^\alpha G_{nn}^\alpha \omega^i \wedge \omega^n \otimes \omega^k \wedge \omega^n.\end{aligned}$$

In particular, (i) only the components $R_{\lambda n\mu n}$ $(1 \leq \lambda, \mu \leq n-1)$ are relevant (and, especially, the first Bianchi identity reduces to the simple symmetry $R_{\lambda n\mu n} = R_{\mu n\lambda n}$); and (ii) the 4-term sum in the global Gauss equations (1.36) (cf. also (1.38)) reduces to the single term

$$\sum_\alpha H_{\lambda\mu}^\alpha G_{nn}^\alpha \qquad (1 \leq \lambda, \mu \leq n-1). \tag{1}$$

It is clear that we may view (1) as giving a vector bundle mapping over $\mathbf{P}V_{\mathbf{C}}^*$, and that for general $H$ the fibre rank of this mapping may be determined.

the hyperplane orthogonal to the line $L_\xi \subset V_C^*$, so that there is an inclusion

$$S^2(\xi^\perp) \subset S^2 V_C .$$

Then, with the identification (3.2) we denote by

$$H_\xi : S^2(\xi^\perp) \to W_C \tag{3.8}$$

the restriction of $H \in \mathrm{Hom}(S^2 V_C, W_C)$.

Finally, using the natural identifications

$$(\xi^\perp)^* = V_C^*/L_\xi$$

$$S^2(\xi^\perp)^* = S^2(V_C^*/L_\xi)$$

we consider the diagram

$$\begin{tikzcd} W_C \otimes S^2(L_\xi) \arrow[rr, "\gamma_{H,\xi}"] \arrow[dr, "H_\xi^* \otimes 1"'] & & K_C \\ & S^2(V_C^*/L_\xi) \otimes S^2(L_\xi) \arrow[ur, "j_\xi"'] & \end{tikzcd} \tag{3.9}$$

where $H_\xi^*$ is the dual of (3.8) using the isomorphism

$$W_C \cong W_C^*$$

given by the Euclidean structure on $W$. We note that both $j_\xi$ and $H_\xi^* \otimes 1$ are standard simple linear algebra maps whereas $\gamma_{H,\xi}$ is the "localized Gauss equations". Accordingly, the main step in the proof of Theorem A is the following

(3.10) PROPOSITION. *The diagram* (3.9) *is commutative.*

*Proof.* It suffices to verify commutatively in some coordinate system. We choose a basis $\omega^1, \ldots, \omega^n$ for $V_C^*$ such that $\xi = [0, \ldots, 0, 1]$ with the dual basis $v_1, \ldots, v_n$ (thus $\xi^\perp = \mathrm{span}\{v_\rho\}$), and an orthonormal basis $w_1, \ldots, w_r$ for $W_C$. Using summation convention we write

$$H = H_{ij}^\mu w_\mu \otimes \omega^i \omega^j$$

and will evaluate

$$\gamma_{H,\xi}\big(w_\mu \otimes (\omega^n)^2\big)(v_i, v_j, v_k, v_l) = \gamma\big(H, w_\mu \otimes (\omega^n)^2\big)(v_i, v_j, v_k, v_l)$$

using (1.36). Since

$$K_C \subset \Lambda^2 V_C^* \otimes \Lambda^2 V_C^*$$

we may assume that $i < j$ and $k < l$. Then by (1.36)

$$\gamma_{H,\xi}\big(w_\mu \otimes (\omega^n)^2\big)(v_i, v_j, v_k, v_l) = \begin{cases} 0 & \text{unless} \quad j = l = n \\ H^\mu_{ik} & \text{when} \quad j = l = n \end{cases} \tag{3.11}$$

(in other words, the only potentially nonzero components in the range of $\gamma_{H,\xi}$ are $T_{\rho n \sigma n}$ where $1 \leqq \rho, \sigma \leqq n-1$). On the other hand, by the definition of the mapping (3.8)

$$(H^*_\xi \otimes 1)\big(w_\mu \otimes (\omega^n)^2\big) = H^\mu_{\rho\sigma}\omega^\rho\omega^\sigma \otimes (\omega^n)^2,$$

and thus by (3.7)

$$j_\xi\Big((H^*_\xi \otimes 1)\big(w_\mu \otimes (\omega^n)^2\big)\Big) = \sum H^\mu_{\rho\sigma}\omega^\rho \wedge \omega^n \otimes \omega^\sigma \wedge \omega^n. \tag{3.12}$$

Comparing (3.11) and (3.12) gives the proposition.

(3.13) COROLLARY. *The complex characteristic variety is given by*

$$\Xi_{H,\mathbf{C}} = \big\{\xi \in \mathbf{P}V^*_{\mathbf{C}} \textit{ such that } H_\xi : S^2(\xi^\perp) \to W_{\mathbf{C}} \textit{ fails to be surjective}\big\}.$$

*Proof.* Using the commutative diagram (3.9) and the fact that $j_\xi$ is an injection, we see that

$$\ker \gamma_{H,\xi} = \operatorname{coker} H_\xi \otimes 1, \tag{3.14}$$

which implies the corollary. Q.E.D.

We now complete the proof of Theorem A. For each $\xi \in \mathbf{P}V^*_{\mathbf{C}}$ we consider the restriction map

$$\operatorname{Hom}\big(S^2V_{\mathbf{C}}, W_{\mathbf{C}}\big) \to \operatorname{Hom}\big(S^2(\xi^\perp), W_{\mathbf{C}}\big). \tag{3.15}$$

Set $N = n(n-1)/2 = \dim S^2(\xi^\perp)$ and note that

$$N - (n-1) = (n-1)(n-2)/2. \tag{3.16}$$

Choosing bases we may think of $\operatorname{Hom}(S^2(\xi^\perp), W_{\mathbf{C}})$ as the space $\mathscr{M}_{N,r}$ of complex $N \times r$ matrices. It is well known that the subvariety

$$\mathscr{M}_{N,r,k} \subset \mathscr{M}_{N,r}$$

of matrices of rank $\leqq r-k$ has *co*dimension given by

$$\operatorname{codim} \mathscr{M}_{N,r,k} = k(N - r + 1). \tag{3.17}$$

For $k = 1$ this is

$$\operatorname{codim} \mathscr{M}_{N,r,1} = N - r + 1. \tag{3.18}$$

We let

$$\Theta_\xi \subset \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$$

be the inverse image of $\mathscr{M}_{N,r,1}$ under the map (3.15). Since this map is *surjective*

$$\operatorname{codim}\Theta_\xi = N - r + 1. \tag{3.19}$$

We note that

$$\Theta_\xi = \left\{ H \in \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}}) : H_\xi \text{ fails to be surjective} \right\}. \tag{3.20}$$

We first prove Theorem B when

$$r \leqq (n-1)(n-2)/2 = N - (n-1).$$

Then by (3.19)

$$\operatorname{codim}\Theta_\xi \geqq (n-1) + 1. \tag{3.21}$$

Since $\xi$ varies over $\mathbf{P}V_{\mathbf{C}}^* \cong \mathbf{P}^{n-1}$, the $\infty^{n-1}$ subvarieties $\Theta_\xi$ cannot fill out $\operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$. More precisely, if

$$\Sigma_{\mathbf{C}} = \left\{ H \in \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}}) : H_\xi \in \Theta_\xi \text{ for some } \xi \in \mathbf{P}V_{\mathbf{C}}^* \right\}$$

then (3.21) implies that

$$\operatorname{codim}\Sigma_{\mathbf{C}} \geqq 1.$$

By Corollary (3.13), this means that

$$\Xi_{H,\mathbf{C}} = \emptyset$$

for a general $H \in \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$, which is Theorem A in this case.

In the general case we define

$$\Theta = \bigcup_{\xi \in \mathbf{P}V_{\mathbf{C}}^*} \Theta_\xi \subset \mathbf{P}V_{\mathbf{C}}^* \times \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}}).$$

Since it is a family of varieties parametrized by $\mathbf{P}V_{\mathbf{C}}^*$, $\Theta$ is an algebraic variety[3]

[3] We will not prove this more or less obvious fact, but remark that

$$\Theta \subset \mathbf{P}V_{\mathbf{C}}^* \times \operatorname{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$$

is defined as the incidence correspondence

$$\Theta = \left\{ (\xi, H) : \dim\ker H_\xi \geqq 1 \right\}.$$

The fibres $\Theta_\xi$ are all isomorphic as algebraic varieties.

and there are tautological maps

$$\begin{array}{l}\Theta \xrightarrow{\ \eta\ } \mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})\\ \big\downarrow \pi \\ \mathbf{P}V_{\mathbf{C}}^*\end{array}$$

By construction and corollary (3.13)

$$\begin{cases}\pi^{-1}(\xi) = \Theta_\xi \\ \pi\big(\eta^{-1}(H)\big) = \Xi_{H,\mathbf{C}}.\end{cases} \tag{3.22}$$

By (3.19) and (3.16)

$$\dim\Theta = r(N+n) + (n-1) - \big[N - r + 1\big].$$

This implies that for a general $H \in \mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$

$$\dim\big(\eta^{-1}(H)\big) = r - 1 - (n-1)(n-2)/2. \tag{3.23}$$

In particular, when $r \leqq (n-1)(n-2)/2$, the mapping $\eta$ is not surjective (in this case we put $-1$ on the right hand side of (3.23)).

(3.24) *Definition.* We define $\Sigma_{\mathbf{C}} \subset \mathrm{Hom}(S^2V_{\mathbf{C}}, W_{\mathbf{C}})$ to be the *proper* subvariety defined by

$$\Sigma_{\mathbf{C}} = \big\{H : \dim\eta^{-1}(H) > r - 1 - (n-1)(n-2)/2\big\}.$$

For $H \notin \Sigma_{\mathbf{C}}$, (3.22) and (3.23) give

$$\dim\Xi_{H,\mathbf{C}} = \max\{r - 1 - (n-1)(n-2)/2, -1\},$$

and this completes the proof of Theorem A.

(b) *Proof of* (iii) *in the Main Theorem.* (i) We begin with some general remarks. Let $(\tilde{M}, ds^2)$ be an abstract Riemannian manifold with curvature tensor $R$ and covariant derivatives $\nabla^k R$ ($\nabla^0 R = R$). Given a point $p \in \tilde{M}$ we let $V = T_p(\tilde{M})$ and $K \subset S^2V^*$ be the space of curvature-like tensors (cf. §1); we may consider $\nabla^k R(p)$ as an element in $K \otimes (\bigotimes^k V^*)$.

*Definitions.* (i) We call

$$R^{\boldsymbol{\cdot}}(p) = \big\{R(p), \nabla R(p), \nabla^2 R(p), \ldots\big\} \tag{3.25}$$

the *curvature sequence* at $p \in \tilde{M}$.

(ii) Two sequences

$$R_i^{\boldsymbol{\cdot}} = \big\{R_i, R_i^{(1)}, R_i^{(2)}, \ldots\big\}, \qquad i = 1, 2,$$

where

$$R_i^{(k)} \in K \otimes (\otimes^k V^*)$$

will be said to be *equivalent* if there is an element of GL($V$) taking $R_1^{\cdot}$ to $R_2^{\cdot}$.

Now suppose that

$$x : \tilde{M} \to M \subset \mathsf{E}^{n+r} \tag{3.26}$$

is an isometric embedding. We fix a point $x \in M$ and, following our general notational conventions, set $V = T_x(M)$ and $W = N_x(M)$. Our considerations will be local in a neighborhood of $p$, and we denote by

$$H \in W \otimes S^2V^*$$

the 2nd fundamental form of $M$ at $p$. We also denote by $j^q(R)(p) = \{R(p), \nabla R(p), \ldots, \nabla^q R(p)\}$ the $q$-jet of a curvature tensor of $\tilde{M}$ at $p$. From the discussion at the end of §1 there are equations

$$\begin{cases} \gamma(H,H) = R(p) \\ \gamma(H,H^{(1)}) = \nabla R(p) \\ \quad\vdots \\ \gamma(H,H^{(q)}) + \{\text{terms involving } H, \ldots, H^{(q-1)}\} = \nabla^q R(p). \end{cases}$$

For each $H$ we denote by

$$\Psi^q(H) \subset \bigoplus_{k=1}^{q} K \otimes (\otimes^k V^*)$$

the range, over all $H^{(1)}, \ldots, H^{(q)}$, of the mapping given by the left hand side of these equations. Then

$$j^q(R)(p) \in \bigcup_H \Psi^q(H). \tag{3.27}$$

It may be shown that, when the codimension $r < n(n-1)/2$ and $q \geqq q(r)$, the right hand side of (3.27) is a *proper* algebraic subvariety of the variety of $q$-jets of curvature tensors of $n$-dimensional Riemannian manifolds.[4] Thus, as previously noted, the isometric embedding system fails to be involutive below the embedding dimension, and (at least in the real analytic case) a necessary condition that $(\tilde{M}, ds^2)$ admit an isometric embedding in $\mathsf{E}^{n+r}$ for $r < n(n-1)/2$ is expressed by algebraic equations on $(j^q R)(p)$ for $q$ sufficiently large.

(ii) We now consider $(\tilde{M}, ds^2)$ for which (3.27) is satisfied. In a little while (cf.

[4]The example of an $M^4 \subset \mathsf{E}^6$ shows that this result is false for just the curvature tensor $R$ (cf. §6 below).

the discussion following (3.53)) we will define a *proper* subvariety

$$\Phi^q \subset \bigcup_H \Psi^q(H)$$

in the space of $q$-jets of curvature tensors of submanifolds $M^n \subset \mathsf{E}^{n+r}$.

*Definition.* We will say that the isometric embedding (3.26) is *general* in case $H \notin \Sigma$ (cf. (3.24)) and $j^{q_2}(R) \notin \Phi^{q_2}$ for an integer $q_2$ to be specified below.

Thus, the general submanifolds $M^n \subset \mathsf{E}^{n+r}$ are open and dense in the $C^{q_2+2}$-topology.

In this section we will prove the following result that provides the main step in the proof of part (iii) in our Main Theorem.

THEOREM B. *Let $M^n \subset \mathsf{E}^{n+r}$ be a general embedding with $r \leqq (n-1)(n-2)/2$. Then for each $p \in M$ the curvature sequence* (3.25) *uniquely determines the* 2*nd fundamental form $H \in W \otimes S^2V^*$ up to* $\mathrm{GL}(W)$.

The proof breaks into several steps.

*Step one.*[5] Using (1.41) we define

$$\gamma_H^{(q)} : W \otimes S^{q+2}V^* \to K \otimes S^qV^* \tag{3.28}$$

by

$$\gamma_H^{(q)}(G) = \left(\frac{q}{q+2}\right)\gamma(H,G)$$

where $\gamma(H,G)$ is given by (1.42) (or, in terms of indices, by (1.43)). The basic property (1.44) is then

$$\frac{\partial}{\partial x^i}\left(\gamma_H^{(q)}(G)\right) = \gamma_H^{(q-1)}\left(\frac{\partial G}{\partial x^i}\right), \qquad G \in W \otimes S^{q+2}V^*. \tag{3.29}$$

In other words, we scale $\gamma_H$ acting on $W \otimes S^{q+2}V^*$ so that we have the commutative diagram

$$\begin{tikzcd} W \otimes S^{q+2}V^* \arrow[r, "d"] \arrow[d, "\gamma_H^{(q)}"] & W \otimes S^{q+1}V^* \otimes V^* \arrow[d, "\gamma_H^{(q-1)}"] \\ K \otimes S^qV^* \arrow[r, "d"] & K \otimes S^{q-1}V^* \otimes V^* \end{tikzcd}$$

[5]This step consists in making explicit the dictionary between modules and sheaves given in [15] in the case of the prolonged Gauss equations, and in interpreting what Theorem A says about this dictionary.

where $d$ is identity $\otimes$ (exterior differentiation). The dual of (3.28) is denoted by

$$\gamma_H^{(q)*} : K^* \otimes S^q V \to W \otimes S^{q+2} V,$$

where we use the metric in $W$ to identify $W^*$ with $W$. Setting

$$\gamma_H^{*\cdot} = \bigoplus_{q \geqq 0} \gamma_H^{(q)*}$$

we infer from (3.29) that

$$\gamma_H^* : K^* \otimes S^\cdot V \to W \otimes S^{\cdot+2} V \tag{3.30}$$

is a *homomorphism of graded* $S = S^\cdot V$-*modules*. Denoting by $P^\cdot$ and $Q^\cdot$ the kernel and cokernel of (3.30), we have an exact sequence of graded $S$-modules

$$0 \to P^\cdot \to K^* \otimes S^\cdot V \to W \otimes S^{\cdot+2} V \to Q^\cdot \to 0. \tag{3.31}$$

By complexifying we obtain the exact sequence

$$0 \to P_{\mathbf{C}}^\cdot \to K_{\mathbf{C}}^* \otimes S^\cdot V_{\mathbf{C}} \to W_{\mathbf{C}} \otimes S^{\cdot+2} V_{\mathbf{C}} \to Q_{\mathbf{C}}^\cdot \to 0 \tag{3.32}$$

of graded $S_{\mathbf{C}} = S^\cdot V_{\mathbf{C}}$-modules.

In algebraic geometry there is a well known "dictionary" between the categories

$$\left\{\begin{array}{l}\text{graded } S_{\mathbf{C}}\text{-modules} \\ \text{of finite type}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{coherent sheaves} \\ \text{over } \mathsf{P} V_{\mathbf{C}}^*\end{array}\right\}.$$

The map $\to$ is obtained by localizing in the sense of algebra. This dictionary is given in [15], and an explanation intended for use in the theory of exterior differential systems is presented in [4]. It will now be used in the proof of Theorem B.

Over the complex projective space $\mathsf{P} = \mathsf{P} V_{\mathbf{C}}^*$ we denote by

$$0 \to \mathscr{P} \to \mathscr{K}^* \to \mathscr{W}(2) \to \mathscr{Q} \to 0 \tag{3.33}$$

the exact sequence of coherent sheaves obtained by localizing (3.32). For large $q$ the maps on cohomology induced from

$$0 \to \mathscr{P}(q) \to \mathscr{K}^*(q) \to \mathscr{W}(q+2) \to \mathscr{Q}(q) \to 0$$

give (this is a consequence of the dictionary discussed in loc. cit.)

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathsf{P}, \mathscr{P}(q)) & \to & H^0(\mathsf{P}, \mathscr{K}^*(q)) & \to & H^0(\mathsf{P}, \mathscr{W}(q+2)) & \to & H^0(\mathsf{P}, \mathscr{Q}(q)) & \to 0 \\
& \| & & \| & & \| & & \| & \\
0 \longrightarrow & P_{\mathbf{C}}^{(q)} & \longrightarrow & K_{\mathbf{C}}^* \otimes S^q V_{\mathbf{C}} & \longrightarrow & W_{\mathbf{C}} \otimes S^{q+2} V_{\mathbf{C}} & \longrightarrow & Q_{\mathbf{C}}^{(q)} & \longrightarrow 0,
\end{array} \tag{3.34}$$

where the bottom row is the $q$th graded piece of (3.32).

