The general theory of superoscillations and supershifts in several variables

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Abstract

In this paper we describe a general method to generate superoscillatory functions of several variables starting from a superoscillating sequence of one variable. Our results are based on the study of suitable infinite order differential operators on holomorphic functions with growth conditions of exponential type, where additional constraints are required when dealing with infinite order differential operators whose symbol is a function that is holomorphic in some open set, but not necessarily entire. The results proved for the superoscillating sequence in several variables are extended to sequences of supershifts in several variables.

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1 Introduction

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. Physical phenomena associated with superoscillatory functions have been known for a long time for example in antennas theory see [31], and in the context of weak values in quantum mechanics, see [1]. In more recent years there has been a wide interest in the theory of superoscillating functions and of supershifts, a notion that generalizes the one of superoscillations, and that was introduced in the literature in order to study the evolution of superoscillations as initial data of the Schrödinger equation of other field equations, like Dirac or Klein-Gordon equations.

An introduction to superoscillatory functions in one variable and some applications to Schrödinger evolution of superoscillatory initial data can be found in [7]. Superoscillatory functions in several variables have been rigorously defined and studied in [6] and in [9] where we have initiated also the theory of supershifts in more then one variable. The aim of this paper is to remove the restrictions in [6, 9] and to obtain a very general theory of superoscillations and supershifts.

Our results are directed to a general audience of physicists, mathematicians, and engineers, and our main tool is the theory of infinite order differential operators acting on spaces of holomorphic functions. The literature on superoscillations is quite large, and without claiming

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completeness we have tried to mention some of the most relevant (and recent) results. Papers [2]-[7], [12], [15], [25], [28] and [29] deal with the issue of permanence of superoscillatory behavior when evolved under a suitable Schrödinger equation; papers [18]-[20], [26]-[27] and [30] are mostly concerned with the physical nature of superoscillations, while papers [10], [11], [13]-[14], [21]-[24] develop in depth the mathematical theory of superoscillations. Finally we have cited [7] as a good reference for the state of the art on the mathematics of superoscillations until 2017, and the *Roadmap on Superoscillations* [17], where the most recent advances in superoscillations and their applications to technology are well explained by the leading experts in this field.

In this paper we extend the results in [9] considering analytic functions in one variable $G_1, \ldots, G_d, d \ge 2$, whose Taylor series at zero have radius of convergence grater than or equal to 1. Thus we define general superoscillating functions of several variables as expressions of the form

$$F_n(x_1, x_2, \dots, x_d) := \sum_{j=0}^n Z_j(n, a) e^{ix_1 G_1(h_j(n))} e^{ix_2 G_2(h_j(n))} \dots e^{ix_d G_d(h_j(n))}$$

where $Z_j(n, a)$, j = 0, ..., n, for $n \in \mathbb{N}_0$ are suitable coefficients of a superoscillating function in one variable as we will see in the sequel. We will give conditions on the functions G_1, \ldots, G_d in order that

$$\lim_{n \to \infty} F_n(x_1, x_2, \dots, x_d) = e^{ix_1 G_1(a)} e^{ix_2 G_2(a)} \dots e^{ix_d G_d(a)}$$

so that, when |a| > 1, $F_n(x_1, x_2, \ldots, x_d)$ is superoscillating. Moreover, we shall also treat the case of sequences that admit a supershift in $d \ge 2$ variables.

The paper is organized in four sections including the introduction. Section 2 contains the preliminary material on superoscillations, the relevant function spaces and their topology, and the study of the continuity of some infinite order differential operators acting on such spaces. Section 3 is the main part of the paper and contains the definition of superoscillating functions in $d \ge 2$ variables as well as some results. Section 4 discusses the notion of supershift in this framework.

2 Preliminary results on infinite order differential operators

We begin this section with some preliminary material on superoscillations and supershifts in one variable. Then we introduce and study some infinite order differential operators that will be of crucial importance to define and study superoscillations and supershifts in several variables.

Definition 2.1. We call generalized Fourier sequence a sequence of the form

$$f_n(x) := \sum_{j=0}^n Z_j(n,a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$
(1)

where $a \in \mathbb{R}$, $Z_j(n, a)$ and $h_j(n)$ are complex and real valued functions of the variables n, a and n, respectively. The sequence (1) is said to be a superoscillating sequence if $\sup_{j,n} |h_j(n)| \leq 1$ and there exists a compact subset of \mathbb{R} , which will be called a superoscillation set, on which $f_n(x)$ converges uniformly to $e^{ig(a)x}$, where g is a continuous real valued function such that |g(a)| > 1.

The classical Fourier expansion is obviously not a superoscillating sequence since its frequencies are not, in general, bounded.

In the recent paper [8] we enlarged the class of superoscillating functions, with respect to the existing literature, and we solved the following problem.

Problem 2.2. Let $h_j(n)$ be a given set of points in [-1,1], j = 0, 1, ..., n, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that |a| > 1. Determine the coefficients $X_j(n)$ of the sequence

$$f_n(x) = \sum_{j=0}^n X_j(n) e^{ih_j(n)x}, \quad x \in \mathbb{R}$$

in such a way that

$$f_n^{(p)}(0) = (ia)^p$$
, for $p = 0, 1, ..., n$

Remark 2.3. The conditions $f_n^{(p)}(0) = (ia)^p$ mean that the functions $x \mapsto e^{iax}$ and $x \mapsto f_n(x)$ have the same derivatives at the origin, for p = 0, 1, ..., n, and therefore the same Taylor polynomial of order n.

