Geophys. J. R. astr. Soc. (1970) **22**, 83–100.

The General Theory of Electromagnetic Induction in a Conducting Half-Space

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(Received 1970 June 6)

Summary

The general theory of electromagnetic induction in a conducting half-space by an external magnetic source is developed in a new way which simplifies and consolidates the classic treatment of A. T. Price. The novel features of the theory are a systematic application of integral transforms and the use of electric and magnetic Hertz vectors aligned normal to the surface of the conductor. It is shown that the solutions associated with the electric Hertz vector correspond only to a free decay of currents within the conductor so that the entire theory of the induction problem is developed in terms of the one scalar component of the magnetic Hertz vector. The general solution of the magnetic Hertz potential corresponding to induction by an arbitrary time-dependent source is obtained in the form of a closed integral involving just one unknown function which is a Fourier transform of the magnetic Hertz potential of the source evaluated at the surface of the conductor. Results corresponding to the special cases of aperiodic and periodic fields are developed and explicit expressions for the electric and magnetic field vectors are also derived. The general theory is illustrated by considering three specific sources: (i) an aperiodic magnetic dipole normal to the surface of the conductor, (ii) a periodic magnetic dipole parallel to the surface of the conductor, and (iii) a periodic line current flowing parallel to the surface of the conductor.

Introduction

Two decades have passed since the publication of his classic paper in which Price (1950) developed the general theory of electromagnetic induction in a semi-infinite conductor with a plane boundary. Over the intervening years it has become the standard reference work for analysing geophysical problems (such as those arising in the magnetic and magnetotelluric methods of exploration geophysics) which involve the induction of earth currents by an external source over sufficiently localized regions of the Earth that its curvature can be neglected. A vast literature has accumulated covering solutions of such problems for different kinds of sources. An extensive bibliography can be found in a recent review article by Ward (1967).

Price's treatment was based on the elementary solutions obtained by separating the variables in the differential equation satisfied by the electromagnetic field vectors. Two distinct types of fundamental solutions emerged. Those of the 'first type' gave the field induced by the time-dependent magnetic field of an external system of currents flowing parallel to the surface of the conductor, while those of the ' second type ' represented freely decaying current systems inside the conductor which had no associated magnetic field outside the conductor. Although the particular case of an electric source field perpendicular to the surface of the conductor was not included in the solutions of the first type, this did not detract from the generality of the solutions since such a source has no effect inside the conductor as explained in a later paper by Price (1962).

While Price's approach provided an excellent description of the physical processes underlying the theory of induction it did not yield a general formula which gave directly the total field corresponding to any prescribed source. Rather it was necessary to construct the solutions for each special case by identifying certain functions of position and time ($P(x, y)$ and $A(t)$ in his notation) appropriate to the given source, and integrating the elementary solutions over a suitable range of values of the separation constant. Moreover, since the elementary solutions were derived directly in terms of the electric and magnetic field vectors (a slightly simpler procedure than using the electromagnetic scalar and vector potentials because, as Price pointed out, the former vectors are solenoidal whereas the vector potential is not), the development of the theory was thereby encumbered with six scalar components.

With a somewhat different approach it is found that these less attractive features of the theory can be avoided and at the same time its development considerably simplified. Thus, in view of the continued interest in induction problems and the increasing importance of their application to geophysical probing, it seems timely to present in this paper the complete theory of electromagnetic induction in a conducting half-space in a more modern form which is quite general and yet concise.

The principal simplification is achieved by developing the theory in terms of Hertz vectors rather than in terms of the electric and magnetic field vectors. There is nothing new in the use of Hertz potentials, of course, but it has been the practice to follow the traditional method applicable to radiation fields by choosing the most appropriate Hertz vector to represent each specific source as it arises. For example, a magnetic dipole parallel to the surface of the conductor would be most conveniently represented by a magnetic Hertz vector along the dipole. The total field, however, would then have to be expressed in terms of a magnetic Hertz vector having components both parallel and normal to the surface. The essential feature of the theory presented in this paper is that both electric and magnetic Hertz vectors are used, and that these are always defined to be normal to the surface of the conductor. It then follows that for *any* source the solutions associated with the electric Hertz vector correspond to Price's solutions of the ' second type ' and merely represent a free decay of currents, while those associated with the magnetic Hertz vector correspond to Price's solutions of the ' first type '. Thus the entire theory of induction by an external magnetic source can be expressed in terms of the one scalar component of the magnetic Hertz potential. Apart from some related ideas set forth by Gordon (1951) in a sequel to Price's paper, this important result appears to have been overlooked by previous authors. It offers obvious advantages for extending the theory to a multilayered conductor.

The other simplification accrues from a systematic use of integral transforms rather than separation of the variables. This allows the general solutions to be expressed directly in terms of an infinite integral involving just one unknown function which is simply the double Fourier transform of the magnetic Hertz potential of the source evaluated at the surface of the conductor. Thus the solution to any induction problem involving the standard types of sources can be written down immediately with the aid of ' Tables of Integral Transforms ' compiled by Erdélyi (1954). For reference purposes these tables will be denoted in the subsequent sections by the abbreviation I.T.

The solutions are derived in the first instance for an arbitrary time-varying source, and the results obtained are then modified to suit the special cases of (i) aperiodic and (ii) periodic sources. Explicit formulas giving the electric and magnetic fields are also quoted. Finally, the usefulness of the general formulas is illustrated by applying them to some specific examples each of which has been the subject of separate investigations in the literature. The first example considered is that of an aperiodic magnetic dipole normal to the surface of the conductor. This has received the attention of Wait (1951), Bhattacharyya (1959) and Meyer (1962). Secondly, a periodic magnetic dipole parallel to the surface is considered, a problem which has been previously analysed by Wait (1953), Quon (1963) and Ward (1967) among others, and finally, we examine the two-dimensional field induced by a periodic infinite line current parallel to the surface of the conductor. This was the example chosen by Price (1950) himself to illustrate the application of his general theory to a specific problem.

2. Electromagnetic field equations

The behaviour of a *quasi-static* electromagnetic field within a uniform, isotropic and source-free medium of conductivity σ and permeability μ is governed by the approximate Maxwell equations

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t, \tag{2.1}$$

$$\operatorname{curl}\mathbf{B} = \alpha\mathbf{E}, \tag{2.2}$$

where $\alpha = \mu\sigma$, $\mathbf{E}$ is the electric field and $\mathbf{B}$ is the magnetic induction. All quantities are measured in MKS units. Defining the scalar and vector potentials ϕ and $\mathbf{A}$ in the usual way,

$$\mathbf{E} = -\operatorname{grad}\phi - \partial\mathbf{A}/\partial t, \tag{2.3}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{2.4}$$

and imposing the condition

$$\operatorname{div}\mathbf{A} + \alpha\phi = 0 \tag{2.5}$$

we can easily verify that ϕ and the components of $\mathbf{A}$ all satisfy a diffusion equation of the form

$$\nabla^2\Phi = \alpha\partial\Phi/\partial t. \tag{2.6}$$

The relation equation (2.5) corresponds to the Lorentz condition for radiation fields. It is automatically satisfied if we introduce electric and magnetic Hertz potentials, $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ respectively, by defining

$$\phi = -\operatorname{div}\boldsymbol{\Pi}, \qquad \mathbf{A} = \alpha\boldsymbol{\Pi} + \operatorname{curl}\boldsymbol{\Gamma}. \tag{2.7}$$

