5tat.

Netherlands school of economics

ECONOMETRIC INSTITUTE

THE GENERAL LINEAR GROUP OF POLYNOMIAL RINGS OVER REGULAR RINGS

A.C.F. VORST

Erasmus

REPORT 8002/M

ERASMUS UNIVERSITY ROTTERDAM. P.O. BOX 1738. ROTTERDAM THE NETHERLANDS

THE GENERAL LINEAR GROUP OF POLYNOMIAL RINGS OVER REGULAR RINGS.

Ton Vorst

ABSTRACT. In this note we shall prove for two types of regular rings A that every element of $GL_r(A[X_1,\ldots,X_n])$ is a product of an element of $E_r(A[X_1,\ldots,X_n])$ (the group of elementary matrices) and an element of $GL_r(A)$, for $r \geq 3$ and n arbitrary. This is a kind of GL_r-analogue of results of Lindel and Mohan-Kumar and is an extension of a result of Suslin.

Contents.

THE GENERAL LINEAR GROUP OF POLYNOMIAL RINGS OVER REGULAR RINGS.

Ton Vorst

Econometric Institute

Erasmus University Rotterdam,

P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

1. Introduction.

In this note we consider rings of the type $A[T_1,\ldots,T_n]$ with A a commutative regular noetherian ring. Suslin ([9]) has proved that if k is a field and $r \geq 3$ every element of $SL_r(k[T_1,\ldots,T_n])$ is an element of $E_r(k[T_1,\ldots,T_n])$ (the group of elementary matrices over $k[T_1,\ldots,T_n]$). This can be seen as a GL_r-analogue of the famous Serre problem, which was solved by Quillen ([7]) and Suslin ([8]). An extension of the Serre problem is the following question, which was first raised by Bass ([1]). Take A as above and let P be a finitely generated projective $A[T_1,\ldots,T_n]$-module. Does there exist a finitely generated projective A-module P' such that $P \cong P' \otimes_A A[T_1,\ldots,T_n]$? If such a P' exists we call P an extended module. The answer to this question is positive in the following two cases for all projective $A[T_1,\ldots,T_n]$-modules.

(i) A is a complete regular equicharacteristic local ring. (See Lindel and Lütkebohmert ([3]) and Mohan Kumar ([6])).

(ii) A is regular and essentially of finite type over a perfect field k. (See Lindel ([5])).

There are a few more cases for which the answer is known to be positive but these are essentially trivial consequences of the first case. (See also remark 2.2). Mohan Kumar has pointed out that one doesn't need the assumption that k is perfect in (ii).

The GL_r-analogue of the question of Bass is the following. Is every element of $GL_r(A[T_1,\dots,T_n])$ a product of an element of $GL_r(A)$ and an element of $E_r(A[T_1,\dots,T_n])$? Since the answer is negative if A is a field, $r = 2$ and $n \geq 2$ (see [9]), we take $r \geq 3$. In this note we shall show that the answer is positive if A is of one of the two types described above. We also don't need the assumption that k is perfect in (ii). For the proofs we shall need an extension of a lemma of Suslin.

In section two we shall start with two lemmas which we will need for the proofs given in section three.

2. Two lemmas.

In this section A will be a commutative ring. Let $S \subset A$ be a multiplicatively closed set. If $\alpha \in GL_r(A)$ (resp. $\alpha(T_1,\dots,T_n) \in GL_r(A[T_1,\dots,T_n])$) we shall mean by α_S (resp. $\alpha_S(T_1,\dots,T_n)$) the image of the element α (resp. $\alpha(T_1,\dots,T_n)$) in $GL_r(S^{-1}A)$ (resp. $GL_r(S^{-1}A[T_1,\dots,T_n])$) under the canonical mapping. We shall use the same notations for elements of A or $A[T_1,\dots,T_n]$. If $h \in A$ is not a nilpotent we shall mean by A_h the ring $S^{-1}A$ where $S = \{h^n\}_{n \geq 0}$ and write α_h

instead of α_S in the notation above. The next lemma is in fact the unstable version of ([10], theorem 1.1.(i)). The proof is essentially the same as in the stable case.

2.1. Lemma. Let $S \subset A$ be a multiplicatively closed set.
If $GL_r(A[T_1,\ldots,T_n]) = GL_r(A).E_r(A[T_1,\ldots,T_n])$ then also $GL_r(S^{-1}A[T_1,\ldots,T_n]) = GL_r(S^{-1}A).E_r(S^{-1}A[T_1,\ldots,T_n])$.