Now assume that $r \leqq (n-1)(n-2)/2$ and that $H$ is general in the sense of definition (3.24) (i.e., $H \notin \Sigma_{\mathrm{C}}$ as defined there). Then by Theorem A,

$$\mathscr{Q} = (0)$$

and (3.33) reduces to

$$0 \to \mathscr{P} \to \mathscr{K}^* \to \mathscr{W}(2) \to 0, \tag{3.35}$$

while the $q$th graded piece of (3.31) becomes

$$0 \to P^{(q)} \to K^* \otimes S^q V \to W \otimes S^{q+2} V \to 0 \tag{3.36}$$

for $q$ sufficiently large.

*Step two.*[6] Let now $H, G \in (W \otimes S^2 V^*) \backslash \Sigma$ (cf. (3.24)). Denote the respective sequences (3.35) by

$$\begin{aligned} &0 \to \mathscr{P}_H \to \mathscr{K}^* \to \mathscr{W}(2) \to 0 \\ &0 \to \mathscr{P}_G \to \mathscr{K}^* \to \mathscr{W}(2) \to 0, \end{aligned} \tag{3.37}$$

and denote the respective sequences (3.36) by

$$\begin{cases} 0 \longrightarrow P_H^{(q)} \longrightarrow K^* \otimes S^q V \xrightarrow{\gamma_H^{(q)*}} W \otimes S^{q+2} V \longrightarrow 0 \\ 0 \longrightarrow P_G^{(q)} \longrightarrow K^* \otimes S^q V \xrightarrow{\gamma_G^{(q)*}} W \otimes S^{q+2} V \longrightarrow 0. \end{cases} \tag{3.38}$$

(3.39) PROPOSITION. *The following conditions are equivalent*:

$$H = A \cdot G \qquad \textit{for some} \quad A \in \mathrm{GL}(W); \tag{3.40}$$

$$P_H^{(q)} = P_G^{(q)} \qquad (\textit{as subspaces of } K^* \otimes S^q V)\ \textit{for} \quad q \geqq q_0. \tag{3.41}$$

*Proof.* If (3.40) holds, then it follows immediately from (1.43) that

$$\gamma_H^{(q)}(W \otimes S^{q+2} V^*) = \gamma_G^{(q)}(W \otimes S^{q+2} V^*)$$

as subspaces of $K \otimes S^q V^*$. Since

$$P_H^{(q)} = \left(\gamma_H^{(q)}(W \otimes S^{q+2} V^*)\right)^{\perp}$$

and similarly for $P_G^{(q)}$, (3.41) follows.

[6]This step consists in expressing in commutative algebra terms what it means that two 2nd fundamental forms be GL($W$) equivalent.

Conversely, assume (3.41). Then, by complexifying and localizing, it follows that $\mathscr{P}_H = \mathscr{P}_G$ as subsheaves of $\mathscr{K}^*$ (loc. cit.; we recall that the basic dictionary is only bijective modulo finite dimensional vector spaces, so that e.g. the sheaves associated to the two graded modules

$$B = \bigoplus_{q \geqq 0} B^q$$

$$B^{[q_0]} = \bigoplus_{q \geqq q_0} B^q$$

are the same). It follows that (3.41) implies a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{P}_H \arrow[r] \arrow[d, equal] & \mathscr{K}^* \arrow[r, "\gamma_H^*"] \arrow[d, equal] & \mathscr{W}(2) \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \mathscr{P}_G \arrow[r] & \mathscr{K}^* \arrow[r, "\gamma_G^*"] & \mathscr{W}(2) \arrow[r] & 0 \end{tikzcd}$$

where

$$\psi : \mathscr{W}(2) \xrightarrow{\sim} \mathscr{W}(2)$$

is a sheaf isomorphism. Consequently, $\psi$ induces an isomorphism of the trivial bundle with fibre $W_{\mathbf{C}}$ over $\mathbf{P}$. Thus $\psi$ is given by $A \in \mathrm{GL}(W_{\mathbf{C}})$ and therefore

$$H = AG.$$

The conjugate of this equation is

$$H = \bar{A}G,$$

which implies that

$$(A - \bar{A})G = 0.$$

In terms of indices this is

$$\left(A_\beta^\alpha - \bar{A}_\beta^\alpha\right)G_{ij}^\beta = 0$$

for all $i, j$. Since $G$ is general this implies that $A = \bar{A}$, which proves the proposition. Q.E.D.

*Step three.* We now show that if (3.40) fails to hold, then $P^\cdot$ and $Q^\cdot$ are "very different".

(3.42) PROPOSITION. *Suppose that $H \neq AG$ for any $A \in \mathrm{GL}(W)$. Then, given*

$N > 0$ *there is a* $q_1 > 0$ *such that for all* $q \geqq q_1$

$$\dim\left(P_H^{(q)^\perp} / P_H^{(q)^\perp} \cap P_G^{(q)^\perp}\right) \geqq N$$

$$\dim\left(P_G^{(q)^\perp} / P_H^{(q)^\perp} \cap P_G^{(q)^\perp}\right) \geqq N.$$

*Proof.* It will suffice to prove the result over $\mathbf{C}$. By proposition (3.39) and the dictionary between graded $S_{\mathbf{C}}$-modules and coherent sheaves over $\mathbf{P}V_{\mathbf{C}}^*$, we infer that

$$\mathscr{P}_H \neq \mathscr{P}_G$$

as coherent subsheaves of $\mathscr{K}^*$. Since $\mathscr{P}_H$ and $\mathscr{P}_G$ both correspond to sub-bundles of the trivial bundle with fibre $\mathscr{K}_{\mathbf{C}}^*$ over $\mathbf{P}$, we infer that for some point $\xi_0 \in \mathbf{P}$ the fibres $(\mathscr{P}_H)_{\xi_0} \subset \mathscr{K}_{\mathbf{C}}^*$ and $(\mathscr{P}_G)_{\xi_0} \subset \mathscr{K}_{\mathbf{C}}^*$ are distinct. It follows that

$$(\mathscr{P}_H)_\xi \neq (\mathscr{P}_G)_\xi$$

for $\xi$ in a neighborhood of $\xi_0$. We thus have an exact sequence of coherent sheaves over $\mathbf{P}$

$$0 \to \mathscr{P}_H \cap \mathscr{P}_G \to \mathscr{P}_H \to \mathscr{F} \to 0 \tag{3.43}$$

where the support

$$\operatorname{supp} \mathscr{F} = \mathbf{P}V_{\mathbf{C}}^*. \tag{3.44}$$

Tensoring (3.43) with $\mathscr{O}(q)$ and taking cohomology gives, for large $q$,

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(\mathbf{P}, (\mathscr{P}_H \cap \mathscr{P}_G)(q)) & \longrightarrow & H^0(\mathbf{P}, \mathscr{P}_H(q)) & \longrightarrow & H^0(\mathbf{P}, \mathscr{F}(q)) & \longrightarrow 0 \\
 & \| & & \| & & \| & \\
0 \longrightarrow & P_H^{(q)} \cap P_G^{(q)} & \longrightarrow & P_H^{(q)} & \longrightarrow & F^{(q)} & \longrightarrow 0
\end{array}$$

where the right hand vertical equality is a definition. From (3.44) together with the discussion of Hilbert functions in [15] (loc. cit.) it follows that

$$\dim F^{(q)} = \frac{Cq^{n-1}}{(n-1)!} + (\text{lower order terms in } q) \tag{3.45}$$

where $C$ is a positive integer. It is clear that (3.45), together with the analogous statement interchanging the roles of $H$ and $G$, gives

$$\left.\begin{array}{l}\dim P_H^{(q)} / \left(P_H^{(q)} \cap P_G^{(q)}\right) \geqq C' q^{n-1} \\ \dim P_G^{(q)} / \left(P_H^{(q)} \cap P_G^{(q)}\right) \geqq C' q^{n-1}\end{array}\right. \qquad q \geqq q_0, \quad C' > 0$$

Choose direct sum decompositions

$$W \otimes S^{q+2}V_{\mathbf{C}} = \left(P_H^{(q)} \cap P_G^{(q)}\right) \oplus U_H^{(q)} \oplus U_G^{(q)} \oplus T^{(q)}$$

where

$$P_H^{(q)} = \left(P_H^{(q)} \cap P_G^{(q)}\right) \oplus U_H^{(q)}$$

$$P_G^{(q)} = \left(P_H^{(q)} \cap P_G^{(q)}\right) \oplus U_G^{(q)}.$$

Then for $q \geqq q_0$,

$$\dim P_H^{(q)\perp}/\left(P_H^{(q)\perp} \cap P_G^{(q)\perp}\right) = \dim U_G^{(q)} \geqq C'q^{n-1}$$

$$\dim P_G^{(q)\perp}/\left(P_H^{(q)\perp} \cap P_G^{(q)\perp}\right) = \dim U_H^{(q)} \geqq C'q^{n-1} \quad \text{Q.E.D.}$$

*Step four.* We will now complete the proof of Theorem B. For this we consider the curvature sequence (3.25) of an $M \subset \mathsf{E}^{n+r}$, and fixing a point $x \in M$ we write the equations (1.48) *at* $x$ in the form

$$\nabla^q R = 2\gamma(H, H^{(q)}) + \phi^{(q)}(H, H^{(1)}, \dots, H^{(q-1)}). \tag{3.46}$$

where $H^{(k)} \in W \otimes S^{k+2}V^*$. Assume that $r \leqq (n-1)(n-2)/2$ and that the 2nd fundamental form of $M$ is general at $x$. Then by the dual of (3.36) we infer that $\gamma_H^{(q)}$ is *injective* for large $q$. This implies that

(3.47) *If the* 2*nd fundamental form of* $M$ *is general and* $r \leqq (n-1)(n-2)/2$, *then for* $q \geqq q_1 + 1$ *the equations* (3.46) *uniquely determine* $H^{(q)}$.

Given $H$ we denote by

$$Z(H) \subset \bigoplus_{k=1}^{q_1} W \otimes S^{k+2}V^*$$

the algebraic variety of all $(H^{(1)}, \dots, H^{(q_1)})$ such that the equations (3.46) are satisfied for $q \leqq q_1$ (keeping in mind that the point $x \in M$ is fixed). For $q \geqq q_1 + 1$ we thus have a *unique* expression

$$H^{(q)} = \psi^{(q)}(H, H^{(1)}, \dots, H^{(q_1)}; R, \nabla R, \dots, \nabla^q R).$$

Now suppose that the general solutions $G \in W \otimes S^2V^* \backslash \Sigma$ (cf. (3.24)) to the Gauss equations

$$\gamma(G, G) = R$$

form a variety $Y$ of dimension $N_1$, and that

$$\max_{G \in Y} \dim Z(G) = N_2.$$

Then the solutions to the infinite sequence of equations

$$\nabla^q R = 2\gamma(G, G^{(q)}) + \phi^{(q)}(G, G^{(1)}, \ldots, G^{(q-1)}) \tag{3.48}$$

where

$$\begin{cases} q = 0, 1, 2, \ldots & \\ G \in W \otimes S^2V^* \backslash \Sigma & \text{is general} \\ G^{(k)} \in W \otimes S^{k+2}V^* & \text{for} \quad k \geqq 1 \end{cases}$$

form a variety of dimension at most $N_1 + N_2$ in $W \otimes S^{\cdot}V^*$.

Now suppose that we have two possible 2nd fundamental forms $H, G \in W \otimes S^2V^* \backslash \Sigma$ such that (3.46) and (3.48) are satisfied for $q = 0, \ldots, q_2 - 1$ where $q_2 \geqq q_1 + 1$. Suppose also that (3.40) is not satisfied and choose $q_2$ sufficiently large that (3.42) holds for $q \geqq q_2$ where $N = 2(N_1 + N_2)$. We observe the following elementary

(3.49) LEMMA. *Let $F_1, F_2$ be two linear subspaces of a vector space $E$ and $\Phi_1, \Phi_2 \subset E$ two algebraic subvarieties. Suppose that*

$$\dim(F_1/F_1 \cap F_2) \geqq 2t$$

$$\dim(F_2/F_1 \cap F_2) \geqq 2t$$

$$\dim \Phi_1 < t, \qquad \dim \Phi_2 < t.$$

*Then we do not have*

$$F_1 \subset F_2 + \Phi_1 - \Phi_2{}^7 \tag{3.50}$$

*Proof.* By projecting to $E/F_1 \cap F_2$ we may reduce to the case

$$E = \mathbf{R}^{2t} \oplus \mathbf{R}^{2t} \oplus \mathbf{R}^s$$

$$F_1 = \mathbf{R}^{2t} \oplus (0) \oplus (0)$$

$$F_2 = (0) \oplus \mathbf{R}^{2t} \oplus (0).$$

Let $\Psi_1, \Psi_2$ be the projections of $\Phi_1$ and $\Phi_2$ for $F_1$. If (3.50) holds then we have

$$F_1 \subset \Psi_1 - \Psi_2. \tag{3.51}$$

[7]This notation means that for every $v_1 \in F_1$ there is $v_2 \in F_2$ and $w_q \in \Phi_1$, $w_2 \in \Phi_2$ with

$$v_1 = v_2 + W_q - w_2.$$

Intuitively, the assumptions on $F_1, F_2$ say that "they cannot differ by a variety of dimension $< 2t$".

But since

$$\dim \Psi_1 < t, \qquad \dim \Psi_2 < t$$
$$\dim F_1 = 2t,$$

(3.51) cannot hold. Q.E.D.

This lemma applies to our situation by using proposition (3.42) and taking

$$E = K \otimes S^{q_2}V^*$$
$$F_1 = \gamma_H^{(q_2)}(W \otimes S^{(q_2+2)}V^*) = P_H^{(q_2)\perp}$$
$$F_2 = \gamma_H^{(q_2)}(W \otimes S^{(q_2+2)}V^*) = P_G^{(q_2)\perp}.$$

We conclude that

$$\gamma_H^{(q_2)}(W \otimes S^{q_2+2}V^*) + \bigcup_{(H^{(1)}, \ldots, H^{(q_2-1)}) \in Z(H)} \phi^{(q_2)}(H, H^{(1)}, \ldots, H^{(q_2-)})$$
$$\neq \gamma_G^{(q_2)}(W \otimes S^{q_2+2}V^*) + \bigcup_{(G^{(1)}, \ldots, G^{(q_2-1)}) \in Z(G)} \phi^{(q_2)}(G, G^{(1)}, \ldots, G^{(q_2-1)}). \tag{3.52}$$

Consequently, there exists $H^{(q_2)} \in W \otimes S^{q_2+2}V^*$ such that *the equation*

$$2\gamma(H, H^{(q_2)}) + \phi^{(q_2)}(H, H^{(1)}, \ldots, H^{(q_2-1)})$$
$$= 2\gamma(G, G^{(q_2)}) + \phi^{(q_2)}(G, G^{(1)}, \ldots, G^{(q_2-1)}) \tag{3.53}$$

*has no solution* $(G, G^{(1)}, \ldots, G^{(q_2)})$ *for* $\underline{any}$ $G \in W \otimes S^2V^* \backslash \Sigma$ *where* $G$, $G^{(1)}, \ldots, G^{(q_2)}$ *satisfies* (3.48) *for* $0 \leqq q_1 \leqq q_2$, *and* (*most importantly*) *where* (3.40) *is not satisfied*. We then determine $M^n \subset \mathsf{E}^{n+r}$ whose curvature sequence is given by[8]

$$\nabla^q R = 2\gamma(H, H^{(q)}) + \phi^{(q)}(H, H^{(1)}, \ldots, H^{(q-1)}), \qquad 0 \leqq q \leqq q_2.$$

For this $M$ the curvature sequence (3.25) then uniquely determines the 2nd fundamental form up to $\mathrm{GL}(W)$.

Finally, by examining the construction we see that the generality conditions on $\tilde{M}$ and on the isometric embedding (3.26) are expressed by $H \notin \Sigma$ (cf. (3.24))

[8]By considering in $\mathsf{E}^{n+r}$ with coordinates $(x^1, \ldots, x^n, y_1, \ldots, y^r)$ submanifolds given by

$$y^\alpha = H_{ij}^\alpha x^i x^j + H_{ijk}^\alpha x^i x^j x^k + \cdots + H_{i_1, \ldots, i_q}^\alpha x^{i_1} \ldots x^{i_q},$$

we may find an $M^n \subset \mathsf{E}^{n+r}$ with arbitrarily prescribed $q$-jet of 2nd fundamental forms at a point.

and by the condition that $j^{(q_2)}(R)$ lie outside some proper algebraic subvariety $\Phi^{(q_2)}$ (which we have no idea how to determine explicitly). This completes the proof of Theorem B.

(b) In this section we will complete the proof of part (iii) of the Main Theorem. We retain the notations introduced at the beginning of §3(a).

(3.54) PROPOSITION. *Suppose that $H \in W \otimes S^2V^*$ is general, and that for some $A \in \mathrm{GL}(W)$ we have*

$$\gamma(H,H) = \gamma(A \cdot H, A \cdot H). \tag{3.55}$$

*Then if either*

$$r \leqq [n^2/8] \tag{3.56}$$

*or*

$$\begin{cases} r \leqq 3 & n = 3 \\ r \leqq 4 & n = 4,5 \\ r \leqq 6 & n = 6,7 \end{cases} \tag{3.57}$$

*it follows that*

$$A \in O(W).$$

*Proof.* Choosing bases $\{w_\mu\}$ for $W$ and $\{\omega^i\}$ for $V^*$ we write

$$H = H^\mu_{ij} w_\mu \otimes \omega^i \omega^j$$

$$A = \|A^\lambda_\mu\|$$

$$B = I - {}^tAA = \|B_{\lambda\mu}\|.$$

The equations (3.55) give

$$(\gamma(H,H) - \gamma(AH,AH))_{ijkl} = 0.$$

By (1.38) these are

$$B_{\lambda\mu}\left(H^\lambda_{ik} H^\mu_{jl} - H^\lambda_{il} H^\mu_{jk}\right) = 0. \tag{3.58}$$

Thus by (1.12)

$$\begin{cases} \text{number of equations (3.58)} = n^2(n^2-1)/12 \\ \text{number of unknowns } B_{\lambda\mu} = r(r+1)/2. \end{cases}$$

Consequently, we can only hope to conclude that

$$B = 0 \tag{3.59}$$

if (approximately)

$$r < n^2/\sqrt{6} \sim n^2/2.45.$$

We shall first prove (3.59) under the much weaker bound (3.56).

For this we set $m = [n/2]$ and consider an $H = (H_{ij})$ where $H_{ij} \in W$ has the particular form

$$H = \begin{Vmatrix} H_{\alpha\beta} & 0 \\ 0 & H_{\rho\sigma} \end{Vmatrix} \qquad 1 \leqq \alpha, \beta \leqq m \quad \text{and} \quad m + 1 \leqq \rho, \sigma \leqq n. \tag{3.60}$$

Then for

$$\begin{cases} i = \alpha, & k = \beta \\ j = \rho, & l = \sigma \end{cases}$$

the equations (3.58) are

$$H^{\lambda}_{\alpha\beta} B_{\lambda\mu} H^{\mu}_{\rho\sigma} = 0. \tag{3.61}$$

If

$$r \leqq m(m+1)/2 \tag{3.62}$$

then for a general $H$ of the form (3.60) we may assume first that the vectors $H_{\rho\sigma}$ span $W$ so that (3.61) implies

$$H^{\lambda}_{\alpha\beta} B_{\lambda\mu} = 0, \tag{3.63}$$

and secondly that the vectors $H_{\alpha\beta}$ span $W$ so that (3.63) implies (3.59). Finally, it is immediate that (3.56) implies (3.62). In sum, under the conditions (3.56) the equations (3.58) (viewed as equations for the $B_{\lambda\mu}$) have maximal rank $r(r+1)/2$ for a general $H \in W \otimes S^2V^*$ (general means in a dense Zariski open subset).