Theorem 2.4 (Solution of Problem 2.2). Let $h_j(n)$ be a given set of points in [-1,1], j = 0, 1, ..., n for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that |a| > 1. If $h_j(n) \neq h_i(n)$, for every $i \neq j$, then the coefficients $X_j(n, a)$ are uniquely determined and given by

$$X_j(n,a) = \prod_{k=0, \ k \neq j}^n \Big(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \Big).$$
(2)

As a consequence, the sequence

$$f_n(x) = \sum_{j=0}^n \prod_{k=0, \ k \neq j}^n \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) \ e^{ixh_j(n)}, \quad x \in \mathbb{R}$$

solves Problem 2.2. Moreover, when the holomorphic extensions of the functions f_n converge in A_1 , we have

$$\lim_{n \to \infty} f_n(x) = e^{iax}, \text{ for all } x \in \mathbb{R}.$$

Our approach to the study of superoscillatory functions in one or several variables makes use of infinite order differential operators. Such operators naturally act on spaces of holomorphic functions. This is the reason for which we consider the holomorphic extension to entire functions of the sequence $f_n(x)$ defined in (2.1) by replacing the real variable x with the complex variable ξ . For the sequences of entire functions we shall consider, a natural notion of convergence is the convergence in the space A_1 or in the space $A_{1,B}$ for some real positive constant B (see the following definition and considerations).

Definition 2.5. The space A_1 is the complex algebra of entire functions such that there exists B > 0 such that

$$\sup_{\xi \in \mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right) < +\infty.$$
(3)

The space A_1 has a rather complicated topology, see e.g. [16], since it is a linear space obtained via an inductive limit. For our purposes, it is enough to consider, for any fixed B > 0, the set $A_{1,B}$ of functions f satisfying (3), and to observe that

$$||f||_B := \sup_{\xi \in \mathbb{C}} \left(|f(\xi)| \exp(-B|\xi|) \right)$$

defines a norm on $A_{1,B}$, called the *B*-norm. One can prove that $A_{1,B}$ is a Banach space with respect to this norm.

Moreover, let f and a sequence $(f_n)_n$ belong to A_1 ; f_n converges to f in A_1 if and only if there exists B such that $f, f_n \in A_{1,B}$ and

$$\lim_{n \to \infty} \sup_{\xi \in \mathbb{C}} \left| f_n(\xi) - f(\xi) \right| e^{-B|\xi|} = 0.$$

With these notations and definitions we can make the notion of continuity explicit (see [14]):

A linear operator $\mathcal{U}: A_1 \to A_1$ is continuous if and only if for any B > 0 there exists B' > 0and C > 0 such that

 $\mathcal{U}(A_{1,B}) \subset A_{1,B'} \text{ and } \qquad \|\mathcal{U}(f)\|_{B'} \le C\|f\|_B, \qquad \text{for any } f \in A_{1,B}.$ (4)

The following result, see Lemma 2.6 in [13], gives a characterization of the functions in A_1 in terms of the coefficients appearing in their Taylor series expansion.

Lemma 2.6. The entire function

$$f(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$$

belongs to A_1 if and only if there exists $C_f > 0$ and b > 0 such that

$$|a_j| \le C_f \frac{b^j}{\Gamma(j+1)}.$$

Remark 2.7. To say that $f \in A_1$ means that $f \in A_{1,B}$ for some B > 0. The computations in the proof of Lemma 2.6 in [13], show that b = 2eB, and that we can choose $C_f = ||f||_B$.

We now define two infinite order differential operators that will be used to study superoscillatory functions and supershifts in several variables. We shall denote by \underline{x} the vector (x_1, \ldots, x_d) in \mathbb{R}^d .

Proposition 2.8. Let d be a positive integer and let $R_{\ell} \in \mathbb{R}_+ \cup \{\infty\}$ for any $\ell = 1, \ldots, d$. Let $(g_{1,m}), \ldots, (g_{d,m})$ be d sequences of complex numbers such that

$$\lim_{m \to \infty} \sup_{m \to \infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_{\ell}}, \quad for \quad \ell = 1, \dots, d.$$
(5)

Let $x_1, \ldots, x_d \in \mathbb{R}$. Denote by $D_{\xi} := \frac{\partial}{\partial \xi}$ the derivative operator with respect to the auxiliary complex variable ξ . We define the formal operator:

$$\mathcal{U}(x_1, x_2, \dots, x_d, D_{\xi}) := \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \frac{D_{\xi}^{k_1}}{i^{k_1}} \tag{6}$$

where we have set

$$y_p := ix_1g_{1,p} + \ldots + ix_dg_{d,p}, \text{ for } p = 1, \ldots r \text{ with } r \in \mathbb{N}.$$

Then, setting

$$R := \min_{\ell=1,\dots,d} R_{\ell},$$

for any real value $0 < B < \frac{R}{4e}$, the operator $\mathcal{U}(x_1, \ldots, x_d, D_{\xi}) : A_{1,B} \to A_{1,4eB}$ is continuous for all $\underline{x} \in \mathbb{R}^d$.

Proof. Let us consider $f \in A_{1,B}$; then we have

$$\mathcal{U}(x_1, \dots, x_d, D_{\xi}) f(\xi) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \frac{D_{\xi}^{k_1}}{i^{k_1}} f(\xi)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \sum_{j=k_1}^{\infty} a_j \frac{j!}{(j-k_1)!} \xi^{j-k_1}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \sum_{j=0}^{\infty} a_{j+k_1} \frac{(j+k_1)!}{j!} \xi^j.$$

Taking the modulus we get

$$|\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f(\xi)| \le \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \cdots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \cdots |y_{k_1-k_2}| \right) \sum_{j=0}^{\infty} |a_{j+k_1}| \frac{(j+k_1)!}{j!} \xi^j.$$

and Lemma 2.6 gives the estimate on the coefficients a_{j+k_1}

$$|a_{j+k_1}| \le C_f \frac{b^{j+k_1}}{\Gamma(j+k_1+1)}.$$

where b = 2eB. Using the well known inequality $(a + b)! \le 2^{a+b}a!b!$ we also have

$$(j+k_1)! \le 2^{j+k_1}j!k_1!$$

so we get

$$\begin{aligned} |\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f(\xi)| &\leq \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \ldots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \ldots |y_{k_1-k_2}| \right) \times \\ &\times C_f \sum_{j=0}^{\infty} \frac{b^{j+k_1}}{\Gamma(j+k_1+1)} \frac{2^{j+k_1}k_1!j!}{j!} |\xi|^j. \end{aligned}$$

