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THE GENERAL ϕ-HERMITIAN SOLUTION TO MIXED PAIRS OF QUATERNION MATRIX SYLVESTER EQUATIONS*

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Abstract. Let $\mathbb{H}^{m\times n}$ be the space of $m\times n$ matrices over $\mathbb{H}$, where $\mathbb{H}$ is the real quaternion algebra. Let A_ϕ be the $n\times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^T, where $A\in\mathbb{H}^{m\times n}$ and ϕ is a nonstandard involution of $\mathbb{H}$. In this paper, some properties of the Moore-Penrose inverse of the quaternion matrix A_ϕ are given. Two systems of mixed pairs of quaternion matrix Sylvester equations $A_1X-YB_1=C_1,\ A_2Z-YB_2=C_2$ and $A_1X-YB_1=C_1,\ A_2Y-ZB_2=C_2$ are considered, where Z is ϕ-Hermitian. Some practical necessary and sufficient conditions for the existence of a solution (X,Y,Z) to those systems in terms of the ranks and Moore-Penrose inverses of the given coefficient matrices are presented. Moreover, the general solutions to these systems are explicitly given when they are solvable. Some numerical examples are provided to illustrate the main results.

Key words. Quaternion, Sylvester-type equations, Moore-Penrose inverse, ϕ-Hermitian solution, Involution, Rank.

AMS subject classifications. 15A09, 15A23, 15A24, 15B33, 15B57, 16R50.

1. Introduction.

Quaternions and quaternion matrices have wide applications in many fields such as signal and color image processing, control theory, orbital mechanics, computer science, and etc (e.g. [1], [23], [25], [32]–[34], [46]). Linear control equations over quaternion algebra have been studied in [25] and [26]. There are various types of linear control equations over quaternion algebra. Sylvester-type equation is one of the important equations in system and control theory and has a huge amount of practical applications in neural network [47], robust control [36], output feedback control [31], the almost noninteracting control ([37], [42]), graph theory [6], and so on. There have been many papers discussing the Sylvester-type matrix equations over a field and quaternion algebra $\mathbb{H}$ (e.g. [2]–[11], [18]–[20], [22], [28]–[30], [39]–[43], [48]).

Rodman [27] considered the standard Sylvester matrix equation $AX-XB=C$ over quaternion algebra. Futorny et al. [8] derived some solvability conditions for the generalized Sylvester equations $AX-\widehat{X}B=C$ and $X-A\widehat{X}B=C$ over $\mathbb{H}$. He et al. [9] gave some solvability conditions and general solution to the system of two-sided coupled generalized Sylvester quaternion matrix equations with four unknowns

$$A_iX_iB_i+C_iX_{i+1}D_i=E_i,\quad i=1,2,3,$$

where A_i,B_i,C_i,D_i,E_i $(i=1,2,3)$ are given quaternion matrices, and $X_1,\dots,X_4$ are unknowns. Very recently, Dmytryshyn et al. [7] gave some solvability conditions for the system of quaternion matrix generalized Sylvester equations

$$A_iX_{i'}^{\varepsilon_i}M_i-N_iX_{i''}^{\delta_i}B_i=C_i,\quad i^{'},i^{''}\in\{1,\dots,t\},\ i=1,\dots,s,$$

where $\varepsilon_i,\delta_i\in\{1,*\}$ and X^* is the quaternion adjoint matrix. There are two forms of mixed pairs of matrix

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Sylvester equations with three variables:

$$\left\{\begin{array}{l} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \end{array}\right. \tag{1.1}$$

and

$$\left\{\begin{array}{l} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \end{array}\right. \tag{1.2}$$

where A_i, B_i and C_i $(i = 1, 2)$ are given matrices and X, Y, Z are unknowns. Lee and Vu [21] presented a solvability condition for the mixed pairs of matrix Sylvester equations (1.1) through the corresponding equivalence relations of the block matrices. Wang and He [39] gave some new computable necessary and sufficient solvability conditions for the system (1.1), and presented the general solution when (1.1) is solvable. He and Wang [16] derived necessary and sufficient solvability conditions and gave the general solution to (1.2).

Quaternion matrix equation and its general solution, especially Hermitian solutions, are important in systems and control theory [27]. ϕ-Hermitian quaternion matrix was first presented by Rodman [27, Definition 2.4] in 2014. To our best knowledge, there has been little information on the ϕ-Hermitian solutions to quaternion matrix Sylvester-type equations. Motivated by the wide application of quaternion matrix equations and Sylvester-type matrix equations and in order to improve the theoretical development of the ϕ-Hermitian solutions to quaternion matrix equations, we consider the mixed pairs of quaternion matrix Sylvester equations (1.1) and (1.2), where Z is ϕ-Hermitian. More specifically,

$$\left\{\begin{array}{l} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \end{array}\right. \quad Z = Z_\phi, \tag{1.3}$$

and

$$\left\{\begin{array}{l} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \end{array}\right. \quad Z = Z_\phi. \tag{1.4}$$

The remainder of the paper is organized as follows. In Section 2, we review some definitions of nonstandard involution ϕ, quaternion matrix A_ϕ and the ϕ-Hermitian quaternion matrix. We also give some numerical examples to illustrate these definitions. In Section 3, we derive some properties of the Moore-Penrose inverse of the quaternion matrix A_ϕ. In Sections 4 and 5, we provide some necessary and sufficient conditions for the existence of a solution (X, Y, Z) to the mixed pairs of quaternion matrix Sylvester equations (1.3) and (1.4), respectively. Furthermore, we present the general solutions to (1.3) and (1.4) when they are solvable.

2. Definition of ϕ-Hermitian quaternion matrix and examples. Let $\mathbb{R}$ and $\mathbb{H}^{m\times n}$ stand, respectively, for the real field and the space of all $m \times n$ matrices over the real quaternion algebra

$$\mathbb{H} = \left\{ a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \,\middle|\, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

It is well known that the quaternion algebra is an associative and noncommutative division algebra. Denoted by $r(A)$ and A^* the rank of a given real quaternion matrix A and its conjugate transpose A^*, respectively. I and 0 are the identity matrix and zero matrix with appropriate sizes, respectively.

The definitions of the nonstandard involution ϕ, quaternion matrix A_ϕ, and the ϕ-Hermitian quaternion matrix were first presented by Rodman [27]. At first, we give the definition of an involution.

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Definition 2.1 (Involution). [27] A map ϕ: $\mathbb{H} \longrightarrow \mathbb{H}$ is called an anti-endomorphism if $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in \mathbb{H}$, and $\phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{H}$. An anti-endomorphism ϕ is called an involution if ϕ^2 is the identity map.

Involutions have matrix representation as given in the following lemma.

Lemma 2.2. [27] Let ϕ be an anti-endomorphism of $\mathbb{H}$. Assume that ϕ does not map $\mathbb{H}$ into zero. Then ϕ is bijective; thus, ϕ is in fact an anti-automorphism. Moreover, ϕ is real linear, and represents ϕ as a 4×4 real matrix with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and ϕ is an involution if and only if

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \tag{2.5}$$

where either $T = -I_3$ or T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

Based on Lemma 2.2, the involutions can be classified into two classes:

- *standard involution* $\phi = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}$;
- *nonstandard involution* $\phi = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$, where T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$.

For $A \in \mathbb{H}^{m \times n}$, we denote by A_ϕ [27] the $n \times m$ matrix obtained by applying ϕ entrywise to the transposed matrix A^T, where ϕ is a nonstandard involution. Here are some examples of A_ϕ, where ϕ is a nonstandard involution.

Example 2.3. The map $\phi : \mathbb{H}^{m \times n} \to \mathbb{H}^{n \times m}$, where $\phi(A) = A^{\eta *} = -\eta A^* \eta$ and $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, is a nonstandard involution. Some properties of this nonstandard involution can be found in [12], [15], [17] and [35]. If $\eta = \mathbf{i}$, we have $\phi(\mathbf{i}) = -\mathbf{i}$, $\phi(\mathbf{j}) = \mathbf{j}$, $\phi(\mathbf{k}) = \mathbf{k}$, and

$$\begin{pmatrix} 1 & -\mathbf{i}+\mathbf{j} & -\mathbf{i}+\mathbf{k} \\ 1+\mathbf{k} & 2-\mathbf{j} & 2+\mathbf{i} \end{pmatrix}_\phi = \begin{pmatrix} 1 & 1+\mathbf{k} \\ \mathbf{i}+\mathbf{j} & 2-\mathbf{j} \\ \mathbf{i}+\mathbf{k} & 2-\mathbf{i} \end{pmatrix}.$$

Example 2.4. The map $\phi : \mathbb{H}^{m \times n} \to \mathbb{H}^{n \times m}$, where $\phi(A) = A^{\xi *} = -\xi A^* \xi$ and $\xi \in \left\{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}), \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})\right\}$, is a nonstandard involution. In particular,

- when $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathbb{H}$,

$$a^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} = a_0 - a_2\mathbf{i} - a_1\mathbf{j} + a_3\mathbf{k}, \text{ if } \xi = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}),$$

$$a^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k})*} = a_0 - a_3\mathbf{i} + a_2\mathbf{j} - a_1\mathbf{k}, \text{ if } \xi = \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k}),$$

$$a^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*} = a_0 + a_1\mathbf{i} - a_3\mathbf{j} - a_2\mathbf{k}, \text{ if } \xi = \frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k}).$$

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- when $\xi = \frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})$, nonstandard involution ϕ is $\phi(\mathbf{i}) = \mathbf{i}$, $\phi(\mathbf{j}) = -\mathbf{k}$, $\phi(\mathbf{k}) = -\mathbf{j}$, and

$$\begin{pmatrix} 1+\mathbf{i}-\mathbf{j}+2\mathbf{k} & \mathbf{i}+2\mathbf{j} & \mathbf{i}+\mathbf{k} \\ 1 & 2\mathbf{j}-3\mathbf{k} & \mathbf{k} \end{pmatrix}_{\phi} = \begin{pmatrix} 1+\mathbf{i}-2\mathbf{j}+\mathbf{k} & 1 \\ \mathbf{i}-2\mathbf{k} & 3\mathbf{j}-2\mathbf{k} \\ \mathbf{i}-\mathbf{j} & -\mathbf{j} \end{pmatrix}.$$

Now we recall the definition of the ϕ-Hermitian matrix.