The proof of the proposition under the condition

$$\begin{cases} r \leqq 3 & \text{when} \quad n = 3 \\ r \leqq 4 & \text{when} \quad n = 4, 5 \\ r \leqq 6 & \text{when} \quad n = 6, 7 \end{cases}$$

consists in simply being careful with the count in the preceding argument.

*Proof of part* (iii) *in the Main Theorem.*

By a *framed embedding*

$$x : \tilde{M} \to \mathsf{E}^{n+r}$$

we shall mean an isometric embedding together with a choice of Darboux frames $(x; e_1, \ldots, e_n; e_{n+1}, \ldots, e_{n+r}) = (x; e_i; e_\mu)$ along $M = x(\tilde{M})$. Suppose that the conditions of (iii) in the Main Theorem are satisfied, and let

$$\begin{tikzcd} M \arrow[r, bend left, "x"] \arrow[r, bend right, "x'"'] & \mathsf{E}^{n+r} \end{tikzcd}$$

be two general isometric embeddings. The main step in the proof is to combine

Theorem B and Proposition (3.54) to draw the following conclusion:

(3.64) *We may consider $x$ and $x'$ as framed embeddings in such a way that* (i) *the tangent frames $e_i, e_i'$ coincide (i.e., $x_*(e_i) = e_i$ and $x'_*(e_i) = e_i'$ for an orthonormal framing $(p; e_i)$ of $\overline{M}$, and* (ii) *the 2nd fundamental forms coincide; i.e.,*

$$H_{ij}^{\mu} = H_{ij}'^{\mu}. \tag{3.65}$$

We now set (cf. (2.27))

$$\begin{cases} \omega^i = x^*(\omega^i), & \omega'^i = x'^*(\omega^i) \\ \omega_j^i = x^*(\psi_j^i), & \omega'^i_j = x'^*(\psi_j^i) \\ \omega_i^{\mu} = x^*(\hbar_i^{\mu}), & \omega'^{\mu}_i = x'^*(\hbar'^{\mu}_i) \\ \omega_{\lambda}^{\mu} = x^*(\kappa_{\lambda}^{\mu}), & \omega'^{\mu}_{\lambda} = x'^*(\kappa_{\lambda}^{\mu}) \end{cases} \tag{3.66}$$

The conditions for a rigid motion taking the framed embedding $x$ to the framed embedding $x'$ are

$$\begin{cases} \omega^i - \omega'^i = 0 \\ \omega_j^i - \omega'^i_j = 0 \\ \omega_i^{\mu} - \omega'^{\mu}_j = 0 \\ \omega_{\lambda}^{\mu} - \omega'^{\mu}_{\lambda} = 0. \end{cases} \tag{3.67}$$

The first two equations in (3.67) follow from (i) in (3.64) (uniqueness of the Levi–Civita connection). The third equation in (3.67) is just (3.65). It remains to show that these imply the last equation in (3.67) (cf. the remark at the end of the proof).

Exterior differentiation of

$$\omega_i^{\mu} - \omega'^{\mu}_i = 0$$

gives, using the Maurer–Cartan equation (1.23),

$$(\omega_{\lambda}^{\mu} - \omega'^{\mu}_{\lambda})\omega_i^{\mu} = 0.$$

Since $r \leq n$ and the embedding is general we may assume that the 1-forms $\omega_i^{\mu}, n+1 \leq \mu \leq n+r$, are linearly independent for each $i$ (this is a very weak form of "general"). The Cartan lemma then gives

$$\omega_{\lambda}^{\mu} - \omega'^{\mu}_{\lambda} = C_{\lambda\tau}^{\mu}\omega_i^{\tau}$$

where

$$C_{\lambda\tau}^{\mu} = -C_{\mu\tau}^{\lambda} = C_{\tau\lambda}^{\mu}.$$

Then the standard argument (cf. (1.2)) implies that

$$C^{\mu}_{\lambda\tau} = 0$$

and we are done. Q.E.D.

*Remark.* The last equation in (3.67) says that the connections in the normal bundles should coincide. Classical "easy" (i.e., using only ODE) embedding and uniqueness theorems say that an embedding is given uniquely up to rigid motion by giving the 1st and 2nd fundamental forms and the connection in the normal bundle subject to various equations (Gauss, Gauss–Codazzi, and Ricci). What we have shown is that, given a Darboux framing for the isometric embedding $x$, there is a unique Darboux framing for the isometric embedding $x'$ such that the 2nd fundamental forms coincide. At this juncture it is essentially a classical result (cf. [8], [16]) that the normal connections must also coincide.

## §4. Nongeneric behavior of the Gauss equations.

(a) *Exteriorly orthogonal forms.* In this section we will begin discussing some of the nongeneric phenomena exhibited by the Gauss equations in low codimension. Our examples will be elaborations on Cartan's theme of exteriorly orthogonal systems of quadratic forms.

We retain the earlier notation, letting $W$ be an Euclidean real vector space of dimension $r$ and letting $V$ be a real vector space of dimension $n$. We define $\gamma$ as in (1.36). We say that a $W$-valued quadratic form on $V$, $H \in W \otimes S^2V^*$ is *exteriorly orthogonal* if

$$\gamma(H,H) = 0 \tag{4.1}$$

Let $X_{r,n} \subset W \otimes S^2V^*$ denote the sub-variety of exteriorly orthogonal forms. In his original paper on the subject, [7] Cartan showed how one could, in principle, completely determine $X_{r,n}$ for every $r$ and $n$. His method becomes quite cumbersome for $r > n$, but, for $r \leq n$ Cartan computed the "generic component" of $X_{r,n}$ explicitly.

We say that an $H \in W \otimes S^2V^*$ is *nondegenerate* if $H \notin W \otimes S^2U$ for any proper subspace $U$ of $V^*$. More generally, we may say that a submanifold $M^n \subset \mathsf{E}^{n+r}$ is *nondegenerate* if $\mathrm{II}_p \in N_pM \otimes S^2(T^*_pM)$ is non-degenerate for every $p \in M$. Geometrically this means that the Gauss map $p \mapsto T_pM$ gives an immersion of $M$ into $G_n(\mathsf{E}^{n+r})$.

(4.2) THEOREM (É. Cartan). *Let $H \in W \otimes S^2V^*$ be exteriorly orthogonal and nondegenerate. Moreover, suppose $r \leq n$. Then we must have $r = n$. In fact, there exists an orthonormal basis $\{w_1, \ldots, w_n\}$ of $W$ unique up to permutation, and a basis of $\{\phi^1, \ldots, \phi^n\}$ of $V^*$ unique up to sign and the same permutation, so that*

$$H = \sum_{i=1}^{n} w_i \otimes (\phi^i)^2 \tag{4.3}$$

*Conversely, every $H$ of the form (5.3) is exteriorly orthogonal.*

The proof can be found in Cartan [7]; but more modern proofs appear in [1] and [16]. It is interesting to note that this theorem depends heavily on the reality of $W$ and $V$. The corresponding assertion over the complexes is false (see [7]). Cartan actually proves more in the low dimensions $n = 2, 3$. In these cases, he shows that the theorem remains true (without the genericity assumption) for arbitrary $r$, where the $\{\phi^1, \ldots, \phi^r\}$ are no longer required to be linearly independent.

The set of $H \in W \otimes S^2V^*$ of the form (4.3) is obviously the quotient of $O(n, \mathbf{R}) \times \mathrm{GL}(n, \mathbf{R})$ by a finite subgroup (of order $2^n n!$) and is therefore a smooth submanifold of $W \otimes S^2V^*$ of dimension $n^2 + n(n-1)/2 = n(3n-1)/2$. Its Zariski closure in $W \otimes S^2V^*$ is an irreducible connected component of $X_{n,n}$ which is certainly not imbedded in any other component. Let us call this component $X^\circ_{n,n} \subseteq X_{n,n}$. If we consider $\gamma$ as a quadratic map from $W \otimes S^2V^*$ to $K$ (as in (2.27)) then dimension count alone shows that $\gamma$ must be singular along $X^\circ_{n,n}$ for $n > 3$.

(4.4) *If $H$ is of the form* (4.3) *then $\Xi_{H,\mathbf{C}}$ consists of the $n(n-1)/2$ lines which pass through two of the points of* $\{[\phi^1], [\phi^2], \ldots, [\phi^n]\} \subset \mathbf{P}V^*_{\mathbf{C}}$.

*Proof.* If $\xi \in V^*_{\mathbf{C}} - \{0\}$, then (3.13) shows that $[\xi] \in \Xi_{H,\mathbf{C}}$ if and only if there exist $\eta \in V^*_{\mathbf{C}} - \{0\}$ and $w \in W$ so that

$$w \cdot H = \eta \circ \xi$$

If we let $r_i = w \cdot w_i$, then this becomes the condition

$$\sum_{i=1}^{n} r_i (\phi^i)^2 = \eta \circ \xi$$

The left hand side of this equation is not the product of two linear factors unless all but two of the $r_i$ vanish, say $r_i = 0$ unless $i = j, k$. But then $\xi = \lambda_j \phi^j + \lambda_k \phi^k$ for some $\lambda_j, \lambda_k$, so the assertion is proved. Q.E.D.

Note that the characteristic variety has $n$ singular points, namely, the $[\phi^i]$. This behavior should be contrasted with the dimension of the characteristic variety for generic $H \in W \otimes S^2V^*$ when $r = n$. By Theorem A, this is $\max\{(n-1) - (n-1)(n-2)/2, -1\}$. Thus, for $n > 3$ the $H$ of the form (5.3) have a larger characteristic variety than the generic $H \in W \otimes S^2V^*$.

According to Theorem IV in [4], the analytic $M^n \subset \mathbf{E}^{2n}$ which satisfy the condition that the second fundamental form at every point is of the form (4.3) (these are, in some sense, the generic flat $n$-manifolds in $\mathbf{E}^{2n}$) form a class of submanifolds depending on at most $n(n-1)/2$ functions of 2 variables. The reason for the "at most" in the above statement is that, if we take $X \subset \mathscr{F}(\tilde{M}) \times \mathscr{F} \times (W \otimes S^2V^*)$ to be the regular points of $\mathscr{F}(M) \times \mathscr{F} \times X^0_{n,n}$, then we have no guarantee that (2.17) is involutive.

In [7], Cartan shows that this system *is* involutive so that the real analytic integrals of (2.17) depend on exactly $n(n-1)/2$ functions of 2 variables.

For our purposes, we will verify this by setting up a slightly different differential system (with essentially the "same" integrals) and proving involutivity. The frame bundle we now introduce will be of use in other problems we shall consider in this section. Let $\mathsf{F}$ denote the bundle of frames $f = (x; e_i, e_\mu)$ of $\mathsf{E}^{n+r}$ which satisfy the conditions

$$\begin{cases} \text{(i)} & e_i \cdot e_\mu = 0 \\ \text{(ii)} & e_\mu \cdot e_\nu = \delta_{\mu\nu} \end{cases}$$

There are canonical functions $g_{ij} = e_i \cdot e_j$ on $\mathsf{F}$ and the submanifold $\mathscr{F} \subset \mathsf{F}$ is defined by the $n(n+1)/2$ equations $g_{ij} = \delta_{ij}$. We denote by $G = (g_{ij})$ the $n \times n$ symmetric positive definite matrix of functions on $\mathsf{F}$. Just as before, on $\mathsf{F}$, we have the equations

$$dx = e'\omega + e''\theta$$

$$d(e' \quad e'') = (e' \quad e'')\begin{pmatrix} \psi & -G^{-1}{}^t h \\ h & \kappa \end{pmatrix}$$

where

$$G\psi + {}^t(G\psi) = dG$$

$$\kappa + {}^t\kappa = 0$$

for uniquely defined matrix valued 1-forms $\omega$, $\theta$, $\psi$, $h$, and $\kappa$. The structure equations

$$d\begin{pmatrix} \omega \\ \theta \end{pmatrix} = -\begin{pmatrix} \psi & -G^{-1}{}^t h \\ h & \kappa \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \theta \end{pmatrix} \tag{4.5}$$

$$d\begin{pmatrix} \psi & -G^{-1}{}^t h \\ h & \kappa \end{pmatrix} = -\begin{pmatrix} \psi & -G^{-1}{}^t h \\ h & \kappa \end{pmatrix} \wedge \begin{pmatrix} \psi & -G^{-1}{}^t h \\ h & \kappa \end{pmatrix} \tag{4.6}$$

hold just as before.

Now (4.2) shows that there are no nondegenerate flat $M^n \subset \mathsf{E}^{n+r}$ for $r < n$. Hence, we assume that $r = n$ and suppose that $M^n \subset \mathsf{E}^{2n}$ is flat and nondegenerate. By (4.2), it follows that there is a unique (up to obvious permutations and signs) local generalized Darboux framing $\Phi : M^n \to \mathsf{F}$ for which the identity

$$\mathrm{II} = \sum_{i=1}^{n} e_{i+n} \otimes (\omega^i)^2 \tag{4.7}$$

holds. (The $\omega^i$ are *not* orthonormal in general!) The image $\Phi(M^n)$ has the property that it is an integral of the system $(I, \omega)$ where $I$ is generated by the 1-forms $\{\theta^\mu, h_j^{i+n} - \delta_j^i \omega^j\}$ and the independence condition is $\omega^1 \wedge \cdots \wedge \omega^n$

$\neq 0$. Conversely, it is clear that an admissible integral of $(I, \omega)$ is $\Phi(M^n) \subseteq \mathsf{F}$ for some $M^n \subseteq \mathsf{E}^{2n}$ on which the induced metric is flat.

(4.8) THEOREM (E. Cartan). *The system $(I, \omega)$ on $\mathsf{F}$ is involutive, with characters $s_1' = n^2$, $s_2' = n(n-1)/2$. Thus, the "generic" flat $M^n \subseteq \mathsf{E}^{2n}$ depend on $n(n-1)/2$ functions of two variables (for $n \geq 2$).*

*Proof.* By (4.5–6), we compute

$$d\theta^\mu \equiv 0 \tag{4.9}$$

$$d\left(\hbar_j^{i+n} - \delta_j^i \omega^j\right) \equiv \sum_{k=2}^{n} \left(\delta_j^i \psi_k^i + \delta_k^i \psi_j^k - \delta_k^i K_{j+n}^{i+n}\right) \wedge \omega^k \tag{4.10}$$

where the congruences are mod $I$. In order to compute the Cartan characters, let $v, \omega \in T_f\mathsf{F}$ be vectors annihilating the 1-forms in $I$ and satisfying

$$\begin{aligned} \omega^k(v) &= \xi^k \\ \omega(w) &= \eta^k \end{aligned} \tag{4.11}$$

The reduced polar equations for these elements are

$$2\psi_j^i \xi^j + \sum_{k \neq j} \psi_j^i \xi^k = 0 \qquad j \tag{4.12a}$$

$$\psi_j^i \xi^i - \kappa_j^i \xi^j = 0 \qquad (i \neq j) \tag{4.12b}$$

$$2\psi_j^i \eta^j + \sum_{k \neq j} \psi_k^i \eta^k = 0 \tag{4.13a}$$

$$\psi_j^i \eta^j - \kappa_j^i \eta^j = 0 \qquad (i \neq j) \tag{4.13b}$$

If none of the $\xi^i$ are zero, equations (4.12) allow us to solve for the $\psi$'s in terms of the $\kappa$'s; consequently these $n^2$ equations are independent. Thus $s_1' = n^2$. If, in addition, we have $\xi^i \eta^j - \xi^j \eta^i \neq 0$ for all $i \neq j$, then equations (4.12b) and (4.13b) allow us to conclude $\kappa_j^i = 0$ and $\psi_j^i = 0$ for $i \neq j$, but equation (4.12a) then forces us to conclude $\psi_i^i = 0$ for all $i$ as well. Thus the equations (4.12–3) (for generic $\xi, \eta$) determine $\psi = \kappa = 0$, i.e., $s_1' + s_2' = n^2 + n(n-1)/2$. We now have

$$\begin{cases} s_1' = n^2 \\ s_2' = n(n-1)/2 \end{cases} \tag{4.14}$$

By Cartan's Test, to establish involutivity it suffices to exhibit an $s_1' + 2s_2'$ parameter family of integral elements at every point. However, if we let $\{(A_j^i) \mid i \neq j\}$ and $\{(B_j^i)\}$ be arbitrary real numbers, one sees that the $n$-plane defined by

$$\begin{cases} \theta^{\mu} = h_j^{i+n} - \delta_j^i \omega^j = 0 & \\ \kappa_j^i = A_j^i \omega^i - A_i^j \omega^j & i \neq j \\ \psi_j^j = \sum_k B_k^j \omega^k & \\ \psi_j^i = 2 B_j^i \omega^i - A_j^i \omega^j & (i \neq j) \end{cases} \tag{4.15}$$

is always an integral element. Since $s_1' + 2s_2' = 2n^2 - n$, we are done. Q.E.D.

*Remark.* From our computations, one sees that an integral two-plane is singular if and only if, for some $i \neq j$, the two form $\omega^i \wedge \omega^j$ vanishes when restricted to that two-plane. Since an integral $(n-1)$ plane is characteristic if and only if *every* sub-two-plane is singular, we see that the characteristics are exactly those on which $\omega^i \wedge \omega^j$ restricts to zero for some $i \neq j$. This recovers the result (4.4) that the characteristic variety at each point of $\mathsf{F}$ is $n(n-1)/2$ lines.

It is interesting to note that there are many compact $M^n \subset \mathsf{E}^{2n}$ in this category. If $\mathscr{C}_i \subseteq \mathsf{E}^2$ $(i = 1, \ldots, n)$ is a smooth closed curve of length 1 and nonvanishing curvature, then $M^n = \mathscr{C}_1 \times \mathscr{C}_2 \times \cdots \times \mathscr{C}_n \subset \mathsf{E}^{2n}$ is an isometric immersion of the standard torus $T^n$ with second fundamental form of type (4.3).

(b) *Isometric embedding of space forms and similar metrics.* Cartan's original motivation for introducing the concept of exterior orthogonality was to study the problem of finding isometric immersions of the space forms into Euclidean space. Recall that a *space form* is a manifold $\tilde{M}^n$ endowed with a metric $d\tilde{s}^2$ of constant sectional curvature $k$. Since we will work locally on $\mathscr{F}(\tilde{M})$, we may as well take $\tilde{M}^n$ to simply connected. The structure equations of such a metric are

$$d\tilde{\omega}^i = -\tilde{\psi}_j^i \wedge \tilde{\omega}^j \tag{4.16}$$

$$d\tilde{\psi}_j^i + \tilde{\psi}_k^i \wedge \tilde{\psi}_j^k = k\tilde{\omega}^i \wedge \tilde{\omega}^j \tag{4.17}$$

We say $ds^2$ is *elliptic*, *parabolic*, or *hyperbolic* depending on whether $k$ is positive, zero, or negative.