Now we use the Gamma function estimate

$$\frac{1}{\Gamma(a+b+2)} \le \frac{1}{\Gamma(a+1)} \frac{1}{\Gamma(b+1)}$$
(7)

to separate the series, and we have

$$\frac{1}{\Gamma(j-\frac{1}{2}+k_1-\frac{1}{2}+2)} \le \frac{1}{\Gamma(j+\frac{1}{2})} \frac{1}{\Gamma(k_1+\frac{1}{2})}$$

and so

$$|\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f(\xi)| \le C_f \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \ldots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \ldots |y_{k_1-k_2}| \right) \times$$

$$\times \frac{(k_1)!(2b)^{k_1}}{\Gamma(k_1 + \frac{1}{2})} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + \frac{1}{2})} (2b|\xi|)^j.$$

Now observe that the latter series satisfies the estimate

$$\sum_{j=0}^{\infty} \frac{1}{\Gamma(k+\frac{1}{2})} (2b|\xi|)^j \le Ce^{4b|\xi|}$$

where C is a positive constant, because of the properties of the Mittag-Leffler function; moreover, the series

$$\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \dots |y_{k_1-k_2}| \right) \frac{(k_1)! (2b)^{k_1}}{\Gamma(k_1 + \frac{1}{2})}$$
(8)

is convergent and is bounded by a positive real constant $C_{\underline{x},G_1,\ldots,G_d}$. In fact, using Stirling formula for the Gamma function, we have

$$m! \sim \sqrt{2\pi m} e^{-m} m^m$$
, for $m \to \infty$

and then we deduce

$$\frac{\Gamma(m+1)}{\Gamma(m+1/2)} \sim \frac{\sqrt{2\pi \, m \, e^{-m} m^m}}{\sqrt{2\pi (m-1/2)} \, e^{-(m-1/2)} \, (m-1/2)^{(m-1/2)}} \sim \sqrt{m-1/2}, \quad \text{for} \quad m \to \infty$$
(9)

so that

$$\frac{k_1!}{\Gamma(k_1 + \frac{1}{2})} \sim \sqrt{k_1 - 1/2}, \text{ for } k_1 \to \infty.$$

Now observe that the series (8) has positive coefficients and so it converges if and only if the series

$$\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{k_1=1}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \dots |y_{k_1-k_2}| \right) (2b)^{k_1} \sqrt{k_1 - 1/2}$$

converges. Given an absolutely convergent series $\sum_{p=0}^{\infty} a_p$, then its *m*-th power can be computed by means of the Cauchy product as follows:

$$\left(\sum_{p=0}^{\infty} a_p\right)^m = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} a_{k_m} a_{k_{m-1}-k_m} \dots a_{k_1-k_2}.$$
 (10)

Using the inequality:

$$\sqrt{k_1 - \frac{1}{2}} \le k_1 \le k_m + (k_{m-1} - k_m) + \ldots + (k_1 - k_2) \le (k_m + 2) \cdot (k_{m-1} - k_m + 2) \cdot (k_1 - k_2 + 2),$$

where $k_1 \ge k_2 \ge \cdots \ge k_m$, we deduce that there exists a positive constant $C_{\underline{x},G_1,\ldots,G_d}$ such that

the following chain of inequalities hold:

$$\begin{split} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{k_1=1}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}| |y_{k_{m-1}-k_m}| \dots |y_{k_1-k_2}| \right) (2b)^{k_1} \sqrt{k_1 - 1/2} \\ &\leq \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{k_1=1}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} |y_{k_m}(k_m + 2)(2b)^{k_m}| |y_{k_{m-1}-k_m}(k_{m-1} - k_m + 2)(2b)^{k_{m-1}-k_m}| \times \\ & \dots \times |y_{k_1-k_2}(k_2 - k_1 + 2)(2b)^{k_1-k_2}| \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{(m)!} \left[\sum_{p=0}^{\infty} |y_p| (p+2)(2b)^p \right]^m \leq \sum_{m=1}^{\infty} \frac{1}{(m)!} \left[\sum_{p=1}^{\infty} |x_1| (p+2)(2b)^p| g_{1,p}| + \\ & \dots + |x_d| (p+2)(2b)^p| g_{d,p}| \right]^m \leq C_{\underline{x},G_1,\dots,G_d} \end{split}$$

where for the equality we used (10), while the last inequality follows by the assumption

$$B < \frac{R}{4e}$$

which implies 2b < R. From the previous estimate we have that the series (8) converges for all $x_1, \ldots, x_d \in \mathbb{R}$. So we finally have

$$|\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f(\xi)| \le C_f C_{\underline{x},G_1,\ldots,G_d} C e^{4b|\xi|}, \quad \underline{x} \in \mathbb{R}^d, \quad \xi \in \mathbb{C}.$$
 (11)

Recalling that b = 2eB, the estimate (11) implies that $\mathcal{U}(x_1, \ldots, x_d,)f \in A_{1,8eB}$, in fact

$$|\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f(\xi)| e^{-8eB|\xi|} \le C_f C_{\underline{x},G_1,\ldots,G_d} C \quad \underline{x} \in \mathbb{R}^d, \ \xi \in \mathbb{C}.$$

Moreover, we deduce that the 8eB-norm satisfies the estimate

$$\|\mathcal{U}(x_1,\ldots,x_d,D_{\xi})f\|_{8eB} \leq C_f C_{\underline{x},G_1,\ldots,G_d} C = C_{\underline{x},G_1,\ldots,G_d} C \|f\|_B.$$

Thus $\mathcal{U}(x_1,\ldots,x_d,D_{\xi}): A_{1,B} \to A_{1,8eB}$ is continuous for all $\underline{x} \in \mathbb{R}^d$.