Proof. Let $\alpha(T_1,\ldots,T_n) \in GL_r(S^{-1}A[T_1,\ldots,T_n])$. We may assume that $\alpha(0,\ldots,0) = I$. Let $f(T_1,\ldots,T_n) = \det(\alpha(T_1,\ldots,T_n)^{-1})$. Since also $f(0,\ldots,0)= 1$ there exist a $g_1 \in S$, an $r \times r$ matrix $\tilde{\alpha}(T_1,\ldots,T_n)$ with coëfficients in $A[T_1,\ldots,T_n]$ and an $\tilde{f}(T_1,\ldots,T_n) \in A[T_1,\ldots,T_n]$ such that

$$\alpha(g_1T_1,\ldots,g_1T_n) = \tilde{\alpha}_S(T_1,\ldots,T_n)$$
$$f(g_1T_1,\ldots,g_1T_n) = \tilde{f}_S(T_1,\ldots,T_n)$$
$$\tilde{\alpha}(0,\ldots,0) = I \text{ and } \tilde{f}(0,\ldots,0) = 1$$

Since $\det(\alpha(g_1T_1,\ldots,g_1T_n)).\ f(g_1T_1,\ldots,g_1T_n) = 1$ there exist a $g_2 \in S$ such that

$$\det(\tilde{\alpha}(g_2T_1,\ldots,g_2T_n)).\tilde{f}(g_2T_1,\ldots,g_2T_n) = 1$$

Hence $\tilde{\alpha}(g_2T_1,\ldots,g_2T_n) \in GL_r(A[T_1,\ldots,T_n])$ and therefore we have

$$\tilde{\alpha}(g_2T_1,\ldots,g_2T_n) = \gamma \prod_{k=1}^{m} e_{i_kj_k}(f_k(T_1,\ldots,T_n)),$$

with $\gamma \in GL_r(A)$ and $f_k(T_1,\ldots,T_n) \in A[T_1,\ldots,T_n]$ for all

$1 \leq k \leq m$. So

$$\alpha(T_1,\ldots,T_n) = \gamma_S. \prod_{k=1}^{m} e_{i_kj_k}(f_k(\frac{T_1}{g_1g_2},\ldots,\frac{T_n}{g_1g_2})_S).$$

q.e.d.

2.2. Remark. An analogue of lemma 2.1 also holds in the projective module case. By this we mean that if every finitely generated projective $A[T_1,\dots,T_n]$ - module is extended from a finitely generated projective A-module then also every finitely generated projective $S^{-1}A[T_1,\dots,T_n]$-module is extended from a finitely generated projective $S^{-1}A$-module. The stable version (K_o-version) of this theorem has been proved in [10] theorem 1.1 (i). One can give the same kind of proof for the unstable case, with which we are concerned here. This shows that Satz 1,1' and 2 of [3] are all easy consequences of the same theorems for rings of the type $B[[X_1,\dots,X_m]]$ with B a regular ring of dimension ≤ 2.

Lemma 2.4 is a generalisation of ([9], lemma 3.7). But for the proof we state first another lemma of Suslin.

2.3. Lemma. (Suslin ([9], lemma 3.3)). Let $h \in A$ be a non-nilpotent and $\delta \in GL_r(A_h)(r \geq 3)$. Let $\sigma(Z) = \delta e_{kl}(Z.f)\delta^{-1}$, where $k \neq l$ and $f \in A_h[Z]$. Then there exist a natural number m and a matrix $\tau \in E_r(A[Z],ZA[Z])$ such that $(\tau)_h = \sigma(h^m Z)$.

2.4. Lemma. Let $B \subset A$ be a subring, $h \in B$ not a nilpotent and $r \geq 3$.

(i) If $Ah + B = A$ then there exist for every $\alpha \in E_r(A_h)$ a $\beta \in E_r(B_h)$ and $\gamma \in E_r(A)$ such that $\alpha = \gamma_h\beta$.

(ii) If moreover $Ah \cap B = Bh$ and h is not a zero-divisor in A then there exist for every $\alpha \in GL_r(A)$ with $\alpha_h \in E_r(A_h)$ a $\beta \in GL_r(B)$ and $\gamma \in E_r(A)$ such that $\alpha = \gamma\beta$.

<u>Proof</u>. (i) Assume that $\alpha = \prod_{k=1}^{m} e_{i_k j_k}(c_k)$ with $c_k \in A_h$.

