

THE GAUSS MAP AND SPACELIKE SURFACES WITH PRESCRIBED MEAN CURVATURE IN MINKOWSKI 3-SPACE

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For an oriented spacelike surface M in Minkowski 3-space L^3 , the Gauss map G is defined to be a mapping of M into the unit pseudosphere H in L^3 , which assigns to each point p of M the point in H obtained by translating the timelike unit normal vector at p to the origin. Our primary object of this paper is to prove a representation formula for spacelike surfaces with prescribed mean curvature in terms of their Gauss maps.

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space R^3 in terms of their Gauss maps and auxiliary holomorphic functions ([8]). More generally, a remarkable representation formula has been discovered by Kenmotsu [3] for arbitrary surfaces in R^3 with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss maps and mean curvature functions. On the other hand, Kobayashi [4, 5] proved the Lorentzian version of the classical Weierstrass-Enneper representation formula for maximal surfaces in Minkowski 3-space L^3 (see also McNertney [10]) and applied it to the study of maximal surfaces with conelike singularities.

Motivated by these results, we shall prove, in §4 of this paper, that arbitrary oriented spacelike surfaces in L^3 satisfy a system of first order partial differential equations involving the mean curvature function H and the Gauss map G of the surface (Theorem 4.1). An interesting feature therein is that the complete integrability condition for the formula then yields a system of nonlinear second order partial differential equations which identifies the gradient of H and the tension field of G (Proposition 5.3). In particular, the condition simply means that the Gauss map G should be a harmonic mapping provided the mean curvature H is constant.

The converse of these observations will be discussed in §6. Our main result is that given a nowhere holomorphic smooth mapping G of a simply connected Riemannian surface M into the pseudosphere H satisfying the complete integrability condition for some nonvanishing smooth function H on M , we can construct explicitly a spacelike immersion of M into L^3 such that the mean curvature of M is H and the Gauss map of M is given by G (Theorem 6.1). This allows us, in particular, to produce a wealth of spacelike surfaces of constant mean curvature in L^3 , and more importantly, to relate

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the geometry of these surfaces to the theory of harmonic mappings through their Gauss maps.

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1. Preliminaries. We begin with fixing our terminology and notation. Let $L^3 = (\mathbf{R}^3, \bar{g})$ denote Minkowski 3-space with flat Lorentzian metric \bar{g} of signature $(+, +, -)$. In terms of the canonical coordinates (x^1, x^2, x^3) of \mathbf{R}^3 , the metric \bar{g} , denoted also by $\langle \cdot, \cdot \rangle$, can be expressed as $\bar{g} = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$. Let M^2 be a connected smooth 2-manifold, and $X: M^2 \rightarrow L^3$ be a smooth immersion of M^2 into L^3 . Throughout this paper, we assume that X is a *spacelike immersion* or M^2 is a *spacelike surface* in L^3 , that is, the pull back $X^*\bar{g}$ of the Lorentzian metric \bar{g} via X is a positive definite metric on M^2 (cf. [1, 7]). Also, we always assume that M is orientable. It should be remarked that there exists no closed spacelike surface in L^3 . Indeed, otherwise the Euclidean normal directions of the surface would all make an angle of more than $\pi/4$ with the horizontal plane, contradicting the fact that a closed surface in \mathbf{R}^3 has Euclidean normals in all directions.

Let $M = (M^2, g)$ denote the Riemannian 2-manifold M^2 with induced metric $g = X^*\bar{g}$ so that $X: M^2 \rightarrow L^3$ is an isometric immersion. By $\xi = (\xi^1, \xi^2)$ we always denote an isothermal coordinates compatible with the orientation on M , by which g is expressed locally as

$$(1.1) \quad g = \lambda^2((d\xi^1)^2 + (d\xi^2)^2), \quad \lambda > 0.$$

It is well-known that (ξ^1, ξ^2) is defined around each point of M , and we may regard M as a Riemann surface by introducing complex coordinates by $z = \xi^1 + \sqrt{-1}\xi^2$.

We shall define a local Lorentzian frame field (e_1, e_2, e_3) adapted to M in L^3 in the following manner. Let $X(\xi) = (X^1(\xi^1, \xi^2), X^2(\xi^1, \xi^2), X^3(\xi^1, \xi^2))$ be a local expression of the immersion X with respect to an isothermal coordinates (ξ^1, ξ^2) on M . For $i=1, 2$, let

$$e_i = \frac{1}{\lambda} \frac{\partial X}{\partial \xi^i} = \frac{1}{\lambda} \left(\frac{\partial X^1}{\partial \xi^i}, \frac{\partial X^2}{\partial \xi^i}, \frac{\partial X^3}{\partial \xi^i} \right).$$

Then (e_1, e_2) defines an orthonormal tangent frame field on M compatible with the orientation. We then define $e_3 = e_1 \times e_2$. Here the exterior product $v \times w$ of two vectors v, w in L^3 is defined by $v \times w = -(\iota_w \iota_v dx^1 \wedge dx^2 \wedge dx^3)^*$, ι_v and $\#$ denoting the interior product with respect to v and the operation of raising indices by the metric \bar{g} , respectively. Note that e_3 is timelike and defines a (Lorentzian) unit normal vector field on M , that is, $\langle e_3, e_3 \rangle = -1$ and $\langle e_3, e_i \rangle = 0$ for $i=1, 2$. In terms of local coordinates, e_3 is given explicitly by

$$(1.2) \quad e_3 = \frac{1}{\lambda^2} \left(\frac{\partial X^3 \partial X^2}{\partial \xi^1 \partial \xi^2} - \frac{\partial X^2 \partial X^3}{\partial \xi^1 \partial \xi^2}, \frac{\partial X^1 \partial X^3}{\partial \xi^1 \partial \xi^2} - \frac{\partial X^3 \partial X^1}{\partial \xi^1 \partial \xi^2}, \frac{\partial X^1 \partial X^2}{\partial \xi^1 \partial \xi^2} - \frac{\partial X^2 \partial X^1}{\partial \xi^1 \partial \xi^2} \right).$$

It should be noted that $\partial/\partial x^1 \times \partial/\partial x^2 = \partial/\partial x^3$, $\partial/\partial x^2 \times \partial/\partial x^3 = -\partial/\partial x^1$ and $\partial/\partial x^3 \times \partial/\partial x^1 = -\partial/\partial x^2$ due to our sign convention for the exterior product.