If ϕ and the components of $\mathbf{A}$, as given by equation (2.7), are substituted in equation (2.6), it is readily seen that the resulting equations are satisfied provided that the components of $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ themselves are also solutions of the diffusion equation (2.6). It then follows by equations (2.3), (2.4) and (2.7) that $\mathbf{E}$ and $\mathbf{B}$ can be expressed in terms of the Hertz potentials by the formulas

$$\mathbf{E} = \operatorname{curl}\operatorname{curl}\boldsymbol{\Pi} - \operatorname{curl}(\partial\boldsymbol{\Gamma}/\partial t), \tag{2.8}$$

$$\mathbf{B} = \operatorname{curl}\operatorname{curl}\boldsymbol{\Gamma} + \alpha\operatorname{curl}\boldsymbol{\Pi}. \tag{2.9}$$

Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors defining the directions of the x, y, and z axes respectively of a rectangular cartesian co-ordinate system. Without loss of generality (Jones

1964) we may choose $\mathbf{\Pi} = \Pi\mathbf{k}$ and $\mathbf{\Gamma} = \Gamma\mathbf{k}$ so that equations (2.8) and (2.9) become

$$\mathbf{E} = \left(\frac{\partial^2 \Pi}{\partial z \partial x} - \frac{\partial^2 \Gamma}{\partial y \partial t}\right)\mathbf{i} + \left(\frac{\partial^2 \Pi}{\partial z \partial y} + \frac{\partial^2 \Gamma}{\partial x \partial t}\right)\mathbf{j} - \left(\frac{\partial^2 \Pi}{\partial x^2} + \frac{\partial^2 \Pi}{\partial y^2}\right)\mathbf{k}, \tag{2.10}$$

$$\mathbf{B} = \left(\frac{\partial^2 \Gamma}{\partial z \partial x} + \alpha\frac{\partial \Pi}{\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 \Gamma}{\partial z \partial y} - \alpha\frac{\partial \Pi}{\partial x}\right)\mathbf{j} - \left(\frac{\partial^2 \Gamma}{\partial x^2} + \frac{\partial^2 \Gamma}{\partial y^2}\right)\mathbf{k}. \tag{2.11}$$

It is clear from equations (2.10) and (2.11) that the electromagnetic fields associated with Π and Γ respectively are distinguished from each other by the fact that the Π-field has no magnetic z-component, while the Γ-field has no electric z-component. Thus all current flow associated with the magnetic Hertz potential is parallel to the xy-plane.

When $\sigma = 0$ (i.e. $\alpha = 0$), both Π and Γ satisfy Laplace's equation

$$\nabla^2 \Phi = 0 \tag{2.12}$$

and the field expressions equations (2.10) and (2.11) reduce to

$$\mathbf{E} = \text{grad}\,(\partial\Pi/\partial z) - \text{curl}\,(\mathbf{k}\partial\Gamma/\partial t), \tag{2.13}$$

$$\mathbf{B} = \text{grad}\,(\partial\Gamma/\partial z). \tag{2.14}$$

Thus the electric Hertz potential Π does not contribute to the magnetic field in a non-conducting medium. Indeed, it follows from equation (2.14) that $-\partial\Gamma/\mu\partial z$ plays the role of a magnetic scalar potential in the region. We note also that the electric field equation (2.13) depends on $\partial\Pi/\partial z$ rather than on Π, so that the complete electromagnetic field in a non-conducting region can be specified in terms of Γ and $\partial\Pi/\partial z$, both of which satisfy equation (2.12) ($\partial\Pi/\partial z$ obviously satisfies equation (2.12) because Π does).

3. The mathematical model and notation

Let μ_0 be the permeability of free space. We shall take the half-space $z > 0$ of the rectangular co-ordinate system (x, y, z) to be occupied by a source-free medium of permeability $\mu = \kappa\mu_0$ and conductivity $\sigma = \alpha/\mu$. At a height h or greater above the conductor, i.e. in the region $z \leqslant -h$ $(h > 0)$, we suppose there exists a given time varying magnetic source in the form of an electric current system of finite extent. The intervening region $-h < z < 0$ between the source currents and the boundary of the conductor is assumed to have a permeability μ_0 and to be non-conducting.

The Hertz potentials Π and Γ, with which we shall be dealing, are, in general, functions of position and time. By defining

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j},$$

we can use the compact notation $\Phi(\mathbf{r}, z, t)$ to denote a function of the four variables x, y, z, t. It will also be convenient to express partial derivatives in the Landau notation by using the appropriate numerical subscript on the function symbol to indicate to which of the ordered variables the differentiation applies. In this convention, we have, for example,

$$\Phi_{13}(\mathbf{r}, +0, \tau) = \lim_{z \to +0} [\partial^2 \Phi/\partial z\, \partial x]_{t=\tau}.$$

It is now necessary to establish a notation for the various integral transforms which will be required later. The double Fourier transform of a function of x, y specified by a Greek letter will always be denoted by a function of ξ, η which is specified by the phonetically related Roman letter. Thus, if we formally define

$$\boldsymbol{\rho} = \xi\mathbf{i}+\eta\mathbf{j},$$

then we write

$$F(\boldsymbol{\rho}, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\mathbf{r}, z, t) \exp(i\boldsymbol{\rho}.\mathbf{r})\, d\mathbf{r}, \tag{3.1}$$

where $\int_{-\infty}^{\infty} \dots d\mathbf{r}$ means $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \dots dx\, dy$. The inverse transform in this notation is given by

$$\Phi(\mathbf{r}, z, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\boldsymbol{\rho}, z, t) \exp(-i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{3.2}$$

In addition we shall require a transform with respect to the variable z first introduced by Churchill (1944). This will be denoted in our notation by changing the function symbol from upper case to lower case, and is defined by

$$f(\boldsymbol{\rho}, \zeta, t) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} F(\boldsymbol{\rho}, z, t)\, K(\zeta, z)\, dz, \tag{3.3}$$

where

$$K(\zeta, z) = \frac{k\rho \sin\zeta z+\zeta\cos\zeta z}{(k^2\rho^2+\zeta^2)^{\frac{1}{2}}}. \tag{3.4}$$

The inverse transform has the symmetric form

$$F(\boldsymbol{\rho}, z, t) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} f(\boldsymbol{\rho}, \zeta, t)\, K(\zeta, z)\, d\zeta. \tag{3.5}$$

If we let $k \to \infty$ in equation (3.4), then equation (3.3) reduces to a Fourier sine transform. This special case will be indicated by a cap on the function symbol, i.e.

$$\hat{f}(\boldsymbol{\rho}, \zeta, t) = \sqrt{\left(\frac{2}{\pi}\right)}\int_{0}^{\infty} F(\boldsymbol{\rho}, z, t) \sin\zeta z\, dz. \tag{3.6}$$