Remark 2.9. Whenever we fix a compact subset $K \subset \mathbb{R}^d$, we have that, for any $\underline{x} \in K$, the constants $C_{\underline{x},G_1,\ldots,G_d}$ appearing in the proof of the previous theorem are bounded by a constant which depends only on K and G_1,\ldots,G_d . Moreover, if $R_{\ell} = \infty$ for any $\ell = 1,\ldots,d$, the continuity of the operator $\mathcal{U}(x_1,\ldots,x_d,D_{\xi})$ holds for any B > 0 and the proof of the previous theorem shows that $\mathcal{U}(x_1,\ldots,x_d,D_{\xi})$ is a continuous operator in A_1 .

Proposition 2.10. Let d be a positive integer and let $R_{\ell} \in \mathbb{R}_+ \cup \{\infty\}$ for any $\ell = 1, \ldots, d$. Let $(g_{1,m}), \ldots, (g_{d,m})$ be d sequences of complex numbers such that

$$\lim \sup_{m \to \infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_{\ell}}, \quad for \quad \ell = 1, \dots, d.$$
(12)

We define the formal operator

$$\mathcal{V}(x_1,\dots,x_d,D_{\xi}) := \sum_{m_1=0}^{\infty} g_{1,m_1}\cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1}\dots x_d^{m_d} \frac{1}{i^{m_1+\dots+m_d}} D_{\xi}^{m_1+\dots+m_d}, \quad (13)$$

where $x_1, \ldots, x_d \in \mathbb{R}, \xi \in \mathbb{C}$. Then, for any real value B > 0, the operator $\mathcal{V}(x_1, \ldots, x_d, D_{\xi})$: $A_{1,B} \to A_{1,8eB}$ is continuous whenever $|x_{\ell}| < \frac{R}{4eB}$ for any $\ell = 1, \ldots, d$ where $R := \min_{\ell=1,\ldots,d} R_{\ell}$. *Proof.* We apply the operator $\mathcal{V}(x_1, \ldots, x_d, D_{\xi})$ to a function f in $A_{1,B}$ for $|\underline{x}| < \frac{R}{4eB}$. We have

$$\begin{aligned} \mathcal{V}(x_1, \dots, x_d, D_{\xi}) f(\xi) &= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1} \dots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} D_{\xi}^{m_1 + \dots + m_2} f(\xi) \\ &= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1} \dots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} D_{\xi}^{m_1 + \dots + m_d} \sum_{j=0}^{\infty} a_j \xi^j \\ &= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1} \dots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} \times \\ &\times \sum_{j=m_1 + \dots + m_d}^{\infty} a_j \frac{j!}{(j - (m_1 + \dots + m_d))!} \xi^{j - (m_1 + \dots + m_d)} \\ &= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1} \dots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} \sum_{k=0}^{\infty} a_{m_1 + \dots + m_d + k} \frac{(m_1 + \dots + m_d + k)!}{k!} \xi^k. \end{aligned}$$

We then have

$$\begin{aligned} |\mathcal{V}(x_1, \dots, x_d, D_{\xi}) f(\xi)| &\leq \sum_{m_1=0}^{\infty} |g_{1,m_1}| \dots \sum_{m_d=0}^{\infty} |g_{d,m_d}| |x_1|^{m_1} \dots |x_d|^{m_d} \times \\ &\times \sum_{k=0}^{\infty} |a_{m_1+\dots+m_d+k}| \frac{(m_1+\dots+m_d+k)!}{k!} |\xi|^k \end{aligned}$$

and using the estimate in Lemma 2.6

$$|a_{m_1+\dots m_d+k}| \le C_f \frac{b^{m_1+\dots m_d+k}}{\Gamma(m_1+\dots+m_d+k+1)},$$

where b = 2eB, we get

$$\begin{aligned} |\mathcal{V}(x_1, \dots, x_d, D_{\xi}) f(\xi)| &\leq \sum_{m_1=0}^{\infty} |g_{1,m_1}| \cdots \sum_{m_d=0}^{\infty} |g_{d,m_d}| |x_1|^{m_1} \dots |x_d|^{m_d} \times \\ &\times C_f \sum_{k=0}^{\infty} \frac{b^{m_1 + \dots + m_d + k}}{\Gamma(m_1 + \dots + m_d + k + 1)} \frac{(m_1 + \dots + m_d + k)!}{k!} |\xi|^k. \end{aligned}$$

With the estimates

$$(m_1 + \dots + m_d + k)! \le 2^{m_1 + \dots + m_d + k} (m_1 + \dots + m_d)!k!$$

and

$$\frac{1}{\Gamma(m_1 + \dots + m_d - \frac{1}{2} + k - \frac{1}{2} + 2)} \le \frac{1}{\Gamma(m_1 + \dots + m_d + \frac{1}{2})} \frac{1}{\Gamma(k + \frac{1}{2})}$$

we separate the series

$$|\mathcal{V}(x_1,\ldots,x_d,D_{\xi})f(\xi)| \leq \sum_{m_1=0}^{\infty} |g_{1,m_1}|\cdots \sum_{m_d=0}^{\infty} |g_{d,m_d}| |x_1|^{m_1} \dots |x_d|^{m_d} \times \sum_{k=0}^{\infty} C_f b^{m_1+\cdots+m_d+k} \frac{1}{\Gamma(m_1+\cdots+m_d+\frac{1}{2})} \frac{1}{\Gamma(k+\frac{1}{2})} \frac{2^{m_1+\cdots+m_d+k}(m_1+\cdots+m_d)!k!}{k!} |\xi|^k.$$