Let h denote the second fundamental form of M in L^3 (cf. [1, 7]). With respect to a Lorentzian frame field (e_1, e_2, e_3) , h is represented by the matrix $(h_{ij})_{1 \leq i,j \leq 2}$, where

$$h_{ij} = -\langle D_{e_i} e_j, e_3 \rangle,$$

D denoting covariant differentiation in L^3 . Then, by an elementary calculation, we see that the fundamental formulas of Gauss and Weingarten for M in L^3 are given as follows:

$$(1.3) \quad \begin{aligned} \frac{\partial^2 X}{\partial \xi^1 \partial \xi^1} &= \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^1} \frac{\partial X}{\partial \xi^1} - \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^2} \frac{\partial X}{\partial \xi^2} + \lambda^2 h_{11} e_3, \\ \frac{\partial^2 X}{\partial \xi^1 \partial \xi^2} &= \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^2} \frac{\partial X}{\partial \xi^1} + \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^1} \frac{\partial X}{\partial \xi^2} + \lambda^2 h_{12} e_3, \\ \frac{\partial^2 X}{\partial \xi^2 \partial \xi^2} &= -\frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^1} \frac{\partial X}{\partial \xi^1} + \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi^2} \frac{\partial X}{\partial \xi^2} + \lambda^2 h_{22} e_3. \\ \frac{\partial e_3}{\partial \xi^1} &= h_{11} \frac{\partial X}{\partial \xi^1} + h_{12} \frac{\partial X}{\partial \xi^2}, \\ (1.4) \quad \frac{\partial e_3}{\partial \xi^2} &= h_{21} \frac{\partial X}{\partial \xi^1} + h_{22} \frac{\partial X}{\partial \xi^2}. \end{aligned}$$

The mean curvature H of M is defined to be $H = (h_{11} + h_{22})/2$. If H vanishes identically on M , then M is said to be *maximal*. It is easy to see from (1.3) that M is maximal if and only if each component function of the immersion X is harmonic on M .

Let $\phi = (1/2)(h_{11} - h_{22}) - \sqrt{-1} h_{12}$, which represents, up to a factor, the (2,0)-part of the complexification of the second fundamental form h of M . Then from (1.4) we have

$$(1.5) \quad \frac{\partial e_3}{\partial z} = H \frac{\partial X}{\partial z} + \phi \frac{\partial X}{\partial \bar{z}},$$

where we set $\partial/\partial z = (1/2)(\partial/\partial \xi^1 - \sqrt{-1} \partial/\partial \xi^2)$ and $\partial/\partial \bar{z} = (1/2)(\partial/\partial \xi^1 + \sqrt{-1} \partial/\partial \xi^2)$. Note that $\phi(p) = 0$ at a point $p \in M$ if and only if p is an umbilical point of M . It is also not difficult to see that the Gaussian curvature K of M is given by

$$(1.6) \quad K = -H^2 + |\phi|^2,$$

for $K = -(h_{11}h_{22} - h_{12}^2)$ by the equation of Gauss.

2. The Gauss map. For a spacelike surface M in L^3 , the *Gauss map* G of M is by definition a mapping of M into L^3 , which assigns to each point $p \in M$ the point in L^3 obtained by translating parallelly the unit normal vector $e_3(p)$ of M at p to the origin of L^3 (cf. [1, 7]). Note that, since $e_3(p)$ is a timelike unit vector at $p \in L^3$, the Gauss map G is in fact a mapping of M into the unit pseudosphere H in L^3 . That is, the image of G is contained in a spacelike surface H in L^3 defined by

$$H = \{(x^1, x^2, x^3) \in L^3 \mid (x^1)^2 + (x^2)^2 - (x^3)^2 = -1\},$$

which is a two-sheeted hyperboloid in L^3 , and has constant Gaussian curvature $K \equiv -1$ with respect to the induced metric.

On H we may define a natural complex structure in the following manner. Let $U_1 = H - \{(0, 0, 1)\}$ and $U_2 = H - \{(0, 0, -1)\}$, and introduce complex coordinates by means of stereographic mappings $\psi_1: U_1 \rightarrow \mathbf{C}$ and $\psi_2: U_2 \rightarrow \mathbf{C}$, which are defined respectively by

$$(2.1) \quad \begin{aligned} \psi_1(x) &= \frac{x^1 + \sqrt{-1}x^2}{1 - x^3}, & x = (x^1, x^2, x^3) \in U_1, \\ \psi_2(x) &= \frac{x^1 - \sqrt{-1}x^2}{1 + x^3}, & x = (x^1, x^2, x^3) \in U_2. \end{aligned}$$

In fact, $\psi_1(x)$ is the intersection of the line joining $x \in U_1$ and the north pole $(0, 0, 1) \in H$, and the (x^1, x^2) -plane identified with \mathbf{C} by setting $\zeta = x^1 + \sqrt{-1}x^2$. Similarly, ψ_2 represents the stereographic mapping from the south pole $(0, 0, -1) \in H$. It should be noted that the images of ψ_1 and ψ_2 are contained in the set $\mathbf{C} - \{|\zeta| = 1\}$, and the inverse mappings ψ_1^{-1} and ψ_2^{-1} of ψ_1 and ψ_2 are given respectively by

$$(2.2) \quad \begin{aligned} \psi_1^{-1}(\zeta) &= \left(\frac{2 \operatorname{Re} \zeta}{1 - |\zeta|^2}, \frac{2 \operatorname{Im} \zeta}{1 - |\zeta|^2}, -\frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right), & \zeta \in \mathbf{C} - \{|\zeta| = 1\}, \\ \psi_2^{-1}(\zeta) &= \left(\frac{2 \operatorname{Re} \zeta}{1 - |\zeta|^2}, -\frac{2 \operatorname{Im} \zeta}{1 - |\zeta|^2}, \frac{1 + |\zeta|^2}{1 - |\zeta|^2} \right), & \zeta \in \mathbf{C} - \{|\zeta| = 1\}. \end{aligned}$$

