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The General Modulus-based Jacobi Iteration Method for Linear Complementarity Problems

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Abstract. For the large sparse linear complementarity problem, by reformulating it as an implicit fixed-point equation problem, Bai propose a class of modulus-based matrix splitting iteration methods in [12]. In this paper, we discuss one form of these methods–the general modulus-based Jacobi iteration method, proved the convergence, and derive the domain and the optimum value of the parameter for one special situation. Numerical results show that this method is superior to some modulus-related methods in computing efficiency and feasible aspects in some situations.

1. Introduction

We are concerned here with the linear complementarity problem, abbreviated as LCP(q, A), for finding a pair of real vectors z and $r \in R^n$ such that

$$\begin{cases} r := Az + q \geq 0, \ z \geq 0, \\ z^T r = 0, \end{cases} \tag{1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is a given large, sparse, and real matrix, $q \in R^n$ is a given real vector.

To compute a numerical solution of the LCP(q, A), many authors utilize matrix splitting to construct feasible and efficient iteration methods. When the system matrix A is large and sparse, the iteration methods have many advantages than the direct method [19]. For example, the projected successive overrelaxation (PSOR) iterations [2,3], the multilevel iteration method [25], the general fixed-point iterations [5~7], and the matrix multisplitting iterations [8~14] are about some standard splitting methods for iteratively solving the LCP(q, A). In these works, some convergence results have been established for the cases that the system matrix A is symmetric positive definite, symmetric positive semi-definite, diagonally dominant, or H-matrix. Some of these methods are complicated. By reformulating the LCP(q, A) as an implicit fixed-point equation, Murty presented a modulus (M) iteration method in [10]. This method is seem to be the most effective iteration method due to avoiding the projections of the iterations used in the projected

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relaxation iterations and the general fixed-point iterations. By generalizing this modulus iteration method with the introduction of an iteration parameter, Dong and Jiang proposed a modified modulus (MM) iteration method and proved its convergence in [11] when the system matrix A is symmetric positive definite, and derived the optimum parameter. Although this method inherits the merits of the modulus iteration method, it is not feasible sometime. Based on matrix splitting, Bai established a competitive class of modulus-based matrix splitting iteration methods in [12]. The convergence theorems are proved when the system matrix A is a positive definite matrix or an H_+-matrix. The numerical examples show that some of these methods are efficient, such as the modulus-based successive overrelaxation (MSOR) method is more efficient to theprove the convergence and derive the domain projected relaxation iteration method and the MM method with proper parameters in some situations. It is not easy in practice for the MSOR and modulus-based accelerated overrelaxation (MAOR) method. See [18~26] for further generalizations and developments about this class of iteration methods.

The modulus-based Jacobi (MJ) iteration method is one of the modulus-based matrix splitting iteration methods for the LCP(q, A). In this paper, we discuss the general form of the MJ method based on a reformulation of the LCP(q, A) to an implicit fixed-point equation and on suitable matrix splitting. We prove the convergence, and derive the domain and the optimum value of the parameter about this method for one special situation. When the system matrix A is an H_+-matrix, we present a larger domain of the parameter. In addition, we use numerical examples to show the feasible and superior properties of this method compared with some of modulus-related iteration methods.

The outline of this paper is as follows. We introduce a conclusion, some results, and the MJ method in Section 2. We present the general modulus-based Jacobi (GMJ) iteration method, prove its convergence, and derive the domain and the optimum value of the parameter for one special situation in Section 3. The numerical results about this method are shown and discussed in Section 4. Finally, in Section 5, some concluding remarks are presented.

2. Preliminaries

In this section, we briefly introduce a conclusion, some basic formulas and the MJ method as follows:

Conclusion[12] Let $A = M - N$ be a splitting of the Matrix $A \in R^{n\times n}$, Ω_1 and Ω_2 be $n\times n$ nonnegative diagonal matrices, and Ω and Γ be $n \times n$ positive diagonal matrices such that $\Omega = \Omega_1 + \Omega_2$. For the LCP$(q, A)$, the following statements hold true:
(i) If (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}(\Gamma^{-1}z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation

$$(M\Gamma + \Omega_1)x = (N\Gamma - \Omega_2)x + (\Omega - A\Gamma)|x| - q. \tag{2}$$

(ii) If x satisfies Eq.(2), then

$$\begin{cases} z = \Gamma(|x| + x) \\ r = \Omega(|x| - x) \end{cases} \tag{3}$$

is a solution of the LCP(q, A).