Definition 2.5 (ϕ-Hermitian Matrix). [27] $A \in \mathbb{H}^{n\times n}$ is said to be ϕ-Hermitian if $A = A_{\phi}$, where ϕ is a nonstandard involution.

Remark 2.6. To have a good understanding of ϕ-Hermitian matrix, we introduce two examples.

- For $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a square real quaternion matrix A is said to be η-Hermitian if $A = A^{\eta *}$, where $A^{\eta *} = -\eta A^* \eta$. For example, $\begin{pmatrix} 1+\mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j} \\ \mathbf{i}-\mathbf{j} & \mathbf{i} \end{pmatrix}$ is a j-Hermitian matrix. η-Hermitian matrix was first proposed in [35], and further discussed in [17]. The η-Hermitian matrices arise in statistical signal processing and widely linear modelling ([32]–[35]).
- For $\xi \in \left\{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}), \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})\right\}$, a square real quaternion matrix A is said to be ξ-Hermitian if $A = A^{\xi *}$, where $A^{\xi *} = -\xi A^* \xi$. For example, $\begin{pmatrix} 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & 2\mathbf{i}+\mathbf{j}-\mathbf{k} \\ -\mathbf{i}-2\mathbf{j}-\mathbf{k} & 2\mathbf{i}-2\mathbf{j} \end{pmatrix}$ is a $\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})$-Hermitian matrix.

3. Properties of the Moore-Penrose inverse of A_{ϕ}. In order to solve (1.3) and (1.4), we will make use of Moore-Penrose inverse so we are going to study the properties of the Moore-Penrose inverse of A_{ϕ}. We first give the algebraic properties of quaternion matrix nonstandard involution.

Property 3.1. [27] *Let ϕ be a nonstandard involution. Then, the following hold:*

(1) $(\alpha A + \beta B)_{\phi} = A_{\phi}\phi(\alpha) + B_{\phi}\phi(\beta),\ \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$.

(2) $(A\alpha + B\beta)_{\phi} = \phi(\alpha)A_{\phi} + \phi(\beta)B_{\phi},\ \alpha, \beta \in \mathbb{H},\ A, B \in \mathbb{H}^{m\times n}$.

(3) $(AB)_{\phi} = B_{\phi}A_{\phi},\ A \in \mathbb{H}^{m\times n},\ B \in \mathbb{H}^{n\times p}$.

(4) $(A_{\phi})_{\phi} = A,\ A \in \mathbb{H}^{m\times n}$.

(5) *If $A \in \mathbb{H}^{n\times n}$ is invertible, then $(A_{\phi})^{-1} = (A^{-1})_{\phi}$.*

(6) $r(A) = r(A_{\phi}), A \in \mathbb{H}^{m\times n}$.

(7) $I_{\phi} = I,\ 0_{\phi} = 0$.

The Moore-Penrose inverse $A^{\dagger}$ of a quaternion matrix A, is defined to be the unique matrix $A^{\dagger}$, such that

$$\text{(i) } AA^{\dagger}A = A, \quad \text{(ii) } A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad \text{(iii) } (AA^{\dagger})^* = AA^{\dagger}, \quad \text{(iv) } (A^{\dagger}A)^* = A^{\dagger}A. \tag{3.6}$$

Furthermore, L_A and R_A stand for the projectors $L_A = I - A^{\dagger}A$ and $R_A = I - AA^{\dagger}$ induced by A, respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$.

Property 3.1 helps us to derive properties of the Moore-Penrose inverse of the quaternion matrix A.

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THEOREM 3.2. *Let $A \in \mathbb{H}^{m\times n}$ be given. Then, the following hold:*

(1) $(A_\phi)^\dagger = (A^\dagger)_\phi$.

(2) $(L_A)_\phi = R_{A_\phi},\ (R_A)_\phi = L_{A_\phi}$.

Proof. (1) It follows from Property 3.1 that

$$A_\phi(A^\dagger)_\phi A_\phi = (AA^\dagger A)_\phi = A_\phi \quad \text{and} \quad (A^\dagger)_\phi A_\phi (A^\dagger)_\phi = (A^\dagger A A^\dagger)_\phi = (A^\dagger)_\phi.$$

Hence, $(A^\dagger)_\phi$ satisfies the first and second equations in (3.6). Now we want to prove that $(A^\dagger)_\phi$ satisfies the third and fourth equations in (3.6). It follows from Lemma 2.2 that the map $A \to A^*$ and nonstandard involution ϕ correspond to the real matrices

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

where the matrix T in Q_2 is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$. Note that $Q_1Q_2 = Q_2Q_1$. Hence, we have

$$(A^*)_\phi = (A_\phi)^*. \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}\left(A_\phi(A^\dagger)_\phi\right)^* &= \left((A^\dagger)_\phi\right)^* (A_\phi)^* = \left((A^\dagger)^*\right)_\phi (A^*)_\phi = \left(A^*(A^\dagger)^*\right)_\phi \\ &= \left((A^\dagger A)^*\right)_\phi = (A^\dagger A)_\phi = A_\phi (A^\dagger)_\phi,\end{aligned}$$

$$\begin{aligned}\left((A^\dagger)_\phi A_\phi\right)^* &= (A_\phi)^* \left((A^\dagger)_\phi\right)^* = (A^*)_\phi \left((A^\dagger)^*\right)_\phi = \left((A^\dagger)^* A^*\right)_\phi \\ &= \left((AA^\dagger)^*\right)_\phi = (AA^\dagger)_\phi = (A^\dagger)_\phi A_\phi.\end{aligned}$$

(2) By the definitions of L_A, R_A and the properties of Moore-Penrose inverse of A_ϕ, it follows that

$$(L_A)_\phi = (I - A^\dagger A)_\phi = I_\phi - (A^\dagger A)_\phi = I - A_\phi (A^\dagger)_\phi = I - A_\phi (A_\phi)^\dagger = R_{A_\phi},$$
$$(R_A)_\phi = (I - AA^\dagger)_\phi = I_\phi - (AA^\dagger)_\phi = I - (A^\dagger)_\phi A_\phi = I - (A_\phi)^\dagger A_\phi = L_{A_\phi},$$

establishing $(L_A)_\phi = R_{A_\phi}$ and $(R_A)_\phi = L_{A_\phi}$. ∎

4. The solution to system (1.3). In this section, using the ranks and generalized inverses of matrices, we give some solvability conditions and the general solution to the mixed pairs of quaternion matrix Sylvester equations (1.3). At first, we review some results which will be used in this paper. The following lemma gives the solvability conditions and general solution to the mixed Sylvester matrix equations

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_3 - X_2B_2 = C_2 \end{cases} \tag{4.8}$$

over $\mathbb{H}$.

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Lemma 4.1. [39] *Let A_i, B_i, and C_i $(i = 1, 2)$ be given matrices over $\mathbb{H}$. Set*

$$D_1 = R_{B_1}B_2, \quad A = R_{A_2}A_1, B = B_2L_{D_1}, \quad C = R_{A_2}(R_{A_1}C_1B_1^{\dagger}B_2 - C_2)L_{D_1}.$$

Then, the following statements are equivalent:

(1) *The mixed Sylvester real quaternion matrix equations (4.8) has a solution.*

(2) $R_{A_1}C_1L_{B_1} = 0,\ R_AC = 0,\ CL_B = 0.$

(3)

$$r\begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1),$$

$$r\begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r\begin{pmatrix} B_2 & B_1 & 0 & 0 \\ C_2 & C_1 & A_1 & A_2 \end{pmatrix} = r(A_1,\ A_2) + r(B_1,\ B_2).$$

In this case, the general solution to (4.8) can be expressed as

$$X_1 = A_1^{\dagger}C_1 + U_1B_1 + L_{A_1}W_1,$$

$$X_2 = -R_{A_1}C_1B_1^{\dagger} + A_1U_1 + V_1R_{B_1},$$

$$X_3 = A_2^{\dagger}(C_2 - R_{A_1}C_1B_1^{\dagger}B_2 + A_1U_1B_2) + W_4D_1 + L_{A_2}W_6,$$

where

$$U_1 = A^{\dagger}CB^{\dagger} + L_AW_2 + W_3R_B,$$

$$V_1 = -R_{A_2}(C_2 - R_{A_1}C_1B_1^{\dagger}B_2 + A_1U_1B_2)D_1^{\dagger} + A_2W_4 + W_5R_{D_1},$$

and $W_1, \ldots, W_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

By applying the Lemma 4.8 with conjugation, we can solve the following system of quaternion matrix equations

$$A_1X - YB_1 = C_1, \quad A_2X - ZB_2 = C_2. \tag{4.9}$$

More specifically,

Lemma 4.2. *Let A_i, B_i, and $C_i (i = 1, 2)$ be given matrices over $\mathbb{H}$. Set*

$$A_{11} = R_{(A_2L_{A_1})}A_2, \quad B_{11} = B_1L_{B_2}, \quad C_{11} = R_{(A_2L_{A_1})}(C_2 - A_2A_1^{\dagger}C_1)L_{B_2}.$$