4. Boundary conditions

Since the source is finite we may assume that Π and Γ and their derivatives vanish as $r \to \infty$, and also, of course, as $z \to \infty$. In addition the usual electromagnetic boundary conditions apply across the plane surface of the conductor. In order to express these conditions in terms of the Hertz potentials it is convenient to define

$$\Gamma(\mathbf{r}, +0, t)-\Gamma(\mathbf{r}, -0, t) = \Lambda(\mathbf{r}, t), \tag{4.1}$$

$$\Gamma_3(\mathbf{r}, +0, t)+\kappa\Gamma_3(\mathbf{r}, -0, t) = \Upsilon(\mathbf{r}, t), \tag{4.2}$$

$$\Pi_3(\mathbf{r}, +0, t)-\Pi_3(\mathbf{r}, -0, t) = \Psi(\mathbf{r}, t). \tag{4.3}$$

Then by equation (2.11), the continuity across the plane $z = 0$ of the normal component of $\mathbf{B}$, and of the tangential components of the magnetic intensity, implies

$$\Lambda_{11}(\mathbf{r}, t)+\Lambda_{22}(\mathbf{r}, t) = 0, \tag{4.4}$$

$$\Upsilon_1(\mathbf{r}, t)+\alpha\Pi_2(\mathbf{r}, +0, t) = 0, \tag{4.5}$$

$$\Upsilon_2(\mathbf{r}, t)-\alpha\Pi_1(\mathbf{r}, +0, t) = 0. \tag{4.6}$$

Likewise, by equation (2.10), the condition that the tangential components of **E** are continuous across $z = 0$, requires

$$\Psi_1(\mathbf{r}, t) - \Lambda_{32}(\mathbf{r}, t) = 0, \tag{4.7}$$

$$\Psi_2(\mathbf{r}, t) + \Lambda_{31}(\mathbf{r}, t) = 0. \tag{4.8}$$

If we differentiate equation (4.5) with respect to y and equation (4.6) with respect to x, and subtract, we obtain

$$\Pi_{11}(\mathbf{r}, +0, t) + \Pi_{22}(\mathbf{r}, +0, t) = 0.$$

Thus $\Pi(\mathbf{r}, +0, t)$ is a two-dimensional harmonic function in the xy-plane. Since it also vanishes as $r \to \infty$, it must identically vanish, i.e.

$$\Pi(\mathbf{r}, +0, t) = 0. \tag{4.9}$$

If instead we differentiate equation (4.5) with respect to x and equation (4.6) with respect to y and add we find

$$\Upsilon_{11}(\mathbf{r}, t) + \Upsilon_{22}(\mathbf{r}, t) = 0. \tag{4.10}$$

A similar procedure applied to equations. (4.7) and (4.8) yields

$$\Psi_{11}(\mathbf{r}, t) + \Psi_{22}(\mathbf{r}, t) = 0. \tag{4.11}$$

Equations (4.4), (4.10) and (4.11) show that Λ, Υ and Ψ are all harmonic functions in x and y which tend to zero as $r \to \infty$. Hence they too must vanish identically.

Substituting these results back into equations (4.1), (4.2) and (4.3), taking their Fouiier transform, and that of equation (4.9) also, we obtain the simple boundary conditions

$$G(\boldsymbol{\rho}, +0, t) = G(\boldsymbol{\rho}, -0, t), \tag{4.12}$$

$$G_3(\boldsymbol{\rho}, +0, t) = \kappa G_3(\boldsymbol{\rho}, -0, t), \tag{4.13}$$

$$P_3(\boldsymbol{\rho}, +0, t) = P_3(\boldsymbol{\rho}, -0, t), \tag{4.14}$$

$$P(\boldsymbol{\rho}, +0, t) = 0. \tag{4.15}$$

We note that the boundary conditions for the magnetic Hertz potential are uncoupled from those for the electric Hertz potential. Thus the solutions for Γ and Π exist quite independently of each other.

5. The field outside the conductor

It was shown in Section 2 that a quasi-static electromagnetic field in the non-conducting region $-h < z < 0$ could be specified in terms of the two scalars Γ and Π_3, both of which satisfy Laplace's equation (2.12) in the region. Applying the Fourier transform (equation (3.1)) to equation (2.12), and assuming that the derivatives of the field vanish as $r \to \infty$, we find that G and P_3 (the Fourier transforms of Γ and Π_3 respectively) both satisfy the differential equation

$$F_{33}(\boldsymbol{\rho}, z, t) = \rho^2 F(\boldsymbol{\rho}, z, t). \tag{5.1}$$

Let us denote the magnetic Hertz potential of the source currents alone by $\Gamma^{(s)}$. Clearly its Fourier transform $G^{(s)}$ will satisfy equation (5.1), and since the field of the source must approach zero when $z \to +\infty$, the appropriate solution for $G^{(s)}$ in $-h < z < 0$ is

$$G^{(s)}(\boldsymbol{\rho}, z, t) = G^{(s)}(\boldsymbol{\rho}, -0, t) \exp(-\rho z). \tag{5.2}$$

If we formally change the sign of z in equation (5.2) and subtract, we obtain

$$G^{(s)}(\boldsymbol{\rho}, z, t) - G^{(s)}(\boldsymbol{\rho}, -z, t) = -2G^{(s)}(\boldsymbol{\rho}, -0, t) \sinh \rho z. \tag{5.3}$$

Likewise we denote the magnetic Hertz potential associated with the currents flowing inside the conductor by $\Gamma^{(i)}$. The field of these currents must vanish as $z \to -\infty$, so that the solution of equation (5.1) which gives $G^{(i)}$ in $-h < z < 0$ is

$$G^{(i)}(\boldsymbol{\rho}, z, t) = G^{(i)}(\boldsymbol{\rho}, -0, t) \exp(\rho z). \tag{5.4}$$

The total potential Γ is the sum of $\Gamma^{(s)}$ and $\Gamma^{(i)}$. Thus we can substitute $G^{(i)} = G - G^{(s)}$ in equation (5.4), and then use equation (5.2) and the boundary condition (equation (4.12)), to obtain

$$G(\boldsymbol{\rho}, z, t) = G(\boldsymbol{\rho}, +0, t) \exp(\rho z) - 2G^{(s)}(\boldsymbol{\rho}, -0, t) \sinh \rho z. \tag{5.5}$$

Hence, substituting equation (5.3) in equation (5.5) and applying the inversion formula (equation (3.2)) we obtain the solution for Γ in $-h < z < 0$ in the form

$$\Gamma(\mathbf{r}, z, t) = \Gamma^{(s)}(\mathbf{r}, z, t) - \Gamma^{(s)}(\mathbf{r}, -z, t) + \Gamma^{*}(\mathbf{r}, z, t), \tag{5.6}$$

where

$$\Gamma^{*}(\mathbf{r}, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\boldsymbol{\rho}, +0, t) \exp(\rho z - i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{5.7}$$

The first two terms in equation (5.6) represent the known potential of the source plus the potential of its image in the plane $z = 0$. The third term is expressed by equation (5.7) in terms of the boundary value of the solution of G in $z > 0$. Thus, if the magnetic Hertz potential within the conductor is known, then equations (5.6) and (5.7) complete the solution in the region $-h < z < 0$.