Finally we get

$$\begin{aligned} |\mathcal{V}(x_1, \dots, x_d, D_{\xi}) f(\xi)| &\leq C_f \sum_{m_1=0}^{\infty} |g_{1,m_1}| \cdots \sum_{m_d=0}^{\infty} |g_{d,m_d}| (2b|x_1|)^{m_1} \cdots (2b|x_d|)^{m_d} \times \\ &\times \frac{(m_1 + \dots + m_d)!}{\Gamma(m_1 + \dots + m_d + \frac{1}{2})} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \frac{1}{2})} (2b|\xi|)^k. \end{aligned}$$

Using (9) we have

$$\frac{(m_1 + \dots + m_d)!}{\Gamma(m_1 + \dots + m_d + \frac{1}{2})} \sim \sqrt{m_1 + \dots + m_d - 1/2}, \text{ for } m_1 + \dots + m_d \to \infty,$$

and, moreover, $\sqrt{m_1 + \cdots + m_d - 1/2} \leq m_1 \cdots m_d$ if $m_\ell \geq 2$ for any $\ell = 1, \ldots, d$. Since $|x_\ell| < \frac{R}{4eB}$ for any $\ell = 1, \ldots, d$ and b = 2eB, the series

$$\sum_{m_\ell=1}^{\infty} m_\ell |g_{\ell,m_\ell}| (2b|x_\ell|)^{m_\ell}$$

converges to a constant which depends on $x_{\ell} \in \mathbb{R}$. Thus there exist constants $C_{x_{\ell}}$ such that

$$|\mathcal{V}(x_1, \dots, x_d, D_{\xi}) f(\xi)| \le C_f C_{x_1} \dots C_{x_d} (2b|\xi|) e^{2b|\xi|} \le C_f C_{x_1, \dots, x_d} e^{4b|\xi|}$$

from which, recalling that $C_f = ||f||_B$, we deduce

$$\|\mathcal{V}(x_1,\ldots,x_d,D_{\xi})f\|_{8eB} \le C_{x_1,\ldots,x_d}\|f\|_B.$$

We conclude that the operator $\mathcal{V}(x_1, \ldots, x_d, D_{\xi}) : A_{1,B} \to A_{1,8eB}$ is continuous.

Remark 2.11. Whenever we fix a compact subset

$$K \subset \{\underline{x} \in \mathbb{R}^d : |x_\ell| < \frac{R}{4eB} \text{ for any } \ell = 1, \dots, d\},$$

we have that, for any $\underline{x} \in K$, the constants $C_{x_{\ell}}$'s, appearing in the proof of the previous theorem are bounded by a constant which depends only on K. Moreover, if $R_{\ell} = \infty$ for any $\ell = 1, \ldots, d$, the continuity of the operator $\mathcal{V}(x_1, \ldots, x_d, D_{\xi})$ holds to be true for any $\underline{x} \in \mathbb{R}^d$ and the proof of the previous theorem shows that $\mathcal{V}(x_1, \ldots, x_d, D_{\xi})$ satisfies the conditions in (4). Thus we conclude that $\mathcal{V}(x_1, \ldots, x_d, D_{\xi})$ is a continuous operator in A_1 .

3 Superoscillating functions in several variables

We recall some preliminary definitions related to superoscillating functions in several variables.

Definition 3.1 (Generalized Fourier sequence in several variables). For $d \in \mathbb{N}$ such that $d \geq 2$, we assume that $(x_1, ..., x_d) \in \mathbb{R}^d$. Let $(h_{j,\ell}(n))$, j = 0, ..., n for $n \in \mathbb{N}_0$, be real-valued sequences for $\ell = 1, ..., d$. We call generalized Fourier sequence in several variables a sequence of the form

$$F_n(x_1, \dots, x_d) = \sum_{j=0}^n c_j(n) e^{ix_1 h_{j,1}(n)} e^{ix_2 h_{j,2}(n)} \dots e^{ix_d h_{j,d}(n)},$$
(14)

where $(c_j(n))_{j,n}$, j = 0, ..., n, for $n \in \mathbb{N}_0$ is a complex-valued sequence.

Definition 3.2 (Superoscillating sequence). A generalized Fourier sequence in several variables $F_n(x_1, \ldots, x_d)$, with $d \in \mathbb{N}$ such that $d \geq 2$, is said to be a superoscillating sequence if

$$\sup_{j=0,...,n, n \in \mathbb{N}_0} |h_{j,\ell}(n)| \le 1, \text{ for } \ell = 1,...,d,$$

and there exists a compact subset of \mathbb{R}^d , which will be called a superoscillation set, on which $F_n(x_1,\ldots,x_d)$ converges uniformly to $e^{ix_1g_1}e^{ix_2g_2}\ldots e^{ix_dg_d}$, where $|g_\ell| > 1$ for $\ell = 1,\ldots,d$.

In the paper [6] we studied the function theory of superoscillating functions in several variables under the additional hypothesis that there exist $r_{\ell} \in \mathbb{N}$, such that

$$p = r_1 q_1 + \ldots + r_d q_d. \tag{15}$$

In that case, we proved that for $p, q_{\ell} \in \mathbb{N}, \ell = 1, \dots, d$ the function

$$F_n(x, y_1, \dots, y_d) = \sum_{j=0}^n C_j(n, a) e^{ix(1-2j/n)^p} e^{iy_1(1-2j/n)^{q_1}} \dots e^{iy_d(1-2j/n)^{q_d}}$$

is superoscillating when |a| > 1, where $C_j(n, a)$ are suitable coefficients. In the paper [9], we were able to remove the condition (15), while here we will show that it is possible to replace the functions $(1-2j/n)^p$ in the terms $e^{ix(1-2j/n)^p}$ with more general holomorphic functions. As we shall see, different function spaces are involved in the proofs according to the fact that the holomorphic functions are entire or not.