Eq.(2) is not applicable in practice, as it involves many arbitrary parameters. By letting $\Omega_2 = 0, \Gamma = \frac{1}{\gamma}I$, and denoting $\Omega := \gamma\Omega$, the above two equations can be simplified as:
(i) If (z, r) is a solution of the LCP(q, A), then $x = \frac{\gamma}{2}(z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation

$$(M + \Omega)x = Nx + (\Omega - A)|x| - \gamma q. \tag{4}$$

(ii) If x satisfies Eq.(4), then

$$\begin{cases} z = \frac{1}{\gamma}I(|x| + x) \\ r = \frac{1}{\gamma}\Omega(|x| - x) \end{cases} \tag{5}$$

is a solution of the LCP(q, A).

So the LCP(q, A) can be turned into Eqs.(4) and (5) simply.

Based on Eqs.(4) and (5), the following modulus-based matrix splitting iteration method for solving the LCP(q, A) was presented in [12].

$$(M + \Omega)x^{(k+1)} = Nx^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q$$

with

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}).$$

We remark that the above iteration formula provides a general framework. It can yield a series of modulus-based matrix splitting iteration methods by suitably choosing the matrix splittings and the iteration parameters, e.g., the MJ, modulus-based Gauss-Seidel (MGS), MSOR and MAOR iteration methods. The M method in [10] and the MM method in [11] are two special cases of this method. The class of methods have several advantages, but not every form has practical value. Numerical examples have shown that some forms of these methods are superior to the projected relaxation iteration method as well as the modified modulus iteration method. The methods used in these examples are the MM method, the MGS method and the MSOR method. They do well in practice except for the parameter-choice. The MJ method is rarely used in practice for some limitations. We give the iteration formula of the MJ method in the following, and will discuss the general form of it in the next section. The detail materials about other modulus-based matrix splitting iteration methods, such as the MGS, MSOR, and MAOR iteration methods can refer to paper [12].

Let $M = \text{diag}(A)$ be the diagonal matrix of A in the above formula and $\gamma = 1$, then the MJ method is

$$(\text{diag}(A) + \Omega)x^{(k+1)} = (\text{diag}(A) - A)x^{(k)} + (\Omega - A)|x^{(k)}| - q$$

with

$$z^{(k+1)} = |x^{(k+1)}| + x^{(k+1)}.$$

3. The GMJ Method

Let M be a symmetric matrix in Eq.(4), that is, let diag(A) be a general symmetric matrix in MJ method, then we obtain the general form of the MJ method, abbreviated as GMJ. In this section, we mainly discuss the convergence property and the parameter selecting problem about this method.

Method(GMJ) Let $A = M - N$ be a splitting of the system matrix $A \in R^{n\times n}$ with M being a symmetric matrix. Given an initial vector $x^{(0)} \in R^n$, compute $x^{(k+1)} \in R^n$ by solving the linear system

$$(M + \Omega)x^{(k+1)} = Nx^{(k)} + (\Omega - A)|x^{(k)}| - \gamma q. \tag{6}$$

Then set

$$z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)}) \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{7}$$

until the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty} \subset R^n$ is convergent. Here, Ω is a selected $n \times n$ positive diagonal matrix, and γ is a positive constant.

Due to the relations Eqs.(4) and (5), the following theorem about the GMJ method can be established easily.

Theorem 1 Let $A \in R^{n\times n}$, if the GMJ method converges for some matrices M, N, Ω, positive number γ, and an initial vector $x^{(0)} \in R^n$, then the LCP$(q; A)$ has solution, the limit must be a solution.

When the LCP(q, A) has solutions, and so Eq.(6) has solutions. However, not all GMJ methods are convergent with different M, N and Ω. It is possible that there are different convergent GMJ methods, with a same initial vector $x^{(0)}$, the limits are different, though all of which are the solutions of the LCP(q, A). In general meaning, for a convergent GMJ method, different initial vectors $x^{(0)}$ should produce the same limit. We know that the LCP(q, A) has a unique solution for all $q \in R^n$ if and only if A has positive principal minors from [24]. In fact, the GMJ method can deal with the situation when the LCP(q, A) exists more solutions, which will be illustrated by the numerical example in section 4. For corresponding the GMJ method with the solution, we assume that the LCP(q, A) has an unique solution, then we can obtain a corollary of Theorem 1 easily, and a convergence theorem for the GMJ method.