If $k$ is positive, then it is well known that $M^n$ can be embedded in $\mathsf{E}^{n+1}$ as the round hypersphere of radius $1/\sqrt{k}$. If $k$ is zero, then $M^n$ can actually be embedded as an open set in $\mathsf{E}^n$. Henceforth, we shall consider only the hyperbolic case $(k < 0)$. Cartan's observation in this case was that, since

$$R_{ijkl} = k\left(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j\right)$$

one could write

$$R = -\gamma\left(\sqrt{-k}\, ds^2, \sqrt{-k}\, ds^2\right)$$

where we have used (1.36) and simply set $W = \mathsf{R}^1$ with the obvious inner product. From this, Cartan deduced

(4.18) *The hyperbolic space form cannot be isometrically immersed in* $\mathsf{E}^{2n-2}$ (*even locally*).

*Proof.* If one has an isometric immersion $x : \tilde{M}^n \to \mathsf{E}^{n+r}$, let $H \in W \otimes S^2(V^*)$ be its second fundamental form at some point of $\tilde{M}$. Let $\hat{W} = W \otimes \mathsf{R}^1$ (orthogonal direct sum) and set $\hat{H} = H \oplus \sqrt{-k}\, ds^2$. The Gauss equation

$$\gamma(H,H) = R = -\gamma\left(\sqrt{-k}\, ds^2, \sqrt{-k}\, ds^2\right)$$

then becomes

$$\gamma(\hat{H},\hat{H}) = 0.$$

Since $ds^2|_p \notin S^2U$ for any proper subspace $U$ of $V^*$, it follows that $\hat{H}$ satisfies the hypotheses of (4.2), whence we must have

$$r + 1 = \dim \hat{W} \geq n \quad \text{Q.E.D.}$$

We define a metric $d\tilde{s}^2$ on $\tilde{M}^n$ to be *quasi-hyperbolic* if there exists a nondegenerate symmetric 2-form $Q$ on $\tilde{M}$ satisfying $\gamma(Q, Q) = -\tilde{R}$. In terms of a co-frame $\tilde{\omega}^i$ on $\tilde{M}^n$, we write these equations as

$$Q = \tilde{Q}_{ij}\omega^i \circ \omega^j$$

$$\tilde{R}_{ijkl} = \tilde{Q}_{ik}\tilde{Q}_{jl} - \tilde{Q}_{il}\tilde{Q}_{jk}$$

If $n = 2$, this concept is not too interesting since any metric with nonvanishing curvature is quasi-hyperbolic. When $n = 3$, quasi-hyperbolicity is an *open* condition since $-\gamma$ is a local diffeomorphism of $S^2V^*$ with an open subset of $K$ away from the locus of degenerate quadrics in $S^2V^*$ (see §5). When $n \geq 4$ this is a *strong* condition on the metric $ds^2$.

Cartan's observation extends to

(4.18′) *If* $ds^2$ *on* $M^n$ *is quasi-hyperbolic then* $(M^n, ds^2)$ *cannot be locally isometrically immersed into* $\mathsf{E}^{n+r}$ *for* $r < n - 1$.

Let us now investigate the structure equations of a quasi-hyperbolic metric. Let $\mathsf{F}(\tilde{M})$ denote the $\mathrm{GL}(n)$-bundle of *all* frames $f = (p; \tilde{e}_1, \ldots, \tilde{e}_n)$. We shall write $\tilde{e}$ for the row of vectors $(\tilde{e}_1, \ldots, \tilde{e}_n)$. Let $\tilde{\pi} : \mathsf{F}(\tilde{M}) \to \tilde{M}$ be the obvious projection. In the usual way, we construct the canonical differential forms on $\mathsf{F}(\tilde{M})$. In particular, there exists a unique column of 1-forms $\tilde{\omega} = (\tilde{\omega}^i)$ satisfying, for all $v \in T\mathsf{F}(\tilde{M})$

$$d\tilde{\pi}(v) = \tilde{e}_i\tilde{\omega}^i(v) \tag{4.19}$$

We write this as $d\tilde{\pi} = \tilde{e}\tilde{\omega}$, the matrix multiplication being understood. Let $\tilde{G} = (\tilde{g}_{ij}) = (\tilde{e}_i \cdot \tilde{e}_j)$ be the matrix of dot products, regarded as a (smooth) function on $\mathsf{F}(\tilde{M})$ with values in positive definite symmetric matrices. We have the formula

$$\tilde{\pi}^*(ds^2) = {}^t\tilde{\omega}\tilde{G}\tilde{\omega} \tag{4.20}$$

(the right hand side is a symmetric product). The Levi–Civita connection on $\mathsf{F}(\tilde{M})$ is the unique $n \times n$ matrix of 1-forms $\tilde{\psi} = (\tilde{\psi}_j^i)$ satisfying

$$d\tilde{\omega} = -\tilde{\psi} \wedge \tilde{\omega} \tag{4.21}$$

$$\tilde{G}\tilde{\psi} + {}^t(\tilde{G}\tilde{\psi}) = d\tilde{G} \tag{4.22}$$

(Note that the $n + n^2$ forms $\{\tilde{\omega}^i, \tilde{\psi}_j^i\}$ form a coframing of $\mathsf{F}(\tilde{M})$.) Setting

$$\tilde{\Omega}^* = \tilde{G}(d\tilde{\psi} + \tilde{\psi} \wedge \tilde{\psi})$$

the first Bianchi identity shows that

$$\tilde{\Omega}^* = \tfrac{1}{2}\tilde{R}\tilde{\omega} \wedge \tilde{\omega}$$

where $\tilde{R}$ is a $K$-valued 0-form as before.

We now use the assumption that the metric is quasi-hyperbolic. Setting $\tilde{\pi}^*(Q) = \tilde{Q}_{ij}\tilde{\omega}^i \circ \tilde{\omega}^j = {}^t\tilde{\omega}\tilde{Q}\tilde{\omega}$, the equation $-\gamma(\tilde{Q}, \tilde{Q}) = \tilde{R}$ becomes

$$\tilde{\Omega}^* = -\tilde{Q}\tilde{\omega} \wedge {}^t(\tilde{Q}\tilde{\omega}). \tag{4.23}$$

Note that if $A \in \mathrm{GL}(n)$ is constant and we let $r_A : \mathsf{F}(\tilde{M}) \to \mathsf{F}(\tilde{M})$ denote the standard right action, then the formulae

$$\begin{cases} r_A^*(\tilde{\omega}) = A^{-1}\tilde{\omega} & r_A^*(\tilde{G}) = {}^tA\tilde{G}A \\ r_A^*(\psi) = A^{-1}\tilde{\psi}A & r_A^*(\tilde{Q}) = {}^tA\tilde{Q}A \end{cases} \tag{4.24}$$

show that (4.23) has the correct frame equivariance.

Since we will be dealing with the Gauss equations, it will be convenient to reduce frames so that $Q$ becomes as simple as possible. Since $Q$ is nondegenerate, (4.24) assures us that, if $Q$ has type $(m, n-m)$, then we may reduce to an $O(m, n-m)$ sub-bundle of $F(\tilde{M})$ on which $Q$ has the form

$$(\tilde{\omega}^1)^2 + \cdots + (\tilde{\omega}^m)^2 - (\tilde{\omega}^{m+1})^2 - \cdots - (\tilde{\omega}^n)^2$$

To fix ideas, *we assume for the remainder of this section that $Q$ is positive definite*. Let $\mathcal{F}'(\tilde{M}) \subset \mathsf{F}(M)$ be the $O(n)$ sub-bundle on which

$$Q = (\tilde{\omega}^1)^2 + \cdots + (\tilde{\omega}^n)^2.$$

Since $\tilde{G}$ on $\mathscr{F}'(\tilde{M})$ is well defined on $\tilde{M}$ up to conjugation by an orthogonal matrix, its eigenvalues are metric invariants which are easily seen to be algebraic functions of the Riemann cui /ature tensor in orthonormal frames. Now (4.23) becomes

$$\tilde{\Omega}^* = -\tilde{\omega} \wedge {}^t\tilde{\omega}. \tag{4.23$'$}$$

Moreover, since $\tilde{\psi}$ satisfies (4.24) (with $A \in O(n)$), we see that the splitting of $\tilde{\psi}$ into symmetric and skew-symmetric parts is $O(n)$-equivariant. Write

$$\tilde{\psi} = \tilde{\nu} + \tilde{\mu}$$

where $\tilde{\nu} + {}^t\tilde{\nu} = 0 = \tilde{\mu} - {}^t\tilde{\mu}$. If we now differentiate (4.23′) using (4.21) and (4.22) we get

$$\tilde{\mu} \wedge \tilde{\omega} \wedge {}^t\tilde{\omega} + \tilde{\omega} \wedge {}^t\tilde{\omega} \wedge \tilde{\mu} = 0. \tag{4.25}$$

Let $\tilde{\Phi} = (\tilde{\Phi}^i) = (\tilde{\mu}^i_j \wedge \tilde{\omega}^j)$. Then (4.25) is simply

$$\tilde{\Phi}^i \wedge \tilde{\omega}^j = \tilde{\Phi}^j \wedge \tilde{\omega}^i \tag{4.25$'$}$$

for all $i, j$.

When $n = 3$, these three exterior equations imply that the $\tilde{\mu}^i_j$ are linear combinations of the $\tilde{\omega}^k$. Writing

$$\tilde{\mu}^i_j = \tilde{\mu}^i_{jk}\tilde{\omega}^k \tag{4.26}$$

and substituting (4.26) into (4.25), we see that the 18 components $\tilde{\mu}^i_{jk} = \tilde{\mu}^j_{ik}$ satisfy the three relations

$$\sum_j \tilde{\mu}^j_{ij} = \sum_j \tilde{\mu}^j_{ji}. \tag{4.25$''$}$$

Thus the tensor

$$\mathrm{III} = \tilde{\mu}^i_{jk}\tilde{\omega}^i \circ \tilde{\omega}^j \otimes \tilde{\omega}^k$$

has fifteen independent components. (Note that this is also the dimension of $K^{(1)}$ when $n = 3$, as it should be since (4.25) is the second Bianchi identity and quasi-hyperbolicity is an open condition when $n = 3$.)

Now suppose $n > 3$. Then, for all $i, j$, (4.25′) gives

$$\tilde{\Phi}^i \wedge \tilde{\omega}^i \wedge \tilde{\omega}^j = -\tilde{\Phi}^j \wedge \tilde{\omega}^i \wedge \tilde{\omega}^i = 0$$

so we must have $\tilde{\Phi}^i \wedge \tilde{\omega}^i$ divisible by every $\tilde{\omega}^k$. Since $\tilde{\Phi}^i \wedge \tilde{\omega}^i$ is only a 3-form, this gives

$$\tilde{\Phi}^i \wedge \tilde{\omega}^i = 0.$$

It follows that there exist $\tilde{\eta}^i$ for which

$$\tilde{\Phi}^i = \tilde{\eta}^i \wedge \tilde{\omega}^i$$

Equation (4.25′) then implies that $\tilde{\eta}^i \wedge \tilde{\omega}^i \wedge \tilde{\omega}^j = \tilde{\eta}^j \wedge \tilde{\omega}^j \wedge \tilde{\omega}^i$ so

$$\tilde{\eta}^i + \tilde{\eta}^j \equiv 0 \qquad \bmod \tilde{\omega}^i, \tilde{\omega}^j$$

for $i \neq j$. If we let $k$ be another index distinct from $i$ and $j$, the similar equations for the pairs $(i,k)$ and $(k,j)$ then allow us to conclude

$$\tilde{\eta}^i \equiv 0 \qquad \bmod \tilde{\omega}^i, \tilde{\omega}^j, \tilde{\omega}^k$$

for *all* $i, j, k$ distinct. This clearly implies

$$\tilde{\eta}^i \equiv 0 \qquad \bmod \tilde{\omega}^i,$$

so $\tilde{\Phi}^i \equiv 0$ for all $i$. However, this equation gives

$$\tilde{\mu}^i_j \wedge \tilde{\omega}^j \equiv 0.$$

By Cartan's Lemma, there exist $\tilde{\mu}^i_{jk} = \tilde{\mu}^i_{kj}$ for which

$$\tilde{\mu}^i_j = \tilde{\mu}^i_{jk}\tilde{\omega}^k. \tag{4.26′}$$

Summarizing, we see that for $n > 3$, the cubic form

$$\mathrm{III} = \tilde{\mu}^i_{jk}\tilde{\omega}^i \circ \tilde{\omega}^j \circ \tilde{\omega}^k$$

contains all of the covariant derivatives of the Riemann curvature tensor. It is interesting to note that, when $n > 3$, $\tilde{\mu} \wedge \tilde{\omega} = 0$ so

$$d\tilde{\omega} = -\tilde{\psi} \wedge \tilde{\omega} = -\tilde{\nu} \wedge \tilde{\omega}.$$

By construction, $\mathscr{F}'(\tilde{M})$ is the orthonormal frame bundle of $\tilde{M}$ with the metric $Q$. The skew symmetry of $\tilde{\nu}$ together with the above equation shows that $\tilde{\nu}$ is the Levi–Civita connection on $\mathscr{F}'(\tilde{M})$.

Now consider the case where $(M^n, ds^2)$ can be isometrically immersed into $\mathsf{E}^{2n-1}$. Let $p \in M$ be arbitrary, $V = T_pM$, $W = N_pM$ and let $H \in W \otimes S^2V^*$ be the second fundamental form at $p$. We have already remarked that $H \oplus Q \in (W \oplus \mathsf{R}^1) \otimes S^2V^*$ satisfies the hypothesis of (4.2) so we may write

$$H \oplus Q = \sum_i w_i \otimes (\phi^i)^2. \tag{4.27}$$

Let $w_i = w_i' + r_i$ where $w_i' \in W$ and $r_i \in \mathsf{R}^1$. Since $Q = r_i(\phi^i)^2$, we see that

$r_i > 0$ for each $i$. Let

$$\tilde{\omega}^i = \sqrt{r_i}\,\phi^i$$

$$b_i = w_i'/r_i$$

to get the equations

$$\begin{cases} H = \sum_i b_i \otimes (\tilde{\omega}^i)^2 \\ Q = \sum_i (\tilde{\omega}^i)^2. \end{cases} \tag{4.28}$$

The equations $w_i \cdot w_j = \delta_{ij}$ then imply

$$b_i \cdot b_j = -1 \qquad \text{for} \quad i \neq j \tag{4.29}$$

Letting $B_i = b_i \cdot b_i + 1$, we obtain the relations

$$\sum_i 1/B_i = 1 \tag{4.30}$$

$$\sum_i b_i/B_i = 0. \tag{4.31}$$

Moreover, (4.31) is the only nontrivial linear relation among the $b_i$.

(4.32) PROPOSITION. *The characteristic variety of an $H \in W \otimes S^2V^*$ of the form* (4.28) (*where the $b_i$ satisfy* (4.29)) *consists of the $n(n-1)$ points* $\{[\sqrt{B_i}\,\tilde{\omega}^i \pm \sqrt{B_j}\,\tilde{\omega}^j] \mid i \neq j\}$.

*Proof.* Just as in the proof of (4.4), we see that $[\xi] \in \Xi_{H,\mathbf{C}}$ if and only if there exist nonzero $w$ and $\eta$ satisfying $w \cdot H = \xi \circ \eta$. Now

$$w \cdot H = (w \cdot b_i)(\tilde{\omega}^i)^2.$$

It follows that $w \cdot H$ is a product of two linear factors if and only if, for some $\{i, j\}$ distinct, $w \cdot b_k = 0$ for $k \neq i, j$. Due to the fact that (4.30) is the only nontrivial linear relation among the $b$'s, we see that (once $i, j$ are chosen) this is $n-2$ linear equations for $w$. The unique solution up to scalar multiples is therefore given by $w = b_i - b_j$. We compute

$$\begin{aligned}(b_i - b_j) \cdot H &= B_i(\tilde{\omega}^i)^2 - B_j(\tilde{\omega}^j)^2 \\ &= \left(\sqrt{B_i}\,\tilde{\omega}^i + \sqrt{B_j}\,\tilde{\omega}^j\right)\left(\sqrt{B_i}\,\tilde{\omega}^i - \sqrt{B_i}\,\tilde{\omega}^j\right) \quad \text{Q.E.D.}\end{aligned}$$

Just as before, Theorem A now implies that the isometric embeddings of a quasi-hyperbolic $\tilde{M}^n$ into $\mathsf{E}^{2n-1}$ depend on at most $n(n-1)$ functions of one variable. In the case that $\tilde{M}^n$ is actually a hyperbolic space form, Cartan showed that this upper bound is attained. We would like to determine the conditions on a quasi-hyperbolic metric which allow us to attain this maximum. The required differential system may be described as follows:

Let $\mathfrak{B} \subseteq W \times \cdots \times W$ ($n$ times) be the submanifold of $n$-tuples $(b_1, \ldots, b_n)$ satisfying (4.29). The fact that $\mathfrak{B}$ *is* a submanifold of dimension $n(n-1)/2$ is an exercise left to the reader. We let $b_i : \mathfrak{B} \to W$ be projection on the $i$'th factor and we let $B_i = b_i \cdot b_i + 1$ denote the associated function. Note that the identities (4.30) and (4.31) automatically hold. On the manifold $\Psi = \mathscr{F}'(\tilde{M}) \times \mathsf{F} \times \mathfrak{B}$, the relevant differential system is given by

$$I\begin{cases} G - \tilde{G} = 0 \\ \omega - \tilde{\omega} = 0 \\ \theta = 0 \\ \psi - \tilde{\psi} = 0 \\ h_i - b_i\omega^i = 0 \end{cases}$$

(we have written $h_i$ for the column $(h_i^\mu)$ of height $n-1$ and we regard the $b_i$ as column vectors). The independence condition is that the $n(n+1)/2$ components of $\tilde{\omega}$ and $\tilde{\psi}$ should all remain independent. (Recall that the symmetric part of $\tilde{\psi}$ is zero mod $\tilde{\omega}$). Because $G : \mathsf{F} \to \{$symmetric positive definite matrices$\}$ is a submersion, we see that the 0-form equation $G - \tilde{G} = 0$ defines a smooth submanifold $\Psi' \subset \Psi$ and we may as well restrict $I$ to this submanifold so as to remove the 0-form equations in $I$.