Since P_3 also satisfies equation (5.1) subject to a boundary condition equation (4.14) which is identical in form to the condition (equation (4.12)) satisfied by G, the solution for Π_3 is analagous to equation (5.6), namely

$$\Pi_3(\mathbf{r}, z, t) = \Pi_3{}^{(s)}(\mathbf{r}, z, t) - \Pi_3{}^{(s)}(\mathbf{r}, -z, t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} P_3(\mathbf{r}, +0, t) \exp(\rho z - i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{5.8}$$

6. Solution of the electric Hertz potential

In $z > 0$, Π satisfies the diffusion equation (2.6) which, when Fourier transformed according to equation (3.1), becomes

$$\alpha P_4(\boldsymbol{\rho}, z, t) = P_{33}(\boldsymbol{\rho}, z, t) - \rho^2 P(\boldsymbol{\rho}, z, t). \tag{6.1}$$

We now apply the sine transform (equation (3.6)) to this equation and make use of the boundary condition (equation (4.15)) for integrating the term involving P_{33} by parts. The resulting equation is

$$\alpha \hat{p}_4(\boldsymbol{\rho}, \zeta, t) = -(\rho^2 + \zeta^2)\, \hat{p}(\boldsymbol{\rho}, \zeta, t),$$

which has the immediate solution

$$\hat{p}(\boldsymbol{\rho}, \zeta, t) = \hat{p}(\boldsymbol{\rho}, \zeta, 0) \exp(-\beta t), \tag{6.2}$$

where

$$\beta = (\rho^2 + \zeta^2)/\alpha. \tag{6.3}$$

The inverse sine transform of equation (6.2) now gives

$$P(\boldsymbol{\rho}, z, t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_{0}^{\infty} \hat{p}(\boldsymbol{\rho}, \zeta, 0) \exp(-\beta t) \sin \zeta z\, d\zeta, \tag{6.4}$$

which is, in turn, inverted by equation (3.2) to yield

$$\Pi(\boldsymbol{\rho}, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\boldsymbol{\rho}, z, t) \exp(-i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho} \tag{6.5}$$

as the solution of the electric Hertz potential in $z > 0$. If equation (6.4) is differentiated with respect to z, we also obtain when $z \to +0$,

$$P_3(\boldsymbol{\rho}, +0, t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \bar{p}(\boldsymbol{\rho}, \zeta, 0) \exp(-\beta t)\, \zeta d\zeta.$$

This result may be substituted into equation (5.8) to complete the solution outside the conductor.

In any given problem the initial distribution of the potential field within the conductor must be specified in order that $\bar{p}(\boldsymbol{\rho}, \zeta, 0)$ may be determined. We note, however, that the field within the conductor as given by equation (6.4) is, in any case, completely unaffected by the source field. In fact, since β is real and positive, the field freely decays in time as does also that part of the field outside the conductor which is given by the third term in equation (5.8). If the external source has been established for a sufficiently long time that such fields may be assumed to have decayed away (or if we consider the prescribed initial potential to be zero everywhere inside the conductor), then we may take $\Pi = 0$ in $z > 0$. In $-h < z < 0$, however, it is clear from equation (5.8) that there still remains a non-vanishing part of the electric Hertz potential which is just the combined field of the source and its image, given by

$$\Pi_3(\mathbf{r}, z, t) = \Pi_3^{(s)}(\mathbf{r}, z, t) - \Pi_3(\mathbf{r}, -z, t). \tag{6.6}$$

The potential (equation (6.6)) contributes only to the electric field outside the conductor, as shown in Section 2, and is merely required to accommodate the electric field of the source and of the electric charge induced on the surface of the conductor.

The behaviour of these solutions is in complete accord with the properties ascribed by Price to his 'solutions of the second type'.

7. Solution of the magnetic Hertz potential

In $z > 0$ the Fourier transform of the magnetic Hertz potential satisfies the same differential equation (6.1) as the electric Hertz potential. This time, however, we apply the transform defined in equations (3.3) and (3.4), with $k = \kappa$. The term involving G_{33} can be integrated by parts as follows:

$$\sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} G_{33}(\boldsymbol{\rho}, z, t)\, K(\zeta, z)\, dz$$

$$= \frac{\zeta[\kappa\rho G(\boldsymbol{\rho}, +0, t) - G_3(\boldsymbol{\rho}, +0, t)]}{[\frac{1}{2}\pi(\kappa^2 \rho^2 + \zeta^2)]^{\frac{1}{2}}} - \zeta^2 g(\boldsymbol{\rho}, \zeta, t). \tag{7.1}$$

The first term on the right-hand side of equation (7.1) can be related to the source field by differentiating equation (5.5) with respect to z, letting $z \to -0$, and substituting the boundary condition (equation (4.13)). We obtain

$$\kappa\rho G(\boldsymbol{\rho}, +0, t) - G_3(\boldsymbol{\rho}, +0, t) = 2\kappa\rho G^{(s)}(\boldsymbol{\rho}, -0, t).$$

Hence the equation satisfied by g becomes

$$g_4(\boldsymbol{\rho}, \zeta, t) = \sqrt{\left(\frac{8}{\pi}\right)} \frac{\kappa\rho\zeta G^{(s)}(\boldsymbol{\rho}, -0, t)}{\alpha(\kappa^2 \rho^2 + \zeta^2)^{\frac{1}{2}}} - \beta g(\boldsymbol{\rho}, \zeta, t), \tag{7.2}$$

where β is given by equation (6.3). The solution of equation (7.2) is readily found to be

$$g(\boldsymbol{\rho}, \zeta, t) = g(\boldsymbol{\rho}, \zeta, 0) \exp(-\beta t)$$

$$+ \sqrt{\left(\frac{8}{\pi}\right)} \int_0^t \frac{\kappa\rho\zeta G^{(s)}(\boldsymbol{\rho}, -0, t-u)}{\alpha(\kappa^2 \rho^2 + \zeta^2)^{\frac{1}{2}}} \exp(-\beta u)\, du. \quad (7.3)$$

The first term in this solution is independent of the source and again describes the free decay of an electromagnetic field initially existing inside the conductor. It is similar in form to the solution of the electric Hertz potential given by equation (6.2), and although the inversion formula (equation (3.5)) will this time include terms involving both $\sin\zeta z$ and $\cos\zeta z$ the final form of the solution is otherwise the same as before. The corresponding magnetic Hertz potential outside the conductor is given by Γ^* defined in equation (5.7) since this is the remaining term in equation (5.6) when $\Gamma^{(s)}$ is set equal to zero.

Physically, the solutions differ from those for the electric Hertz potential in that they produce a magnetic field in the non-conducting region, but do not contribute to the component of the electric field normal to the surface of the conductor, as can be seen from equation (2.10). Thus that part of Γ which is independent of the source represents the free decay of currents flowing parallel to the bounding surface of the conductor and corresponds to Price's 'free modes of decay of the first type'.

Solutions which describe a free decay of the field can usually be disregarded in induction problems where we are concerned with the field induced in the conductor by an external magnetic source. Henceforth, therefore, we shall suppose that $p(\boldsymbol{\rho}, \zeta, 0) = g(\boldsymbol{\rho}, \zeta, 0) = 0$ so that the complete theory of induction in a conducting half-space is contained in the second term of the solution (equation (7.3)) (together with equation (6.6) if the electric Hertz potential outside the conductor is required). It should be pointed out, however, that inherent in the neglect of the initial field is the assumption that the external source is not established until $t = 0$. For if a time-dependent source were to exist before this instant, it would have already induced a current system in the conductor thereby contradicting the supposition that the initial field is zero.