Corollary 1 Suppose that the LCP(q, A) exists an unique solution. If the GMJ method converges, then Given any initial vector $x^{(0)} \in R^n$, the limit must be the solution.

Theorem 2 Let $A = M - N$ be a splitting of the matrix $A \in R^{n\times n}$ with M being a symmetric matrix. Suppose that $\Omega \in R^{n\times n}$ is a positive diagonal matrix, and γ is a positive constant. Let $|||.|||$ be the matrix norm induced by the monotonous vector norm $||.||$, $\delta(\Omega) = 2|||(M+\Omega)^{-1}N|||+|||(M+\Omega)^{-1}(M-\Omega)|||$. If $\delta < 1$, and the LCP(q, A) exists an unique solution, then given any initial vector $x^{(0)} \in R^n$, the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty}$ generated by GMJ method converges to the solution.

Proof. Let z_* be the solution of the LCP(q, A), then

$$x_* = \frac{\gamma}{2}(z_* - \Omega^{-1}r)$$

satisfies Eq. (4), namely

$$(M + \Omega)x_* = Nx_* + (\Omega - A)|x_*| - \gamma q. \tag{8}$$

After subtracting Eq.(8) from Eq.(6), we have

$$(M + \Omega)(x^{(k+1)} - x_*) = N(x^{(k)} - x_*) + (\Omega - A)(|x^{(k)}| - |x_*|),$$

or equivalently

$$\begin{aligned}(x^{(k+1)} - x_*) &= (M + \Omega)^{-1}N(x^{(k)} - x_*) + (M + \Omega)^{-1}(\Omega - A)(|x^{(k)}| - |x_*|)\\ &= (M + \Omega)^{-1}N(x^{(k)} - x_*) + (M + \Omega)^{-1}(-(M - \Omega) + N)(|x^{(k)}| - |x_*|).\end{aligned}$$

We have

$$\begin{aligned}||(x^{(k+1)} - x_*)|| &\le (2|||(M + \Omega)^{-1}N||| + |||(M + \Omega)^{-1}(M - \Omega)|||)||(x^{(k)} - x_*)||\\ &= \delta(\Omega)||(x^{(k)} - x_*)||.\end{aligned}$$

So, when $\delta(\Omega) < 1$, $\{x^{(k)}\}_{k=0}^{\infty}$ converges to x_*, and $\{z^{(k)}\}_{k=0}^{\infty}$ converges to the solution z_* from Eq.(7). □

From this theorem, we know that when the LCP(q, A) has only one solution, both the matrix M and the Ω have many choices. Once the conditions are satisfied, the GMJ method can be used. The monotonous vector norms have many forms, such as $||.||_p, p \in N, ||.||_\infty$, etc. The convergence theorem is more general than the similar conclusion shown in [12]. Next, we mainly discuss one special situation, that is, the vector norm is 2-norm, M is a symmetric positive definite matrix and $\Omega = \omega I$, where ω is a positive constant, then we derive the following theorem about the domain and the optimum value of the parameter.

Theorem 3 Suppose that the LCP(q,A) exists an unique solution, $A = M - N$ is a splitting of the matrix $A \in R^{n\times n}$ with M being a symmetric positive definite matrix, $\Omega = \omega I \in R^{n\times n}$, ω is a positive parameter, and γ is a positive constant. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and the largest eigenvalue of the matrix M, respectively, and define $\tau = |||M^{-1}N|||_2$. If $\tau < 1$ and and $\tau\mu_{max} < \mu_{min}$ then when $\omega \in (\tau\mu_{\max}, +\infty)$, the iteration sequence $\{z^{(k)}\}_{k=0}^{\infty}$ generated by the GMJ method converges to the solution for any initial vector $x^{(0)} \in R^n$. Moreover, the optimum parameter is: $\omega_{opt} = \sqrt{\mu_{\min}\mu_{\max}}$.