Then, the following statements are equivalent:

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(1) *The system of quaternion matrix equations (4.9) has a solution.*

(2) $R_{A_1}C_1L_{B_1} = 0,\ R_{A_{11}}C_{11} = 0,\ C_{11}L_{B_{11}} = 0.$

(3)

$$r\begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1),$$

$$r\begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r\begin{pmatrix} C_1 & A_1 \\ C_2 & A_2 \\ B_1 & 0 \\ B_2 & 0 \end{pmatrix} = r\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

In this case, the general solution to (4.9) can be expressed as

$$X = A_1^{\dagger}C_1 + V_2B_1 + L_{A_1}U_2,$$

$$Y = -R_{A_1}C_1B_1^{\dagger} + A_1V_2 + W_6R_{B_1},$$

$$Z = -R_{(A_2L_{A_1})}(C_2 - A_2A_1^{\dagger}C_1 - A_2V_2B_1)B_2^{\dagger} + A_2L_{A_1}W_1 + W_3R_{B_2},$$

where

$$V_2 = A_{11}^{\dagger}C_{11}B_{11}^{\dagger} + L_{A_{11}}W_4 + W_5R_{B_{11}},$$

$$U_2 = (A_2L_{A_1})^{\dagger}(C_2 - A_2A_1^{\dagger}C_1 - A_2U_1B_1) + W_1B_2 + L_{(A_2L_{A_1})}W_2,$$

and $W_1, \ldots, W_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

The following real quaternion matrix equation

$$A_1X_1 + X_2B_1 + C_3X_3D_3 + C_4X_4D_4 = E_1 \tag{4.10}$$

is needed in solving the systems (1.3) and (1.4). The solution to (4.10) over any arbitrary division rings with involutional anti-automorphisms are given in [14]. Note that quaternion algebra is a special case of an arbitrary division ring. Hence, we can also give the solvability conditions and the general solution to real quaternion matrix equation (4.10).

Lemma 4.3. [14, 38] *Let $A_1, B_1, C_3, D_3, C_4, D_4$, and E_1 be given matrices over $\mathbb{H}$. Set*

$$A = R_{A_1}C_3, \quad B = D_3L_{B_1}, \quad C = R_{A_1}C_4, \quad D = D_4L_{B_1},$$

$$E = R_{A_1}E_1L_{B_1}, \quad M = R_AC, \quad N = DL_B, \quad S = CL_M.$$

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Then the quaternion matrix equation (4.10) has a solution if and only if

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution can be expressed as

$$X_1 = A_1^{\dagger}(E_1 - C_3X_3D_3 - C_4X_4D_4) - A_1^{\dagger}T_7B_1 + L_{A_1}T_6,$$

$$X_2 = R_{A_1}(E_1 - C_3X_3D_3 - C_4X_4D_4)B_1^{\dagger} + A_1A_1^{\dagger}T_7 + T_8R_{B_1},$$

$$X_3 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}ST_2R_NDB^{\dagger} + L_AT_4 + T_5R_B,$$

$$X_4 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_ST_1 + L_MT_2R_N + T_3R_D,$$

where $T_1, \ldots, T_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

To simplify the solution of (1.3), we introduce the following lemma.

Lemma 4.4. [24] *Given $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times k}$, and $C \in \mathbb{H}^{l\times n}$, we have:*

(1) $r(A) + r(R_AB) = r(B) + r(R_BA) = r(A,\ B)$.

(2) $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{pmatrix} A \\ C \end{pmatrix}$.

In the following theorem, we will give two solvability conditions and the general solution to the mixed pairs of quaternion matrix Sylvester equations (1.3). For simplicity, put

$$A_{11} = R_{B_2}B_1, \quad B_{11} = R_{A_1}A_2, \quad C_{11} = B_1L_{A_{11}}, \quad D_{11} = R_{A_1}(R_{A_2}C_2B_2^{\dagger}B_1 - C_1)L_{A_{11}}, \tag{4.11}$$

$$A_{22} = (L_{A_2}, -(R_{C_{11}}B_2)_{\phi}), \quad B_{22} = \begin{pmatrix} R_{C_{11}}B_2 \\ -(L_{A_2})_{\phi} \end{pmatrix}, \quad D_{22} = [R_{A_1}(-C_1 + C_2B_2^{\dagger}B_1)L_{A_{11}}]_{\phi}, \tag{4.12}$$

$$C_{22} = (A_2^{\dagger}C_2L_{B_2})_{\phi} + (B_2)_{\phi}(C_{11}^{\dagger})_{\phi}D_{22}(B_{11}^{\dagger})_{\phi} - A_2^{\dagger}C_2 - B_{11}^{\dagger}D_{11}C_{11}^{\dagger}B_2, \tag{4.13}$$

$$A = R_{A_{22}}L_{B_{11}}, \quad B = B_2L_{B_{22}}, \quad C = -R_{A_{22}}(B_2)_{\phi}, \quad D = (L_{B_{11}})_{\phi}L_{B_{22}}, \tag{4.14}$$

$$E = R_{A_{22}}C_{22}L_{B_{22}}, \quad M = R_AC, \quad N = DL_B, \quad S = CL_M. \tag{4.15}$$

Theorem 4.5. *Let A_i, B_i and C_i $(i = 1, 2)$ be given matrices over $\mathbb{H}$. Then the following statements are equivalent:*

(1) *The mixed pairs of quaternion matrix Sylvester equations (1.3) has a solution (X, Y, Z).*

(2)

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad i = 1, 2, \tag{4.16}$$

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$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1,\ A_2) + r(B_1,\ B_2), \tag{4.17}$$

$$r\begin{pmatrix} C_1 & C_2(A_2)_\phi - A_2(C_2)_\phi & A_1 & A_2(B_2)_\phi \\ B_1 & B_2(A_2)_\phi & 0 & 0 \end{pmatrix} = r(A_1,\ A_2(B_2)_\phi) + r(B_1,\ B_2(A_2)_\phi), \tag{4.18}$$

$$r\begin{pmatrix} C_2(A_2)_\phi - A_2(C_2)_\phi & A_2(B_2)_\phi \\ B_2(A_2)_\phi & 0 \end{pmatrix} = 2r(A_2(B_2)_\phi), \tag{4.19}$$

$$r\begin{pmatrix} C_1 & C_2(A_2)_\phi - A_2(C_2)_\phi & A_1 & A_2(B_2)_\phi \\ 0 & -(C_1)_\phi & 0 & (B_1)_\phi \\ B_1 & B_2(A_2)_\phi & 0 & 0 \\ 0 & (A_1)_\phi & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} A_1 & A_2(B_2)_\phi \\ 0 & (B_1)_\phi \end{pmatrix}. \tag{4.20}$$

(3)

$$R_{A_2}C_2L_{B_2} = 0, \quad D_{11}L_{C_{11}} = 0, \quad R_{B_{11}}D_{11} = 0, \tag{4.21}$$

$$R_M R_A E = 0, \quad R_C E L_B = 0, \quad R_A E L_D = 0. \tag{4.22}$$

In this case, the general solution to (1.3) can be expressed as

$$X = \frac{X_1 + (X_5)_\phi}{2}, \quad Y = \frac{X_2 + (X_4)_\phi}{2}, \quad Z = Z_\phi = \frac{X_3 + (X_3)_\phi}{2}, \tag{4.23}$$

where

$$X_1 = A_1^\dagger(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6, \tag{4.24}$$

$$X_2 = -R_{A_2}C_2B_2^\dagger + A_2U_1 + V_1R_{B_2}, \tag{4.25}$$

$$X_4 = -(C_2B_2^\dagger)_\phi + V_2(A_2)_\phi + (R_{B_2})_\phi U_2, \tag{4.26}$$

$$X_5 = -[(A_1)^\dagger(-C_1 + C_2B_2^\dagger B_1 - A_2(V_2)_\phi B_1)L_{A_{11}}]_\phi + (A_{11})_\phi T_1 + T_3(L_{A_1})_\phi, \tag{4.27}$$

$$X_3 = A_2^\dagger C_2 + U_1B_2 + L_{A_2}W_1, \tag{4.28}$$

or

$$X_3 = (A_2^\dagger C_2 L_{B_2})_\phi + (B_2)_\phi V_2 + T_6(L_{A_2})_\phi, \tag{4.29}$$

$$U_1 = B_{11}^\dagger D_{11} C_{11}^\dagger + L_{B_{11}}W_2 + W_3R_{C_{11}}, \tag{4.30}$$

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$$V_1 = -R_{A_1}(C_1 - R_{A_2}C_2B_2^{\dagger}B_1 + A_2U_1B_1)A_{11}^{\dagger} + A_1W_4 + W_5R_{A_{11}}, \tag{4.31}$$

$$V_2 = (C_{11}^{\dagger})_{\phi}D_{22}(B_{11}^{\dagger})_{\phi} + (R_{C_{11}})_{\phi}T_4 + T_5(L_{B_{11}})_{\phi}, \tag{4.32}$$

$$U_2 = [(-C_1 + C_2B_2^{\dagger}B_1 - A_2(V_2)_{\phi}B_1)(A_{11})^{\dagger}]_{\phi} + T_1(A_1)_{\phi} + (R_{A_{11}})_{\phi}T_2, \tag{4.33}$$

$$W_1 = (I_{p_1},\ 0)[A_{22}^{\dagger}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi}) - A_{22}^{\dagger}Z_7B_{22} + L_{A_{22}}Z_6], \tag{4.34}$$

$$T_4 = (0,\ I_{p_2})[A_{22}^{\dagger}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi}) - A_{22}^{\dagger}Z_7B_{22} + L_{A_{22}}Z_6], \tag{4.35}$$

$$W_3 = [R_{A_{22}}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi})B_{22}^{\dagger} + A_{22}A_{22}^{\dagger}Z_7 + Z_8R_{B_{22}}]\begin{pmatrix} I_{p_2} \\ 0 \end{pmatrix}, \tag{4.36}$$

$$T_6 = [R_{A_{22}}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi})B_{22}^{\dagger} + A_{22}A_{22}^{\dagger}Z_7 + Z_8R_{B_{22}}]\begin{pmatrix} 0 \\ I_{p_1} \end{pmatrix}, \tag{4.37}$$

$$W_2 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}SZ_1R_NDB^{\dagger} + L_AZ_2 + Z_3R_B, \tag{4.38}$$

$$T_5 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_SZ_4 + L_MZ_1R_N + Z_5R_D, \tag{4.39}$$

and the remaining W_j, T_j, Z_j *are arbitrary matrices over* $\mathbb{H}$, p_1 *is the column number of* A_2, p_2 *is the row number of* B_1.