Because the Gauss equations have already been solved, we easily compute that

$$\begin{cases} d(\omega - \tilde{\omega}) \equiv 0 \\ d\theta \equiv 0 \\ d(\psi - \tilde{\psi}) \equiv 0 \end{cases} \qquad \mathrm{mod}\ I$$

We now compute

$$\begin{aligned} d\left(h_i - b_i\omega^i\right) &\equiv -h_j \wedge \psi_i^j - \kappa \wedge h_i - db_i \wedge \omega^i + b_i\psi_j^i \wedge \omega^j \\ &\equiv -\left(\delta_{ij}\beta_j - (b_i - b_j)\psi_j^i\right) \wedge \omega^j \qquad \mathrm{mod}\ I \end{aligned}$$

(where we have written $\beta_j = db_j + \kappa b_j$). If we fix $i$, the fact that (4.31) is the only linear relation among the $b_j$ implies that $\{b_i - b_j \mid j \neq i\}$ is a basis for $W$. Thus we may write

$$\beta_i = \sum_j (b_i - b_j)\pi_{ij} \tag{4.33}$$

for unique 1-forms $\{\pi_{ij} \mid i \neq j\}$. The relations (4.29) differentiate to give

$$d(b_i \cdot b_j) = b_i \cdot \beta_j + b_j \cdot \beta_i = 0$$

so (4.33) implies

$$B_j \pi_{ij} + B_i \pi_{ji} = 0 \qquad i \neq j \tag{4.34}$$

Thus, at most $n(n-1)/2$ of the $\pi_{ij}$ are linearly independent. The fact that $\mathfrak{B}$ has dimension $n(n-1)/2$ shows that at least $n(n-1)/2$ of the $\pi_{ij}$ are linearly independent; consequently (4.34) constitutes the full set of linear relations among the $\{\pi_{ij} \mid i \neq j\}$. We may now write

$$d(\not{h}_i - b_i \omega^i) \equiv -\sum_j (b_i - b_j)(\pi_{ij} \wedge \omega^i - \tilde{\psi}_j^i \wedge \omega^j) \qquad \text{mod } I$$

$$\equiv -\sum_j (b_i - b_j)(\pi_{ij} \wedge \omega^i - \tilde{\nu}_j^i \wedge \omega^j - \tilde{\mu}_j^i \wedge \omega^j) \qquad \text{mod } I.$$

Now the terms $(\pi_{ij} \wedge \omega^i - \tilde{\nu}_j^i \wedge \omega^j)$ constitute the symbol part of the differential system while the terms $(\tilde{\mu}_j^i \wedge \omega^i)$ constitute the torsion since $\tilde{\mu} \equiv 0 \bmod \tilde{\omega}$. If the system is to be in involution, there must exist admissable integral elements of $I$ at every point of $\Psi'$. If $\xi^n$ is such an admissable integral element and we restrict all the forms to $\xi^n$ we see that we must have

$$\pi_{ij} \wedge \omega^i - \tilde{\nu}_j^i \wedge \omega^j = \tilde{\mu}_j^i \wedge \omega^j \qquad (i \neq j) \tag{4.35}$$

on $\xi^n$ (because for $i$, fixed, the $\{b_i - b_j \mid j \neq i\}$ form a basis of $W$). Wedging with $\omega^i$ on both sides, we see that

$$\tilde{\nu}_j^i + \tilde{\mu}_j^i \equiv 0 \qquad \text{mod } \omega^i, \omega^j \quad (i \neq j)$$

Since $\tilde{\nu}_j^i$ is skew symmetric in $i, j$ and $\tilde{\mu}_j^i$ is symmetric in $i, j$, we see that, on the integral element $\xi^n$, we have

$$\tilde{\nu}_j^i \equiv \tilde{\mu}_j^i \equiv 0 \qquad \text{mod } \tilde{\omega}^i, \tilde{\omega}^j \quad \forall i \neq j. \tag{4.36}$$

In particular, this places strong restrictions on the tensor $\mathrm{III} = \tilde{\omega}^i \circ \tilde{\omega}^j \otimes \tilde{\mu}_j^i$: We have the equation

$$\frac{\partial^3 \mathrm{III}}{\partial \tilde{\omega}^i \partial \tilde{\omega}^j \partial \tilde{\omega}^k} = 0 \qquad \{i, j, k\} \text{ distinct} \tag{4.36$'$}$$

which must hold on III in every framing in $\mathscr{F}'(M)$. By invariant theory, this set of equations (for every co-frame in $\mathscr{F}'(M)$) must define an $O(n)$-invariant subspace of the tensor bundle in which III takes values.

If $n > 3$, then III takes values in the symmetric cubic forms. Under $O(n)$,

$S^3(T^*)$ decomposes into two irreducible pieces (see Weyl, [20]):

$$S^3(T^*) \cong T^* \oplus H^3(T^*)$$

The injection $T^* \hookrightarrow S^3(T^*)$ is given by symmetric multiplication by $Q \in S^2(T^*)$ while the subspace $H^3(T^*) \subset S^3(T^*)$ consists of the so-called "harmonic forms," i.e., those who trace with respect to $Q$ is zero. Since (4.36′) implies that III must lie in *one* of these spaces and since $H^3(T^*)$ contains terms of the form $\tilde{\omega}^i \circ \tilde{\omega}^j \circ \tilde{\omega}^k$ ($i, j, k$ distinct) we see that (4.36′) implies that

$$\mathrm{III} = Q \cdot \lambda$$

where $\lambda = (\lambda_i \tilde{\omega}^i)$.

If $n = 3$, then III takes values in a bundle constructed from a representation space of $O(3)$ of dimension (15) which contains the 10 dimensional representation space $S^3(T^*)$. In fact, we have

$$V^{15} \simeq T^* \oplus H^2(T^*) \oplus H^3(T^*)$$

where $H^2 \subset S^2$ is the harmonic quadratic form space. Again by invariant theory, one shows that the linear conditions (4.36′) can hold in all frames if and only if III takes values in the bundle constructed from the $T^*$-piece. Just as before, this implies that III is cubic (i.e., symmetric in all three indices, a condition which is *not* automatic when $n = 3$). Thus (4.36′) implies that

$$\mathrm{III} = Q \cdot \lambda$$

for some $\lambda = (\lambda_i \tilde{\omega}^i)$.

Now suppose that this necessary condition is satisfied. Then we must have

$$\begin{cases} \tilde{\mu}^i_j = \lambda_i \tilde{\omega}^j + \lambda_j \tilde{\omega}^i & i \neq j \\ \tilde{\mu}^i_i = 2\lambda_i \tilde{\omega}^i + \sum\limits_j \lambda_j \tilde{\omega}^j \end{cases} \tag{4.37}$$

(to avoid fractions, we have replaced $\lambda$ by $3\lambda$).

We may now rewrite the structure equations in the form (all equations mod $I$)

$$\begin{cases} d(\omega - \tilde{\omega}) \equiv 0 \\ d\theta \equiv 0 \\ d(\psi - \tilde{\psi}) \equiv 0 \\ d(\hbar_i - b_i \omega^i) \equiv -\sum\limits_{j \neq i} (b_i - b_j)\big(\pi_{ij} \wedge \omega^i - \hat{\nu}^i_j \wedge \omega^j\big) \end{cases} \tag{4.38}$$

where, for convenience, we have set

$$\hat{\nu}^i_j = \tilde{\nu}^i_j - \lambda_i \tilde{\omega}^j + \lambda_j \tilde{\omega}^i.$$

The symbol relations, in addition to the relations forced by the form of (4.38), are

$$B_j\pi_{ij} + B_i\pi_{ji} = 0 \tag{4.39}$$

$$\hat{v}_j^i + \hat{v}_i^j = 0 \tag{4.40}$$

In other words, the "torsion has been absorbed."

We will now verify Cartan's test for involution. Let $v \in T_x\Psi'$ be a vector annihilating the 1-forms of $I$ and satisfying

$$\omega^i(v) = \xi^i \tag{4.41}$$

By (4.38), the reduced polar equations are

$$\xi^i\pi_{ij} - \xi^j\hat{v}_j^i = 0 \qquad i \neq j \tag{4.42}$$

The equations (4.39), (4.40) and (4.42) will imply the equations

$$\pi_{ij} = \hat{v}_j^i = 0 \tag{4.43}$$

so long as, for every $i \neq j$, we have

$$(\xi^i)^2 B_i \neq (\xi^j)^2 B_j. \tag{4.44}$$

In turn, this is equivalent to, for $(i \neq j)$

$$\left(\sqrt{B_i}\,\tilde{\omega}^i \pm \sqrt{B_j}\,\tilde{\omega}^j\right)(v) \neq 0 \tag{4.44$'$}$$

i.e., that $V$ not be characteristic. This is in accordance with the general theory.

Thus, we have $s_1 = n(n-1)$. Since there are no more "free" differentials in the two-forms of $I$ (remember that $\pi_{ii}$ and $\hat{v}_i^i$ do not appear) we see that $s'_\alpha = 0$ for $\alpha > 1$.

To complete Cartan's test, it is sufficient to exhibit an $n(n-1)$ parameter family of integral elements at each point of $\Psi'$. However, if $\{A_j^i \mid i \neq j\}$ is any set of $n(n-1)$ numbers, then the $n$-plane defined by the relations

$$\begin{cases} \omega - \tilde{\omega} = \theta = \psi - \tilde{\psi} = h_i - b_i\omega^i = 0 \\ \hat{v}_j^i = A_j^iB_j\tilde{\omega}^j - A_i^jB_i\tilde{\omega}^i \qquad i \neq j \\ \pi_{ij} = A_i^jB_i\omega^j - A_j^iB_i\omega^i \qquad i \neq j \end{cases} \tag{4.45}$$

is clearly an integral element. Thus, by Cartan's test, the system is involutive. We record this as

(4.46) *Let $(\tilde{M}^n, d\tilde{s}^2)$ $(n \geq 3)$ be a quasi-hyperbolic Riemannian manifold with $Q$ positive definite. The isometric embedding system for $M^n \hookrightarrow \mathbb{E}^{2n-1}$ (the lowest* *dimension possible) is involutive so that the real analytic local solutions depend on $n(n-1)$ functions of one variable (the maximum possible) if and only if the form* III *is symmetric cubic and satisfies*

$$\mathrm{III} \equiv LQ$$

*where $L$ is a linear factor.*

Of course, this raises the question of the existence of such $(\tilde{M}^n, d\tilde{s}^2)$. Obviously the hyperbolic space forms have this property, since they satisfy $\mathrm{III} = 0$. By studying the structure equations derived above for such systems it is possible to characterize these metrics completely:

(4.47) *Let $\mathsf{L}^{n+1}$ be $n+1$ dimensional Lorentz space, i.e., $\mathsf{R}^{n+1}$ endowed with an inner product of type $(+, \ldots, +, -)$. A simply-connected quasi-hyperbolic $(\tilde{M}^n, d\tilde{s}^2)$ can be isometrically immersed (uniquely up to Lorentz transformations) as a space-like hypersurface if $n > 3$ or $n = 3$ and* III *is symmetric cubic. Moreover $Q$ is positive definite if and only if the image is convex in $\mathsf{L}^{n+1}$. Finally,* $\mathrm{III} = Q.L$ *if and only if the image lies in a quadratic hypersurface.*

For example, the standard "round" hyperquadric $H^n(k) = \{x \in \mathsf{L}^{n+1} \mid \langle x, x\rangle = -1/k^2\}$ yields the classical embedding of the space forms. The other convex space-like hyperquadrics are (generally) not complete. Obviously, they define an $(n+1)$ parameter family of metrics satisfying the condition $\mathrm{III} = Q.L$.

We will omit the proof of (4.47) since it is not of direct concern to us and would require an excursion into affine geometry too lengthy to include here.

We will conclude this section by making a few remarks about the case $n = 3$. In this case, quasi-hyperbolicity is an *open* condition on the metric $d\tilde{s}^2$, equivalent to the condition that all sectional curvatures of $d\tilde{s}^2$ be negative. One can show without undue difficulty that the differential system $I$ on $\Psi' = \{\chi \in \Psi \mid (G - \tilde{G})(\chi) = 0\}$ is diffeomorphic to the standard system defined in an earlier section for isometrically embedding $\tilde{M}^3 \subset \mathsf{E}^5$ where $\tilde{M}^3$ is a negatively sectionally curved Riemannian manifold.

Our discussion has shown

(4.48) *The symbol of* (4.38) *is always involutive when $n = 3$, and that the characteristic variety of such a symbol is always six distinct points.*

This is in spite of the fact that the natural embedding dimension is one higher: The system for $\tilde{M}^3 \subset \mathsf{E}^6$ is determined.

One does not expect the general $d\tilde{s}^2$ on $\tilde{M}^3$ to embed locally in $\mathsf{E}^5$, of course, and the conditions (4.36) show that one could easily specify a metric on a neighborhood of a point in $\tilde{M}^3$ for which the equations for which the equations

$$\tilde{\mu}^1_{23} = \tilde{\mu}^2_{31} = \tilde{\mu}^3_{12} = 0$$

have no solution on the fiber in $F'(M)$ over that point. It follows that no integral elements of $I$ (which are admissible) pass through any point of $\Xi'$ lying over this point and hence that there is no local isometric embedding of $d\tilde{s}^2$ into $\mathsf{E}^5$ on a neighborhood of this point.

Thus one sees the role of the torsion equations (4.36′) in the study of isometric embeddings $M^3 \subseteq \mathsf{E}^5$. One could probably pursue this calculation to verify Cartan's claim, in [6], that the generic $M^3 \subseteq \mathsf{E}^5$ is rigid, but the relevant calculations would be quite long and tedious.

In any case, (4.46) and (4.47) clearly identify the four parameter family of metrics (with negative curvature) on an $\tilde{M}^3$ which have the maximum isometric deformability in $\mathsf{E}^5$.

## 5. The Gauss equations and the $\mathrm{GL}(n)$ representation theory for tensors.

(a) *Introduction.* In this section we study the Gauss equations (1.37):

$$\gamma(H,H)=R \quad \text{where } \gamma: W\otimes S^2V^*\otimes W\otimes S^2V^* \longrightarrow K\subset \Lambda^2V^*\otimes\Lambda^2V^*$$

and also the prolonged Gauss equations (1.48):

$$\gamma(H,G)=\nabla^qR \quad \text{where } \gamma: W\otimes S^2V^*\otimes W\otimes S^{q+2}V^* \longrightarrow K^{(q)}\subset K\otimes S^qV^*.$$

The first result, needed in a previous chapter, states that the spaces

$$K=K^{(0)},K^{(1)},K^{(2)},\dots$$

are all $\mathrm{GL}(V^*)$-irreducible and gives that

$$K^{(q)}=(K\otimes S^qV^*)\cap(K^{(1)}\otimes S^{(q-1)}V^*)$$

$$\dim K^{(q)}=\left(\frac{q+1}{q+3}\right)\binom{n}{2}\binom{n+q+1}{q+2}$$

where $n=\dim V^*$.

These results follow from the $\mathrm{GL}(V^*)$-representation theory of $\bigotimes^q V^*$. A rapid review of the basic results of that theory is the content of sections (b)–(f). This theory also yields the $\mathrm{GL}(V^*)$-decomposition

$$\mathrm{Sym}^2(S^2V^*)\cong S^4V^*\oplus K.$$

This decomposition and the $\mathrm{GL}(V^*)$-equivariance of the maps $\gamma$ provide the basis for analyzing the Gauss equations.

Our main result (Theorem H) includes the following: *Let* $R\in K$, $\dim W\geq \binom{n-1}{2}+2$. *Then there exists* $H\in W\otimes S^2V^*$ *such that* $\gamma(H,H)=R$.

Finally, classical proofs of rigidity theorems employ the following method of proof: If $H_1,H_2\in W\otimes S^2V^*$ are such that $\gamma(H_1,H_1)=\gamma(H_2,H_2)$, then $H_1=A\cdot H_2$ where $A\in \mathrm{O}(W)$, the orthogonal group for the inner product space

$W$. We give an example where a rigidity theorem holds, by the Main Theorem, but it cannot be proved by the above approach.

(b) (GL)($n$) *and symmetric group actions.* Let $V^*$ be an $n$-dimensional vector space over the field $\mathsf{F}$, where $\mathsf{F} = \mathsf{Q}$ or $\mathsf{F} =$ some field extension of $\mathsf{Q}$. Let

$$\bigotimes^q V^* = V^* \otimes V^* \otimes \cdots \otimes V^* \qquad (q \text{ copies}).$$

We shall write the action of $\mathrm{GL}(V^*)$ as a left action on $V^*$:

$$A \in \mathrm{GL}(V^*), \qquad v \in V^*, \quad A \cdot v \in V^*.$$

This induces an action of $\mathrm{GL}(V^*)$ on $\bigotimes^q V^*$:

$$\rho : \mathrm{GL}(V^*) \longrightarrow \mathrm{Aut}\Big(\bigotimes^q V^*\Big)$$

$$A \longmapsto \Big(v_1 \otimes \cdots \otimes v_q \overset{\rho(A)}{\longmapsto} A \cdot (v_1 \otimes \cdots \otimes v_q) = (Av_1) \otimes \cdots \otimes (Av_q)\Big)$$

where the action is defined on decomposable tensors and is extended by linearity.

The symmetric group $S_q$ acts on $\bigotimes^q V^*$ by permutation of the factors of a decomposable tensor in $\bigotimes^q V^*$. For example, for $q = 2$ and the transposition $(1\ 2) \in S_2$,

$$(1\ 2) \cdot (v \otimes w) = w \otimes v.$$

For general $q$ we define the action of $S_q$ on $\bigotimes^q V^*$:

$$\mathscr{O} : S_q \longrightarrow \mathrm{Aut}\Big(\bigotimes^q V^*\Big)$$

$$\pi \longmapsto (v_1 \otimes \cdots \otimes v_q) \overset{\mathscr{O}(\pi)}{\longrightarrow} \pi \cdot (v_1 \otimes \cdots \otimes v_q) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}}(q)$$

CLAIM. *The actions of* $\mathrm{GL}(V^*)$ *and* $S_q$ *on* $\bigotimes^q V^*$ *commute.*

*Proof.* Let $A \in \mathrm{GL}(V^*)$, $\pi \in S_q$

$$\begin{aligned} \pi \cdot A \cdot (v_1 \otimes \cdots \otimes v_q) &= \pi \cdot ((A\underbrace{v_1}) \otimes \cdots \otimes (Av_q)) \\ &= (Av_{\pi^{-1}\underbrace{(1)}}) \otimes \cdots \otimes (Av_{\pi^{-1}(q)}) \\ &= A \cdot (v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(q)}) \\ &= A \cdot \pi \cdot (v_1 \otimes \cdots \otimes v_q). \quad \text{Q.E.D.} \end{aligned}$$

(c) *Representations of algebras.* Let $\mathfrak{A}$ be an $\mathsf{F}$-algebra. A *representation of* $\mathfrak{A}$ is an $\mathsf{F}$-linear ring homomorphism

$$\Phi : \mathfrak{A} \dashrightarrow \operatorname{End}(W) \tag{5.1}$$

where $W$ is a vector space over $\mathsf{F}$.

The representation (5.1) is *reducible* if there exists a nontrivial, proper subspace $W' \subset W$ such that

$$(\Phi(a))(W') \subset W' \qquad \text{for all} \quad a \in \mathfrak{A}.$$

$W'$ is said to be a *$\Phi$-invariant subspace* and $\Phi$ restricts to a representation on $W'$ which we denote by $\Phi_{W'}$.

If the representation (5.1) is not reducible we say it is *irreducible.*

$\Phi$ is said to be *fully reducible* if there exist $\Phi$-invariant subspaces $W_i \subset W$, $i = 1, 2, \ldots, p$ such that each $\Phi_{W_i}$ is irreducible and $W = W_1 \oplus W_2 \oplus \cdots \oplus W_p$.

We say the representation (5.1) is *degenerate* [20] if there exists a proper subspace $W' \subset W$ such that

$$(\Phi(a))(W) \subset W' \qquad \text{for all} \quad a \in \mathfrak{A},$$

(i.e., all operators $\Phi(a)$ "push" the full space $W$ into $W'$.)

The algebra homomorphism

$$\begin{aligned} &\mathfrak{A} \dashrightarrow \operatorname{End}(\mathfrak{A}) \\ &a \longmapsto (l_a : x \longmapsto ax) \end{aligned}$$

sending an element into the operator $l_a$ (left multiplication by $a$) is called the *regular representation of* $\mathfrak{A}$.

For a finite group $\gamma$ we define the regular representation of $\gamma$ as the regular representation of the group ring $\mathsf{F}[\gamma]$, which is itself an algebra. (Our only application of this definition will be for $\gamma = S_q$.)