Define $\sigma_p = \prod_{k=1}^{p-1} e_{i_k j_k}(c_k)\ (1 \leq p \leq m)$.

By lemma 2.3 there exist a natural number s and matrices $\tau_p = \tau_p(Z) \in E_r(A[Z], ZA[Z])$ such that

$$\tau_p(Z)_h = \sigma_p e_{i_p j_p}(h^s Z)\sigma_p^{-1}$$

From the assumption it follows that $Ah^n + B = A$ for all n. Hence for all $1 \leq k \leq m$ we can find $a_k \in A$, $t_k \in B$ and a natural number m_k such that

$$c_k = a_k h^s + \frac{b_k}{h^{m_k}}$$

So we have $\alpha = \prod_{k=1}^{m} e_{i_k j_k}(a_k h^s) e_{i_k j_k}(\frac{b_k}{h^{m_k}}) =$

$$\prod_{k=m}^{1} \sigma_k e_{i_k j_k}(a_k h^s)\sigma_k^{-1} \prod_{k=1}^{m} e_{i_k j_k}(\frac{b_k}{h^{m_k}})$$

Now take $\gamma = \prod_{k=m}^{1} \tau_k(a_k) \in E_r(A)$ and

$$\beta = \prod_{k=1}^{m} e_{i_k j_k}(\frac{b_k}{h^{m_k}}) \in E_r(B) \text{ and we are done.}$$

(ii) From the assumptions it follows that $Ah^n \cap B = Bh^n$ for all n. Hence $B_h \cap A = B$. Using (i) we can write $\alpha_h = \gamma_h . \beta$ with $\gamma \in E_r(A)$ and $\beta \in E_r(B_h)$. Now $\gamma^{-1}\alpha \in GL_r(A)$ and $\beta \in GL_r(B_h)$ and moreover $(\gamma^{-1}\alpha)_h = \beta$. But this implies that $\gamma^{-1}\alpha \in GL_r(B)$. Hence $\alpha = \gamma(\gamma^{-1}\alpha) \in E_r(A)GL_r(B)$.

2.5. Remark. The conditions in 2.4 (ii) are equivalent to the conditions that h is not a zero-divisor and

$\lim\limits_{\leftarrow n} B/h^nB = \lim\limits_{\leftarrow n} A/h^nA$.

3. GL_r for polynomial rings over regular rings.

In this section all rings will be commutative and $r \geq 3$. We shall prove that $GL_r(A[T_1,\ldots,T_n]) = GL_r(A)E_r(A[T_1,\ldots,T_n])$ for two types of regular rings. By [9], theorem 3.1 we know that it is enough to study local rings.

3.1. Theorem. If A is an equicharacteristic regular complete local ring and $\alpha(T_1,\ldots,T_n) \in GL_r(A[T_1,\ldots,T_n])$ is such that $\alpha(0,\ldots,0) = I$ then $\alpha(T_1,\ldots,T_n) \in E_r(A[T_1,\ldots,T_n])(r \geq 3)$.

Proof. By Cohen's structure theorem we know that $A \cong k[[X_1,\ldots,X_m]]$, where k is the residue class field of A. We shall prove the theorem by induction on m. The case m = 0 has been proved by Suslin ([9], corollary 6.7). If n = 0 there is nothing to prove. So we may assume $m \geq 1$ and $n \geq 1$.

Since $\det(\alpha(T_1,\ldots,T_n)) = 1$ and invertible matrices with determinant one over a field are elementary, there exists a non-zero $f \in k[[X_1,\ldots,X_m]][T_1,\ldots,T_n]$ such that $\alpha(T_1,\ldots,T_n)_f \in E_r(k[[X_1,\ldots,X_m]][T_1,\ldots,T_n]_f)$.

Now there exists a transformation of the variables $T_1,\ldots,T_n$ such that f gets the following form:

$$f = hT_n^s + f_{s-1}T_n^{s-1} + \ldots + f_o ,$$

with $f_i \in k[[X_1,\dots,X_m]][T_1,\dots,T_{n-1}]$ and $h \in k[[X_1,\dots,X_m]]$. Since the leading coëfficient of f is a unit in $k[[X_1,\dots,X_m]]_h[T_1,\dots,T_{n-1}]$ we have (by [9], corollary 5.7)