Theorem 3.3 (The general case of $d \ge 2$ variables). Let d be a positive integer and let $R_{\ell} \in \mathbb{R}_+ \cup \{\infty\}$ be such that $R_{\ell} \ge 1$ for any $\ell = 1, \ldots, d$. Let G_1, \ldots, G_d be holomorphic functions whose series expansion at zero is given by

$$G_{\ell}(\lambda) = \sum_{m_{\ell}=0}^{\infty} g_{\ell,m} \lambda^{m_{\ell}}, \quad \forall \ell = 1, \dots, d$$
(16)

and, moreover, the sequences $(g_{\ell,m})$ satisfy the condition

$$\lim \sup_{m \to \infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_{\ell}}, \quad \forall \ell = 1, \dots, d.$$

Let

$$f_n(x) := \sum_{j=0}^n Z_j(n,a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$
(17)

be superoscillating functions as in Definition 2.1 and assume that their entire extensions to the functions $f_n(\xi)$ converge to $e^{ia\xi}$ in $A_{1,B}$ for some positive real value $0 < B < \frac{R}{4e}$ where $R := \min_{\ell=1,...,d} R_{\ell}$. We define

$$F_n(x_1,\ldots,x_d) := \sum_{j=0}^n Z_j(n,a) e^{ix_1 G_1(h_j(n))} e^{ix_2 G_2(h_j(n))} \ldots e^{ix_d G_d(h_j(n))}$$

Then, whenever |a| < R we have

$$\lim_{n \to \infty} F_n(x_1, x_2, \dots, x_d) = e^{ix_1 G_1(a)} e^{ix_2 G_2(a)} \dots e^{ix_d G_d(a)},$$

uniformly on compact subsets of \mathbb{R}^d . In particular, $F_n(x_1, x_2, \ldots, x_d)$ is superoscillating when |a| > 1.

Proof. Since $R_{\ell} \geq 1$ for any $\ell = 1, \ldots, d$ and $|h_j(n)| < 1$, using (10) we have the chain of equalities

$$\begin{split} F_n(x_1, x_1, \dots, x_d) &= \sum_{j=0}^n Z_j(n, a) e^{ix_1 G_1(h_j(n)) + ix_2 G_2(h_j(n)) + \dots + ix_d G_d(h_j(n))} \\ &= \sum_{j=0}^n Z_j(n, a) \sum_{m=0}^\infty \frac{1}{m!} \Big[ix_1 G_1(h_j(n)) + ix_2 G_2(h_j(n)) + \dots + ix_d G_d(h_j(n)) \Big]^m \\ &= \sum_{j=0}^n Z_j(n, a) \sum_{m=0}^\infty \frac{1}{m!} \Big[ix_1 \sum_{p=1}^\infty g_{1,p}(h_j(n))^p + \dots + ix_d \sum_{p=1}^\infty g_{d,p}(h_j(n))^p \Big]^m \\ &= \sum_{j=0}^n Z_j(n, a) \sum_{m=0}^\infty \frac{1}{m!} \Big[\sum_{p=1}^\infty (ix_1 g_{1,p} + \dots + ix_d g_{d,p})(h_j(n))^p \Big]^m \\ &= \sum_{m=0}^\infty \frac{1}{m!} \sum_{k_1=0}^\infty \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) (h_j(n))^{k_1}, \end{split}$$

where we have set

$$y_p := ix_1g_{1,p} + \ldots + ix_dg_{d,p}, \text{ for } p = 1, \ldots r \text{ with } r \in \mathbb{N}.$$

We define the infinite order differential operator

$$\mathcal{U}(x_1, x_2, \dots, x_d, D_{\xi}) := \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \frac{D_{\xi}^{k_1}}{i^{k_1}}.$$
 (18)

Since $0 < B < \frac{R}{4e}$, Proposition 2.8 implies that the operator

$$\mathcal{U}(x_1, x_2, \dots, x_d, D_{\xi}) : A_{1,B} \mapsto A_{1,8eB}$$

is continuous. We observe that

$$F_n(x_1, x_2, \dots, x_d) = \mathcal{U}(x_1, x_2, \dots, x_d, D_{\xi}) \sum_{j=0}^n Z_j(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}$$

The explicit computation of the term $\mathcal{U}(x_1, \ldots, x_d, D_{\xi})e^{i\xi a}$ gives

$$\begin{aligned} \mathcal{U}(x_1, \dots, x_d, D_{\xi}) e^{i\xi a} &= \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) \frac{D_{\xi}^{k_1}}{i^{k_1}} e^{i\xi a} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) a^{k_1} e^{i\xi a}, \end{aligned}$$

so we finally get

$$\lim_{n \to \infty} F_n(x_1, \dots, x_d) =$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} y_{k_m} y_{k_{m-1}-k_m} \dots y_{k_1-k_2} \right) a^{k_1} e^{i\xi a} \Big|_{\xi=0}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1=0}^{\infty} \left(\sum_{k_2=0}^{k_1} \dots \sum_{k_m=0}^{k_{m-1}} (y_{k_m} a^{k_m}) (y_{k_{m-1}-k_m} a^{k_{m-1}-k_m}) \dots (y_{k_1-k_2} a^{k_1-k_2}) \right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{p=1}^{\infty} y_p a^p \right)^m = \sum_{m=0}^{\infty} \frac{1}{m!} (ix_1 G_1(a) + \dots + ix_d G_d(a))^m = e^{ix_1 G_1(a) + \dots + ix_d G_d(a)}$$

where the third equality is due to the formula (10) and the fourth equality holds because we are assuming |a| < R. The previous limit is uniform over the compact subset of \mathbb{R}^d because of Remark 2.9.