*Examples.* (1) Let $\mathfrak{A} \subset \operatorname{End}(\bigotimes^q V^*)$ be the subalgebra generated by the operators $\{\rho(A) \mid A \in \mathrm{GL}(V^*)\}$. $\mathfrak{A}$ is called the *enveloping algebra* of the representation $\bigotimes^q V^*$ of $\mathrm{GL}(V^*)$. Clearly a decomposition into irreducibles with respect to a group representation will also be a decomposition into irreducibles with respect to the representation of the enveloping algebra (and conversely).

Let $\mathfrak{B} \subset \operatorname{End}(\bigotimes^q V^*)$ be the enveloping algebra for the representation of the symmetric group $S_q$ on $\bigotimes^q V^*$.

(2) Let $\mathfrak{C} \subset \operatorname{End}(\bigotimes^q V^*)$ be the subalgebra defined by

$$\mathfrak{C} = \left\{ T \in \operatorname{End}(\textstyle\bigotimes^q V^*) \mid TS = ST \text{ for all } S \in \mathfrak{A} \right\}$$

$\mathfrak{C}$ is called the *commutator algebra of* $\mathfrak{A}$.

*Remarks.* The group homomorphism

$$\mathfrak{D} : S_q \dashrightarrow \mathrm{Aut}\Big( \bigotimes^q V^* \Big)$$

extends by linearity to an algebra representation of the group ring:

$$\Theta : \mathsf{F}[S_q] \dashrightarrow \mathrm{End}\Big( \bigotimes^q V^* \Big). \tag{5.2}$$

Clearly Image$(\Theta) = \mathfrak{B}$, the enveloping algebra for $S_q$. $\Theta$ is not injective, in general. As an example consider

$$\varphi \in \mathrm{End}\Big( \bigotimes^q V^* \Big), \qquad \varphi : v_1 \otimes \cdots \otimes v_q \longmapsto v_1 \wedge \cdots \wedge v_q .$$

Clearly $\varphi \equiv 0$ for $q > n$. On the other hand,

$$\varphi = n!\, \Theta\Big( \sum_{\pi \in S_q} \mathrm{sgn}(\pi)\pi \Big).$$

We are now in a position to state the sequence of results which lead to the full decomposition of $\bigotimes^q V^*$ into irreducibles with respect to $\mathfrak{A}$, the enveloping algebra of $\bigotimes^q V^*$ as a $\mathrm{GL}(V^*)$ representation. Throughout $\lambda$, $\mu, \nu$ will be subalgebras of $\mathrm{End}(W)$, $W$ an $\mathsf{F}$-vector space, $\dim W = n$.

We shall need the following construction of a representation $\lambda_m$ from a given representation $\lambda$, for $m \in \mathsf{Z}^+$: Let $\lambda_m$ be the subalgebra of $\mathrm{End}(\bigoplus_1^m W)$ consisting of operators of the form

$$T : (w_1, \ldots, w_m) \dashrightarrow \Big( \sum_j \lambda_{1j} w_j , \sum_j \lambda_{2j} w_j, \ldots, \sum_j \lambda_{mj} w_j \Big)$$

where $\lambda_{ij} \in \lambda$.

LEMMA ([20], p. 86). *If* $\lambda \subset \mathrm{End}(W)$ *is irreducible*, $\lambda \not\equiv 0$, $m \in \mathsf{Z}^+$, *then* $\lambda_m \subset \mathrm{End}(\bigoplus_1^m W)$ *is irreducible.*

*Definition* ([20], p. 90). Let $\lambda$ be an (abstract) algebra. The *inverse algebra* $\lambda'$ differs from $\lambda$ in that the multiplication of two elements $a$ and $b$ is now defined as $ba$ rather than $ab$.

The following result includes the "double commutator theorem" as well as the explicit correspondence between the decomposition of an algebra and the decomposition of its commutator algebra.

THEOREM ([20], p. 95). *Suppose* $\lambda \subset \mathrm{End}(W)$ *is a fully reducible* $\mathsf{F}$-*algebra, with commutator algebra* $\mu$. *Then* $\mu$ *is also fully reducible and* $\lambda$ *is the commutator* *of $\mu$. Moreover, their decompositions are given by:*

$$\lambda = \sum_{i=1}^{r} l_i(\nu^i)_{m_i}, \qquad \nu = \sum_{i=1}^{r} m_i((\nu^i)')_{l_i}$$

*where $\nu^i, (\nu^i)'$ are inverse (abstract) division algebras, and $l_i, m_i \in \mathbf{Z}^+$, $i = 1, \ldots, r$.*

*Remark* ([20], p. 87). The above theorem assumes the regular representation of the (abstract) division algebra, and this is irreducible.

The significance of the above theorem is that in order to decompose a fully reducible (matrix) algebra it is (essentially) sufficient to decompose its commutator algebra (i.e., it is equivalent to know $\nu^i$ or the inverse $(\nu^i)'$).

What is the cummutator algebra of $\mathfrak{A}$, the enveloping algebra of $\bigotimes^q V^*$ as a $\mathrm{GL}(V^*)$-representation? The answer is given by the following theorem.

THEOREM ([20], p. 98, p. 130). *The commutator of $\mathfrak{A}$ is $\mathfrak{B}$ where*

$$\mathfrak{B} = \mathrm{Image}\left(\Theta : \mathsf{F}[S_q] \dashrightarrow \mathrm{End}\left(\bigotimes^q V^*\right)\right).$$

Is $\mathfrak{B}$ fully reducible? The answer is yes and follows from the following two useful theorems.

THEOREM ([20], p. 89). *If the regular representation of an algebra $\lambda$ is fully reducible, with irreducible parts $\lambda_1, \lambda_2, \ldots,$ then every (nondegenerate) representation of $\lambda$ is fully reducible, and splits into irreducible parts each of which is equivalent to one of the $\lambda_i$.*

By definition, $\Theta$ is a nondegenerate representation of $\mathsf{F}[S_q]$, and hence the above theorem can be applied to conclude that $\Theta$ is fully reducible, once we have the following:

THEOREM ([20], p. 101ff). *The regular representation of the group ring $\mathsf{F}[S_q]$ is fully reducible.*

(This theorem holds more generally for the regular representation of any finite group.)

In summary, we can essentially determine the decomposition of $\mathfrak{A}$ by decomposing its commutator algebra $\mathfrak{B}$. The irreducible parts of $\mathfrak{B}$ must belong to the irreducible parts of the regular representation of $\mathsf{F}[S_q]$. Thus, we are led to consider the regular representation of $\mathsf{F}[S_q]$.

(d) *The regular representation of* $\mathsf{F}[S_q]$. Suppose

$$\mathsf{F}[S_q] = W_1 \oplus W_2 \oplus \cdots \oplus W_t$$

is the decomposition of $\mathsf{F}[S_q]$ into irreducible subspaces, and

$$P_i : \mathsf{F}[S_q] \dashrightarrow W_i, \qquad P_i P_i = P_i$$

are the projection operators onto the subspaces $W_i$, $i = 1, 2, \ldots, t$. Then

$$Id = P_1 + P_2 + \cdots + P_t, \qquad P_i P_j = \begin{cases} 0, & i \neq j \\ P_i, & i = j \end{cases} \tag{5.3}$$

The idea behind the decomposition of the regular representation of $\mathsf{F}[S_q]$ is: Can we solve (5.3) with $P_i \in \mathrm{Image}(\mathsf{F}[S_q] \dashrightarrow \mathrm{End}(\mathsf{F}[S_q]))$? Equivalently, are the projection operators $P_i$ linear combinations of left-multiplication operators? As we shall see, the answer is yes. The $P_i$ are usually called *Young Tableaux* (or *Young Symmetrizers*).

We illustrate the situation by a simple example.

*Example.* Decomposition of $\mathsf{F}[S_2]$ $(q = 2)$.

Since the decomposition of $\mathsf{F}[S_2]$ corresponds to the decomposition of $V^* \otimes V^*$ we begin with the well-known decomposition:

$$V^* \otimes V^* = S^2(V^*) \oplus \Lambda^2(V^*)$$

with projection operators

$$\pi_1 : v \otimes w \longmapsto \tfrac{1}{2}(v \otimes w + w \otimes v) \in S^2(V^*) \qquad \text{(symmetric tensors)}$$

$$\pi_2 : v \otimes w \longmapsto \tfrac{1}{2}(v \otimes w - w \otimes v) \in \Lambda^2(V^*) \qquad \text{(alternating tensors).}$$

Note that

$$\pi_1 = \Theta(\tfrac{1}{2}(Id + (1\ 2))), \qquad \pi_2 = \Theta(\tfrac{1}{2}(Id - (1\ 2))).$$

Setting

$$P_1 = \tfrac{1}{2}(Id + (1\ 2)), \qquad P_2 = \tfrac{1}{2}(Id - (1\ 2))$$

solves (5.3) for the case $q = 2$.

A crucial remark must be added. The operators $P_i$ in (5.3) must be *primitive* ([20], p. 102); i.e., we cannot further decompose

$$P_i = Q_1 + Q_2, \qquad Q_1, Q_2 \text{ nonzero projection operators.}$$

The condition $P_i$ primitive corresponds to $W_i$ irreducible.

THEOREM ([20], p. 102, p. 110). *For each positive integer $q$ there exist primitive projection operators $P_i$ in the enveloping algebra of the regular representation of $\mathsf{F}[S_q]$ solving* (5.3). *Moreover, setting*

$$\overline{W}_i = \mathrm{Image}\Big(\Theta(P_i) : \overset{q}{\otimes} V^* \dashrightarrow \overset{q}{\otimes} V^*\Big) \tag{5.4}$$

*$\overline{W}_i$ is an irreducible, invariant subspace of $\bigotimes^q V^*$ with respect to* $\mathrm{GL}(V^*)$. *Any* $\mathrm{GL}(V^*)$*-invariant subspace of $\bigotimes^q V^*$ is decomposable into irreducible subspaces, each of which is similar to one of the spaces $\overline{W}_i$.*

We now describe explicitly the Young Symmetrizers (5.3).

*Definition.* (1) Let $q \in Z^+$. A *partition* $\lambda$ of $q$ is a tuple $\lambda = (\lambda_1, \dots, \lambda_m)$, $\lambda_i \in Z^+$ such that

$$\sum_{i=1}^{m} \lambda_i = q, \qquad \lambda_1 \geq \cdots \geq \lambda_m > 0.$$

(2) If $\lambda$ is a partition of $q$ then $\lambda'$ is a partition of $q$. (Take the transpose of the $\lambda$-diagram given below.)

(3) If $\lambda$ is a partition of $q$, a $\lambda$-*Symmetrizer* is a diagram of the form

$$T_\sigma^\lambda = \begin{array}{|c|c|c|c|c|}
\hline
i_1 & i_2 & \cdots & \cdots & i_{\lambda_1} \\
\hline
i_{\lambda_1+1} & \cdots & i_{\lambda_1+\lambda_2} \\
\cline{1-3}
i_{\lambda_1+\lambda_2+1} & \\
\cline{1-2}
 & \\
\cline{1-2}
 \\
\cline{1-1}
i_q \\
\cline{1-1}
\end{array}$$

where $\sigma = \binom{1\ 2\ \cdots\ q}{i_1 i_2 \cdots i_q}$ is a permutation of $\{1, 2, \dots, q\}$, i.e., $\sigma \in A_q$, the permutation group on $\{1, \dots, q\}$.

*Example.*

$$q = 9, \qquad \lambda = (4, 3, 1, 1), \qquad \sigma = \begin{pmatrix} 123456789 \\ 256431798 \end{pmatrix}$$

$$T_\sigma^\lambda = \begin{array}{|c|c|c|c|}
\hline
2 & 5 & 6 & 4 \\
\hline
3 & 1 & 7 \\
\cline{1-3}
9 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}$$

We now associate to $T_\sigma^\lambda$ a projection operator.

Given a $\lambda$-symmetrizer $T_\sigma^\lambda$, we shall define subgroups of the symmetric group $S_q$ (where $q = \sum_i \lambda_i$; $\lambda$ a partition of $q$). Let $r_i$ be the set of elements in the $i$th row of $T_\sigma^\lambda$. Let $c_j$ be the set of elements in the $j$th column of $T_\sigma^\lambda$. Thus, if

$$\sigma = \begin{pmatrix} 1\ 2 \cdots q \\ i_1 i_2 \cdots i_q \end{pmatrix}$$

then

$$r_1 = \{i_1, i_2, \ldots, i_{\lambda_1}\},$$
$$r_2 = \{i_{\lambda_1+1}, \ldots, i_{\lambda_1+\lambda_2}\}, \ldots \text{ etc.}$$

and

$$c_1 = \{i_1, i_{\lambda_1+1}, \ldots, i_{\lambda_1+\cdots+\lambda_{(m-1)}+1}\},$$
$$c_2 = \{i_2, i_{\lambda_1+2}, \ldots\}, \ldots \text{ etc.}$$

We define the $\{R_i\}$ to be the subgroups of $S_q$ given by permutations of the elements in the $i$th row, all other elements being left fixed

$$R_i = \text{symmetric group on } r_i \subset S_q .$$

We define the *row stabilizer* $R = R(\lambda, \sigma)$ to be the subgroup of $S_q$ generated by $\{R_i\}$.

Similarly we define the $\{C_j\}$ to be the subgroups of $S_q$ given by permutation of the elements of the $j$th column.

$$C_j = \text{symmetric group on } c_j \subseteq S_q$$

and we define the *column stabilizer* $C = C(\lambda, \sigma)$ to be the subgroup of $S_q$ generated by $\{C_j\}$.

Define

$$\tilde{T}_\sigma^\lambda \in \mathsf{F}[S_q]$$

$$\tilde{T}_\sigma^\lambda = \sum_{\substack{\tau \in R \\ \pi \in C}} (-1)^{sgn(\pi)} \pi \circ \tau \in \mathsf{F}[S_q]$$

where $sgn(\pi)$ = the sign of the permutation $\pi$. We remark that this definition is equivalent to the following:

$$T_\sigma^\lambda = \Big( \sum_{\pi \in C} (-1)^{sgn(\pi)} \pi \Big) \circ \Big( \sum_{\tau \in R} \tau \Big) \tag{5.5}$$

where composition is in the sense of $\mathsf{F}[S_q]$.

*Example.*

$$q = 4, \qquad \lambda = (\lambda_1, \lambda_2) = (2,2), \qquad \sigma = \begin{pmatrix} 1234 \\ 1324 \end{pmatrix}$$

$$T_\sigma^\lambda = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$R = I, (13), (24), (13)(24)$$

$$C = I, (12), (34), (12)(34).$$

Hence

$$\begin{aligned}\tilde{T}_\sigma^\lambda &= (I-(12)-(34)+(12)(34))(I+(13)+(24)+(13)(24))\\ &= I+(13)+(24)-(12)-(34)\\ &\quad -(132)-(124)-(143)-(234)\\ &\quad +(12)(34)+(13)(24)+(14)(23)\\ &\quad +(1432)+(1234)-(1324)-(1423).\end{aligned}$$

It can be shown ([20], p. 124) that

$$\tilde{T}_\sigma^\lambda \circ \tilde{T}_\sigma^\lambda = c\tilde{T}_\sigma^\lambda, \qquad c \neq 0.$$

Hence $(1/c)\tilde{T}_\sigma^\lambda$ is a projection operator.

THEOREM ([20], p. 127). *$P$ is a Young Symmetrizer (see (5.3)) if and only if $cP$ is of the form (5.5) for some constant $c \neq 0$.*

(e) $\mathrm{GL}(V^*)$*-irreducible subspaces of* $\bigotimes^q V^*$. Using (5.4) we shall write

$$V^{*(\lambda)} = V^{*(\lambda_1,\lambda_2,\ldots,\lambda_m)} = \mathrm{Image}\left(\Theta(\tilde{T}_\sigma^\lambda) : \bigotimes^q V^* \dashrightarrow \bigotimes^q V^*\right). \tag{5.6}$$

We have omitted to mention the permutation $\sigma$ in the left-hand side of (5.6). If $\sigma$ is not clear from context we shall write $V_\sigma^{*(\lambda)}$. Part (iv) of the following theorem states that up to equivalence, $\sigma$ can be disregarded. Even with this notation two representations are unambiguously defined, namely

(a) $\lambda = (\lambda_1) = (q) \Rightarrow V^{*(\lambda)} = \mathrm{Sym}^q(V^*) = \{\text{symmetric tensors}\}$

(b) $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q) = (1, 1, \ldots, 1) \Rightarrow V^{*(\lambda)} = \Lambda^q V^* = \{\text{alternating tensors}\}$

(Compare the following theorem with (5.4).)

THEOREM ([20], p. 133). *Let $\lambda$ be a partition of $q$.*

(i) *$V^{*(\lambda)}$ is a $\mathrm{GL}(V^*)$-irreducible subspace of $\bigotimes^q V^*$, etc.*

(ii) *Any $\mathrm{GL}(V^*)$-irreducible subspace of $\bigotimes^q V^*$ is of the form $V^{*(\lambda)}$ for some partition $\lambda$ of $q$.*

(iii) *If $V^{*(\lambda)} = V_\sigma^{*(\lambda)}$, then there exists a constant $c > 0$ such that*

$$c\Theta(\tilde{T}_\sigma^\lambda)$$

*is a projection operator onto $V^{*(\lambda)}$.*

(iv) *If $\sigma, \pi \in S_q$, then $V_\sigma^{*(\lambda)}$ and $V_\pi^{*(\lambda)}$ are equivalent $\mathrm{GL}(V^*)$-representations.*

(v)

$$\dim V^{*(\lambda)} = \left(\prod_{(i,j)\in\mathfrak{F}_\lambda} (n+j-1)\right)\left(\prod_{(i,j)\in\mathfrak{F}_\lambda} (d_{ij})\right)$$

*where* $\mathfrak{F}_\lambda = \{(i,j) \mid (i,j)$ *is a position in the diagram* $\lambda\}$ *and* $d_{ij} = \lambda_i + (\lambda')_j - (i + j) + 1$.

Example of (v): $\lambda = (2,2) = \lambda'$, $\mathfrak{F}_\lambda = \{(1,1),(1,2),(2,1),(2,2)\}$

$$\begin{aligned}\dim V^{*(\lambda)} &= n(n+1)(n-1)n/(3\cdot 2\cdot 2\cdot 1)\\ &= n^2(n^2-1)/12.\end{aligned}$$

(f) *Decomposition of the tensor product: The Littlewood–Richardson Rule.* If $\rho_i : G \dashrightarrow \mathrm{Aut}(W_i)$, $i = 1,2$ are two representations of the group $G$, we can construct the tensor product representation

$$\rho_1 \otimes \rho_2 : G \dashrightarrow \mathrm{Aut}(W_1 \otimes W_2)$$

$$a \longmapsto (w_1 \otimes w_2) \xrightarrow{(\rho_1 \otimes \rho_2)(a)} (\rho_1(a)(w_1) \otimes \rho_2(a)(w_2))$$

Let $W_1 \otimes W_2$ denote this representation.