Proof. From Theorem 2, we know

$$\delta(\Omega) = \delta(\omega I) = 2|||(M + \omega I)^{-1}N|||_2 + |||(M + \omega I)^{-1}(M - \omega I)|||_2,$$

where

$$\begin{aligned} |||(M + \omega I)^{-1}N|||_2 &\le |||(M + \omega I)^{-1}M^{-1}|||_2|||M^{-1}N|||_2 \\ &= \max_{\lambda\in sp(M)} \frac{\lambda\tau}{\lambda + \omega} = \frac{\mu_{\max}\tau}{\mu_{\max} + \omega}, \end{aligned}$$

and

$$\begin{aligned} |||(M + \omega I)^{-1}(M - \omega I)|||_2 &= \max_{\lambda\in sp(M)} \frac{|\lambda - \omega|}{\lambda + \omega} \\ &= \max\{\frac{|\mu_{\min} - \omega|}{\mu_{\min} + \omega}, \frac{|\mu_{\max} - \omega|}{\mu_{\max} + \omega}\} \\ &= \begin{cases} \dfrac{\mu_{\max} - \omega}{\mu_{\max} + \omega}, & \text{for} \quad \omega \le \sqrt{\mu_{\min}\mu_{\max}}, \\ \dfrac{\omega - \mu_{\min}}{\mu_{\min} + \omega}, & \text{for} \quad \omega \ge \sqrt{\mu_{\min}\mu_{\max}}. \end{cases} \end{aligned}$$

Hence, it holds that

$$\delta(\Omega) = \delta(\omega I) \le \begin{cases} \dfrac{(1 + 2\tau)\mu_{\max} - \omega}{\mu_{\max} + \omega}, & \text{for} \quad \omega \le \sqrt{\mu_{\min}\mu_{\max}}, \\ \dfrac{\omega - \mu_{\min}}{\mu_{\min} + \omega} + \dfrac{2\mu_{\max}\tau}{\mu_{\max} + \omega}, & \text{for} \quad \omega \ge \sqrt{\mu_{\min}\mu_{\max}}. \end{cases}$$

With solving the systems respectively,

$$\begin{cases} \omega \le \sqrt{\mu_{\min}\mu_{\max}}, \\ \dfrac{(1 + 2\tau)\mu_{\max} - \omega}{\mu_{\max} + \omega} < 1, \end{cases}$$

and

$$\begin{cases} \omega \ge \sqrt{\mu_{\min}\mu_{\max}}, \\ \dfrac{\omega - \mu_{\min}}{\mu_{\min} + \omega} + \dfrac{2\mu_{\max}\tau}{\mu_{\max} + \omega} < 1, \end{cases}$$

we can obtain the domain of the parameter ω.

For the first system, when $\omega \le \sqrt{\mu_{\min}\mu_{\max}}$, for $\delta(\omega I) < 1$, need add the condition $\tau\mu_{\max} < \mu_{\min}$. So this case can be summarized as:

(i) When $\tau^2\mu_{\max} < \mu_{\min}$, $\tau\mu_{\max} < \omega \le \sqrt{\mu_{\min}\mu_{\max}}$, then $\delta(\omega I) < 1$.

For the second system, when $\omega \ge \sqrt{\mu_{\min}\mu_{\max}}$, there are three cases:

(1) If $\mu_{\min} < \tau\mu_{\max}$, for $\delta(\omega I) < 1$, need add another condition $\omega < \frac{(1-\tau)\mu_{\min}\mu_{\max}}{\tau\mu_{\max}-\mu_{\min}}$. So this case can be summarized as:

(ii) When $\tau^2\mu_{\max} < \mu_{\min} < \tau\mu_{\max}$, $\sqrt{\mu_{\min}\mu_{\max}} \le \omega < \frac{(1-\tau)\mu_{\min}\mu_{\max}}{\tau\mu_{\max}-\mu_{\min}}$, then $\delta(\omega I) < 1$.

(2) If $\mu_{\min} > \tau\mu_{\max}$, ω needs no other conditions.

(3) If $\mu_{\min} = \tau\mu_{\max}$, ω also needs no other conditions.

So, the two cases (2) and (3) can be summarized as:

(iii) When $\mu_{\min} \ge \tau\mu_{\max}$, $\omega \ge \sqrt{\mu_{\min}\mu_{\max}}$, then $\delta(\omega I) < 1$.