Proof. We first prove that the mixed pairs of quaternion matrix Sylvester equations (1.3) has a solution (X, Y, Z) if and only if the following system of quaternion matrix equations

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_3 - X_2B_2 = C_2, \\ (B_2)_{\phi}X_4 - X_3(A_2)_{\phi} = -(C_2)_{\phi}, \\ (B_1)_{\phi}X_4 - X_5(A_1)_{\phi} = -(C_1)_{\phi} \end{cases} \tag{4.40}$$

has a solution. If (1.3) has a solution, say, (X_0, Y_0, Z_0), then (4.40) clearly has a solution $(X_1, X_2, X_3, X_4, X_5)$ $= (X_0, Y_0, Z_0, (Y_0)_{\phi}, (X_0)_{\phi})$. Conversely, if (4.40) has a solution $(X_1, X_2, X_3, X_4, X_5)$, then

$$(X, Y, Z) = \left(\frac{X_1 + (X_5)_{\phi}}{2}, \frac{X_2 + (X_4)_{\phi}}{2}, \frac{X_3 + (X_3)_{\phi}}{2}\right)$$

is a solution of (1.3). Now we want to solve (4.40).

The main idea for solving (4.40) is that (4.40) has a solution if and only if the systems

$$\begin{cases} A_2X_3 - X_2B_2 = C_2, \\ A_1X_1 - X_2B_1 = C_1, \end{cases} \tag{4.41}$$

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and

$$\left\{\begin{array}{l}(B_2)_\phi X_4 - X_3(A_2)_\phi = -(C_2)_\phi,\\ (B_1)_\phi X_4 - X_5(A_1)_\phi = -(C_1)_\phi\end{array}\right. \tag{4.42}$$

are solvable, and the X_3 in (4.41) is the same as in (4.42).

It follows from Lemma 4.1 that (4.41) is consistent if and only if

$$r\begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1), \quad r\begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2), \tag{4.43}$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1,\ A_2) + r(B_1,\ B_2), \tag{4.44}$$

or

$$R_{A_2}C_2L_{B_2} = 0, \quad D_{11}L_{C_{11}} = 0, \quad R_{B_{11}}D_{11} = 0, \tag{4.45}$$

where B_{11}, C_{11}, and D_{11} are defined in (4.11). In this case, the general solution to (4.41) can be expressed as

$$X_3 = A_2^\dagger C_2 + U_1B_2 + L_{A_2}W_1, \tag{4.46}$$

$$X_2 = -R_{A_2}C_2B_2^\dagger + A_2U_1 + V_1R_{B_2},$$

$$X_1 = A_1^\dagger(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6,$$

where

$$U_1 = B_{11}^\dagger D_{11}C_{11}^\dagger + L_{B_{11}}W_2 + W_3R_{C_{11}},$$

$$V_1 = -R_{A_1}(C_1 - R_{A_2}C_2B_2^\dagger B_1 + A_2U_1B_1)A_{11}^\dagger + A_1W_4 + W_5R_{A_{11}},$$

A_{11}, B_{11}, C_{11}, and D_{11} are defined in (4.11), and $W_1, \ldots, W_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Next we consider (4.42). It follows from Lemma 4.2 that (4.42) is solvable if and only if

$$r\begin{pmatrix} -(C_1)_\phi & (B_1)_\phi \\ (A_1)_\phi & 0 \end{pmatrix} = r(A_1)_\phi + r(B_1)_\phi,$$

$$r\begin{pmatrix} -(C_2)_\phi & (B_2)_\phi \\ (A_2)_\phi & 0 \end{pmatrix} = r(A_2)_\phi + r(B_2)_\phi,$$

$$r\begin{pmatrix} -(C_1)_\phi & (B_1)_\phi \\ -(C_2)_\phi & (B_2)_\phi \\ (A_1)_\phi & 0 \\ (A_2)_\phi & 0 \end{pmatrix} = r\begin{pmatrix} (A_1)_\phi \\ (A_2)_\phi \end{pmatrix} + r\begin{pmatrix} (B_1)_\phi \\ (B_2)_\phi \end{pmatrix},$$

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i.e., (4.43) and (4.44). In this case, the general solution to (4.42) can be expressed as

$$X_3 = (A_2^{\dagger}C_2L_{B_2})_{\phi} + (B_2)_{\phi}V_2 + T_6(L_{A_2})_{\phi}, \tag{4.47}$$

$$X_4 = -(C_2B_2^{\dagger})_{\phi} + V_2(A_2)_{\phi} + (R_{B_2})_{\phi}U_2,$$

$$X_5 = -[(A_1)^{\dagger}(-C_1 + C_2B_2^{\dagger}B_1 - A_2(V_2)_{\phi}B_1)L_{A_{11}}]_{\phi} + (A_{11})_{\phi}T_1 + T_3(L_{A_1})_{\phi},$$

where

$$V_2 = (C_{11}^{\dagger})_{\phi}D_{22}(B_{11}^{\dagger})_{\phi} + (R_{C_{11}})_{\phi}T_4 + T_5(L_{B_{11}})_{\phi},$$

$$U_2 = [(-C_1 + C_2B_2^{\dagger}B_1 - A_2(V_2)_{\phi}B_1)(A_{11})^{\dagger}]_{\phi} + T_1(A_1)_{\phi} + (R_{A_{11}})_{\phi}T_2,$$

A_{11}, B_{11}, C_{11}, and D_{22} are defined in (4.11) and (4.12), and $T_1, \dots, T_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Equating X_3 in (4.46) and X_3 in (4.47) gives

$$\begin{aligned}&(L_{A_2}, -(R_{C_{11}}B_2)_{\phi})\begin{pmatrix}W_1\\T_4\end{pmatrix} + (W_3,\ T_6)\begin{pmatrix}R_{C_{11}}B_2\\-(L_{A_2})_{\phi}\end{pmatrix} + L_{B_{11}}W_2B_2 - (B_2)_{\phi}T_5(L_{B_{11}})_{\phi}\\&= (A_2^{\dagger}C_2L_{B_2})_{\phi} + (B_2)_{\phi}(C_{11}^{\dagger})_{\phi}D_{22}(B_{11}^{\dagger})_{\phi} - A_2^{\dagger}C_2 - B_{11}^{\dagger}D_{11}C_{11}^{\dagger}B_2,\end{aligned}$$

i.e.,

$$A_{22}\begin{pmatrix}W_1\\T_4\end{pmatrix} + (W_3,\ T_6)B_{22} + L_{B_{11}}W_2B_2 - (B_2)_{\phi}T_5(L_{B_{11}})_{\phi} = C_{22}, \tag{4.48}$$

where A_{22}, B_{22}, C_{22} are defined in (4.12) and (4.13). It follows from Lemma 4.3 that the equation (4.48) is consistent if and only if

$$R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0.$$

In this case, the general solution to the equation (4.48) can be expressed as

$$\begin{pmatrix}W_1\\T_4\end{pmatrix} = A_{22}^{\dagger}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi}) - A_{22}^{\dagger}Z_7B_{22} + L_{A_{22}}Z_6,$$

$$(W_3,\ T_6) = R_{A_{22}}(C_{22} - L_{B_{11}}W_2B_2 + (B_2)_{\phi}T_5(L_{B_{11}})_{\phi})B_{22}^{\dagger} + A_{22}A_{22}^{\dagger}Z_7 + Z_8R_{B_{22}},$$

$$W_2 = A^{\dagger}EB^{\dagger} - A^{\dagger}CM^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}DB^{\dagger} - A^{\dagger}SZ_1R_NDB^{\dagger} + L_AZ_2 + Z_3R_B,$$

$$T_5 = M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger} + L_ML_SZ_4 + L_MZ_1R_N + Z_5R_D,$$

where A, B, C, D, E, M, N, S are defined in (4.14) and (4.15), $Z_1, \dots, Z_7$ are arbitrary matrices over $\mathbb{H}$.