It is then immediate to see that $\psi_1(x)\psi_2(x) = -1$ for $x \in U_1 \cap U_2$, and $\{\psi_1, \psi_2\}$ defines a complex structure on H , since $\psi_2 \circ \psi_1^{-1}(\zeta) = -1/\zeta$ and $\psi_1 \circ \psi_2^{-1}(\zeta) = -1/\zeta$. It is also not difficult to see that ψ_1 and ψ_2 are conformal with respect to the induced metric on H and the flat metric on \mathbf{C} . (Indeed, the induced metric on H can be written as $4|d\zeta|^2/(1 - |\zeta|^2)^2$, ζ being complex coordinates defined by stereographic mappings.)

In consequence, we obtain the following sequence of mappings:

$$M \xrightarrow{G} H \subset L^3 \xrightarrow{\psi_i} \mathbf{C} - \{|\zeta| = 1\}, \quad i = 1, 2.$$

We often refer to the composite mapping $\Psi_i = \psi_i \circ G$ for $i = 1, 2$ also as the Gauss map

of M (into \mathbf{C}). Moreover, we omit the subscript i in Ψ_i , and write simply as Ψ , if there is no confusion or if the statement under consideration holds for both Ψ_i .

3. Beltrami equation. Let M be a spacelike surface immersed in L^3 by a mapping $X: M \rightarrow L^3$, and Ψ denote the Gauss map of M into \mathbf{C} as in §2. The goal of this section is to prove that Ψ satisfies a Beltrami equation. To start with, we prove the following lemma.

LEMMA 3.1. *If $X = (X^1, X^2, X^3): M \rightarrow L^3$ is a spacelike immersion, then*

$$(3.1) \quad \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial z} = -\Psi_1 \frac{\partial X^3}{\partial z},$$

$$(3.2) \quad \frac{\partial X^3}{\partial z} = -\Psi_1 \frac{\partial(X^1 - \sqrt{-1}X^2)}{\partial z},$$

$$(3.3) \quad \frac{\partial X^3}{\partial z} \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}} = -\frac{\lambda^2 \Psi_1}{(1 - |\Psi_1|^2)^2}.$$

PROOF. Since $z = \xi^1 + \sqrt{-1}\xi^2$ for which (ξ^1, ξ^2) is an isothermal coordinates on M , it follows from (1.1) that

$$(3.4) \quad \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial \bar{z}} \right\rangle = \frac{\lambda^2}{2}, \quad \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial z} \right\rangle = \left\langle \frac{\partial X}{\partial \bar{z}}, \frac{\partial X}{\partial \bar{z}} \right\rangle = 0.$$

On the other hand, if (e_1, e_2, e_3) is a Lorentzian frame field adapted to M in L^3 , then we see from (1.2)

$$(3.5) \quad e_3 = -\frac{2\sqrt{-1}}{\lambda^2} \left(\frac{\partial X^3}{\partial z} \frac{\partial X^2}{\partial \bar{z}} - \frac{\partial X^2}{\partial z} \frac{\partial X^3}{\partial \bar{z}}, \frac{\partial X^1}{\partial z} \frac{\partial X^3}{\partial \bar{z}} - \frac{\partial X^3}{\partial z} \frac{\partial X^1}{\partial \bar{z}}, \frac{\partial X^1}{\partial z} \frac{\partial X^2}{\partial \bar{z}} - \frac{\partial X^2}{\partial z} \frac{\partial X^1}{\partial \bar{z}} \right),$$

and also from (2.1)

$$(3.6) \quad \Psi_1 = \frac{e_3^1 + \sqrt{-1}e_3^2}{1 - e_3^3},$$

$$(3.7) \quad (1 - |\Psi_1|^2)(1 - e_3^3) = 2,$$

where we put $e_3 = (e_3^1, e_3^2, e_3^3)$. On substituting (3.5) into (3.6), and making use of (3.4) and (3.7), we can then check (3.1), (3.2) and (3.3) without difficulty by a straightforward calculation.

We shall now compute the derivatives of the Gauss map Ψ . First we prove:

PROPOSITION 3.2. *The complex derivatives of the Gauss map Ψ_1 are given by*

$$(3.8) \quad \frac{\partial \Psi_1}{\partial \bar{z}} = \frac{H}{2} (1 - |\Psi_1|^2)^2 \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}},$$

$$(3.9) \quad \frac{\partial \Psi_1}{\partial z} = \frac{\phi}{2} (1 - |\Psi_1|^2)^2 \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}}.$$

PROOF. Differentiating (3.6) with respect to \bar{z} and applying (1.5), we get

$$\frac{\partial \Psi_1}{\partial \bar{z}} = \frac{1}{1 - e_3^3} \left[H \frac{\partial X^1}{\partial \bar{z}} + \bar{\phi} \frac{\partial X^1}{\partial z} + \sqrt{-1} \left(H \frac{\partial X^2}{\partial \bar{z}} + \bar{\phi} \frac{\partial X^2}{\partial z} \right) \right] + \frac{1}{1 - e_3^3} \Psi_1 \left[H \frac{\partial X^3}{\partial \bar{z}} + \bar{\phi} \frac{\partial X^3}{\partial z} \right].$$

Then, by (3.1) and (3.2) together with (3.7), it is verified that

$$\begin{aligned} \frac{\partial \Psi_1}{\partial \bar{z}} &= \frac{1 - |\Psi_1|^2}{2} \left[H \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}} + \bar{\phi} \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial z} \right] \\ &\quad - \frac{1 - |\Psi_1|^2}{2} \left[|\Psi_1|^2 H \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}} + \bar{\phi} \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial z} \right] \\ &= \frac{H}{2} (1 - |\Psi_1|^2)^2 \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}}, \end{aligned}$$

thus proving (3.8). (3.9) can be proved in a similar fashion.

