Bull. Austral. Math. Soc. 47H99, 47J20, 49J40
Vol. 73 (2006) [307–317]

THE GENERALISED f-PROJECTION OPERATOR WITH AN APPLICATION

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In this paper, we introduce a new concept of generalised f-projection operator which extends the generalised projection operator $\pi_K : B^* \to K$, where B is a reflexive Banach space with dual space B^* and K is a nonempty, closed and convex subset of B. Some properties of the generalised f-projection operator are given. As an application, we study the existence of solution for a class of variational inequalities in Banach spaces.

1. Introduction and preliminaries

Let B be a Banach space with dual space B^*. As usual, $\langle \varphi, x \rangle$, denotes the duality pairing of B^* and B, where $\varphi \in B^*$ and $x \in B$. If B is a Hilbert space, $\langle \varphi, x \rangle$ denotes an inner product on B. Let K be a nonempty, closed and convex subset of B. The metric projection operator $P_K : B^* \to K$ has been used in many areas such as optimisation theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, et cetera (see, for example, [**10, 11, 12, 13, 14, 16, 17, 20**] and the references therein).

In 1994, Alber [**1**] introduced the generalised projections $\pi_K : B^* \to K$ and $\Pi_K : B \to K$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [**2**], Alber presented some applications of the generalised projections to approximate solving variational inequalities and Von-Neumann intersection problem in Banach spaces. Recently, Li [**17**] extended the definition of the generalised projection operator $\pi_K : B^* \to K$, where B is a reflexive Banach space with dual space B^* and K is a nonempty, closed and convex subset of B and studied some properties of the generalised projection operator with applications to solving the variational inequality in Banach spaces. Some related works, we refer to [**3, 4, 5, 6, 7, 8, 9**] and the references therein.

Received 9th January, 2006
This work was supported by the National Natural Science Foundation of China, the Applied Research Project of Sichuan Province (05JY029-009-1) and the Educational Science Foundation of Chongqing (KJ051307).

Motivated and inspired by the above works, in this paper, we introduce and study a new class of generalised f-projection operator in Banach spaces, which extends the definition of the generalised projection operators introduced by Alber [**1**] and Li [**17**]. Some properties of the generalised f-projection operator are given. As an application, we study the existence of solution to a class of variational inequalities.

Let B be a Banach space with dual space B^*. The normalised duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \left\{ j(x) \in B^* : \langle j(x), x \rangle = \|j(x)\| \cdot \|x\| = \|x\|^2 = \|j(x)\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of B^* and B. Without confusion, one understands that $\|j(x)\|$ is the B^*-norm and $\|x\|$ is the B-norm. Many properties of the normalised duality mapping J can be found in Takahashi [**18**] or Vainberg [**19**]. We list some properties below for easy reference:

(i) J is a monotone and bounded operator in arbitrary Banach spaces;

(ii) J is a strictly monotone operator in strictly convex Banach spaces;

(iii) J is a continuous operator in smooth Banach spaces;

(iv) J is a uniformly continuous operator on each bounded set in uniformly smooth Banach spaces;

(v) J is the identity operator in Hilbert spaces, that is, $J = I_H$;

(vi) $J(x) = \partial(\|x\|^2/2)$, where $\partial(\|x\|^2/2)$ denotes subdifferential of $\|x\|^2/2$ at x.

Let $G : B^* \times B \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2 + f(x),$$

where $\varphi \in B^*$, $x \in B$ and $f : B \to R \cup \{+\infty\}$ is proper, convex, lower semi-continuous, and bounded from below. It is easy to see that

$$G(\varphi, x) \geqslant \left(\|\varphi\| - \|x\|\right)^2 + f(x).$$

From the definitions of G and f, it is easy to see that the G has the following properties:

(vii) $G(\varphi, x)$ is convex and continuous with respect to φ when x is fixed;

(viii) $G(\varphi, x)$ is convex and lower-semi-continuous with respect to x when φ is fixed;

(ix) $\left(\|\varphi\| - \|x\|\right)^2 + f(x) \leqslant G(\varphi, x) \leqslant \left(\|\varphi\| + \|x\|\right)^2 + f(x)$.

Lemma 1.1. ([**15**, p. 94].) *Let X be a Banach space. The following conditions are equivalent.*

(1) *X is strictly convex;*

(2) *If $x, y \in X$ and $\|x+y\| = \|x\| + \|y\|$, then $x = 0$ or $y = 0$ or $y = \alpha x$ for some $\alpha > 0$.*

DEFINITION 1.1: Let B be a Banach space with dual space B^*. Let K be a nonempty, closed and convex subset of B. We say that $\pi_K^f : B^* \to 2^K$ is a generalised f-projection operator if

$$\pi_K^f \varphi = \{u \in K : G(\varphi, u) = \inf_{y \in K} G(\varphi, y)\} \quad \forall \varphi \in B^*. \tag{1.1}$$

REMARK 1.1. If $f(x) = 0$ for all $x \in B$, then the generalised f-projection operator reduces to the generalised projection operator defined by Alber [**1**] and Li [**17**].