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Now we want to show the equivalence (4.22) $\Leftrightarrow$ (4.18)–(4.20). It follows from Lemma 4.4 that

$$R_M R_A E = 0 \Leftrightarrow r(R_A E,\ R_A C) = r(R_A C) \Leftrightarrow r(E,\ A,\ C) = r(A,\ C)$$

$$\Leftrightarrow r\left(R_{A_{22}} C_{22} L_{B_{22}},\ R_{A_{22}} L_{B_{11}},\ R_{A_{22}} (B_2)_\phi\right) = r\left(R_{A_{22}} L_{B_{11}},\ R_{A_{22}} (B_2)_\phi\right)$$

$$\Leftrightarrow r\begin{pmatrix} C_{22} & L_{B_{11}} & (B_2)_\phi & A_{22} \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}},\ (B_2)_\phi,\ A_{22}) + r(B_{22}). \tag{4.49}$$

Note that the systems (4.41) and (4.42) are consistent, under the equalities in (4.45), i.e., (4.43) and (4.44). In this case,

$$X_3^1 := A_2^\dagger C_2 + B_{11}^\dagger D_{11} C_{11}^\dagger B_2 \quad \text{and} \quad X_3^2 := (A_2^\dagger C_2 L_{B_2})_\phi + (B_2)_\phi (C_{11}^\dagger)_\phi D_{22} (B_{11}^\dagger)_\phi$$

are special solutions to (4.41) and (4.42), respectively. Then, we have

$$C_{22} = X_3^2 - X_3^1. \tag{4.50}$$

Substituting (4.50) into (4.49) yields

$$R_M R_A E = 0 \Leftrightarrow r\begin{pmatrix} X_3^2 - X_3^1 & L_{B_{11}} & (B_2)_\phi & A_{22} \\ B_{22} & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}},\ (B_2)_\phi,\ A_{22}) + r(B_{22})$$

$$\Leftrightarrow r\begin{pmatrix} X_3^2 - X_3^1 & L_{B_{11}} & (B_2)_\phi & L_{A_2} \\ R_{C_{11}} B_2 & 0 & 0 & 0 \\ R_{(A_2)_\phi} & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}},\ (B_2)_\phi,\ L_{A_2}) + r\begin{pmatrix} R_{C_{11}} B_2 \\ R_{(A_2)_\phi} \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} X_3^2 - X_3^1 & I & (B_2)_\phi & 0 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & B_1 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & (A_2)_\phi & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 \end{pmatrix} = r\begin{pmatrix} I & (B_2)_\phi & 0 & 0 \\ A_2 & 0 & 0 & A_1 \\ 0 & 0 & A_2 & 0 \end{pmatrix} + r\begin{pmatrix} B_2 & B_1 & 0 \\ I & 0 & (A_2)_\phi \end{pmatrix}$$

$$\Leftrightarrow r\begin{pmatrix} C_1 & C_2(A_2)_\phi - A_2(C_2)_\phi & A_1 & A_2(B_2)_\phi \\ B_1 & B_2(A_2)_\phi & 0 & 0 \end{pmatrix} = r(A_1,\ A_2(B_2)_\phi) + r(B_1,\ B_2(A_2)_\phi).$$

Similarly, it can be shown that

$$R_C E L_B = 0 \Leftrightarrow (4.19) \quad \text{and} \quad R_A E L_D = 0 \Leftrightarrow (4.20). \qquad \square$$

We remark that both Conditions (2) and (3) in Theorem 4.5 are practical. In fact, the proof of Theorem 4.5 reveals that Condition (2) is the result of applying Lemma 4.4 on Condition (3) and is more straightforward than Condition (3) in terms of checking the solvability of (1.3).

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Remark 4.6. We can also give some solvability conditions and general solution to the following system of quaternion matrix equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Z - YB_2 = C_2, \end{cases} \quad X = X_\phi$$

by exchanging X and Z in the system (1.3).

Remark 4.7. The study on the η-Hermitian solutions to quaternion matrix equations has drawn more attention in recent years (e.g. [12], [13], [15], [44], [45]).

- As a special case of the mixed pairs of quaternion matrix Sylvester equations (1.3), we can give some necessary and sufficient conditions for the existence of a solution (X, Y, Z) to the systems (1.3), where Z is η-Hermitian, i.e., $Z = Z^{\eta*} = -\eta Z^*\eta,\ \eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
- As another special case of the mixed pairs of quaternion matrix Sylvester equations (1.3), we can give some necessary and sufficient conditions for the existence of a solution (X, Y, Z) to the systems (1.3), where Z is ξ-Hermitian, i.e., $Z = Z^{\xi*} = -\xi Z^*\xi,\ \xi \in \{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j}), \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})\}$.

We give two examples to illustrate Theorem 4.5 for the two special cases, respectively.

Example 4.8. Let

$$A_1 = \begin{pmatrix} 2-\mathbf{i}+\mathbf{j} & \mathbf{i}+\mathbf{k} & 1+\mathbf{j}+\mathbf{k} \\ 1+2\mathbf{i}-\mathbf{j} & -1-\mathbf{k} & \mathbf{i}-\mathbf{j}-\mathbf{k} \\ 1-3\mathbf{i}+2\mathbf{j} & 1+\mathbf{i}+2\mathbf{k} & 1-\mathbf{i}+2\mathbf{j}+2\mathbf{k} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \mathbf{i}+2\mathbf{k} & 1+\mathbf{j}-\mathbf{k} & \mathbf{j} \\ -1-2\mathbf{j} & \mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{k} \\ -1+\mathbf{i}-2\mathbf{j}+2\mathbf{k} & 1+\mathbf{i}+2\mathbf{j} & \mathbf{j}+\mathbf{k} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2+2\mathbf{j}+\mathbf{k} & -\mathbf{i}+\mathbf{j}-\mathbf{k} & 1+2\mathbf{i}+\mathbf{j} \\ -1-\mathbf{k} & \mathbf{i}+\mathbf{k} & -2\mathbf{i}-\mathbf{j} \\ 1+2\mathbf{j} & \mathbf{j} & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \mathbf{i}+\mathbf{j} & \mathbf{i}+\mathbf{k} & 1+2\mathbf{j} \\ -1+\mathbf{i}-\mathbf{j} & -1+\mathbf{i}-\mathbf{k} & 1+\mathbf{i}-2\mathbf{j} \\ -1 & -1 & \mathbf{i} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 5-7\mathbf{i}+11\mathbf{j}+9\mathbf{k} & 2+8\mathbf{i}-\mathbf{j}-\mathbf{k} & 6-6\mathbf{i}+4\mathbf{j}-5\mathbf{k} \\ -2+7\mathbf{i}-3\mathbf{j} & -1-4\mathbf{i}-9\mathbf{j}+6\mathbf{k} & 3+6\mathbf{i} \\ 17-18\mathbf{i}+14\mathbf{j}+\mathbf{k} & -5+14\mathbf{i}+2\mathbf{j}-9\mathbf{k} & 3-8\mathbf{i}-7\mathbf{k} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 4-2\mathbf{i}+11\mathbf{j}-2\mathbf{k} & 5-4\mathbf{i}+3\mathbf{j}+\mathbf{k} & -1-2\mathbf{i}+\mathbf{j}+3\mathbf{k} \\ -2+\mathbf{i}-5\mathbf{j}+2\mathbf{k} & -2+2\mathbf{j}+\mathbf{k} & -1-\mathbf{i}-4\mathbf{j}-2\mathbf{k} \\ 5+3\mathbf{j}-4\mathbf{k} & 4-7\mathbf{i}+2\mathbf{j}-2\mathbf{k} & 4-\mathbf{i}+\mathbf{j}-2\mathbf{k} \end{pmatrix}.$$

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We consider the mixed pairs of quaternion matrix Sylvester equations (1.3), where Z is $\mathbf{k}$-Hermitian. Direct computations yield

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 3, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \end{cases}$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1,\ A_2) + r(B_1,\ B_2) = 6,$$

$$r\begin{pmatrix} C_1 & C_2A_2^{\mathbf{k}*} - A_2C_2^{\mathbf{k}*} & A_1 & A_2B_2^{\mathbf{k}*} \\ B_1 & B_2A_2^{\mathbf{k}*} & 0 & 0 \end{pmatrix} = r(A_1,\ A_2B_2^{\mathbf{k}*}) + r(B_1,\ B_2A_2^{\mathbf{k}*}) = 6,$$

$$r\begin{pmatrix} C_2A_2^{\mathbf{k}*} - A_2C_2^{\mathbf{k}*} & A_2B_2^{\mathbf{k}*} \\ B_2A_2^{\mathbf{k}*} & 0 \end{pmatrix} = 2r(A_2B_2^{\mathbf{k}*}) = 4,$$

$$r\begin{pmatrix} C_1 & C_2A_2^{\mathbf{k}*} - A_2C_2^{\mathbf{k}*} & A_1 & A_2B_2^{\mathbf{k}*} \\ 0 & -C_1^{\mathbf{k}*} & 0 & B_1^{\mathbf{k}*} \\ B_1 & B_2A_2^{\mathbf{k}*} & 0 & 0 \\ 0 & A_1^{\mathbf{k}*} & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} A_1 & A_2B_2^{\mathbf{k}*} \\ 0 & B_1^{\mathbf{k}*} \end{pmatrix} = 10.$$

All the rank equalities in (4.16)–(4.20) hold. Hence, the mixed pairs of quaternion matrix Sylvester equations (1.3) has a solution (X, Y, Z), where Z is $\mathbf{k}$-Hermitian. Also, it is easy to show that

$$X = \begin{pmatrix} 1+2\mathbf{i}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{j}-\mathbf{k} \\ 1-2\mathbf{i}+2\mathbf{j}-\mathbf{k} & \mathbf{i}-\mathbf{j}+\mathbf{k} & 0 \\ 2+2\mathbf{j} & 2\mathbf{i}+2\mathbf{k} & 1+\mathbf{j}-\mathbf{k} \end{pmatrix},$$

$$Y = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 1+\mathbf{k} & 2\mathbf{j} \\ 1 & \mathbf{j} & 1+\mathbf{k} \\ 1+\mathbf{i}+\mathbf{j} & 1-2\mathbf{j} & 1+\mathbf{j}+\mathbf{k} \end{pmatrix},$$

and

$$Z = Z^{\mathbf{k}*} = \begin{pmatrix} 1+\mathbf{i}+\mathbf{j} & 1-\mathbf{k} & \mathbf{k} \\ 1+\mathbf{k} & \mathbf{i} & 0 \\ -\mathbf{k} & 0 & \mathbf{j} \end{pmatrix}$$

satisfy (1.3).