By the same argument we can also prove the following

PROPOSITION 3.3. *The complex derivatives of the Gauss map Ψ_2 are given by*

$$(3.10) \quad \frac{\partial \Psi_2}{\partial \bar{z}} = \frac{H}{2} (1 - |\Psi_2|^2)^2 \frac{\partial(X^1 - \sqrt{-1}X^2)}{\partial \bar{z}},$$

$$(3.11) \quad \frac{\partial \Psi_2}{\partial z} = \frac{\phi}{2} (1 - |\Psi_2|^2)^2 \frac{\partial(X^1 - \sqrt{-1}X^2)}{\partial \bar{z}}.$$

From these propositions the following theorem is now immediate.

THEOREM 3.4. *The Gauss map Ψ of a spacelike surface M in L^3 satisfies a Beltrami equation*

$$(3.12) \quad H \frac{\partial \Psi}{\partial z} = \phi \frac{\partial \Psi}{\partial \bar{z}}.$$

It is well-known that the Gauss map of a minimal surface in Euclidean 3-space is a holomorphic mapping into the Riemann sphere (cf. [8]). In connection with this, we may point out the following

PROPOSITION 3.5. *Let M be a spacelike surface in L^3 . Then at $p \in M$*

$$(3.13) \quad H(p)=0 \iff \frac{\partial \Psi}{\partial \bar{z}}(p)=0,$$

$$(3.14) \quad \phi(p)=0 \iff \frac{\partial \Psi}{\partial z}(p)=0.$$

PROOF. It is verified from Lemma 3.1 that

$$(3.15) \quad \left| \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}} \right| = \frac{\lambda}{|1 - |\Psi_1|^2|}.$$

Hence from Proposition 3.2 we get

$$(3.16) \quad \left| \frac{\partial \Psi_1}{\partial \bar{z}} \right| = \alpha |H|, \quad \left| \frac{\partial \Psi_1}{\partial z} \right| = \alpha |\phi|,$$

where $\alpha = \lambda |1 - |\Psi_1|^2|/2$. Since $\alpha \neq 0$, this proves the proposition when $p \in \Psi_1^{-1}(\mathbf{C})$. The proof for the case $p \in \Psi_2^{-1}(\mathbf{C})$ is similar.

4. Representation formula. Given a spacelike surface M in L^3 , we shall now prove a representation formula for M in terms of the Gauss map Ψ and the mean curvature H of M .

THEOREM 4.1. *Let M be a spacelike surface immersed in L^3 by a mapping $X = (X^1, X^2, X^3) : M \rightarrow L^3$. Let H and Ψ_i ($i = 1, 2$) denote the mean curvature function of M and the Gauss map of M into \mathbf{C} defined in §2, respectively. Then the following hold.*

(1) *On $\Psi_1^{-1}(\mathbf{C})$, we have*

$$(4.1) \quad \begin{aligned} H \frac{\partial X^1}{\partial z} &= \frac{1 + \Psi_1^2}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z}, \\ H \frac{\partial X^2}{\partial z} &= \sqrt{-1} \frac{1 - \Psi_1^2}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z}, \\ H \frac{\partial X^3}{\partial z} &= -2 \frac{\Psi_1}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z}. \end{aligned}$$

(2) *On $\Psi_2^{-1}(\mathbf{C})$, we have*

$$(4.2) \quad \begin{aligned} H \frac{\partial X^1}{\partial z} &= \frac{1 + \Psi_2^2}{(1 - |\Psi_2|^2)^2} \frac{\partial \bar{\Psi}_2}{\partial z}, \\ H \frac{\partial X^2}{\partial z} &= -\sqrt{-1} \frac{1 - \Psi_2^2}{(1 - |\Psi_2|^2)^2} \frac{\partial \bar{\Psi}_2}{\partial z}, \end{aligned}$$

$$H \frac{\partial X^3}{\partial z} = 2 \frac{\Psi_2}{(1 - |\Psi_2|^2)^2} \frac{\partial \bar{\Psi}_2}{\partial z}.$$

PROOF. (1) Recall that by (3.8) we have

$$(4.3) \quad \frac{\partial \bar{\Psi}_1}{\partial z} = \frac{H}{2} (1 - |\Psi_1|^2)^2 \frac{\partial (X^1 - \sqrt{-1}X^2)}{\partial z}$$

on $\Psi_1^{-1}(C)$. From (3.1) and (3.2) it then follows that

$$(4.4) \quad \Psi_1^2 \frac{\partial \bar{\Psi}_1}{\partial z} = \frac{H}{2} (1 - |\Psi_1|^2)^2 \frac{\partial (X^1 + \sqrt{-1}X^2)}{\partial z}.$$

Hence, by adding (4.4) to (4.3), we get

$$(1 + \Psi_1^2) \frac{\partial \bar{\Psi}_1}{\partial z} = H (1 - |\Psi_1|^2)^2 \frac{\partial X^1}{\partial z},$$

and, by subtracting (4.4) from (4.3),

$$(1 - \Psi_1^2) \frac{\partial \bar{\Psi}_1}{\partial z} = -\sqrt{-1} H (1 - |\Psi_1|^2)^2 \frac{\partial X^2}{\partial z}.$$

Since $1 - |\Psi_1|^2 \neq 0$, it follows from these that on $\Psi_1^{-1}(C)$

$$(4.5) \quad H \frac{\partial X^1}{\partial z} = \frac{1 + \Psi_1^2}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z},$$

$$(4.6) \quad H \frac{\partial X^2}{\partial z} = \sqrt{-1} \frac{1 - \Psi_1^2}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z}.$$

Now note that from (3.2) we also have

$$(4.7) \quad H \frac{\partial X^3}{\partial z} = -\Psi_1 H \frac{\partial (X^1 - \sqrt{-1}X^2)}{\partial z}.$$

It then follows from (4.3) and (4.7) that on $\Psi_1^{-1}(C)$

$$(4.8) \quad H \frac{\partial X^3}{\partial z} = -2 \frac{\Psi_1}{(1 - |\Psi_1|^2)^2} \frac{\partial \bar{\Psi}_1}{\partial z},$$

for $1 - |\Psi_1|^2 \neq 0$.