$$\alpha_h(T_1,\dots,T_n) \in E_r(k[[X_1,\dots,X_m]]_h[T_1,\dots,T_n]).$$

By the Weierstrass preparation theorem ([2],§3.8) we know that h is the product of a unit u and a distinguished polynomial (i.e. $h = u.(X_m^r+a_{r-1}X_m^{r-1}+\dots+a_o)$ with $a_i \in (X_1,\dots,X_{m-1})k[[X_1,\dots,X_{m-1}]]$ for all $0 \leq i \leq r-1$). So we may assume that h is a distinguished polynomial and hence every $g \in k[[X_1,\dots,X_m]]$ can be uniquely written as $g = qh + r$ with $q \in k[[X_1,\dots,X_m]]$ and $r \in k[[X_1,\dots,X_{m-1}]][X_m]$ a polynomial in X_m of degree $\leq r-1$ (see [2],§3.8 proposition 5).

Hence if one takes $B = k[[X_1,\dots,X_{m-1}]][X_m, T_1,\dots,T_n]$ and $A' = k[[X_1,\dots,X_m]][T_1,\dots,T_n]$ one has $A'h + B = A'$, $A'h \cap B = Bh$ and h is not a zero-divisor in A'. So by lemma 2.4 we have

$$\alpha(T_1,\dots,T_n) = \gamma(T_1,\dots,T_n)\beta(X_m,T_1,\dots,T_n)$$

with $\gamma(T_1,\dots,T_n) \in E_r(A')$ and $\beta(X_m,T_1,\dots,T_n) \in GL_r(B)$. Since $\beta(X_m,0,\dots,0)^{-1} = \gamma(0,\dots,0) \in E_r(k[[X_1,\dots,X_m]])$ also $\beta(X_m,0,\dots,0)^{-1} \in$ $\in E_r(k[[X_1,\dots,X_m]][T_1,\dots,T_n])$. Now

$\alpha(T_1,\dots,T_n) = \gamma(T_1,\dots,T_n)\beta(X_m,0,\dots,0)(\beta(X_m,0,\dots,0)^{-1}$ $\beta(X_m,t_1,\dots,T_n)) \in E_r(k[[X_1,\dots,X_m]][T_1,\dots,T_n])$ since $\beta(X_m,0,\dots,0)^{-1}\beta(X_m,T_1,\dots,T_n) \in E_r(k[[X_1,\dots,X_{m-1}]][X_m,T_1,\dots,T_n])$ by the induction hypothesis. q.e.d.

Let k be a field. Recall that we call A of essentially finite type over k if $A = S^{-1}C$ with S a multiplicatively closed subset of C and $C = k[X_1,\dots,X_m]/I$, a quotient ring of a polynomial ring over k. For the proof of the next theorem we need a proposition of Lindel ([5], propositions 2 and 3).

3.2. Proposition (Lindel). Let A be a regular local ring of essentially finite type over a perfect field k. Then there exists a subring B of A with an element $h \in B$ such that

(i) $B = k[X_1,\dots,X_p]_{\underline{p}}$, where $\underline{p}$ is a prime ideal of $k[X_1,\dots,X_p]$

(ii) $Ah + B = A$ and $Ah \cap B = Bh$.

We are now ready for the next theorem.

3.3. Theorem. Let A be a regular ring of essentially finite type over a field k. If $\alpha(T_1,\dots,T_n) \in GL_1(A[T_1,\dots,T_n])$ is such that $\alpha(0,\dots,0) = I$ then $\alpha(T_1,\dots,T_n) \in E_r(A[T_1,\dots,T_n])$ $(r \geq 3)$.

Proof. We first consider the case in which k is a perfect field. The proof goes by induction on dim A. By [9] theorem 3.1 it is in each step enough to prove the theorem for local rings. If dim A = 0 we have a field and we are again in Suslin's case ([9], corollary 6.7). Hence we assume dim $A \geq 1$.

Take a ring B and an $h \in B$ as in proposition 3.2. Since $\dim A_h < \dim A$ we have that $\alpha_h(T_1,\dots,T_n) \in E_r(A_h[T_1,\dots,T_n])$ by the induction hypothesis. Since A is local and regular we know that h is not a zero-divisor in $A[T_1,\dots,T_n]$ and we can apply lemma 2.4 (ii) to $\alpha(T_1,\dots,T_n)$. So

$$\alpha(T_1,\ldots,T_n) = \gamma(T_1,\ldots,T_n)\beta(T_1,\ldots,T_n),$$

with $\beta(T_1,\ldots,T_n) \in GL_r(B[T_1,\ldots,T_n])$ and

$$\gamma(T_1,\ldots,T_n) \in E_r(A[T_1,\ldots,T_n]).$$

Hence we have

$$\alpha(T_1,\ldots,T_n) = \gamma(T_1,\ldots,T_n)\gamma(0,\ldots,0)^{-1}(\beta(0,\ldots,0)^{-1}\beta(T_1,\ldots,T_n)),$$