Taking $g(\boldsymbol{\rho}, \zeta, 0) = 0$, and applying the inverse transform (equation (3.5)) to the remaining part of equation (7.3) we obtain

$$G(\boldsymbol{\rho}, z, t) = \frac{4\kappa}{\pi\alpha} \int_0^\infty \int_0^t \frac{\rho\zeta \exp(-\beta u)(\kappa\rho \sin\zeta z + \zeta \cos\zeta z)}{\kappa^2 \rho^2 + \zeta^2} \times G^{(s)}(\boldsymbol{\rho}, -0, t-u)\, du\, d\zeta. \quad (7.4)$$

Equation (7.4) completes the theory of electromagnetic induction in a conducting half-space by an external magnetic source. For we can invert it by equation (3.2) to give

$$\Gamma(\mathbf{r}, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\boldsymbol{\rho}, z, t) \exp(-i\mathbf{r}\cdot\boldsymbol{\rho})\, d\boldsymbol{\rho}, \quad (7.5)$$

which totally specifies the electromagnetic field in $z > 0$, and we can let $z \to +0$ in equation (7.4) and substitute it in equation (5.7) to obtain the magnetic Hertz potential in $-h < z < 0$ as given by equation (5.6).

The general solutions may be simplified still further when the source possesses certain specified time variations. Two simple cases merit special treatment: (i) an aperiodic source which is suddenly created at the instant $t = 0$ but is static thereafter, and (ii) a periodic source.

8. Aperiodic source

Suppose that $\Gamma^{(s)}$ is stationary for $t > 0$, but vanishes for $t < 0$ (which assumption is already inherent in the general solution equation (7.4)). Then we may write

$$\Gamma^{(s)}(\mathbf{r}, z, t) = \Gamma^{(s)}(\mathbf{r}, z, +0)H(t), \tag{8.1}$$

where $H(t)$ is the Heaviside function, and hence for $t > 0$

$$G^{(s)}(\boldsymbol{\rho}, -0, t) = G^{(s)}(\boldsymbol{\rho}, -0, +0). \tag{8.2}$$

It follows that

$$\int_0^t G^{(s)}(\boldsymbol{\rho}, -0, t-u) \exp(-\beta u)\, du = \frac{1-\exp(-\beta t)}{\beta} G^{(s)}(\boldsymbol{\rho}, -0, +0),$$

which, when substituted into equation (7.4), gives

$$G(\boldsymbol{\rho}, z, t) = \frac{4\kappa\rho}{\pi} G^{(s)}(\boldsymbol{\rho}, -0, +0) \int_0^\infty \frac{\kappa\rho\zeta \sin\zeta z + \zeta^2 \cos\zeta z}{(\rho^2+\zeta^2)(\kappa^2\rho^2+\zeta^2)} [1-\exp(-\beta t)]\, d\zeta. \tag{8.3}$$

By expressing the integrand in partial fractions and using standard Fourier sine and cosine transforms (I.T. Section 1.2 (11), Section 1.4 (15), Section 2.2 (15), Section 2.4 (26)), it is a straightforward exercise to show that

$$G(\boldsymbol{\rho}, z, t) = \frac{\kappa \exp(-\rho z)}{\kappa+1} [2-\chi(\rho\lambda, z/\lambda)]\, G^{(s)}(\boldsymbol{\rho}, -0, +0), \tag{8.4}$$

where $\lambda = \sqrt{(t/\alpha)}$ and

$$\chi(v, w) = \operatorname{erfc}(v-\tfrac{1}{2}w) - \frac{\exp(2vw)}{\kappa-1} \{(\kappa+1) \operatorname{erfc}(v+\tfrac{1}{2}w)$$
$$-2\kappa \operatorname{erfc}(\kappa v+\tfrac{1}{2}w) \exp[v(\kappa-1)(\kappa v+v+w)]\}. \tag{8.5}$$

By equation (5.2) it is clear that

$$\exp(-\rho z)\, G^{(s)}(\boldsymbol{\rho}, -0, +0) = G^{(s)}(\boldsymbol{\rho}, z, +0) \tag{8.6}$$

so that when equation (8.4) is substituted in equation (7.5) we may write the solution of the magnetic Hertz potential (for $z > 0$, $t > 0$) in the form

$$\Gamma(\mathbf{r}, z, t) = \frac{2\kappa}{\kappa+1} \Gamma^{(s)}(\mathbf{r}, z, +0)$$
$$- \frac{\kappa}{2\pi(\kappa+1)} \int_{-\infty}^{\infty} G^{(s)}(\boldsymbol{\rho}, z, +0)\, \chi(\rho\lambda, z/\lambda) \exp(-i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{8.7}$$

The limiting value of the field as $t \to \infty$ is given by the first term of equation (8.7), since, by equation (8.5), $\chi(v, w) \to 0$ as $v \to \infty$. It represents, of course, the static field within a permeable half-space as given by magnetic image theory.

To find the field outside the conductor we first let $z \to +0$ in equation (8.4), which yields

$$G(\boldsymbol{\rho}, +0, t) = \frac{\kappa}{\kappa+1} [2-\chi(\rho\lambda, 0)]\, G^{(s)}(\boldsymbol{\rho}, -0, +0), \tag{8.8}$$

where, by equation (8.5),

$$\chi(v, 0) = \frac{2}{\kappa-1}\{\kappa \exp[v^2(\kappa^2-1)] \operatorname{erfc} \kappa v - \operatorname{erfc} v\}. \tag{8.9}$$

If we now replace z by $-z$ in equation (8.6) and substitute in equation (8.8) we have

$$\exp(\rho z)\, G(\boldsymbol{\rho}, +0, t) = \frac{\kappa}{\kappa+1}\{2-\chi(\rho\lambda, 0)\}\, G^{(s)}(\boldsymbol{\rho}, -z, +0).$$

This can be substituted into equation (5.7) and the first term inverted immediately to give (for $-h < z < 0$, $t > 0$)

$$\Gamma^*(\mathbf{r}, z, t) = \frac{2\kappa}{\kappa+1}\Gamma^{(s)}(\mathbf{r}, -z, +0) - \frac{\kappa}{2\pi(\kappa+1)}\int_{-\infty}^{\infty} G^{(s)}(\boldsymbol{\rho}, -z, +0)\,\chi(\rho\lambda, 0)\exp(-i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho}. \tag{8.10}$$

The leading term of equation (8.10) represents its limiting value as $t \to \infty$. Thus when Γ^* is substituted in equation (5.6) we find that

$$\Gamma(\mathbf{r}, z, \infty) = \Gamma^{(s)}(\mathbf{r}, z, +0) + \frac{\kappa-1}{\kappa+1}\Gamma^{(s)}(\mathbf{r}, -z, +0),$$

which is again in accordance with magnetic image theory for the static field outside a permeable half-space.