(2) can be proved in a similar fashion, or one can derive it from (1) by means of the relation $\Psi_1 \cdot \Psi_2 = -1$ valid on $\Psi_1^{-1}(C) \cap \Psi_2^{-1}(C)$.

REMARK 4.1. The Euclidean counterpart of Theorem 4.1, namely, the corresponding representation formula for surfaces in Euclidean 3-space has been proved

in Kenmotsu [3].

REMARK 4.2. If we carry out the same argument, utilizing the equations (3.9), (3.11) instead of (3.8), (3.10), then we obtain the following representation formula in terms of Ψ and ϕ : On $\Psi_1^{-1}(\mathbf{C})$,

$$(4.9) \quad \begin{aligned} \bar{\phi} \frac{\partial X^1}{\partial z} &= \frac{1 + \Psi_1^2}{(1 - |\Psi_1|^2)^2} \overline{\frac{\partial \Psi_1}{\partial z}}, \\ \bar{\phi} \frac{\partial X^2}{\partial z} &= \sqrt{-1} \frac{1 - \Psi_1^2}{(1 - |\Psi_1|^2)^2} \overline{\frac{\partial \Psi_1}{\partial z}}, \\ \bar{\phi} \frac{\partial X^3}{\partial z} &= -2 \frac{\Psi_1}{(1 - |\Psi_1|^2)^2} \overline{\frac{\partial \Psi_1}{\partial z}}. \end{aligned}$$

(The corresponding formula also holds on $\Psi_2^{-1}(\mathbf{C})$.)

Now, let M be a spacelike surface immersed in L^3 by $X = (X^1, X^2, X^3)$: $M \rightarrow L^3$, and assume that $\phi \neq 0$. If we set $F = [\bar{\phi}(1 - |\Psi_1|^2)^2]^{-1}(\overline{\partial \Psi_1 / \partial z})$, then it follows from (4.9) that

$$(4.10) \quad \left(\frac{\partial X^1}{\partial z}, \frac{\partial X^2}{\partial z}, \frac{\partial X^3}{\partial z} \right) = (F(1 + \Psi_1^2), \sqrt{-1}F(1 - \Psi_1^2), -2F\Psi_1),$$

and, in consequence,

$$(4.11) \quad F = \frac{1}{2} \left(\frac{\partial X^1}{\partial z} - \sqrt{-1} \frac{\partial X^2}{\partial z} \right).$$

Recall that if M is assumed to be maximal in L^3 , then each component function of the immersion X is harmonic on M . It then follows from (4.11) that F is holomorphic in this case. This fact implies that (4.10) gives a Lorentzian counterpart of the classical Weierstrass-Enneper formula for minimal surfaces in Euclidean 3-space (cf. [8]). To be more precise, the following has been proved.

PROPOSITION 4.2 (Kobayashi [4], McNertney [10]). *Any simply connected maximal spacelike surface M in L^3 can be represented in the form*

$$(4.12) \quad X(z) = 2 \operatorname{Re} \int^z (F(1 + \Psi_1^2), \sqrt{-1}F(1 - \Psi_1^2), -2F\Psi_1) dz + c,$$

where $z \in M$ and $c \in L^3$, the integral being taken along an arbitrary path from a fixed point to the point z .

PROOF. Here we remark only on the following matters. For more details, see [4], [10]. First, F is defined by (4.11), which is a holomorphic function on M . Ψ_1 is given as

$\Psi_1 = -(1/2F)(\partial X^3/\partial z)$ by virtue of (4.10), which defines a meromorphic function on M such that $F\Psi_1^2$ is holomorphic on M . (The exceptional case where $F \equiv 0$ corresponds to the (x^1, x^2) -plane in L^3 , but it can be obtained by setting $F \equiv 1$ and $\Psi_1 \equiv 0$ in (4.12).)

5. Integrability condition. In this section we shall show that the Gauss map Ψ of an arbitrary spacelike surface M in L^3 satisfies a nonlinear second order partial differential equation in Ψ and H . The equation we obtain will then turn out to be the complete integrability condition of the first order PDE system in Theorem 4.1 with given data H and Ψ . First we prove:

PROPOSITION 5.1. *Let M be a spacelike surface in L^3 . Then the mean curvature function H of M and the Gauss map Ψ of M into C satisfy the following second order partial differential equation*

$$(5.1) \quad H \left(\frac{\partial^2 \Psi}{\partial z \partial \bar{z}} + \frac{2\bar{\Psi}}{1 - |\Psi|^2} \frac{\partial \Psi}{\partial z} \frac{\partial \Psi}{\partial \bar{z}} \right) = \frac{\partial H}{\partial z} \frac{\partial \Psi}{\partial \bar{z}}.$$

PROOF. We shall prove (5.1) for Ψ_1 . To do this, we may consider only the case where $H \neq 0$. Indeed, if $H(p) = 0$ at $p \in \Psi_1^{-1}(C)$, then $\partial \Psi_1 / \partial \bar{z}(p) = 0$ by (3.13), and hence (5.1) holds trivially there.