What is the decomposition into irreducibles of the $\mathrm{GL}(V^*)$-representation $V^{*(\lambda)} \otimes V^{*(\mu)}$ where $\lambda$, $\mu$ are partitions of $r, q-r$ respectively? Using the isomorphism

$$\left(\bigotimes^{r} V^*\right) \otimes \left(\bigotimes^{q-r} V^*\right) \xrightarrow{\approx} \bigotimes^{q} V^*$$

it is clear that in principle the answer can be stated in terms of Young Symmetrizers of $\bigotimes^q V^*$. The *Littlewood–Richardson Rule* ([14], pp. 60ff) is an algorithm that computes the decomposition of $V^{*(\lambda)} \otimes V^{*(\mu)}$. In practice, it is carried out with the aid of diagrams. To best convey the algorithm an example is calculated step-by-step, alongside the description of the algorithm steps.

We assume given two partitions

$$\lambda = (\lambda_1, \ldots, \lambda_k), \qquad \mu = (\mu_1, \ldots, \mu_l)$$

of $r, q-r$ respectively, which for our example we take to be

$$\lambda = (3,2), \qquad \mu = (2,2) \qquad (\text{Note } l = 2.)$$

*Step* 1. Draw the diagram of $\lambda$ with $x$'s in each position. Draw the diagram of $\mu$ with each position in the $j$'th row filled by the integer $j$.

$$\lambda \dashrightarrow \begin{matrix} xxx \\ xx \end{matrix}, \qquad \mu \dashrightarrow \begin{matrix} 11 \\ 22 \end{matrix}.$$

*Step* 2. Create new diagrams by adjoining the 1's in the diagram for $\mu$ in step 1 onto the diagram of $\lambda$ in step 1, subject to the restrictions

(a) each resulting diagram must correspond to a partition

(i.e., row-lengths are nonincreasing)

(b) no column can have more than a single “1”.

| | | | | |
|---|---|---|---|---|
| *xxx*11, | *xxx*1, | *xxx*1, | *xxx*, | *xxx* |
| *xx* | *xx*1 | *xx* | *xx*1 | *xx* |
| | | 1 | 1 | 11 |

(Note that

*xxx*
*xx*11

would violate (a) and

*xxx*
*xx*
1
1

would violate (b).)

*Step* $k+1$ $(k \leq l-1)$. For each diagram $D$ created in step $k$, create new diagrams by adjoining the $(k+1)$'s in the diagram for $\mu$ in step 1 onto the diagram $D$ subject to the restrictions

(a) each resulting diagram must correspond to a partition
(b) no column can have more than a single “$k+1$”.

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | √ | √ | √ |
| *xxx*11 ----→ | *xxx*1122, | *xxx*112, | *xxx*1112, | *xxx*11, | *xxx*11, | *xxx*11 |
| *xx* | *xx* | *xx*2 | *xx* | *xx*22 | *xx*2 | *xx* |
| | | | 2 | | 2 | 22 |

| | | | | | |
|---|---|---|---|---|---|
| | | | | √ | √ |
| *xxx*1 -----→ | *xxx*122, | *xxx*12, | *xxx*12, | *xxx*1, | *xxx*1 |
| | *xx*1 | *xx*12 | *xx*1 | *xx*12 | *xx*1 |
| | | | 2 | 2 | 22 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | √ | √ |
| *xxx*1 -----→ | *xxx*122, | *xxx*12, | *xxx*12, | *xxx*12, | *xxx*1, | *xxx*1, | *xxx*1, | *xxx*1 |
| *xx* | *xx* | *xx*2 | *xx* | *xx* | *xx*22 | *xx*2 | *xx*2 | *xx* |
| 1 | 1 | 1 | 12 | 1 | 1 | 12 | 1 | 12 |
| | | | | 2 | | | 2 | 2 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | √ |
| *xxx* ------→ | *xxx*22, | *xxx*2, | *xxx*2, | *xxx*2, | *xxx*, | *xxx* |
| *xx*1 | *xx*1 | *xx*12 | *xx*1 | *xx*1 | *xx*1 | *xx*1 |
| 1 | 1 | 1 | 12 | 1 | 122 | 12 |
| | | | | 2 | | 2 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | √ |
| *xxx* ------→ | *xxx*22, | *xxx*2, | *xxx*2, | *xxx*, | *xxx*, | *xxx* |
| *xx* | *xx* | *xx*2 | *xx* | *xx*2 | *xx*2 | *xx* |
| 11 | 11 | 11 | 11 | 112 | 11 | 11 |
| | | | 2 | | 2 | 22 |

*Step* $l+1$. For each diagram $D$ created in step $l$, construct the sequence $d(D) = (d_1, d_2, \ldots, d_q)$, consisting of integers and $x$'s, obtained by reading the first row of $D$ from *right to left*, then the second row of $D$ from right to left, . . . etc. Since the $x$'s will be ignored in the consideration of these sequences we will use a shorthand that omits them. e.g.

$$\begin{array}{l} xxx1122 \dashrightarrow (2,2,1,1,x,x,x,x,x) \dashrightarrow (2,2,1,1) \\ xx \end{array}$$

$$\begin{array}{l} xxx11 \dashrightarrow (1,1,x,x,x,2,2,x,x) \dashrightarrow (1,1,2,2) \\ xx22 \end{array}$$

$$\begin{array}{l} xxx \dashrightarrow (x,x,x,1,x,x,2,1,2) \dashrightarrow (1,2,1,2) \\ xx1 \\ 12 \\ 2 \end{array}$$

(We leave out the rest of the sequences. The procedure is clear.) *Disqualify* $D$ if there are some integers $m$, $p$ such that $d_m(D) = p > 1$ and

$$\#\{d_j(D) \mid j < m \text{ and } d_j(D) = p-1\} \leq \#\{d_j(D) \mid j < m \text{ and } d_j(D) = p\}.$$

If $D$ is not disqualified, we say $D$ is *retained*. e.g.

$$\begin{array}{ll} D = xxx1122 \dashrightarrow (2,2,1,1) & \text{Disqualify } D \\ \quad\;\; xx & \end{array}$$

$$\begin{array}{ll} \quad\;\; xxx11 \dashrightarrow (1,1,2,2) & \text{Retain } D \\ \quad\;\; xx22 & \end{array}$$

$$\begin{array}{ll} D = xxx \dashrightarrow (1,2,1,2) & \text{Retain } D \\ \quad\;\; xx1 & \\ \quad\;\; 12 & \end{array}$$

We have placed a "$\surd$" above the retained diagrams (see step 2 above.)

*Step* $l+2$. For each diagram $D$ in step 1, let $\nu_D$ denote the partition corresponding to the diagram. Each retained $D$ contributes $V^*(\nu_D)$ to the irreducible $\mathrm{GL}(V^*)$-decomposition of $V^{*(\lambda)} \otimes V^{*(\mu)}$.

$$\begin{aligned} V^{*(3,2)} \otimes V^{*(2,2)} \cong\; & V^{*(5,4)} \oplus V^{*(5,3,1)} \oplus V^{*(5,2,2)} \oplus V^{*(4,4,1)} \\ & \oplus V^{*(4,3,2)} \oplus V^{*(4,3,1,1)} \oplus V^{*(4,2,2,1)} \\ & \oplus V^{*(3,3,2,1)} \oplus V^{*(3,2,2,2)} \end{aligned}$$

(g) *The spaces* $K = K^{(0)}, K^{(1)}, \ldots$

The space of curvature tensors, $K$, has been defined as

$$K = \mathrm{Ker}(\partial : \Lambda^2 V^* \otimes \Lambda^2 V^* \dashrightarrow V^* \otimes \Lambda^3 V^*)$$

$$(u \wedge v) \otimes (z \wedge w) \dashrightarrow u \otimes (v \wedge z \wedge w) - v \otimes (u \wedge z \wedge w).$$

Clearly $\partial$ is a $\mathrm{GL}(V^*)$-equivariant, linear map; i.e., for any $A \in \mathrm{GL}(V^*)$ the following is a commutative diagram.

$$\begin{tikzcd} \Lambda^2 V^* \otimes \Lambda^2 V^* \arrow[r, dashed, "\partial"] \arrow[d, "A\cdot"'] & V^* \otimes \Lambda^3 V^* \arrow[d, "A\cdot"] \\ \Lambda^2 V^* \otimes \Lambda^2 V^* \arrow[r, dashed, "\partial"] & V^* \otimes \Lambda^3 V^*. \end{tikzcd}$$

PROPOSITION. *The following are irreducible decompositions with respect to* $\mathrm{GL}(V^*)$:

(a) $\Lambda^2 V^* \otimes \Lambda^2 V^* \cong V^{*(2,2)} \oplus V^{*(2,1,1)} \oplus V^{*(1,1,1,1)}$

(b) $V^* \otimes \Lambda^3 V^* \cong V^{*(2,1,1)} \oplus V^{*(1,1,1,1)}$

*Proof.* Note that the partition $\Lambda = (1, 1, \ldots, 1)$ of $q$ corresponds to the irreducible representation $\Lambda^q(V^*)$. Now apply the Littlewood–Richardson Rule. Q.E.D.

COROLLARY. $K \cong V^{*(2,2)}$; *hence $K$ is irreducible*, $\dim K = n^2(n^2-1)/12$.

*Proof.* We can rewrite $\partial$ as

$$\partial : V^{*(2,2)} \oplus V^{*(2,1,1)} \oplus V^{*(1,1,1,1)} \dashrightarrow V^{*(2,1,1)} \oplus V^{*(1,1,1,1)}.$$

Since $\partial$ is equivariant it respects these irreducible decompositions; i.e., $\partial$ restricted to an irreducible subspace is either identically zero or is an isomorphism to an equivalent irreducible space. The formula

$$u \otimes (v \wedge w \wedge z) = \tfrac{1}{2}(\partial((u \wedge v) \otimes (z \wedge w) - (u \wedge w) \otimes (v \wedge z)$$
$$- (v \wedge w) \otimes (u \wedge z)))$$

shows that $\partial$ is surjective onto $V^* \otimes \Lambda^3 V^*$ and hence

$$K = \operatorname{Ker} \partial \cong V^{*(2,2)}.$$

The dimension statement follows from part (v) in the theorem in section e. Q.E.D.

The space $K^{(1)}$ has been defined as

$$\begin{aligned} K^{(1)} = K \otimes V^* \cap \operatorname{Ker}\{\partial^{(1)} &: \Lambda^2 V^* \otimes \Lambda^2 V^* \otimes V^* \dashrightarrow \Lambda^2 V^* \otimes \Lambda^3 V^*\} \\ (u \wedge v) \otimes (z \wedge w) \otimes t &\longmapsto (u \wedge v) \otimes (z \wedge w \wedge t). \end{aligned} \tag{5.7}$$

By abuse of notation let $\partial^{(1)}$ also denote $\partial^{(1)}$ restricted to $K \otimes V^*$. Then

$$K^{(1)} = \operatorname{Ker}\{\partial^{(1)} : K \otimes V^* \dashrightarrow \Lambda^2 V^* \otimes \Lambda^3 V^*\}. \tag{5.8}$$

By the Littlewood–Richardson Rule, we rewrite (5.8) as

$$K^{(1)} = \mathrm{Ker}\{\partial^{(1)} : V^{*(3,2)} \oplus V^{*(2,2,1)} \dashrightarrow V^{*(2,2,1)} \oplus V^{*(2,1,1,1)} \oplus \Lambda^5 V^*\}.$$

Proposition. *$K^{(1)} \cong V^{*(3,2)}$; hence $K^{(1)}$ is irreducible,*

$$\dim K^{(1)} = \frac{n^2(n^2-1)(n+2)}{24}.$$

*Proof.* Following the reasoning of the preceding proposition, it suffices to show $K^{(1)} \neq \{0\}$. We assume $\dim V^* = n \geq 3$. Let $u, v, w \in V^*$ be linearly independent. Then $(u \wedge v) \otimes (u \wedge v) \in K$, and

$$\partial^{(1)}((u \wedge v) \otimes (u \wedge v) \otimes w) = (u \wedge v) \otimes (u \wedge v \wedge w) \neq 0. \quad \text{Q.E.D.}$$

In general, we define $K^{(q)}$ as the image of the $\mathrm{GL}(V^*)$-equivariant map

$$\gamma : S^2V^* \otimes S^{q+2}V^* \dashrightarrow K \otimes S^qV^* \subset \bigotimes^{q+4} V^* \tag{5.9}$$

$$\begin{aligned}(\gamma(H \otimes G))(w_1, \ldots, w_4, v_1, \ldots, v_q) = \tfrac{1}{2}H(w_1, w_3)G(w_2, w_4, v_1, \ldots, v_q)\\ + H(w_2, w_4)G(w_1, w_3, v_1, \ldots, v_q)\\ - H(w_1, w_4)G(w_2, w_3, v_1, \ldots, v_q)\\ - H(w_2, w_3)G(w_1, w_4, v_1, \ldots, v_q)\end{aligned}$$

The Littlewood–Richardson Rule implies

$$S^2V^* \otimes S^{q+2}V^* \cong S^{q+4}V^* \oplus V^{*(q+3,1)} \oplus V^{*(q+2,2)}$$

and

$$K \otimes S^qV^* \cong S^qV^* \otimes V^{*(2,2)} \cong V^{*(q+2,2)} \oplus V^{*(q+1,2,1)} \oplus V^{*(q,2,2)}.$$

Thus, it suffices to show $\gamma \neq 0$ to conclude that $K^{(q)} \cong V^{*(q+2,2)}$. Consider $H = u^* \otimes u^*$, $G = v^* \otimes v^* \otimes \cdots \otimes v^*$, $u, v \in V$ such that $u^*(u) = v^*(v) = 1$, $u^*(v) = v^*(u) = 0$. Then

$$\begin{aligned}\gamma(H \otimes G)(u, v, u, v, v, \ldots, v) &= \tfrac{1}{2}((u^* \otimes u^*)(u, u)(v^* \otimes \cdots \otimes v^*)(v, \ldots, v))\\ &\neq 0.\end{aligned}$$

Thus $(H \otimes G) \neq 0$ in $\bigotimes^{q+4} V^*$, hence $\neq 0$. We have thus proved the first part of the following proposition.

PROPOSITION. $K^{(q)} \cong V^{*(q+2,2)}$; *hence* $K^{(q)}$ *is irreducible and*

$$\dim K^{(q)} = \left(\frac{q+1}{q+3}\right)\binom{n}{2}\binom{n+q+1}{q+2}.$$

*Moreover,*

$$K^{(q)} = (K \otimes S^q V^*) \cap (K^{(1)} \otimes S^{q-1} V^*)$$

For the second part we decompose

$$\begin{aligned} K^{(1)} \otimes S^{q-1}V^* &= V^{*(3,2)} \otimes S^{q-1}V^* \\ &= V^{*(q+2,2)} \oplus V^{*(q+1,3)} \oplus V^{*(q+1,2,1)}. \end{aligned}$$

Comparing this with the decomposition for $K \otimes S^q V^*$ we find that $V^{*(q+2,2)}$ is the only common factor. Q.E.D.

(h) *The Gauss equations*: *An equivariant approach.* Let $(W, \langle\, , \rangle)$ be a vector space of $\mathbb{R}$, of dimension $r$, with inner product $\langle\, , \rangle$. Recall equation (1.36)

$$\gamma : (W \otimes S^2V^*) \times (W \otimes S^2V^*) \dashrightarrow K \subset \bigotimes^4 V^* \tag{5.10}$$

defined for $H, G \in W \otimes S^2V^*$ by the condition

$$\begin{aligned} (\gamma(H,G))(v_1, v_2, v_3, v_4) = \tfrac{1}{2}\big\langle H(v_1,v_3), G(v_2,v_4)\big\rangle + \big\langle H(v_2,v_4), G(v_1,v_3)\big\rangle \\ - \big\langle H(v_1,v_4), G(v_2,v_3)\big\rangle + \big\langle H(v_2,v_3), G(v_1,v_4)\big\rangle \end{aligned} \tag{5.10'}$$

where the $v_i$ are arbitrary elements of $V$.

At the first prolongation of the differential system for the isometric embedding problem $(\tilde{M}, ds_M^2) \dashrightarrow E^{n+r}$, the torsion consists of the *Gauss equations*

$$\gamma(H,H) = R_M \tag{5.11}$$

where $R_M \in K \subset \bigotimes^4 T^*M$ is the Riemann curvature tensor (all indices lowered).

Part (ii) of Proposition (2.29) shows that (5.11) can always be solved for $H$ when $r \geq \binom{n}{2}$; i.e., (5.10) is surjective for $r \geq \binom{n}{2}$. In this section we shall show that (5.10) is surjective when $r \geq \binom{n-1}{2} + 2$. The proof uses the equivariance of $\gamma$ by decomposing the spaces appearing in (5.10) into $GL(V^*)$-irreducibles. These decompositions also yield a qualitative description of the solution space of (5.11).

Finally, the special case of $r = 2$ is considered. We show that

$$\operatorname{codim}\big(\{\gamma(H,H) \in K \mid H \in W \otimes S^2V^*\}\big) = \begin{cases} 2, & n = 4 \\ 1, & n \geq 5. \end{cases}$$

The result for $n = 4$ is somewhat surprising, since a näive dimension count would predict a codimension of 1.

Our discussion proceeds in two parts.

(h.1) *Equivariant factoring of* $\gamma$. Define

$$\begin{aligned} f : W \otimes S^2V^* &\dashrightarrow \mathrm{Sym}^2(S^2V^*) \\ w \otimes h &\mapsto \langle w, w\rangle h \otimes h. \end{aligned} \tag{5.12}$$

Then $f$ is $\mathrm{GL}(V^*)$-equivariant, quadratic, and $O(W)$-invariant. The Littlewood–Richardson Rule shows that

$$S^2V^* \otimes S^2V^* \cong V^{*(4)} \oplus V^{*(2,2)} \oplus V^{*(3,1)}. \tag{5.13}$$

Note that

$$(u \otimes u) \otimes (u \otimes u) \in \mathrm{Sym}^2(S^2V^*) \cap S^4V^*. \tag{5.14}$$

Also, if $T$ denotes the Young Symmetrizer $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ (see the example in 5(d), then

$$T((u \otimes u) \otimes (v \otimes v) + (v \otimes v) \otimes (u \otimes u)) = (u \wedge v) \otimes (u \wedge v).$$

Thus

$$T : \mathrm{Sym}^2(S^2V^*) \dashrightarrow K \qquad \text{is surjective.} \tag{5.15}$$

From (5.13)–(5.15) we conclude

$$\mathrm{Sym}^2(S^2V^*) \cong S^4V^* \oplus K. \tag{5.16}$$

**Proposition.** *The following diagram is commutative*

$$\begin{tikzcd} W \otimes S^2V^* \arrow[rr, dashed, "\gamma"] \arrow[dr, "f"'] & & K \subset S^2V^* \otimes S^2V^* \\ & \mathrm{Sym}^2(S^2V^*) \cong S^4V^* \oplus V^{*(2,2)} \arrow[ur, "cT"'] & \end{tikzcd} \tag{5.17}$$

*for some constant* $c \neq 0$, *where $T$ is linear*, $\mathrm{GL}(V^*)$-*equivariant*, $T|_{S^4V^*} \equiv 0$, *and $T$ is surjective*.

(By abuse of notation we have used $\gamma(H)$ here to mean $\gamma(H, H)$—see (5.10).)