Combine (i), (ii) and (iii), and notice $\tau < 1$, we learn that situation(ii) implies situation(i), situation(iii) implies situation(i) too, situation(ii) and situation(iii) are complementary. Therefore, we obtain the domain for the parameter ω, that is, if $\tau < 1$ and $\tau\mu_{max} < \mu_{min}$, then when $\omega \in (\tau\mu_{\max}, +\infty)$, the GMJ method is convergent. Note that when $\omega \in (\tau\mu_{\max}, \sqrt{\mu_{\min}\mu_{\max}}]$, the function $\delta(\Omega) = \frac{(1+2\tau)\mu_{\max}-\omega}{\mu_{\max}+\omega}$ is a monotonous decreasing function about ω, and when $\omega \in [\sqrt{\mu_{\min}\mu_{\max}}, +\infty)$, the function $\delta(\Omega) = \frac{\omega-\mu_{\min}}{\mu_{\min}+\omega} + \frac{2\mu_{\max}\tau}{\mu_{\max}+\omega}$ is a monotonous increasing function about ω. So, the optimum parameter is:

$$\omega_{opt} = \sqrt{\mu_{\min}\mu_{\max}} \quad \text{when} \quad \omega \in (\tau\mu_{\max}, +\infty). \tag{9}$$

Summarize the above conclusions, we can obtain the theorem. □

The proof of the theorem refers partly to the proof of Theorem 4.1 presented in [12] and the results are more general. From this theorem, we know that the domain of the parameter ω is sufficient but not necessary, which means that there probably exists ω out of the domain which makes the GMJ method converge. In addition, the meaning of the optimum parameter here is that $\delta(\Omega) = \delta(\omega_{opt}I)$ is only a boundary of $\delta(\Omega)$, which is independent with the true value of $\delta(\Omega)$. The parameter ω_{opt} is equivalent to a threshold, that is, $\delta(\Omega)$ must be smaller on this point, but may not be the smallest one compared with others around the ω_{opt}. The parameter ω which makes the $\delta(\Omega)$ be the smallest is near the ω_{opt}, which will be illustrated by the numerical example in section 4. It is not good to calculate the ω_{opt} with this formula directly for the large complicated matrix M. For some special cases, we may estimate the approximate value by some ways. In a word, when the LCP(q, A) has solutions, we can try to construct a better GMJ method to solving it by the above theorem.

Next, we discuss a particular case, that is, the system matrix $A \in R^{n\times n}$ in the LCP(q, A) is an H_+-matrix. Denote by diag(A) the diagonal matrix of A, $\Omega = \omega I$, we can obtain the following corollary of Theorem 3 easily.

Corollary 2 Let $A \in R^{n\times n}$ be an H_+-matrix, and $M = \text{diag}(A)$. Assume that $\Omega = \omega I$, ω is a positive parameter, and γ is a positive constant. Denote by $\mu_{\min}$ and $\mu_{\max}$ the smallest and the largest diagonal element of the matrix M, respectively. If $\tau = |||M^{-1}(M - A)|||_2 < 1$ and $\tau^2\mu_{\max} < \mu_{\min}$, then when $\omega \in (\tau\mu_{\max}, +\infty)$, the GMJ method is convergent. In addition, the optimum parameter is: $\omega_{opt} = \sqrt{\mu_{\min}\mu_{\max}}$.

The domain of the parameter is larger than the one in Theorem 4.3 proposed in paper [12] when $\tau < \frac{1}{2}$, and is smaller when $\frac{1}{2} \leq \tau < 1$.

4. Numerical Results

In general, the modulus-based relaxation iteration methods are superior to the projected relaxation iteration method and the modified modulus iteration method, we can learn this from paper [12]. Both the MSOR method and the MAOR method need select proper parameters, some parameters will provide good convergence, but not for others. It is not easy to choose proper parameters in practice, therefore, we here only compare the GMJ method with other modulus-related iteration methods. It is obvious that the GMJ method is superior to the MJ method. We mainly use examples to examine the numerical effectiveness of the GMJ method compared with the M method, the MM method and the MGS method, from the aspects of the number of iteration steps(IT), elapsed CPU time in seconds (CPU), and norm of absolute residual vectors (RES). RES is defined as the absolute value of the inner product:

$$\text{RES}(z^{(k)}) := abs(< z^{(k)}, Az^{(k)} + q >),$$

where $z^{(k)}$ is the k-th approximate solution to the LCP(q, A). In our computation, because the limit is independent with the initial vectors for a convergent method, for convenience, all initial vectors in the first three examples are chosen to be $x^{(0)} = (1, 0, 1, 0, \ldots, 1, 0, \ldots)^T \in R^n$. Let m be a prescribed positive integer and $n = m^2$.