This being remarked, recall that from (3.7) and (3.8) we have

$$(5.2) \quad \frac{\partial \Psi_1}{\partial \bar{z}} = 2H \frac{1}{(1 - e_3^3)^2} \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial \bar{z}}.$$

On the other hand, a simple calculation using (1.3), (3.6) and (3.7) yields

$$(5.3) \quad \frac{\partial(X^1 + \sqrt{-1}X^2)}{\partial z \partial \bar{z}} = \lambda^2 H \frac{\Psi_1}{1 - |\Psi_1|^2}.$$

Hence, differentiating (5.2) with respect to z and applying (1.5) and (5.3), we get

$$(5.4) \quad \frac{\partial^2 \Psi_1}{\partial z \partial \bar{z}} = \frac{1}{H} \frac{\partial H}{\partial z} \frac{\partial \Psi_1}{\partial \bar{z}} + (1 - |\Psi_1|^2) \left[H \frac{\partial X^3}{\partial z} + \phi \frac{\partial X^3}{\partial \bar{z}} \right] \frac{\partial \Psi_1}{\partial \bar{z}} + \frac{\lambda^2 H^2}{2} (1 - |\Psi_1|^2) \Psi_1.$$

Substituting (4.1) and (4.9) into (5.4) and applying (3.16), we then obtain

$$(5.5) \quad \frac{\partial^2 \Psi_1}{\partial z \partial \bar{z}} + \frac{2\bar{\Psi}_1}{1 - |\Psi_1|^2} \frac{\partial \Psi_1}{\partial z} \frac{\partial \Psi_1}{\partial \bar{z}} = \frac{1}{H} \frac{\partial H}{\partial z} \frac{\partial \Psi_1}{\partial \bar{z}},$$

thus proving (5.1) for Ψ_1 . We also get the same equation for Ψ_2 by the same argument, or from (5.5) by using the relation $\Psi_1 \cdot \Psi_2 = -1$.

COROLLARY 5.2 (Milnor [6]). *The mean curvature of a spacelike surface M in L^3 is constant if and only if the Gauss map G of M is a harmonic mapping into H .*

PROOF. It is not difficult to observe from (3.13) as well as (5.1), which is in fact

a nonlinear elliptic system in Ψ , that H is constant if and only if Ψ satisfies

$$\frac{\partial^2 \Psi}{\partial z \partial \bar{z}} + \frac{2\bar{\Psi}}{1 - |\Psi|^2} \frac{\partial \Psi}{\partial z} \frac{\partial \Psi}{\partial \bar{z}} = 0,$$

which shows that G , whose coordinates expression is Ψ , is a harmonic mapping into H (cf. [2]).

REMARK 5.1. (1) Equation (5.1) does not depend on the metric on M , but depends only on the complex structure on M .

(2) It should be noted that geometrically (5.5) means the following: The tension field $\tau(G)$ (see [2] for definition) of the Gauss map G coincides, up to translations in L^3 , with the gradient ∇H of the mean curvature function H (cf. [9]).

REMARK 5.2. Corollary 5.2 gives a Lorentzian counterpart of a theorem of Ruh and Vilms [9] that the mean curvature of a hypersurface in Euclidean n -space is constant if and only if its Gauss map is harmonic.

In what follows, let M be a Riemann surface, and H denote, as before, the unit pseudosphere in L^3 with the induced metric of constant negative Gaussian curvature and natural complex structure defined in §2. Given a *nonvanishing* smooth function $H: M \rightarrow \mathbf{R}$ and a smooth mapping $G: M \rightarrow H$, let us now look at the following system of first order partial differential equations:

$$(5.6) \quad \begin{aligned} \frac{\partial X^1}{\partial z} &= \frac{1}{H(1 - |\Psi_i|^2)^2} \frac{\partial \Psi_i}{\partial z} \\ \frac{\partial X^2}{\partial z} &= (-1)^{i-1} \frac{\sqrt{-1}}{H} \frac{1 - \Psi_i^2}{(1 - |\Psi_i|^2)^2} \frac{\partial \bar{\Psi}_i}{\partial z} \quad \text{on } \Psi_i^{-1}(\mathbf{C}) \\ \frac{\partial X^3}{\partial z} &= (-1)^i \frac{2}{H(1 - |\Psi_i|^2)^2} \frac{\Psi_i}{\partial \bar{\Psi}_i}. \end{aligned}$$

Here Ψ_i denotes the composition $\Psi_i = \psi_i \circ G$ of G and the stereographic mapping ψ_i defined by (2.1), and $i = 1, 2$. It should be noted that owing to the relation $\Psi_1 \cdot \Psi_2 = -1$, the right sides of (5.6) for $i = 1, 2$ are compatible on $\Psi_1^{-1}(\mathbf{C}) \cap \Psi_2^{-1}(\mathbf{C})$, and hence (5.6) defines a system defined globally on M .

With these prepared, we now prove the following

PROPOSITION 5.3. *Equation (5.1) is the complete integrability condition of the system (5.6).*

PROOF. Let P denote the right side of (5.6), that is,

$$(5.7) \quad P = (f_i(1 + \Psi_i^2), (-1)^{i-1} \sqrt{-1} f_i(1 - \Psi_i^2), (-1)^i 2 f_i \Psi_i),$$

where $f_i = [H(1 - |\Psi_i|^2)^2]^{-1} (\partial \bar{\Psi}_i / \partial z)$. Assuming that H and Ψ_i satisfy (5.1), we shall

show that (5.6) is a completely integrable system. To do this, it suffices to see that $\partial P/\partial \bar{z} \in \mathbf{R}^3$. But this is immediate; in fact, by a direct calculation we can easily see that if (5.1) is satisfied,

$$(5.8) \quad \frac{\partial P}{\partial \bar{z}} = \frac{H}{2} \lambda^2 \left(\frac{2 \operatorname{Re} \Psi_i}{1 - |\Psi_i|^2}, (-1)^{i-1} \frac{2 \operatorname{Im} \Psi_i}{1 - |\Psi_i|^2}, (-1)^i \frac{1 + |\Psi_i|^2}{1 - |\Psi_i|^2} \right),$$

where $\lambda = 2[H(1 - |\Psi_i|^2)]^{-1} |\partial \Psi_i / \partial \bar{z}|$.