*Proof.* Without loss of generality we can assume $r = \dim W = 1$. We shall use the fact that $h \in S^2V^*$ implies there exist $\varphi^i \in V^*$, $a_i \in \mathsf{F}$ such that $h$

$= \sum_i a_i \varphi^i \otimes \varphi^i$; i.e., $h$ can be "diagonalized". Now we compute

$$
\begin{aligned}
(Tf)(h)(v_1, v_2, v_3, v_4) &= T(h \otimes h)(v_1, v_2, v_3, v_4) \\
&= T\Big(\sum_{i,j} a_i a_j \varphi^i \otimes \varphi^i \otimes \varphi^j \otimes \varphi^j\Big)(v_1, v_2, v_3, v_4) \\
&= -8 \sum_{i,j} a_i a_j (\varphi^i \wedge \varphi^j) \otimes (\varphi^i \wedge \varphi^j)(v_1, v_2, v_3, v_4) \\
&= C \sum_{i,j} a_i a_j \big(\varphi^i(v_1)\varphi^j(v_2) - \varphi^i(v_2)\varphi^j(v_1)\big) \cdot \\
&\qquad \cdot \big(\varphi^i(v_3)\varphi^j(v_4) - \varphi^i(v_4)\varphi^j(v_3)\big) \\
&= C \sum_{i,j} a_i a_j \big(\big[\varphi^i(v_1)\varphi^i(v_3)\big]\big[\varphi^j(v_2)\varphi^j(v_4)\big] - \cdots \\
&= C\big[h(v_1, v_3)h(v_2, v_4) - h(v_1, v_4)h(v_2, v_3)\big] \\
&= C \cdot \gamma(h, h) \qquad \text{(using the notation of (5.10))} \\
&= C \cdot \gamma(h) \qquad \text{(using the notation of the statement of the proposition)} \qquad \text{Q.E.D.}
\end{aligned}
$$

(h.2) THEOREM H. *$\gamma$. $W \otimes S^2V^* \dashrightarrow K$ is surjective if* $\dim W \geq \binom{n-1}{2} + 2$. *Each fiber contains a point where $\gamma$ has maximal rank.*

*Proof.* Without loss of generality we fix $r = \binom{n-1}{2} + 2$. Since $\gamma$ is a quadratic map, the image of $\gamma$ is a positive cone in $K$, hence it suffices to show Image($\gamma$) contains a neighborhood of 0 in $K$. This will follow from the implicit function theorem, if we can find a point $H_0 \in W \otimes S^2V^*$ satisfying

(i) $\gamma(H_0) = 0$

(ii) $\operatorname{rank}(d\gamma(H_0)) = \dim K = n^2(n^2 - 1)/12$.

By abuse of notation, we identify the tangent space of a linear space with the space itself. Differentiating (5.17) thus leads to the commutative diagram

$$
\begin{tikzcd}
W \otimes S^2V^* \arrow[rr, dashed, "d\gamma(H_0)"] \arrow[dr, "df(H_0)"'] & & K \\
& \operatorname{Sym}^2(S^2V^*) \simeq S^4V^* \oplus V^{*(2,2)} \arrow[ur, "cT"'] &
\end{tikzcd}
\tag{5.17'}
$$

The idea of the proof is to show $d\gamma(H_0)$ surjective by showing

$$\big(df(H_0)(W \otimes S^2V^*)\big) + S^4V^* = \operatorname{Sym}^2(S^2V^*).$$

Let $W$ have orthonormal basis $\{t_\mu\}$, $1 \leqq \mu \leqq r$. Given $H_0, G \in W \otimes S^2V^*$ we can then write

$$H_0 = \sum_{\mu} t_\mu \otimes v^\mu, \qquad G = \sum_{\mu} t_\mu \otimes z^\mu$$
$$(df(H_0))(G) = 2\sum_{\mu} v^\mu \circ z^\mu \tag{5.18}$$

where we have identified $W \otimes S^2V^*$ and $T_{H_0}(W \otimes S^2V^*)$. Let $\{e^i\}$, $1 \leqq i \leqq n$, be a basis for $V^*$. Let

$$\beta^{ij,kl} = (e^i \circ e^j) \otimes (e^k \circ e^l) + (e^k \circ e^l) \otimes (e^i \circ e^j).$$

Then

$$\mathrm{Sym}^2(S^2V^*) = \mathrm{span}\{\ \beta^{ij,kl}\ \} \tag{5.19a}$$

$$\beta^{ij,kl} = \beta^{ji,kl} = \beta^{ij,lk} = \beta^{ji,lk} = \beta^{kl,ij}$$
$$= \beta^{lk,ij} = \beta^{kl,ji} = \beta^{lk,ji} \tag{5.19b}$$

$$\beta^{ij,kl} + \beta^{ik,jl} + \beta^{il,jk} \in S^4V^*. \tag{5.19c}$$

Relabel the indices $\mu = 1, 2, \ldots, r$ by the $r$ labels

$$\{(ij) \mid 1 \leq i < j \leq n-1\} \cup \{n-1, n\}.$$

Define the following vectors $v^\mu \in S^2V^*$

$$v^{(ij)} = e^i \circ e^j, \qquad 1 \leq i < j \leq n-1$$
$$v^{n-1} = e^1 \otimes e^1 + e^2 \otimes e^2 + \cdots + e^{n-1} \otimes e^{n-1}$$
$$v^n = \left(\sum_{i=2}^{n} e^i\right) \circ \left(\sum_{j=2}^{n} e^j\right).$$

Set

$$H_0 = \sum_{\mu} t_\mu \otimes v^\mu.$$

(i)

$$\gamma(H_0) = \gamma\left(\sum_{\mu} t_\mu \otimes v^\mu\right) = \sum \gamma(v^\mu) \qquad \text{(because of the orthonormality of the } \{t_\mu\})$$

$$= \sum_{1 \leq i < j \leq n} \gamma(e^i \circ e^j) + \gamma(v^{n-1}) = Id_* + Id_* = 0 \tag{5.20}$$

where $Id_* \in \Lambda^2 V^* \otimes \Lambda^2 V^*$,

$$(Id_*)_{(ij,kl)} = \begin{cases} \delta^{(ij)}_{(kl)} & \text{for} \quad 1 \le i < j \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Id_* = \sum_{\substack{1 \le i < j \le n \\ 1 \le k < l \le n}} (Id_*)_{(ij,kl)} (e^i \wedge e^j) \otimes (e^k \wedge e^l)$$

(ii) As remarked above, it remains to show

$$\big(df(H_0)(W \otimes S^2V^*)\big) + S^4V^* = \mathrm{Sym}^2(S^2V^*) \tag{5.21}$$

and by (5.19), it suffices to show

$$\beta^{ij,kl} \in (df(H_0)(W \otimes S^2V^*) + S^4V^*, \qquad i \ne j, \quad k \ne l. \tag{5.22}$$

*1st Case.* $(5.22)_{ij,kl}$ when $n \notin \{i,j\}$ or $n \notin \{k,l\}$.
Say $n \notin \{i,j\}$. Then $\beta^{ij,kl} = (e^i \circ e^j) \otimes (e^k \circ e^l) = v^{(ij)} \otimes (e^k \circ e^l)$, i.e., $\beta^{ij,kl} = (df(H_0))(G)$, if $G = \frac{1}{2} t_{(ij)} \otimes e^k \circ e^l$ (no summation).

*2nd Case.* $(5.22)_{ij,kl}$ when $ij, kl = in, kn$, $n \notin \{i,k\}$, $i \ne k$.
In this case $\beta^{in,kn} \equiv -\frac{1}{2}\beta^{ik,nn} \pmod{S^4V^*}$ by (5.19), and the 1st case applies.

*3rd Case.* $(5.22)_{in,in}$ when $i \ne n$.

$$\begin{aligned} \beta^{in,in} &\equiv -\tfrac{1}{2}\beta^{nn,ii} && \pmod{S^4V^*} \\ &\equiv v^n \otimes \left(-\tfrac{1}{2} e^i \otimes e^i\right) && \big(\mathrm{mod}\big(S^4V^* + \text{span of 1st and 2nd cases}\big)\big). \end{aligned}$$

This completes the proof of the theorem H. Q.E.D.

(h.3) The differentiated version of (5.17), namely (5.17)′, has an interesting application in the local description of the fiber of $\gamma$, $\gamma^{-1}(\gamma(H))$, for $H$ a general element in $W \otimes S^2V^*$. (Recall that the $O(W)$-invariance of $\gamma$ has the geometric interpretation that the Gauss equations have built in the $O(W)$-choice of orthonormal frame in the normal bundle, the so-called "spinning in the normal bundle"). Explicitly, let $\{t_\mu\}$ be an orthonormal basis for $W$,

$$H \in W \otimes S^2V^*, \qquad H = \sum_\mu t_\mu \otimes v^\mu, \qquad 1 \le \mu \le r \le \binom{n}{2}$$

and assume the $\{v^\mu\}$ are linearly independent in $S^2V^*$. Choose $\{v^s\} \subset S^2V^*$,

$\binom{n}{2} + 1 \leq s \leq \binom{n+1}{2}$, such that $\{v^\mu, v^s\}$ form a basis for $S^2V^*$. Let

$$G \in W \otimes S^2V^*, \qquad G = \sum_\mu t_\mu \otimes z^\mu, \qquad z^\mu = \sum_\nu a_\nu^\mu v^\nu + \sum_s a_s^\mu v^s. \tag{5.23}$$

Then, straightforward calculation shows

$$(df(H))(G) = 0 \Leftrightarrow \begin{cases} a_\nu^\mu + a_\mu^\nu = 0, & 1 \leq \mu, \nu \leq r \\ a_s^\mu = 0, & r+1 \leq s \leq \binom{n+1}{2} \end{cases}$$

$$\text{i.e.,} \quad \ker(df(H)) \cong \text{Lie algebra of } O(W). \tag{5.24}$$

Thus, (5.24) shows that $\ker(df(H))$ accounts for the "spinning in the normal bundle". However, we clearly have

$$\ker(d\gamma(H)) \supset \ker(df(H)). \tag{5.25}$$

More precisely,

$$\ker(d\gamma(H)) = \{ G \,|\, (df(H))(G) \in S^4V^* \}. \tag{5.26}$$

We show that the distinction between the right and left hand sides of (5.25) is meaningful by considering the case $r = 2$.

Let $H \in W \otimes S^2V^*$, $H = t_1 \otimes v^1 + t_2 \otimes v^2$, $\{t_1, t_2\}$ orthonormal in $W$. Since we are interested in comparing dimensions of the linear spaces in (5.25), we can complexify and compute dimensions over C. Now $\gamma$ is $GL(V^*, \mathbf{C})$-equivariant, and in particular since $H$ is general we can "simultaneously diagonalize" $v^1, v^2$; i.e., there exists a basis $\varphi^i$ of $V^* \otimes \mathbf{C}$, such that

$$v^k = \sum_i a_i \varphi^i \otimes \varphi^i, \qquad v^2 = \sum_j b_j \varphi^j \otimes \varphi^j.$$

*Note.* It is not always the case that a pair of quadratic forms $v^1, v^2$ on $\mathbf{C}^n$ can be simultaneously diagonalized (e.g., $x_1^2$ and $x_1 x_2$ where $(x_1, \ldots, x_n) \in \mathbf{C}^n$). If, however, $v^1$ is nonsingular (or, more generally, if $\det\|t_1 v^1 - t_2 v^2\| \not\equiv 0$) then this is possible. Since we are interested only in generic $H$ (because the fibre dimension of $\gamma$ at most increases under specialization) we may assume $v^1, v^2$ simultaneously diagonalizable.

We want to compute

$$\dim \ker\big(d\gamma(H) : W \otimes S^2V^* \dashrightarrow K\big)$$

$$= \dim\{ G \in W \otimes S^2V^* \,|\, (df(H))(G) \in S^4V^* \}.$$

Let $G = t_1 \otimes k^1 + t_2 \otimes k^2$, $k^\mu = k^\mu_{rs}\varphi^r \circ \varphi^s$, $k^\mu_{rs} = k^\mu_{sr}$, $\mu = 1, 2$. Then

$$(df(H))(G) = v^1 \circ k^1 + v^2 \circ k^2$$

$$= \sigma_{pqrs}\varphi^p \otimes \varphi^q \otimes \varphi^r \otimes \varphi^s \qquad \text{(summation convention)}$$

where

$$\sigma_{pqrs} = \delta_{pq}\left(a_p k^1_{rs} + b_{\frac{1}{2}} k^2_{rs}\right) + \delta_{rs}\left(a_r k^1_{pq} + b_r k^2_{pq}\right).$$

LEMMA.
(a) $\sigma_{pqrs} = \sigma_{qprs} = \sigma_{pqsr} = \sigma_{qpsr} = \sigma_{rspq} = \sigma_{srpq} = \sigma_{rsqp} = \sigma_{srqp}$
(b) $(\sigma_{pqrs}) \in S^4V^* \Leftrightarrow \sigma_{pqrs} = \sigma_{prqs} = \sigma_{psqr}$ $\qquad (*)_{pqrs}$
(c) *If $H$ is generic, $n \geq 4$, and $(\sigma_{pqrs}) \in S^4V^*$, then $k^1, k^2$ are both diagonal.*

*Proof.* Only (c) is not obvious.

Suppose $1 \leq r, s \leq n$, $r \neq s$. Since $n \geq 4$, there exist $p$, $p'$ such that $p$, $p'$, $r$, $s$ are all distinct. Now suppose

$$(\sigma_{pqrs}) \in S^4V^*.$$

This implies

$$\begin{cases} 0 = \sigma_{prps} = \sigma_{pprs} = a_p k^1_{rs} + b_p k^2_{rs} \\ 0 = \sigma_{p'rp's} = \sigma_{p'p'rs} = a_{p'} k^1_{rs} + b_{p'} k^2_{rs}\,. \end{cases}$$

We rewrite this as

$$\begin{pmatrix} a_p & b_p \\ a_{p'} & b_{p'} \end{pmatrix} \cdot \begin{pmatrix} k^1_{rs} \\ k^2_{rs} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $H$ is generic, we conclude that

$$k^1_{rs} = k^2_{rs} = 0. \quad \text{Q.E.D.}$$

COROLLARY. $(\sigma_{ijkl}) \in S^4V^* \Leftrightarrow (\sigma_{ijkl})$ *satisfies* $(*)_{pqrs}$ *where* $(pqrs)$ *ranges over the following five possibilities*:

$$\begin{cases} \text{(i)} & (pqrs) = (iiii) \\ \text{(ii)} & (pqrs) = (iiij) \\ \text{(iii)} & (pqrs) = (iijj) \\ \text{(iv)} & (pqrs) = (iijk) \\ \text{(v)} & (pqrs) = (ijkl) \end{cases} \tag{5.27}$$

By the lemma $\mathrm{Ker}(\mathrm{def}(H))$ consists only of elements $k = (k^1, k^2)$, $k^i$ diagonal. We interpret the system (5.27) as conditions on $\{k_{jj}^{\mu}\}$. It is clear that equations of types (i), (ii), (iv) and (v) are automatically satisfied. It suffices to consider equations of type (iii)

$$\begin{aligned} &(\mathrm{iii}) \leftrightarrow (*)_{pqrs} = (*)_{iijj} \leftrightarrow \sigma_{iijj} = \sigma_{ijij} = 0 \\ &\text{i.e.,} \qquad a_i k_{jj}^1 + b_i k_{jj}^2 + a_j k_{ii}^1 + b_j k_{ii}^2 = 0 \end{aligned} \qquad (*)_{ij}$$

Thus we get $\binom{n}{2}$ equations $\{(*)_{ij} \mid i < j\}$ in the $2n$ variables $\{k_{ii}^1, k_{jj}^2\}$.

It can be shown (e.g., using a symbolic manipulation computer language such as MIT's MACSYMA) that the rank of the system (5.27) for $n = 5$, is $g$; i.e., the system has corank one. This also shows that for $n \geq 5$ the system has corank one, since we can work with five variables at a time. (To make this last statement more precise consider $n > 5$. Then the $n = 5$ system is embedded in the larger system by identifying the first five variables. This fixes, up to scalar, ten components of a solution to the larger system. By exchanging the roles of the variables this determines the remaining components.)

For the case $n = 4$, the type (iii) equations yield a system which is clearly of maximal rank in general. *Since the system has six equations in eight variables, the solution space is two-dimensional.*

We emphasize that this result for $n = 4$ is somewhat unexpected for the following reason. The Gauss map

$$\gamma : W \otimes S^2V^* \to K$$

is a quadratic map with

$$\begin{cases} \dim(\mathrm{domain}(\gamma)) = r \cdot \binom{n+1}{2} & \left(1 \leq r \leq \binom{n}{2}\right) \\ \dim(\mathrm{range}(\gamma)) = \dim(K) = \frac{1}{12}\left(n^2(n^2-1)\right) \end{cases} \qquad (5.28)$$

If $\gamma$ were a generic quadratic map, one would expect its generic fiber to have a dimension dictated by the dimensions of (5.28) plus the "spinning in the normal bundle" dimension. When $r = 2$ and $n \geq 5$, we see

$$\dim(W \otimes S^2V^*) = n^2 + n < \frac{n^4 - n^2}{12} = \dim K.$$

Hence one would expect the Gauss map to have one-dimensional fibers arising from the $O(2)$-action, and we have seen this is what happens.

For $n = 4$,

$$\dim(W \otimes S^2V^*) = 20 = \dim K$$

and the same conclusion would not be unexpected. However, in this case the image of the Gauss map has codimension two. We stress that the tangent vectors

$G \in \ker(d\gamma(H))$ that are not infinitesimal generators of the $O(2)$-action are characterized by

$$\gamma(H, G) \in S^4V^* - \{0\}.$$

## References


1. C. B. Allendoerfer, Amer. J. Math. **61** (1939), 633–644.
2. E. Berger, R. Bryant, and P. Griffiths, Proc. Nat. Acad. Sci. **78** (1981), 4657–4660.
3. R. Bryant, S. S. Chern, and P. Griffiths, Proceedings of the Beijing Symposium (1980).
4. R. Bryant and P. Griffiths, *Characteristic varieties of exterior differential systems*, to appear.
5. R. Bryant, P. Griffiths, and D. Yang, *Characteristics and existence of isometric embeddings*, to appear.
6. E. Cartan, *Les systemes differentielles exterieurs et leur applications geometriques*, Hermann (1945).
7. ———, Bull. Soc. Math. France **47** (1919), 125–160 and **48** (1920), 132–208.
8. S. S. Chern and R. Osserman, *Remarks on the Riemannian metric of a numerical submanifold*, to appear.
9. J. Gasqui, Jour. Diff. Geom. **10** (1975), 61–84.
10. H. Goldschmidt, Annals of Math. **86** (1967), 246–270.
11. R. Greene and H. Jacobowitz, Annals of Math. **93** (1971), 189–204.
12. M. Gromov and V. Roklin, Russian Math. Surveys **25** (1970), 1–57.
13. E. Kaneda and N. Tanaka, J. Math. Kyoto Univ. **18** (1978), 1–70.
14. D. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, Oxford (1940).
15. J. P. Serre, Annals of Math. **61** (1955), 197–278.
16. M. Spivak, *A Comprehensive Introduction to Differential Geometry* (Vol. 5), Publish or Perish (1975).
17. T. I. Thomas, Acta Math. **67** (1936), 164–211.
18. J. Vilms, J. Diff. Geom. **12** (1977), 197–202.
19. ———, Trans. Amer. Math. Soc. **260** (1980), 595–605.
20. H. Weyl, *The Classical Groups*, Princeton Press (1946).


Berger and Griffiths: Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

Bryant: Department of Mathematics, Rice University, Houston, Texas 77251