9. Periodic source

We assume that the source varies harmonically in time with an angular frequency ω. Thus we write

$$\Gamma^{(s)}(\mathbf{r}, z, t) = \Gamma^{(s)}(\mathbf{r}, z, +0)\exp(i\omega t), \tag{9.1}$$

whence

$$G^{(s)}(\boldsymbol{\rho}, -0, t) = G^{(s)}(\boldsymbol{\rho}, -0, +0)\exp(i\omega t) \tag{9.2}$$

and

$$\int_0^t G^{(s)}(\boldsymbol{\rho}, -0, t-u)\exp(-\beta u)\, du = \frac{\exp(i\omega t)-\exp(-\beta t)}{\beta+i\omega} G^{(s)}(\boldsymbol{\rho}, -0, +0). \tag{9.3}$$

The term involving $\exp(-\beta t)$ represents a transient field which is a consequence of the assumption that the source is not initiated until $t = 0$. The steady-state periodic field containing the factor $\exp(i\omega t)$ becomes dominant after the source has been established for some time. We shall, therefore, retain only the first term of equation (9.3) in deriving the periodic solution.

By equation (7.4), we have for $z > 0$

$$G(\boldsymbol{\rho}, z, t) = \frac{4\kappa\rho \exp(i\omega t)}{\pi} G^{(s)}(\boldsymbol{\rho}, -0, +0)\int_0^{\infty} \frac{\kappa\rho\zeta \sin \zeta z + \zeta^2 \cos \zeta z}{(\rho^2+\zeta^2+i\omega\alpha)(\kappa^2\rho^2+\zeta^2)}\, d\zeta,$$

which can be evaluated immediately by comparison with the corresponding result for equation (8.3), i.e.

$$G(\boldsymbol{\rho}, z, t) = \frac{2\kappa\rho \exp[i\omega t - z\nu(\rho)]}{\kappa\rho + \nu(\rho)} G^{(s)}(\boldsymbol{\rho}, -0, +0), \tag{9.4}$$

where $$\nu(\rho) = \sqrt{(\rho^2 + i\omega\alpha)}.$$

Hence, following the same procedure used in Section 8 we find that the general solution of a periodic magnetic Hertz potential in $z > 0$ is given by

$$\Gamma(\mathbf{r}, z, t) = \frac{\kappa \exp(i\omega t)}{\pi} \int_{-\infty}^{\infty} \frac{\rho G^{(s)}(\boldsymbol{\rho}, z, +0)}{\kappa\rho + \nu(\rho)} \exp\{z[\rho - \nu(\rho)] - i\mathbf{r}.\boldsymbol{\rho}\}\, d\boldsymbol{\rho}, \tag{9.5}$$

while in the region $-h < z < 0$ we obtain

$$\Gamma^*(\mathbf{r}, z, t) = \frac{\kappa \exp(i\omega t)}{\pi} \int_{-\infty}^{\infty} \frac{\rho G^{(s)}(\boldsymbol{\rho}, -z, +0)}{\kappa\rho + \nu(\rho)} \exp(-i\mathbf{r}.\boldsymbol{\rho})\, d\boldsymbol{\rho} \tag{9.6}$$

for substitution in equation (5.6). Note that when $\omega \to 0$, the integrals in equations (9.5) and (9.6) reduce to $2\kappa(\kappa+1)^{-1}\,\Gamma^{(s)}(\boldsymbol{\rho}, \pm z, 0)$, respectively. Thus the correct magnetostatic limit, given by the first terms in equations (8.7) and (8.10), is again obtained.

10. Solutions for E and B

Since (for an induction problem) the electromagnetic field inside the conductor is completely specified by the magnetic Hertz potential defined by equations (7.4) and (7.5), we find that by equations (2.10), (2.11) and (2.6) the field in $z > 0$ is

$$\mathbf{E}_{||} = -\operatorname{curl}(\mathbf{k}\partial\Gamma/\partial t), \qquad E_z = 0, \tag{10.1}$$

$$\mathbf{B} = \operatorname{grad}(\partial\Gamma/\partial z) - \alpha\mathbf{k}\,\partial\Gamma/\partial t, \tag{10.2}$$

where $\mathbf{E}_{||} = \mathbf{E} - E_z\mathbf{k}$ is the component of the electric field parallel to the surface of the conductor.

Outside the conductor both Γ in equation (5.6) and Π_3 in equation (6.6) are, in general, non-vanishing. In this region the source and image potentials each satisfy equations (2.13) and (2.14). Thus we may use these formulas to express the electric and magnetic fields of the source and of its image in terms of the corresponding Hertz potentials, noting only that a differentiation of the image potentials with respect to z introduces a change in sign. Denoting the electric and magnetic vectors of the source by $\mathbf{E}^{(s)}$ and $\mathbf{B}^{(s)}$ respectively and of the image by $\mathbf{E}'$ and $\mathbf{B}'$ respectively, we can show by equations (2.13), (2.14), (5.6), (5.7) and (6.6) that

$$\left.\begin{aligned} \mathbf{E}_{||} &= \mathbf{E}_{||}{}^{(s)} - \mathbf{E}_{||}{}' - \operatorname{curl}(\mathbf{k}\,\partial\Gamma^*/\partial t), \\ E_z &= E_z{}^{(s)} + E_z{}', \end{aligned}\right\} \tag{10.3}$$

$$\mathbf{B} = \mathbf{B}^{(s)} + \mathbf{B}' + \operatorname{grad}(\partial\Gamma^*/\partial z) - 2B_z{}'\,\mathbf{k} \tag{10.4}$$

in the region $-h < z < 0$.

The above formulas provide a quick means of calculating the electric and magnetic fields everywhere once the magnetic Hertz potential has been found for any given source. We note that no reference to the electric Hertz potential is required for computing **E** and **B**.

We shall now illustrate the general theory with some examples involving specific magnetic sources.

11. Aperiodic dipole normal to the surface

Suppose that at $t = 0$ a steady current $4\pi M/\mu_0\ \Delta A$ starts to flow around a small, closed loop of area ΔA, which lies in the plane $z = -h$ and is centred on the point $(0, 0, -h)$. The current constitutes an aperiodic magnetic dipole whose moment is $4\pi M\mathbf{k}H(t)$. The magnetic induction of this source is

$$\mathbf{B}^{(s)} = -MH(t)\,\mathrm{grad}\,\{(h+z)/R^3\}, \tag{11.1}$$

where

$$R = \{r^2+(h+z)^2\}^{\frac{1}{2}} \tag{11.2}$$

is the distance from the dipole to the field point. Since $H'(t) = \delta(t)$, the Dirac generalized function, it follows by equation (2.1) that the time-dependent electric field of the source will consist of an impulse at $t = 0$, but will vanish thereafter. Hence for $t > 0$ we may take $\mathbf{E}^{(s)} = 0$, and

$$\Gamma^{(s)}(\mathbf{r}, z, t) = \Gamma^{(s)}(\mathbf{r}, z, +0) = M/R, \tag{11.3}$$

the last result following from equations (11.1) and (2.14).

Since the source is dependent only on r and z, it is convenient to introduce cylindrical co-ordinates (r, θ, z) and (ρ, ψ, z) where

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \xi = \rho\cos\psi, \qquad \eta = \rho\sin\psi. \tag{11.4}$$

The double Fourier transform (equation (3.1)) of equation (11.3) can then be put in the form

$$G^{(s)}(\boldsymbol{\rho}, z, t) = \frac{M}{2\pi}\int_0^\infty\int_{-\pi}^{\pi}\frac{r}{R}\exp\left[i\rho r\cos(\theta-\psi)\right]d\theta\,dr. \tag{11.5}$$

Using the Bessel integral of order zero

$$2\pi J_0(u) = \int_{-\pi}^{\pi}\exp\left[iu\cos(\theta-\psi)\right]d\theta \tag{11.6}$$

we can simplify equation (11.5) to

$$G^{(s)}(\boldsymbol{\rho}, z, +0) = M\int_0^\infty\frac{r}{R}J_0(\rho r)\,dr = \frac{M\exp\left[-\rho(h+z)\right]}{\rho}, \tag{11.7}$$

the last result following by a standard Hankel transform (I.T., Section 8.2 (4)). The solution to the problem is now found by substituting equations (11.3) and (11.7) into equations (8.7) and (8.10).