6. Spacelike surfaces with prescribed mean curvature. We shall now prove a converse of Theorem 4.1. Namely, by solving the PDE system (5.6), we shall construct a spacelike surface M in L^3 with prescribed nonvanishing mean curvature H and Gauss map G . To be precise, we are going to prove the following

THEOREM 6.1. *Let M be a simply connected Riemann surface, $H: M \rightarrow \mathbf{R}$ be a nonvanishing real smooth function on M , and $G: M \rightarrow \mathbf{H}$ be a nowhere holomorphic smooth mapping of M into the unit pseudosphere \mathbf{H} in L^3 . For $i = 1, 2$, let Ψ_i denote the composition $\Psi_i = \psi_i \circ G$ of G and the stereographic mapping ψ_i defined by (2.1). Suppose that H and Ψ_i satisfy the differential equation (5.1). Then there exists a spacelike immersion $X: M \rightarrow L^3$ with the following properties:*

- (1) *The mean curvature of M is H , and the Gauss map of M is given by G .*
- (2) *$X = (X^1, X^2, X^3)$ is given explicitly as*

$$(6.1) \quad \begin{aligned} X^1(z) &= 2 \operatorname{Re} \int^z \frac{1}{H(1 - |\Psi_i|^2)^2} \frac{\partial \bar{\Psi}_i}{\partial z} dz + c^1, \\ X^2(z) &= 2 \operatorname{Re} \int^z (-1)^{i-1} \frac{\sqrt{-1}}{H} \frac{1 - \Psi_i^2}{(1 - |\Psi_i|^2)^2} \frac{\partial \bar{\Psi}_i}{\partial z} dz + c^2, \\ X^3(z) &= 2 \operatorname{Re} \int^z (-1)^i \frac{2}{H(1 - |\Psi_i|^2)^2} \frac{\Psi_i}{\partial z} \frac{\partial \bar{\Psi}_i}{\partial z} dz + c^3, \end{aligned}$$

where $z \in \Psi_i^{-1}(\mathbf{C})$ and $c = (c^1, c^2, c^3) \in L^3$, the integral being taken along an arbitrary path from a fixed point to the point z .

PROOF. For given function H and given mapping G , we shall look at the complex PDE system (5.6) defined on M . Note that, on account of Proposition 5.3, the system (5.6) is completely integrable, since H and Ψ_i satisfy (5.1). Moreover, any real solution $X = (X^1, X^2, X^3)$ of the system (5.6) can be represented as

$$(6.2) \quad X(z) = 2 \operatorname{Re} \int^z P dz + c,$$

where P is defined by (5.7) and $c \in \mathbf{R}^3$. Indeed, since M is simply connected and $\partial P/\partial \bar{z} \in \mathbf{R}^3$ by (5.8), the right side of (6.2), where the integral is taken along an arbitrary path in

M from a fixed point to a variable point z , defines a single-valued mapping, and satisfies (5.6) with given H and Ψ_i . Thus we define a mapping $X: M \rightarrow L^3$ by (6.2), and shall prove that X has the desired properties.

It is easy to see from (5.6) that X satisfies

$$(6.3) \quad \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial \bar{z}} \right\rangle = \frac{\lambda^2}{2}, \quad \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial z} \right\rangle = \left\langle \frac{\partial X}{\partial \bar{z}}, \frac{\partial X}{\partial \bar{z}} \right\rangle = 0,$$

where $\lambda = 2[H(1 - |\Psi_i|^2)]^{-1}|\partial\Psi_i/\partial\bar{z}|$. Note that, since G is nowhere holomorphic, $\partial\Psi_i/\partial\bar{z} \neq 0$ everywhere. Then it follows from (6.3) that X defines a spacelike immersion with induced metric $g = \lambda^2|dz|^2$, and by setting $z = \xi^1 + \sqrt{-1}\xi^2$, we get an isothermal coordinates on M with respect to g . On the other hand, from (5.6) together with (3.5) and (2.2), it is immediate to verify that the Gauss map of M coincides with G and the mean curvature of M is given by H .

REMARK 6.1. In Theorem 6.1, if we merely assume $G: M \rightarrow H$ to be a smooth mapping which satisfies the complete integrability condition (5.1) with given H , then the mapping $X: M \rightarrow L^3$ given by (6.2) is, in general, not a spacelike immersion but have singularities which occur where $\partial\Psi_i/\partial\bar{z} = 0$.

COROLLARY 6.2. *Let $X: M \rightarrow L^3$ be a spacelike immersion in Theorem 6.1. Then the following hold.*

(1) *The induced metric g on M is given by*

$$g = \left[\frac{2}{H(1 - |\Psi|^2)} \left| \frac{\partial \Psi}{\partial \bar{z}} \right| \right]^2 |dz|^2.$$

(2) *The Gaussian curvature K of M is given by*

$$K = H^2 \left[\left| \frac{\Psi_z}{\Psi_{\bar{z}}} \right|^2 - 1 \right].$$

PROOF. (1) is already proved. (2) can be obtained by substituting (3.12) into (1.6).

As in the case of minimal surfaces in Euclidean 3-space, it is not difficult to see from Proposition 4.2 that two noncongruent maximal spacelike surfaces may have the same Gauss map (cf. [4]). However, for spacelike surfaces with nonvanishing mean curvature in Theorem 6.1, we have the uniqueness in the following sense.

PROPOSITION 6.3. *Let X (resp. \tilde{X}) be a spacelike immersion in Theorem 6.1 of a simply connected Riemann surface M into L^3 with nonvanishing mean curvature function H (resp. \tilde{H}) and Gauss map G (resp. \tilde{G}) into H . Then the following statements are equivalent:*

(1) *There exist a holomorphic diffeomorphism ϕ on M and an orientation preserving isometry τ of L^3 such that for $z \in M$*

$$(6.5) \quad \tau \circ X(z) = \tilde{X} \circ \phi(z).$$

(2) *There exist a holomorphic diffeomorphism φ on M and an orientation preserving isometry σ of H such that for $z \in M$*

$$(6.6) \quad \begin{aligned} \sigma \circ G(z) &= \tilde{G} \circ \varphi(z), \\ H(z) &= \tilde{H} \circ \varphi(z). \end{aligned}$$