Consider the LCP(q, A), $A \in R^{n\times n}$ is given by

$$A(\mu, \eta, \zeta) = \widehat{A} + \mu I + \eta B + \zeta C,$$

Where μ, η, and ζ are constants, $q = -A(\mu, \eta, \zeta)z_*$,

$$\widehat{A} = \text{Tridiag}(-I, S, -I) = \begin{pmatrix} S & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & \cdots & 0 & 0 \\ 0 & -I & S & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & S & -I \\ 0 & 0 & \cdots & \cdots & -I & S \end{pmatrix} \in R^{n\times n}$$

is a block-tridiagonal matrix,

$$B = \text{Tridiag}(0, 0, I) = \begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & I \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \in R^{n\times n}$$

has the same structure with $\widehat{A}$,

$$S = \text{tridiag}(-1, 4, -1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdots & 4 & -1 \\ 0 & 0 & \cdot & \cdots & -1 & 4 \end{pmatrix} \in R^{m\times m}$$

is a tridiagonal matrix, and $C = \text{diag}(1, 2, 1, 2, ...) \in R^{n\times n}$ is a diagonal matrix. In the first three examples, the vector $z_* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T \in R^n$ is the unique solution of the LCP$(q, A(\mu, \eta, \zeta))$.

Example 4.1

Suppose that $\omega = \sqrt{\lambda_{\min}(A)\lambda_{\max}(A)}$ in the MM method by [11], $M = \text{diag}(A), \Omega = \omega_{opt}I = (4 + \mu)I$ in the GMJ method, $\Omega = (1/2)\text{diag}(M)$ in the MGS method, $\gamma = 1$, and $\zeta = 0$. We discuss two cases: $\mu = 4, \eta = 0$ and $\mu = 4, \eta = 1$ in the following:

(i) When $\mu = 4, \eta = 0$, the matrix A is a symmetric positive definite matrix, and is an H_+-matrix too, then the LCP(q, A) has an unique solution. The M method, the MM method, the MGS method, the MSOR method and the GMJ method all can be used by Theorem 4.3 in paper [12] and Theorem 3 presented in this paper.

(ii) When $\mu = 4, \eta = 1$, the matrix A is not a symmetric positive definite matrix, but an H_+-matrix and a diagonal dominant matrix. Then there is an unique solution for the LCP(q, A). The M method, the MGS method, the MSOR method and the GMJ method can be used by Theorem 4.3 in paper [12] and Corollary 2 in this paper.

We obtain the following Table 1.

From the table, we can find that the GMJ method is superior to some modulus-related methods in some situations. This method is likely not good as well as the MGS method just because the matrix A has the same diagonal elements and the matrix Ω has the limitations of selecting scalar matrices in this example. In the following example, we will find the different results.

Example 4.2

In this example, we discuss the matrix A with different diagonal elements. Let $\mu = 4, \eta = 1, \zeta = 1$, and $M = \text{diag}(A)$, we obtain the following Table 2.

From the table, we find the GMJ method is more better than the MGS method. Moreover, the GMJ method with parameter matrix $\Omega = \omega M$ $(\omega > 0)$ also provides very good results.

Table 1: Numerical results

	Method	n=900 IT	n=900 CPU	n=900 RES($z^{(k)}$)	n=2500 IT	n=2500 CPU	n=2500 RES($z^{(k)}$)
	M	69	25.4	7.5e-06	71	316	7.7e-06
$\mu = 4$	MM	15	1.07	5.2e-06	16	11.0	4.3e-06
$\eta = 0$	MGS	30	0.03	8.1e-06	31	0.09	9.5e-06
	GMJ	28	0.32	7.0e-06	30	1.65	5.6e-06
	M	71	30.6	8.2e-06	73	534	8.7e-06
$\mu = 4$	MM						
$\eta = 1$	MGS	20	0.01	7.6e-06	21	0.06	6.2e-06
	GMJ	20	0.06	6.3e-06	21	0.53	8.0e-06