EXAMPLE 4.9. Let

$$A_1 = \begin{pmatrix} \mathbf{i}+\mathbf{j}-\mathbf{k} & 2+\mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{j} \\ -1+\mathbf{j}+\mathbf{k} & -1+2\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{k} \\ -1+\mathbf{i}+2\mathbf{j} & 1+3\mathbf{i}+2\mathbf{k} & \mathbf{j}+\mathbf{k} \end{pmatrix},$$

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$$B_1 = \begin{pmatrix} 1+\mathbf{i}+\mathbf{k} & \mathbf{j}-3\mathbf{k} & 3\mathbf{i} \\ -2+\mathbf{j}+\mathbf{k} & \mathbf{i}-\mathbf{j}+2\mathbf{k} & -2\mathbf{i}+\mathbf{j} \\ -1+\mathbf{i}+\mathbf{j}+2\mathbf{k} & \mathbf{i}-\mathbf{k} & \mathbf{i}+\mathbf{j} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2\mathbf{i}+\mathbf{k} & 3+\mathbf{j}+\mathbf{k} & 1-\mathbf{i}+\mathbf{j}-\mathbf{k} \\ 1+\mathbf{i} & \mathbf{j} & 1+\mathbf{k} \\ \mathbf{k} & \mathbf{i}+2\mathbf{j}+\mathbf{k} & \mathbf{j} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \mathbf{j} & 2\mathbf{i}+\mathbf{k} & 3 \\ \mathbf{k} & -2-\mathbf{j} & 3\mathbf{i} \\ \mathbf{j}+\mathbf{k} & -2+2\mathbf{i}-\mathbf{j}+\mathbf{k} & 3+3\mathbf{i} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 5+2\mathbf{i}+9\mathbf{j}+5\mathbf{k} & 2-\mathbf{j}+7\mathbf{k} & -5+2\mathbf{i}+5\mathbf{j}-9\mathbf{k} \\ 8+2\mathbf{i}-8\mathbf{j}+9\mathbf{k} & -5+3\mathbf{i}-8\mathbf{j} & -2\mathbf{i}+2\mathbf{j}+7\mathbf{k} \\ 4+\mathbf{i}+2\mathbf{j}+11\mathbf{k} & -2+4\mathbf{i}+2\mathbf{j}+2\mathbf{k} & -9+\mathbf{i}+9\mathbf{j}-\mathbf{k} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -1-3\mathbf{i}-8\mathbf{j}+2\mathbf{k} & -2-3\mathbf{i}+7\mathbf{k} & -5-\mathbf{j}+\mathbf{k} \\ 5+6\mathbf{i}-4\mathbf{j}-\mathbf{k} & 9-6\mathbf{i}-2\mathbf{j}+8\mathbf{k} & -6-8\mathbf{i}-12\mathbf{j}-8\mathbf{k} \\ 5+\mathbf{i}-2\mathbf{j}+3\mathbf{k} & 2-4\mathbf{i}+5\mathbf{j}-\mathbf{k} & -5-3\mathbf{i}-2\mathbf{j}+4\mathbf{k} \end{pmatrix}.$$

Now we consider the mixed pairs of quaternion matrix Sylvester equations (1.3), where $Z = Z^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*}$. Direct computations yield

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 3, & \text{if } i=1, \\ 4, & \text{if } i=2, \end{cases}$$

$$r\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1,\ A_2) + r(B_1,\ B_2) = 5,$$

$$r\begin{pmatrix} C_1 & C_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} - A_2C_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & A_1 & A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \\ B_1 & B_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & 0 & 0 \end{pmatrix}$$
$$= r\left(A_1,\ A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*}\right) + r\left(B_1,\ B_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*}\right) = 4,$$

$$r\begin{pmatrix} C_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} - A_2C_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \\ B_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & 0 \end{pmatrix} = 2r\left(A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*}\right) = 2,$$

$$r\begin{pmatrix} C_1 & C_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} - A_2C_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & A_1 & A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \\ 0 & -C_1^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & 0 & B_1^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \\ B_1 & B_2A_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & 0 & 0 \\ 0 & A_1^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} & 0 & 0 \end{pmatrix} = 2r\begin{pmatrix} A_1 & A_2B_2^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \\ 0 & B_1^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} \end{pmatrix} = 6.$$

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All the rank equalities in (4.16)–(4.20) hold. Hence, the mixed pairs of quaternion matrix Sylvester equations (1.3) has a solution (X, Y, Z), where $Z = Z^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*}$. Moreover, it is easy to show that

$$X = \begin{pmatrix} -\mathbf{i} & 1+\mathbf{k} & 2+\mathbf{i}+2\mathbf{j}+\mathbf{k} \\ 2+\mathbf{j} & 0 & \mathbf{i}-\mathbf{k} \\ 2-\mathbf{i}+\mathbf{j} & 1+\mathbf{k} & 2+2\mathbf{i}+2\mathbf{j} \end{pmatrix},$$

$$Y = \begin{pmatrix} 2+\mathbf{k} & \mathbf{i}+\mathbf{j}+2\mathbf{k} & 0 \\ 1+\mathbf{i}+\mathbf{j} & 1-\mathbf{i} & \mathbf{i}+3\mathbf{k} \\ 0 & 0 & 1+\mathbf{j} \end{pmatrix},$$

and

$$Z = Z^{\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{j})*} = \begin{pmatrix} 1+\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{i}+2\mathbf{j} & \mathbf{i} \\ -2\mathbf{i}-\mathbf{j} & \mathbf{i}-\mathbf{j} & \mathbf{j} \\ -\mathbf{j} & -\mathbf{i} & \mathbf{k} \end{pmatrix}$$

satisfy (1.3).

5. The solution to system (1.4). In this section, we consider some solvability conditions and the general solution to the mixed pairs of quaternion matrix Sylvester equations (1.4). The following lemma gives the solvability conditions and general solution to the mixed Sylvester matrix equations

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_2 - X_3B_2 = C_2 \end{cases} \tag{5.51}$$

over $\mathbb{C}$.

LEMMA 5.1. *[16] Let A_i, B_i, and C_i $(i = 1, 2)$ be given matrices over $\mathbb{H}$. Set*

$$A_{11} = R_{(A_2A_1)}A_2, \quad B_{11} = R_{B_1}L_{B_2}, \quad C_{11} = R_{(A_2A_1)}(A_2R_{A_1}C_1B_1^{\dagger} + C_2)L_{B_2}.$$

Then the following statements are equivalent:

(1) *The mixed generalized Sylvester quaternion matrix equations (5.51) is solvable.*

(2)

$$R_{A_1}C_1L_{B_1} = 0, \quad R_{A_{11}}C_{11} = 0, \quad C_{11}L_{B_{11}} = 0.$$

(3)

$$r\begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1), \quad r\begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r\begin{pmatrix} A_2A_1 & A_2C_1 + C_2B_1 \\ 0 & B_2B_1 \end{pmatrix} = r(A_2A_1) + r(B_2B_1).$$

If any of the above condition is satisfied, then the general solution to (5.51) can be expressed as

$$X_1 = A_1^{\dagger}C_1 + U_1B_1 + L_{A_1}W_1,$$

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$$X_2 = -R_{A_1}C_1B_1^\dagger + A_1U_1 + V_1R_{B_1},$$

$$X_3 = -R_{(A_2A_1)}(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1})B_2^\dagger + A_2A_1W_4 + W_5R_{B_2},$$

where

$$V_1 = A_{11}^\dagger C_{11}B_{11}^\dagger + L_{A_{11}}W_2 + W_3R_{B_{11}},$$

$$U_1 = (A_2A_1)^\dagger(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1}) + W_4B_2 + L_{(A_2A_1)}W_6,$$

and $W_1, \ldots, W_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Using the similar method, we can extend Lemma 5.1 to $\mathbb{H}$. For simplicity, put

$$A_{11} = R_{(A_2A_1)}A_2, \quad B_{11} = R_{B_1}L_{B_2}, \quad C_{11} = R_{(A_2A_1)}(A_2R_{A_1}C_1B_1^\dagger + C_2)L_{B_2}, \tag{5.52}$$

$$A_{22} = (B_1L_{(B_2B_1)})_\phi, \quad B_{22} = (R_{A_1}L_{A_2})_\phi, \quad C_{22} = -[R_{A_1}(C_1 + A_2^\dagger C_2L_{B_2}B_1)L_{(B_2B_1)}]_\phi, \tag{5.53}$$

$$A_{33} = (A_2A_1,\ (R_{B_2})_\phi), \quad B_{33} = R_{B_{11}}R_{B_1}B_2^\dagger, \tag{5.54}$$

$$C_{33} = -\,[C_2B_2^\dagger + A_2(C_1 + A_2^\dagger C_2L_{B_2}B_1)(B_2B_1)^\dagger]_\phi + R_{(A_2A_1)}(A_2R_{A_1}C_1B_1^\dagger + C_2 - A_2A_{11}^\dagger C_{11}B_{11}^\dagger R_{B_1})B_2^\dagger, \tag{5.55}$$

$$A = R_{A_{33}}A_{11}, \quad B = B_{33}(R_{A_{33}})_\phi, \quad C = R_{A_{33}}(R_{(B_2B_1)})_\phi, \quad D = (R_{A_{33}}A_2)_\phi, \tag{5.56}$$

$$E = R_{A_{33}}C_{33}(R_{A_{33}})_\phi, \quad M = R_AC, \quad N = DL_B, \quad S = CL_M. \tag{5.57}$$

Then we have the following theorem with two solvability conditions and the general solution to (1.4).