PROOF. [(1) \Rightarrow (2)] Putting $w = \varphi(z)$ and differentiating (6.5), we have $\tau_*(\partial X/\partial z)(z) = (\partial \tilde{X}/\partial w)(\varphi(z)) \cdot \varphi'(z)$ and $\tau_*(\partial X/\partial \bar{z})(z) = (\partial \tilde{X}/\partial \bar{w})(\varphi(z)) \cdot \overline{\varphi'(z)}$ for $z \in M$, τ_* being extended C -linearly. Denoting by (e_A) (resp. (\tilde{e}_A)), $A = 1, 2, 3$, a Lorentzian frame field adapted to X (resp. \tilde{X}) in L^3 , we then get

$$(\tilde{e}_1 + \sqrt{-1}\tilde{e}_2)(\varphi(z)) = |\varphi'(z)| \overline{\varphi'(z)}^{-1} \tau_*(e_1 + \sqrt{-1}e_2)(z),$$

and hence

$$\begin{aligned} 2\tilde{e}_3(\varphi(z)) &= \sqrt{-1}(\tilde{e}_1 + \sqrt{-1}\tilde{e}_2)(\varphi(z)) \times (\tilde{e}_1 - \sqrt{-1}\tilde{e}_2)(\varphi(z)) \\ &= \tau_*(\sqrt{-1}(e_1 + \sqrt{-1}e_2)(z) \times (e_1 - \sqrt{-1}e_2)(z)) = 2\tau_*(e_3(z)), \end{aligned}$$

since τ is orientation preserving. Therefore, by setting $\sigma = \tau_*$, we obtain an orientation preserving isometry σ of H such that $\tilde{G} \circ \varphi(z) = \sigma \circ G(z)$ for $z \in M$. Now differentiating $\tilde{e}_3(\varphi(z)) = \tau_*(e_3(z))$ and substituting (1.5), it can be checked without difficulty that $\tilde{H}(\varphi(z)) = H(z)$ for $z \in M$, thus proving (6.6).

[(2) \Rightarrow (1)] Denote also by σ the extension of σ to an orientation preserving isometry of L^3 . To show (6.5), we may assume σ =identity, considering $\sigma \circ X$ instead of X if necessary. Then we have $G(z) = \tilde{G}(\varphi(z))$, that is, $\Psi(z) = \tilde{\Psi}(\varphi(z))$, since σ is orientation preserving. It then follows from (6.1) that

$$\partial(X^A(z) - \tilde{X}^A(\varphi(z))) / \partial z = 0, \quad A = 1, 2, 3.$$

Therefore, $X(z) = \tilde{X}(\varphi(z)) + c$ for some $c \in \mathbf{R}^3$. This means that $\sigma \circ X(z) = \tilde{X}(\varphi(z)) + c$, and hence there exists an orientation preserving isometry τ of L^3 such that $\tau \circ X(z) = \tilde{X} \circ \varphi(z)$ for $z \in M$.

In the case where a given H in Theorem 6.1 is constant, the complete integrability condition (5.1) requires simply that a given G should be a harmonic mapping. Consequently, given a nonzero real constant H and nowhere holomorphic harmonic mapping G of a simply connected Riemann surface M into H , we can construct, by (6.1), a spacelike immersion $X: M \rightarrow L^3$ with constant mean curvature H and prescribed Gauss map G .

REMARK 6.2. More generally, given a nonzero real constant H and a nonholomorphic harmonic mapping $G: M \rightarrow H$, the mapping $X: M \rightarrow L^3$ given by (6.1) defines a spacelike immersion except for possible isolated singular points, which has, away from these singular points, constant mean curvature H and prescribed Gauss map G . Indeed, it follows from a standard result in the theory of harmonic mappings (cf.

[2, (10.5)]) that if $G: M \rightarrow H$ is a nonholomorphic harmonic mapping, then $\partial\Psi_i/\partial\bar{z}$ has at most isolated zeros where singularities of X occur.

From this point of view, we shall next exhibit some examples of spacelike surfaces of constant mean curvature in L^3 .

EXAMPLE 6.1. Let $D = \{z \in C \mid |z| < 1\}$ be the unit disk in C . Take $H = -1$, and define $\Psi_1: D \rightarrow C$ by $\Psi_1(z) = -\bar{z}$. Then Ψ_1 satisfies (5.1), and the spacelike immersion X defined by (6.1) is written as

$$X(z) = \left(\frac{2 \operatorname{Re} z}{1 - |z|^2}, -\frac{2 \operatorname{Im} z}{1 - |z|^2}, \frac{1 + |z|^2}{1 - |z|^2} \right).$$

This is the standard immersion of the hyperboloid or the upper sheet of H in L^3 .

EXAMPLE 6.2. Take $H = -1/2$, and define $\Psi_1: C \rightarrow C$ by $\Psi_1(z) = (e^{z+\bar{z}} - 1)/(e^{z+\bar{z}} + 1)$. Then Ψ_1 satisfies (5.1); indeed $\Psi_1(C)$ is a geodesic in D . The spacelike immersion X defined by (6.1) is written as

$$X(z) = \left(-\frac{1}{2}(e^{z+\bar{z}} - e^{-(z+\bar{z})}), \sqrt{-1}(\bar{z} - z), \frac{1}{2}(e^{z+\bar{z}} + e^{-(z+\bar{z})}) \right).$$

This is the standard immersion of the hyperbolic cylinder, the surface defined by $(x^3)^2 - (x^1)^2 = 1$ with $x^3 > 0$, in L^3 .

EXAMPLE 6.3. Let M be a closed Riemann surface of genus ≥ 2 . Then each homotopy class of mappings $M \rightarrow M$ contains a harmonic mapping, with respect to the hyperbolic metric of constant negative Gaussian curvature (cf. [2, (6.11)]). Lifting these to the universal covering \tilde{M} of M , we get harmonic mappings $G: \tilde{M} \rightarrow H$. With each of these and a nonzero real constant H , there is associated by (6.1) a spacelike immersion with possible isolated singular points $X: \tilde{M} \rightarrow L^3$, which has, away from singular points, constant mean curvature H and the Gauss map G . (Take the conjugate mapping \bar{G} of G , if G is holomorphic.)

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