We consider first the field inside the conductor. Introducing the dimensionless parameters $\bar{r} = r/\lambda$, $\bar{z} = z/\lambda$, $\bar{h} = h/\lambda$, and transforming the integral in equation (8.7) into cylindrical co-ordinates with the aid of Bessel's integral (equation (11.6)), we find that the solution of Γ in $z > 0$ is

$$\Gamma(\mathbf{r}, z, t) = \frac{\kappa M}{\kappa+1}\left\{\frac{2}{R} - \sqrt{\left(\frac{\alpha}{t}\right)}\int_0^\infty\exp\left[-v(\bar{h}+\bar{z})\right]J_0(\bar{r}v)\,\chi(v,\bar{z})\,dv\right\}. \tag{11.8}$$

An asymptotic expansion of equation (11.8) valid for large $\bar{h}$, i.e. for $t \ll \mu\sigma h^2$, can be found by expanding $\chi(v, \bar{z})$ in a Taylor series about $v = 0$ and integrating each term separately with the aid of the well-known result

$$\int_0^\infty\exp\left[-v(\bar{h}+\bar{z})\right]J_0(\bar{r}v)\,v^m\,dv = \frac{m!}{(\bar{h}+\bar{z})^{m+1}}\,{}_2F_1\left(\tfrac{1}{2}m+\tfrac{1}{2}, \tfrac{1}{2}m+1;\ 1;\ -\frac{\bar{r}^2}{(\bar{h}+\bar{z})^2}\right),$$

where ${}_2F_1$ denotes the hypergeometric function in the usual notation. It follows that

$$\Gamma(\mathbf{r}, z, t) \sim \frac{\kappa M}{\kappa+1}\left\{\frac{2}{R} - \frac{\chi(0,\bar{z})}{R} - \frac{(h+z)\chi_1(0,\bar{z})}{R^3}\sqrt{\left(\frac{t}{\alpha}\right)} + \dots\right\}. \quad (11.9)$$

Now by equation (8.5), we have $\chi(0, \bar{z}) = 2$ and

$$\chi_1(0,\bar{z}) = 2(\kappa+1)\{\bar{z}\,\text{erfc}\,\tfrac{1}{2}\bar{z} - (2/\sqrt{\pi})\exp(-\bar{z}^2/4)\}, \quad (11.10)$$

so that substitution in equation (11.9) gives the first term asymptotic representation of equation (11.8) in the form

$$\Gamma(\mathbf{r}, z, t) \sim \frac{2\kappa M(h+z)}{R^3}\left\{\sqrt{\left(\frac{4t}{\pi\alpha}\right)}\exp(-\alpha z^2/4t) - z\,\text{erfc}\,[\tfrac{1}{2}z\sqrt{(\alpha/t)}]\right\}. \quad (11.11)$$

To obtain the field outside the conductor we apply the same transformation used in deriving equation (11.8) above to the integral appearing in equation (8.10). The solution of Γ in $-h < z < 0$ is then given by equation (5.6) with

$$\Gamma^*(\mathbf{r}, z, t) = \frac{2\kappa M}{(\kappa+1)R'} - \frac{\kappa M}{\kappa+1}\sqrt{\left(\frac{\alpha}{t}\right)}\int_0^\infty \chi(v, 0)\exp[-v(\bar{h}-\bar{z})]\,J_0(\bar{r}v)\,dv, \quad (11.12)$$

where $\chi(v, 0)$ is given by equation (8.9) and where

$$R' = \{r^2 + (h-z)^2\}^{\frac{1}{2}}$$

is the distance from the image dipole to the field point.

For small values of t, i.e. $t \ll \mu\sigma h^2$, an asymptotic representation of equation (11.12) can be found by the same procedure used to derive equation (11.11). It is obvious from equations (8.9) and (11.10) that $\chi(0, 0) = 2$ and

$$\chi_1(0, 0) = -4(\kappa+1)/\sqrt{\pi}.$$

Hence it follows that

$$\Gamma^*(\mathbf{r}, z, t) \sim \frac{4\kappa M(h-z)}{R'^3}\sqrt{\frac{t}{\pi\alpha}}. \quad (11.13)$$

This completes the solution in $-h < z < 0$.

Because of the symmetry about the z-axis, it is convenient to express the electromagnetic field in cylindrical components $\mathbf{E} = (0, E_\theta, 0)$, $\mathbf{B} = (B_r, 0, B_z)$. Hence for this example, the appropriate formulas for computing the electromagnetic field in $z > 0$ are, by equations (10.1) and (10.2),

$$E_\theta = \partial^2\Gamma/\partial r\,\partial t, \qquad B_r = \partial^2\Gamma/\partial z\,\partial r, \qquad B_z = -\partial(r\partial\Gamma/\partial r)/r\partial r, \quad (11.14)$$

while in the region $-h < z < 0$ they are

$$E_\theta = \partial^2\Gamma^*/\partial r\,\partial t,$$

$$B_r = 3Mr\left(\frac{h+z}{R^5} + \frac{h-z}{R'^5}\right) - \frac{\partial^2\Gamma^*}{\partial z\,\partial r},$$

$$B_z = M\left\{\frac{2(h+z)^2 - r^2}{R^5} - \frac{2(h-z)^2 - r^2}{R'^5}\right\} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Gamma^*}{\partial r}\right),$$

by equations (10.3) and (10.4)

A peculiar feature of the solutions requires explanation. By equations (11.11) and (11.14) it is clear that as $t \to 0$, $B_r \to B_z \to 0$ everywhere inside the conductor and $E_\theta \to 0$ also, provided $z \neq +0$. At the surface $z = +0$ itself however, we have

$$E_\theta \sim -\frac{6\kappa Mhr}{(r^2+h^2)^{\frac{5}{2}}\sqrt{(\pi\alpha t)}}, \qquad (11.15)$$

which becomes infinite as $t \to 0$. This implies that the impulsive electric field associated with the sudden creation of the dipole source causes an instantaneous *surface* current to flow at time $t = 0$, the effect of which is to completely screen the inside of the conductor from the external electromagnetic field. In fact, since for $t = 0$, $B_r = 0$ at $z = +0$, and by equation (11.13),

$$B_r = 6Mhr(r^2+h^2)^{-\frac{5}{2}}$$

at $z = -0$, the well-known boundary condition gives the *surface* current density as

$$\hat{J}_\theta = -\frac{6Mhr}{\mu_0(r^2+h^2)^{\frac{5}{2}}}. \qquad (11.16)$$

On the other hand, if we denote the component of the *volume* current density by J_θ, then for small t we have

$$J_\theta = \sigma E_\theta \sim \hat{J}_\theta/\lambda\sqrt{\pi}$$

by equations (11.15) and (11.16). This shows that the surface current can be regarded, as the limiting value as $t \to 0$ of the volume current flowing in a layer of thickness $\lambda\sqrt{\pi}$. In other words, while the asymptotic formulas equations (11.11) and (11.13) are valid for $t \ll \mu\sigma h^2$, caution must be exercised in interpreting the solutions when t is so near zero that $\lambda\sqrt{\pi}$ is no longer of macroscopic magnitude.