THEOREM 5.2. *Let A_i, B_i, and C_i $(i = 1, 2)$ be given matrices over $\mathbb{H}$. Then the following statements are equivalent:*

(1) *The mixed pairs of quaternion matrix Sylvester equations (1.4) has a solution (X, Y, Z).*

(2)

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad i = 1, 2, \tag{5.58}$$

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$$r\begin{pmatrix} A_2A_1 & A_2C_1+C_2B_1 \\ 0 & B_2B_1 \end{pmatrix} = r(A_2A_1)+r(B_2B_1), \tag{5.59}$$

$$r\begin{pmatrix} (B_2)_\phi A_2C_1+(B_2)_\phi C_2B_1-(C_2)_\phi B_2B_1 & (B_2)_\phi A_2A_1 \\ (A_2)_\phi B_2B_1 & 0 \end{pmatrix} = r((B_2)_\phi A_2A_1)+r((A_2)_\phi B_2B_1), \tag{5.60}$$

$$r\begin{pmatrix} (B_2B_1)_\phi A_2C_1+(B_2B_1)_\phi C_2A_1-(C_2B_1)_\phi A_2A_1-(B_2C_1)_\phi A_2A_1 & (B_2B_1)_\phi A_2A_1 \\ (A_2A_1)_\phi B_2B_1 & 0 \end{pmatrix} = 2r((A_2A_1)_\phi B_2B_1), \tag{5.61}$$

$$r\begin{pmatrix} (B_2)_\phi C_2-(C_2)_\phi B_2 & (B_2)_\phi A_2 \\ (A_2)_\phi B_2 & 0 \end{pmatrix} = 2r((A_2)_\phi B_2). \tag{5.62}$$

(3)

$$R_{A_1}C_1L_{B_1}=0, \quad R_{A_{11}}C_{11}=0, \quad C_{11}L_{B_{11}}=0, \tag{5.63}$$

$$R_MR_AE=0, \quad R_CEL_B=0, \quad R_AEL_D=0. \tag{5.64}$$

The general solution to (1.4) can be expressed as

$$X=\frac{X_1+(X_5)_\phi}{2}, \quad Y=\frac{X_2+(X_4)_\phi}{2}, \quad Z=Z_\phi=\frac{X_3+(X_3)_\phi}{2}, \tag{5.65}$$

where

$$X_1=A_1^\dagger C_1+U_1B_1+L_{A_1}W_1, \tag{5.66}$$

$$X_2=-R_{A_1}C_1B_1^\dagger+A_1U_1+V_1R_{B_1}, \tag{5.67}$$

$$X_4=(A_2^\dagger C_2L_{B_2})_\phi+(B_2)_\phi U_2+V_2(L_{A_2})_\phi, \tag{5.68}$$

$$X_5=[A_1^\dagger(C_1+A_2^\dagger C_2L_{B_2}B_1+L_{A_2}(V_2)_\phi B_1)L_{(B_2B_1)}]_\phi-(B_2B_1)_\phi T_4+T_5(L_{A_1})_\phi, \tag{5.69}$$

$$X_3=-R_{(A_2A_1)}(C_2+A_2R_{A_1}C_1B_1^\dagger-A_2V_1R_{B_1})B_2^\dagger+A_2A_1W_4+W_5R_{B_2}, \tag{5.70}$$

or

$$X_3=-(C_2B_2^\dagger)_\phi+U_2(A_2)_\phi-(R_{B_2})_\phi T_1, \tag{5.71}$$

$$V_1=A_{11}^\dagger C_{11}B_{11}^\dagger+L_{A_{11}}W_2+W_3R_{B_{11}}, \tag{5.72}$$

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$$U_1 = (A_2A_1)^\dagger(C_2 + A_2R_{A_1}C_1B_1^\dagger - A_2V_1R_{B_1}) + W_4B_2 + L_{(A_2A_1)}W_6, \tag{5.73}$$

$$V_2 = A_{22}^\dagger C_{22}B_{22}^\dagger + L_{A_{22}}T_2 + T_3R_{B_{22}}, \tag{5.74}$$

$$U_2 = -[(C_1 + A_2^\dagger C_2L_{B_2}B_1 + L_{A_2}(V_2)_\phi B_1)(B_2B_1)^\dagger]_\phi - T_4(A_1)_\phi - (R_{(B_2B_1)})_\phi T_6, \tag{5.75}$$

$$W_4 = (I_{p_1},\ 0)[A_{33}^\dagger(C_{33} - A_{11}W_3B_{33} - (R_{(B_2B_1)})_\phi T_6(A_2)_\phi) - A_{33}^\dagger Z_7(A_{33})_\phi + L_{A_{33}}Z_6], \tag{5.76}$$

$$T_1 = (0,\ I_{p_2})[A_{33}^\dagger(C_{33} - A_{11}W_3B_{33} - (R_{(B_2B_1)})_\phi T_6(A_2)_\phi) - A_{33}^\dagger Z_7(A_{33})_\phi + L_{A_{33}}Z_6], \tag{5.77}$$

$$T_4 = [R_{A_{33}}(C_{33} - A_{11}W_3B_{33} - (R_{(B_2B_1)})_\phi T_6(A_2)_\phi)(A_{33}^\dagger)_\phi + A_{33}A_{33}^\dagger Z_7 + Z_8(L_{A_{33}})_\phi]\begin{pmatrix} I_{p_1} \\ 0 \end{pmatrix}, \tag{5.78}$$

$$W_5 = [R_{A_{33}}(C_{33} - A_{11}W_3B_{33} - (R_{(B_2B_1)})_\phi T_6(A_2)_\phi)(A_{33}^\dagger)_\phi + A_{33}A_{33}^\dagger Z_7 + Z_8(L_{A_{33}})_\phi]\begin{pmatrix} 0 \\ I_{p_2} \end{pmatrix}, \tag{5.79}$$

$$W_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1R_NDB^\dagger + L_AZ_2 + Z_3R_B, \tag{5.80}$$

$$T_6 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_ML_SZ_4 + L_MZ_1R_N + Z_5R_D, \tag{5.81}$$

and the remaining W_j, T_j, Z_j are arbitrary matrices over $\mathbb{H}$, p_1 is the column number of A_1, p_2 is the row number of B_2.

Proof. The proof is similar to that for Theorem 4.5. It can be shown that (1.4) has a solution (X, Y, Z) if and only if the following system of quaternion matrix equations

$$\begin{cases} A_1X_1 - X_2B_1 = C_1, \\ A_2X_2 - X_3B_2 = C_2, \\ (B_2)_\phi X_3 - X_4(A_2)_\phi = -(C_2)_\phi, \\ (B_1)_\phi X_4 - X_5(A_1)_\phi = -(C_1)_\phi \end{cases} \tag{5.82}$$

has a solution. ∎

Remark 5.3. We can also give some solvability conditions and general solution to the following system of quaternion matrix equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2Y - ZB_2 = C_2, \end{cases} \quad X = X_\phi$$

by taking ϕ to the first and second matrix equations in (1.4).

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Based on Remark 4.7, we can also give two examples with respect to the two special cases.

Example 5.4. Given the quaternion matrices:

$$A_1 = \begin{pmatrix} 2\mathbf{i}+\mathbf{j}+\mathbf{k} & -1+\mathbf{j}-\mathbf{k} & \mathbf{j} \\ -2-\mathbf{j}+\mathbf{k} & -\mathbf{i}+\mathbf{j}+\mathbf{k} & \mathbf{k} \\ 2+\mathbf{i}+2\mathbf{j} & -1+\mathbf{i}-2\mathbf{k} & \mathbf{j}-\mathbf{k} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \mathbf{i}+2\mathbf{j}+3\mathbf{k} & \mathbf{i} & 1+\mathbf{k} \\ 1+\mathbf{i}+3\mathbf{j}+3\mathbf{k} & \mathbf{i}+\mathbf{k} & 2+\mathbf{j}+\mathbf{k} \\ 1+\mathbf{j} & \mathbf{k} & 1+\mathbf{j} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1+\mathbf{k} & \mathbf{i}+\mathbf{j}-\mathbf{k} & \mathbf{i}+2\mathbf{j}+\mathbf{k} \\ \mathbf{i}-\mathbf{j} & -1+\mathbf{j}+\mathbf{k} & -1-\mathbf{j}+2\mathbf{k} \\ 2+2\mathbf{k} & 2\mathbf{i}+2\mathbf{j}-2\mathbf{k} & 2\mathbf{i}+4\mathbf{j}+2\mathbf{k} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \mathbf{i}+2\mathbf{j}+\mathbf{k} & 1+\mathbf{j}-\mathbf{k} & -1+\mathbf{j} \\ -1 & \mathbf{i}+\mathbf{j} & -\mathbf{i} \\ 1+\mathbf{j} & -\mathbf{i}+\mathbf{k} & \mathbf{i} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 2-9\mathbf{i}-4\mathbf{j} & -3-\mathbf{j}+6\mathbf{k} & -3-\mathbf{i}-\mathbf{j}-6\mathbf{k} \\ 6-6\mathbf{i}-7\mathbf{j}-8\mathbf{k} & -3-7\mathbf{i}-4\mathbf{j}-3\mathbf{k} & -2-7\mathbf{i}+2\mathbf{j} \\ 5-6\mathbf{i}-8\mathbf{j}+5\mathbf{k} & 3\mathbf{i}+8\mathbf{k} & -2+5\mathbf{i}-10\mathbf{j}-9\mathbf{k} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 5+3\mathbf{i}-\mathbf{j} & -7-5\mathbf{i} & -5+6\mathbf{i}+2\mathbf{j}+4\mathbf{k} \\ -5+\mathbf{i}-8\mathbf{j}-3\mathbf{k} & 1-6\mathbf{i}-\mathbf{j}+4\mathbf{k} & -2-7\mathbf{i}-8\mathbf{j}+3\mathbf{k} \\ 3+7\mathbf{i}+\mathbf{j}+5\mathbf{k} & -9-7\mathbf{i}+6\mathbf{j}-\mathbf{k} & -14+5\mathbf{i}+5\mathbf{j}+9\mathbf{k} \end{pmatrix}.$$