where the first two factors are contained in $E_r(A[T_1,\ldots,T_n])$. Since the theorem is true for a polynomial ring ([9], corollary 6.7) and B is a localisation of a polynomial ring, the theorem is also true for B by lemma 2.1. Hence

$$\beta(0,\ldots,0)^{-1}\beta(T_1,\ldots,T_n) \in E_r(B[T_1,\ldots,T_n]) \subset E_r(A[T_1,\ldots,T_n]).$$

The case in which k is not a perfect field can be reduced to the perfect field case as follows. We may assume that $A = (k[X_1,\ldots,X_m]/I)_{\underline{p}}$, with $\underline{p}$ a prime ideal of $k[X_1,\ldots,X_m]/I$ and $\alpha(T_1,\ldots,T_n) \in GL_r(A[T_1,\ldots,T_n])$. If Char(k) = p one can find $\alpha_1,\ldots,\alpha_s \in k$ such that a set of generators of I already exists in $A' = \mathbb{F}_p[\alpha_1,\ldots,\alpha_s][X_1,\ldots,X_m]$, a minimal set of generators of $\underline{p}$ already exists in A' and $\alpha(T_1,\ldots,T_n) \in GL_r((A'/I')_{\underline{p}'}[T_1,\ldots,T_n])$ where I' is the ideal generated in A' by the set of generators of I and where $\underline{p}'$ is the ideal generated in A'/I' by the generators of $\underline{p}$. By adjoining a few more elements of k to A' and then a suitable localisation one can construct a ring $B \subset A$ which is regular and of essentially finite type over $\mathbb{F}_p$ such that $\alpha(T_1,\ldots,T_n) \in GL_r(B[T_1,\ldots,T_n])$. Hence

$\alpha(T_1,\ldots,T_n) \in E_r(B[T_1,\ldots,T_n]) \subset E_r(A[T_1,\ldots,T_n])$ since $\mathbb{F}_p$ is perfect.

3.4. Remark. If $A = k\{X_1,\ldots,X_n\}$ is the ring of convergent power series over a field with a valuation then theorem 3.1 also holds for this ring since one has again a Weierstrass preparation theorem.

ACKNOWLEDGEMENTS

I like to thank N. Mohan Kumar for pointing out to me that the case of a perfect field in theorem 3.3 implies the case of an arbitrary field.

REFERENCES

1. H. Bass, Some Problems in "Classical" Algebraic K-theory, Lecture Notes in Mathematics, Vol. 342, Springer Verlag, Berlin, Heidelberg, New York, 1973.
2. N. Bourbaki, Algèbre Commutative, Chap. VII, Diviseurs, Hermann, Paris, 1965.
3. H. Lindel and W. Lütkebohmert, Projective Moduln über Polynomial Erweiterungen von Potenzreihenalgebren, Arch. Math. 28, 1977, 51-54.
4. H. Lindel, Projektive Moduln über Polynomringen $A[T_1,\ldots,T_n]$ mit einem regulären Grundring A, Manuscripta Math. 23, 1978, 143-154.
5. H. Lindel, On a Question of Bass, Quillen and Suslin concerning Projective Modules over Polynomial Rings, Preprint.

6. N. Mohan Kumar, On a Question of Bass and Quillen, Preprint.

7. D. Quillen, Projective Modules over Polynomial rings, Inv. Math. 36, 1976, 166-172.

8. A.A. Suslin, On Projective Modules over Polynomial Rings, Soviet Math. Dokl. 17, 1976, 1160-1164.

9. A.A. Suslin, On the Structure of the Special Linear Group over Polynomial Rings, Izv. Akad. Nauk. S.S.S.R. Ser. Mat. 41, 1977, 235-252.

10. T. Vorst, Polynomial Extensions and Excision for K_1, Math. Ann. 244, 1979, 193-204.

LIST OF REPORTS 1980

8000 "List of Reprints, nos 241-260, Abstracts of Reports Second Half 1979".

8001/O "A Stochastic Method for Global Optimization", by C.G.E. Boender, A.H.G. Rinnooy Kan, L. Stougie and G.T. Timmer.

8002/M "The General Linear Group of Polynomial Rings over Regular Rings", by A.C.F. Vorst.