Remark 3.4. From the inspection of the proof we observe that:

(I) The space of the entire functions on which the infinite order differential operator $\mathcal{U}(x_1, \ldots, x_d, D_{\xi})$ acts is the space $A_{1,B}$ in one complex variable, for some positive real value $0 < B < \frac{R}{4e}$. (III) The variables (x_1, x_1, \ldots, x_d) become the coefficients of the infinite order differential operator $\mathcal{U}(x_1, x_2, \ldots, x_d, D_{\xi})$, defined in (18), that still acts on the space $A_{1,B}$.

4 Supershifts in several variables

The procedure to define superoscillating functions can be extended to the case of supershift. Recall that the supershift property of a function extends the notion of superoscillation and that this concept, that we recall below, turned out to be a crucial ingredient for the study of the evolution of superoscillatory functions as initial conditions of the Schrödinger equation.

Definition 4.1 (Supershift). Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval with $[-1,1] \subset \mathcal{I}$ and let $\varphi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$, be a continuous function on \mathcal{I} . We set

$$\varphi_h(x) := \varphi(h, x), \quad h \in \mathcal{I}, \quad x \in \mathbb{R}$$

and we consider a sequence of points $(h_i(n))$ such that

$$h_j(n) \in [-1, 1]$$
 for $j = 0, ..., n$ and $n \in \mathbb{N}_0$.

We define the functions

$$\psi_n(x) = \sum_{j=0}^n c_j(n)\varphi_{h_j(n)}(x),$$
(19)

where $(c_j(n))$ is a sequence of complex numbers for j = 0, ..., n and $n \in \mathbb{N}_0$. If

$$\lim_{n \to \infty} \psi_n(x) = \varphi_a(x)$$

for some $a \in \mathcal{I}$ with |a| > 1, we say that the function $\psi_n(x)$, for $x \in \mathbb{R}$, admits a supershift.

Remark 4.2. The term supershift comes from the fact that the interval \mathcal{I} can be arbitrarily large (it can also be \mathbb{R}) and that the constant a can be arbitrarily far away from the interval [-1, 1] where the functions $\varphi_{h_{i,n}}(\cdot)$ are indexed, see (19).

Problem 2.2, for the supershift case, is formulated as follows.

Problem 4.3. Let $h_j(n)$ be a given set of points in [-1,1], j = 0, 1, ..., n, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that |a| > 1. Suppose that for every $x \in \mathbb{R}$ the function $h \mapsto G(hx)$ extends to a holomorphic and entire function in h. Consider the functions

$$f_n(x) = \sum_{j=0}^n Y_j(n,a) G(h_j(n)x), \quad x \in \mathbb{R}$$

where $h \mapsto G(hx)$ depends on the parameter $x \in \mathbb{R}$. Determine the coefficients $Y_j(n)$ in such a way that

 $f_n^{(p)}(0) = (a)^p G^{(p)}(0) \quad for \quad p = 0, 1, ..., n.$ (20)

The solution of Problem 4.3, obtained in [8], is summarized in the following theorem.

Theorem 4.4. Let $h_j(n)$ be a given set of points in [-1,1], j = 0, 1, ..., n for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that |a| > 1. If $h_j(n) \neq h_i(n)$ for every $i \neq j$ and $G^{(p)}(0) \neq 0$ for all p = 0, 1, ..., n, then there exists a unique solution $Y_j(n, a)$ of the linear system (20) and it is given by

$$Y_{j}(n,a) = \prod_{k=0, \ k \neq j}^{n} \Big(\frac{h_{k}(n) - a}{h_{k}(n) - h_{j}(n)} \Big),$$

so that

$$f_n(x) = \sum_{j=0}^n \prod_{k=0, \ k \neq j}^n \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) G(h_j(n)x), \quad x \in \mathbb{R}.$$

Remark 4.5. In the sequel, we shall move from the real to the complex setting and we will consider those functions G and sequences $h_j(n)$ for which the holomorphic extension $f_n(z)$ of $f_n(x)$ converges in A_1 to G(az).

We can now extend the notion of supershift of a function in several variables.

Definition 4.6 (Supershifts in several variables). Let |a| > 1. For $d \in \mathbb{N}$ with $d \geq 2$, we assume that $(x_1, ..., x_d) \in \mathbb{R}^d$. Let $(h_{j,\ell}(n))$, j = 0, ..., n for $n \in \mathbb{N}_0$, be real-valued sequences for $\ell = 1, ..., d$ such that for

$$\sup_{j=0,...,n, \ n \in \mathbb{N}_0} \ |h_{j,\ell}(n)| \le 1, \ \text{ for } \ell = 1,...,d$$

and let $G_{\ell}(\lambda)$, for $\ell = 1, ..., d$, be entire holomorphic functions. We say that the sequence

$$F_n(x_1, \dots, x_d) = \sum_{j=0}^n c_j(n) G_1(x_1 h_{j,1}(n)) G_2(x_2 h_{j,2}(n)) \dots G_d(x_d h_{j,d}(n)),$$
(21)

where $(c_j(n))_{j,n}$, j = 0, ..., n, for $n \in \mathbb{N}_0$ is a complex-valued sequence, admits the supershift property if

$$\lim_{n \to \infty} F_n(x_1, \dots, x_d) = G_1(x_1 a) G_2(x_2 a) \dots G_d(x_d a).$$

Theorem 4.7 (The case of $d \ge 1$ variables). Let |a| > 1 and let

$$f_n(x) := \sum_{j=0}^n Z_j(n,a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$
(22)

be a superoscillating function as in Definition 2.1 and assume that its holomorphic extension to the entire functions $f_n(z)$ converges to e^{iaz} in the space $A_{1,B}$ for some positive real value B. Let d be a positive integer and let $R_{\ell} \in \mathbb{R}_+ \cup \{\infty\}$ for any $\ell = 1, \ldots, d$. Let G_1, \ldots, G_d be holomorphic functions whose series expansion at zero is given by

$$G_{\ell}(\lambda) = \sum_{m_{\ell}=0}^{\infty} g_{\ell,m} \lambda^{m_{\ell}}, \quad \forall \ell = 1, \dots, d.$$
(23)