Direct computations yield

$$r\begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 3, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \end{cases}$$

$$r\begin{pmatrix} A_2A_1 & A_2C_1 + C_2B_1 \\ 0 & B_2B_1 \end{pmatrix} = r(A_2A_1) + r(B_2B_1) = 3,$$

$$r\begin{pmatrix} (B_2)^{\mathbf{j}*}A_2C_1 + (B_2)^{\mathbf{j}*}C_2B_1 - (C_2)^{\mathbf{j}*}B_2B_1 & (B_2)^{\mathbf{j}*}A_2A_1 \\ (A_2)^{\mathbf{j}*}B_2B_1 & 0 \end{pmatrix} = r((B_2)^{\mathbf{j}*}A_2A_1) + r((A_2)^{\mathbf{j}*}B_2B_1) = 2,$$

$$r\begin{pmatrix} (B_2B_1)^{\mathbf{j}*}A_2C_1 + (B_2B_1)^{\mathbf{j}*}C_2A_1 - (C_2B_1)^{\mathbf{j}*}A_2A_1 - (B_2C_1)^{\mathbf{j}*}A_2A_1 & (B_2B_1)^{\mathbf{j}*}A_2A_1 \\ (A_2A_1)^{\mathbf{j}*}B_2B_1 & 0 \end{pmatrix}$$
$$= 2r((A_2A_1)^{\mathbf{j}*}B_2B_1) = 2,$$

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$$r\begin{pmatrix}(B_2)^{\mathbf{j}*}C_2-(C_2)^{\mathbf{j}*}B_2 & (B_2)^{\mathbf{j}*}A_2\\ (A_2)^{\mathbf{j}*}B_2 & 0\end{pmatrix}=2r((A_2)^{\mathbf{j}*}B_2)=2.$$

All the rank equalities in (5.58)–(5.62) hold. Hence, (1.4) has a solution (X,Y,Z), where Z is $\mathbf{j}$-Hermitian. It is easy to show that

$$X=\begin{pmatrix}\mathbf{i}+2\mathbf{j} & 1+\mathbf{j}+2\mathbf{k} & \mathbf{i}\\ -1 & \mathbf{j} & 1+\mathbf{k}\\ -1+\mathbf{i}+2\mathbf{k} & 1 & 1+\mathbf{i}+3\mathbf{j}\end{pmatrix},$$

$$Y=\begin{pmatrix}2-\mathbf{k} & -1+2\mathbf{j} & \mathbf{i}+\mathbf{j}+\mathbf{k}\\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 1 & \mathbf{j}\\ 1-\mathbf{i}+\mathbf{j} & 2\mathbf{j} & 1+2\mathbf{j}+\mathbf{k}\end{pmatrix},$$

and

$$Z=Z^{\mathbf{j}*}=\begin{pmatrix}1+\mathbf{i}+\mathbf{k} & 2+\mathbf{j} & \mathbf{i}\\ 2-\mathbf{j} & \mathbf{i} & 0\\ \mathbf{i} & 0 & \mathbf{k}\end{pmatrix}$$

satisfy (1.4).

Example 5.5. Let

$$A_1=\begin{pmatrix}2\mathbf{i}+\mathbf{j} & 1+\mathbf{k} & 2-\mathbf{i}\\ -2+\mathbf{k} & \mathbf{i}-\mathbf{j} & 1+2\mathbf{i}\end{pmatrix},\quad B_1=\begin{pmatrix}1+\mathbf{i}+3\mathbf{k} & \mathbf{j} & 1\\ 0 & 1+2\mathbf{j} & \mathbf{k}\end{pmatrix},$$

$$A_2=\begin{pmatrix}1+\mathbf{i}+\mathbf{k} & -2\mathbf{i}+\mathbf{k}\\ -1+\mathbf{i}-\mathbf{j} & 2-\mathbf{j}\end{pmatrix},\quad B_2=\begin{pmatrix}1+2\mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{j}-\mathbf{k}\\ -2+\mathbf{i}-\mathbf{j}+\mathbf{k} & \mathbf{i}+\mathbf{j}+\mathbf{k}\end{pmatrix},$$

$$C_1=\begin{pmatrix}-1-5\mathbf{i}+\mathbf{j}+2\mathbf{k} & 5\mathbf{i}-4\mathbf{j}-\mathbf{k} & -1+3\mathbf{j}-\mathbf{k}\\ 2\mathbf{i}-3\mathbf{j}-2\mathbf{k} & -7-7\mathbf{i}-\mathbf{k} & -1-\mathbf{i}+\mathbf{j}+\mathbf{k}\end{pmatrix},$$

$$C_2=\begin{pmatrix}1-4\mathbf{i}+10\mathbf{j}+5\mathbf{k} & 1-\mathbf{i}-\mathbf{j}+4\mathbf{k}\\ -3-2\mathbf{i}+7\mathbf{j}+4\mathbf{k} & -8\mathbf{i}-2\mathbf{j}+3\mathbf{k}\end{pmatrix}.$$

Now we consider (1.4), where $Z=Z^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}$. Direct computations yield

$$r\begin{pmatrix}C_i & A_i\\ B_i & 0\end{pmatrix}=r(A_i)+r(B_i)=\begin{cases}3, & \text{if } i=1,\\ 2, & \text{if } i=2,\end{cases}$$

$$r\begin{pmatrix}A_2A_1 & A_2C_1+C_2B_1\\ 0 & B_2B_1\end{pmatrix}=r(A_2A_1)+r(B_2B_1)=2,$$

$$\begin{aligned}&r\begin{pmatrix}(B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2C_1+(B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}C_2B_1-(C_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2B_1 & (B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2A_1\\ (A_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2B_1 & 0\end{pmatrix}\\ &=r((B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2A_1)+r((A_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2B_1)=0,\end{aligned}$$

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$$r\begin{pmatrix} \Omega & (B_2B_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2A_1 \\ (A_2A_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2B_1 & 0 \end{pmatrix} = 2r((A_2A_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2B_1) = 0,$$

$$r\begin{pmatrix} (B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}C_2 - (C_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2 & (B_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2 \\ (A_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2 & 0 \end{pmatrix} = 2r((A_2)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}B_2) = 0,$$

where

$$\Omega = (B_2B_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2C_1 + (B_2B_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}C_2A_1 - (C_2B_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2A_1 - (B_2C_1)^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}A_2A_1.$$

All the rank equalities in (5.58)–(5.62) hold. Hence, (1.4) has a solution (X, Y, Z), where $Z = Z^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*}$. It is easy to show that

$$X = \begin{pmatrix} \mathbf{i}+\mathbf{j}+2\mathbf{k} & 2+2\mathbf{j}+\mathbf{k} & -1 \\ 1-\mathbf{k} & -1-2\mathbf{j} & \mathbf{i}+\mathbf{j} \\ 1+\mathbf{i}+\mathbf{j}+\mathbf{k} & 1+\mathbf{k} & -1+\mathbf{i}+\mathbf{j} \end{pmatrix},$$

$$Y = \begin{pmatrix} 1+\mathbf{i}+2\mathbf{j}-\mathbf{k} & 1+2\mathbf{i}+\mathbf{k} \\ \mathbf{k} & 1 \end{pmatrix},$$

and

$$Z = Z^{\frac{\sqrt{2}}{2}(\mathbf{j}+\mathbf{k})*} = \begin{pmatrix} \mathbf{i}+\mathbf{j}-\mathbf{k} & \mathbf{j} \\ -\mathbf{k} & 1+2\mathbf{j}-2\mathbf{k} \end{pmatrix}$$

satisfy (1.4).

6. Conclusions. We have derived some practical necessary and sufficient conditions for the existence of a solution (X, Y, Z) (with a ϕ-Hermitian Z) to the mixed pairs of quaternion matrix Sylvester equations (1.3) and (1.4) and presented general solutions to (1.3) and (1.4) when they are solvable. Moreover, some numerical examples have been provided.

A more challenging problem is to give the solvability conditions and general solutions to the mixed pairs of quaternion matrix Sylvester equations (1.3) and (1.4), where all of the unknowns X, Y, Z are ϕ-Hermitian.

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