The reason for the appearance of an infinite electric field and its associated surface current is, of course, our assumption that changes in the field are sufficiently slow that displacement currents may be neglected. Clearly an aperiodic source of the type considered here does not satisfy this requirement near $t = 0$, although the subsequent diffusion of the field does. The physical explanation of this phenomenon has been discussed in greater detail by Gordon (1951).

12. Periodic dipole parallel to the surface

In this example we suppose that the magnetic moment of the dipole is directed parallel to the x-axis and that the current flowing round the loop has an harmonic time variation of angular frequency ω. In the same notation as in Section 8 the quasi-static field of this source is

$$\mathbf{B}^{(s)} = -M \exp(i\omega t) \operatorname{grad}(x/R^3)$$

$$\mathbf{E}^{(s)} = i\omega M \exp(i\omega t)\{(h+z)\,\mathbf{j} - y\mathbf{k}\}/R^3.$$

It follows from equation (2.14) that the appropriate magnetic Hertz potential for this field is

$$\Gamma^{(s)}(\mathbf{r}, z, t) = \frac{Mx \exp(i\omega t)}{R(R+h+z)}. \qquad (12.1)$$

The Fourier transform of equation (12.1) can be expressed in cylindrical coordinates by equation (11.4) and simplified with the aid of the formula

$$2\pi i J_1(u) \cos\psi = \int_{-\pi}^{\pi} \cos\theta \exp[iu\cos(\theta-\psi)]\, d\theta, \qquad (12.2)$$

which follows at once from Bessel's integral of order one. We obtain

$$G^{(s)}(\boldsymbol{\rho}, z, t) = iM \exp(i\omega t) \cos\psi \int_0^\infty \frac{J_1(\rho r)\, r^2\, dr}{R(R+h+z)}. \tag{12.3}$$

Noting that $r^2/(R+h+z) = R-h-z$, and using two Hankel transforms (I.T. Section 8.4 (4) and Section 8.5 (3)), we deduce from equation (12.3) that

$$G^{(s)}(\boldsymbol{\rho}, z, +0) = iM\xi\rho^{-2} \exp[-\rho(h+z)]. \tag{12.4}$$

Substituting equation (12.4) in equations (9.5) and (9.6), and using equation (12.2) again, we find that the magnetic Hertz potential in cylindrical co-ordinates is given by

$$\Gamma(\mathbf{r}, z, t) = 2\kappa M \exp(i\omega t) \cos\theta \int_0^\infty \frac{\exp[-\rho h - z\nu(\rho)]}{\kappa\rho + \nu(\rho)} J_1(r\rho)\, \rho\, d\rho \tag{12.5}$$

for the region $z > 0$, and by equation (5.6) for the region $-h < z < 0$, with

$$\Gamma^*(\mathbf{r}, z, t) = 2\kappa M \exp(i\omega t) \cos\theta \int_0^\infty \frac{\exp[-\rho(h-z)]}{\kappa\rho + \nu(\rho)} J_1(r\rho)\, \rho\, d\rho. \tag{12.6}$$

The electromagnetic field components are readily calculated by substituting equation (12.5) into equations (10.1) and (10.2), and equation (12.6) into equations (10.3) and (10.4).

13. Periodic line current parallel to the surface

Finally we consider the inducing field produced by a periodic linear current $(2\pi I/\mu_0) \exp(i\omega t)$ flowing in the positive y-direction along the line $x = 0$, $z = -h$. This example shows how a strictly two-dimensional source can be incorporated in the general theory.

The quasi-static magnetic field of the line current is well known to be

$$\mathbf{B}^{(s)}(\mathbf{r}, z, t) = I \exp(i\omega t) \operatorname{grad}\left(\arctan\frac{x}{h+z}\right), \tag{13.1}$$

whence by equation (2.1)

$$\mathbf{E}^{(s)}(\mathbf{r}, z, t) - \mathbf{E}^{(s)}(0, 0, t) = i\omega I \exp(i\omega t) \log(S/h)\, \mathbf{j}, \tag{13.2}$$

where $S = \{x^2 + (h+z)^2\}^{\frac{1}{2}}$. By equation (2.14) we immediately deduce that

$$\Gamma_3^{(s)}(\mathbf{r}, z, t) = I \exp(i\omega t) \arctan\frac{x}{h+z},$$

which has as its Fourier transform the generalized function

$$G_3^{(s)}(\boldsymbol{\rho}, z, t) = i\pi I \delta(\eta)\, \xi^{-1} \exp[i\omega t - (h+z)\, |\xi|].$$

It is the two dimensional nature of the problem (independence of y) which gives rise to the Dirac delta function $\delta(\eta)$. By integration, we now have

$$G^{(s)}(\boldsymbol{\rho}, z, +0) = -i\pi I \delta(\eta)\, \xi^{-1}\, |\xi|^{-1} \exp[-(h+z)\, |\xi|]. \tag{13.3}$$

Substituting equation (13.3) into equation (9.5) and simplifying, we obtain

$$\Gamma(\mathbf{r}, z, t) = -2\kappa I \exp(i\omega t) \int_0^\infty \frac{\exp[-h\xi - z\nu(\xi)]}{\kappa\xi + \nu(\xi)} \sin\xi x \frac{d\xi}{\xi} \tag{13.4}$$

for the magnetic Hertz potential in $z > 0$. Similarly, substitution of equation (13.3) in equation (9.6) leads to

$$\Gamma^*(\mathbf{r}, z, t) = -2\kappa I \exp(i\omega t) \int_0^\infty \frac{\exp[-\xi(h-z)]}{\kappa\xi + \nu(\xi)} \sin \xi x \frac{d\xi}{\xi}, \tag{13.5}$$

which gives the field in $-h < z < 0$.

Hence, writing $\Xi(\xi) = \exp[-\xi(h-z)]/\{\kappa\xi + \nu(\xi)\}$, and substituting equations (13.1), (13.2), and (13.5) in equations (10.3) and (10.4) we find, for example, that the electric and magnetic fields outside the conductor are

$$E_y = i\omega I \exp(i\omega t) \left\{\log \frac{S}{S'} - 2\kappa \int_0^\infty \Xi(\xi) \cos \xi x \, d\xi\right\},$$

$$B_x = I \exp(i\omega t) \left\{\frac{h+z}{S^2} + \frac{h-z}{S'^2} - 2\kappa \int_0^\infty \xi \Xi(\xi) \cos \xi x \, d\xi\right\},$$

$$B_z = I \exp(i\omega t) \left\{\frac{h+z}{S^2} - \frac{h-z}{S'^2} - 2\kappa \int_0^\infty \xi \Xi(\xi) \sin \xi x \, d\xi\right\},$$

where $S' = \{x^2 + (h-z)^2\}^{\frac{1}{2}}$.

Acknowledgments

This work was supported by a National Research Council of Canada Grant No. A3830.

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