Moreover, we suppose the sequences $(g_{l,m})$'s satisfy the condition

$$\lim \sup_{m \to \infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_{\ell}}, \quad \forall \ell = 1, \dots, d.$$

We define

$$F_n(x_1, \dots, x_d) = \sum_{j=0}^n Z_j(n, a) G_1(x_1 h_j(n)) \cdots G_d(x_d h_j(n)),$$

where $Z_j(n,a)$ are given as in (22). Then, $F_n(x_1,\ldots,x_d)$ admits the supershift property that is

$$\lim_{n \to \infty} F_n(x_1, \dots, x_d) = G_1(x_1 a) \cdots G_d(x_d a)$$

uniformly on compact subsets of $\{\underline{x} \in \mathbb{R}^d : |x_\ell| < R' \text{ for any } \ell = 1, \ldots, d\}$ where

$$R' := \min\left(\frac{R}{|a|}, \frac{R}{4eB}, R\right) \qquad \text{where} \quad R := \min_{\ell=1,\dots,d} R_{\ell}$$

Proof. Since $|x_{\ell}| < R$ for any $\ell = 1, \ldots, d$, we have

$$F_n(x_1, \dots, x_d) = \sum_{j=0}^n Z_j(n, a) G_1(x_1 h_j(n)) \dots G_d(x_d h_j(n))$$

= $\sum_{j=0}^n Z_j(n, a) \sum_{m_1=0}^\infty g_{m_1} \dots \sum_{m_d=0}^\infty g_{m_d} x_1^{m_1} \dots x_d^{m_d} (h_j(n))^{m_1 + \dots + m_d}.$

We now consider the auxiliary complex variable ξ and we note that

$$\lambda^{\ell} = \frac{1}{i^{\ell}} D_{\xi}^{\ell} e^{i\xi\lambda} \Big|_{\xi=0} \quad \text{for} \quad \lambda \in \mathbb{C}, \quad \ell \in \mathbb{N},$$
(24)

where D_{ξ} is the derivative with respect to ξ and $|_{\xi=0}$ denotes the restriction to $\xi = 0$. We have

$$F_n(x_1, \dots, x_d) = \sum_{j=0}^n Z_j(n, a) \sum_{m_1=0}^\infty g_{m_1} \cdots \sum_{m_d=0}^\infty g_{m_d} x_1^{m_1} \cdots x_d^{m_d} [h_j(n)]^{m_1 + \dots + m_d}$$

$$= \sum_{j=0}^n Z_j(n, a) \sum_{m_1=0}^\infty g_{m_1} \cdots \sum_{m_d=0}^\infty g_{m_d} x_1^{m_1} \cdots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} D_{\xi}^{m_1 + \dots + m_d} e^{i\xi h_j(n)} \Big|_{\xi=0}$$

$$= \sum_{m_1=0}^\infty g_{m_1} \cdots \sum_{m_d=0}^\infty g_{m_d} x_1^{m_1} \cdots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} D_{\xi}^{m_1 + \dots + m_d} \sum_{j=0}^n Z_j(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}.$$

We define the operator

$$\mathcal{V}(x_1, \dots, x_d, D_{\xi}) := \sum_{m_1=0}^{\infty} g_{m_1} \cdots \sum_{m_d=0}^{\infty} g_{m_d} x_1^{m_1} \cdots x_d^{m_d} \frac{1}{i^{m_1 + \dots + m_d}} D_{\xi}^{m_1 + \dots + m_d}$$

so that we can write

$$F_n(x_1, \dots, x_d) = \mathcal{V}(x_1, \dots, x_d, D_{\xi}) \sum_{j=0}^n Z_j(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}$$

Since $|x_{\ell}| < \frac{R}{4eB}$ for any $\ell = 1, ..., d$, we can use Proposition 2.10 in order to compute the following limit

$$\lim_{n \to \infty} F_n(x_1, \dots, x_d) = \mathcal{V}(x_1, \dots, x_d, D_{\xi}) \lim_{n \to \infty} \sum_{j=0}^n Z_j(n, a) e^{i\xi h_j(n)} \Big|_{\xi=0}$$

$$= \mathcal{V}(x_1, \dots, x_d, D_{\xi}) e^{i\xi a} \Big|_{\xi=0}$$

$$= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} x_1^{m_1} \dots x_d^{m_2} \frac{1}{i^{m_1+\dots+m_d}} D_{\xi}^{m_1+\dots+m_2} e^{i\xi a} \Big|_{\xi=0}$$

$$= \sum_{m_1=0}^{\infty} g_{1,m_1} \cdots \sum_{m_d=0}^{\infty} g_{d,m_d} (ax_1)^{m_1} \dots (ax_d)^{m_2} = G_1(ax_1) \cdots G_d(ax_d)$$

where the last equality holds because we are assuming $|x_{\ell}| < \frac{R}{|a|}$ for any $\ell = 1, \ldots, d$. The previous limit is uniform over the compact subset of $\{\underline{x} \in \mathbb{R}^d : |x_{\ell}| < R' \text{ for any } \ell = 1, \ldots, d\}$ because of Remark 2.11.

Remark 4.8. A special case of the previous theorem occurs when the holomorphic functions G_{ℓ} 's are entire functions. Moreover, differently from Theorem 3.3, in Theorem 4.7 the parameters x_{ℓ} appear in the arguments of the functions G_{ℓ} 's. This implies that the hypothesis of Theorem 4.7 imposes more constraints on the parameters x_{ℓ} 's, namely $|x_{\ell}| < R'$ for any $\ell = 1, \ldots, d$.

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