Table 2: Numerical results

	Method	n=900 IT	n=900 CPU	n=900 RES($z^{(k)}$)	n=2500 IT	n=2500 CPU	n=2500 RES($z^{(k)}$)
	MGS	27	0.03	6.9e-06	28	2.73	7.4e-06
$\Omega = 9.5I$	GMJ	23	0.01	5.8e-06	24	0.03	7.9e-06
$\Omega = 1M$	GMJ	23	0.01	6.5e-06	24	0.03	8.7e-06

Example 4.3

Because the matrix $M + \Omega$ is a symmetric positive definite matrix, the conjugate gradient method (CG) in [28] can be used in the inner iterating processes. In the following, we investigate the value of the true optimum parameter, and connect the GMJ method with the CG method in examples. We show two cases both with $\zeta = 0$ and $\Omega = \omega I(\omega > 0)$.

(i) When $M = \text{Tridiag}(0, S, 0) \in R^{n\times n}$, then $\omega_{opt} = 7.7$, the true optimum parameter is about $\omega_t = 6.2 < \omega_{opt}$.

(ii) When $M = 1.3(B + B^T) + 7I$, then $\omega_{opt} = 6.5$, the true optimum parameter is about $\omega_t = 7.7 > \omega_{opt}$.

We obtain the following Table 3.

Table 3: Numerical results

	Method	n=900 IT	n=900 CPU	n=900 RES($z^{(k)}$)	n=2500 IT	n=2500 CPU	n=2500 RES($z^{(k)}$)
$\omega_t = 6.2$	GMJ_t	19	0.25	2.1e-06	19	1.48	8.2e-06
$\mu = 4$	GMJ	22	0.20	7.9e-06	24	1.51	4.3e-06
$\eta = 0$	GMJ_{CG}	22	0.15	7.9e-06	24	0.68	4.1e-06
$\omega_t = 7.7$	GMJ_t	11	0.30	3.9e-06	11	2.28	4.8e-06
$\mu = 4$	GMJ	16	0.35	8.2e-06	17	2.37	6.8e-06
$\eta = 1$	GMJ_{CG}	16	0.26	8.2e-06	17	1.35	6.0e-06

From the table, we find that the optimum parameter ω_{opt} obtained by the formula $\sqrt{\mu_{\min}\mu_{\max}}$ is not completely equal to the true optimum parameter, but is close to it in most cases. Besides a few more iteration steps, the ω_{opt} has the same efficiency almost. In addition, the GMJ method connected with the CG method is more efficient than the original GMJ method.

Example 4.4

In the end of this section, we give a special example where the system matrix A in the LCP(q, A) contains with zero diagonal elements, and show the GMJ method to handle with more solutions' situation.

Consider the LCP(q, A), in which $A = M - N$, where

$$M = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdots & 2 & 2 \\ 0 & 0 & \cdot & \cdots & 2 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdot & \cdots & 1 & 0 \\ 0 & 0 & \cdot & \cdots & 0 & 1 \end{pmatrix},$$

then M is not a symmetric positive definite matrix. Suppose the order of the matrix A is 10, $z_* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T \in R^{10}$, $q = -Az_*$, then the last diagonal element of A is 0, and the LCP(q, A) has solution z_*. If $\Omega = 2I$, we have $|||M^{-1}N|||_2 = 1.78$, $\delta(\Omega) = 2|||(M+\Omega)^{-1}N|||_2 + |||(M+\Omega)^{-1}(M-\Omega)|||_2 = 3.17$. We can obtain that the GMJ method is convergent for many initial vectors $x^{(0)} \in R^{10}$, and the limit is $z_{**} = (1, 2, 1, \ldots, 1, 2, 5, 0)^T \in R^{10}$, which is also a solution of the LCP(q, A). Note that this example does not satisfy Theorem 2 and Theorem 3. This implies the conditions in the two theorems are only sufficient.

5. Concluding Remarks

In paper [12], most examples show the MM method and the MGS method provide the best results for solving the LCP(q, A) than some modulus-related methods and the projected relaxation iteration methods in most cases. The GMJ is the general form of the MJ method. From the above examples, we can find the GMJ method has some good properties, which is better than the M method, the MJ method, the MM method and the MGS method for the same problem in most cases. In addition, the GMJ method connected with the CG method has more high efficiency, and the GMJ method can also handle with the LCP(q, A) where the matrix A has non-positive diagonal elements. In a word, the GMJ method is an efficient iteration method for the LCP(q, A).

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