2. PROPERTIES OF THE GENERALISED f-PROJECTION π_K^f

The following theorem shows that the operator π_K^f is well defined for reflexive Banach spaces.

THEOREM 2.1. *If B is a reflexive Banach space with dual space B^* and K is a nonempty closed convex subset of B, then $\pi_K^f \varphi \neq \emptyset$ for all $\varphi \in B^*$.*

PROOF: For any given $\varphi \in B^*$ and $x \in B$, we have

$$\left(\|\varphi\| - \|x\|\right)^2 + f(x) \leqslant G(\varphi, x) \leqslant \left(\|\varphi\| + \|x\|\right)^2 + f(x).$$

Since f is bounded from below, it follows that, for any given $\varphi \in B^*$, $\inf_{y \in K} G(\varphi, y)$ is finite and so there exist a sequence $\{x_n\} \in K$ such that

$$\lim_{n \to \infty} G(\varphi, x_n) = \inf_{y \in K} G(\varphi, y).$$

Let $\inf_{y \in K} G(\varphi, y) = a$. Then for any given $\varepsilon > 0$, there exists $N > 0$ such that

$$\left|G(\varphi, x_n) - a\right| < \varepsilon$$

for all $n \geqslant N$. Thus,

$$\left(\|\varphi\| - \|x_n\|\right)^2 + f(x_n) - a < \varepsilon.$$

Since $f(x)$ is bounded from below, there exists $L \in R$ such that

$$\left(\|\varphi\| - \|x_n\|\right)^2 + L - a < \varepsilon. \tag{2.2}$$

On the other hand,

$$\begin{aligned} a = \inf_{y \in K} G(\varphi, y) &= \inf_{y \in K} \left\{\|\varphi\|^2 - 2\langle \varphi, x_n \rangle + \|x_n\|^2 + f(x_n)\right\} \\ &\geqslant \inf_{y \in K} \left\{\|\varphi\|^2 - 2\langle \varphi, x_n \rangle + \|x_n\|^2\right\} + L \\ &\geqslant \inf_{y \in K} \left(\|\varphi\| - \|x_n\|\right)^2 + L \\ &\geqslant L. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we know that $\{x_n\}$ is bounded. Since B is reflexive, there exist a subsequence of $\{x_n\}$, which after relabelling we again denote by $\{x_n\}$, and a point $x_0 \in K$ such that $\{x_n\}$ converges weakly to x_0. For each given φ, since $G(\varphi, x)$ is convex and lower-semi-continuous with respect to x, we know that $G(\varphi, x)$ is weakly lower-semi-continuous with respect to x. Thus, we have

$$G(\varphi, x_0) \leqslant \liminf_{n\to\infty} G(\varphi, x_n) = \lim_{n\to\infty} G(\varphi, x_n) = \inf_{y\in K} G(\varphi, y)$$

and so $x_0 \in \pi_K^f \varphi$. Therefore, $\pi_K^f \varphi \neq \emptyset$. This completes the proof. ∎

THEOREM 2.2. *If B is a reflexive Banach space with dual space B^* and K is a nonempty, closed and convex subset of B, then the following properties hold:*

- (f_1) *For any given $\varphi \in B^*$, $\pi_K^f \varphi$ is a nonempty, closed, bounded, and convex subset of K;*
- (f_2) *π_K^f is monotone, that is, for any $\varphi_1, \varphi_2 \in B^*$, $x_1 \in \pi_K^f \varphi_1$ and $x_2 \in \pi_K^f \varphi_2$,*

$$\langle x_1 - x_2, \varphi_1 - \varphi_2 \rangle \geqslant 0;$$

- (f_3) *If B is smooth, then for any given $\varphi \in B^*$, $x \in \pi_K^f \varphi$ if and only if*

$$2\big\langle \varphi - J(x), x - y \big\rangle + f(y) - f(x) \geqslant 0$$

for all $y \in K$;

- (f_4) *If K is a closed convex cone and $f : K \to R \cup \{+\infty\}$ is positively homogeneous, that is, $f(tx) = tf(x)$ for all $t > 0$ and $x \in K$, then for any $\varphi \in B^*$ and $x_1, x_2 \in \pi_K^f \varphi$, we have $x_1 \neq \mu x_2$ for all $\mu \in (0, +\infty)$ with $\mu \neq 1$;*
- (f_5) *If K is a closed convex cone, $f : K \to R \cup \{+\infty\}$ is positively homogeneous and B is strictly convex, then the operator $\pi_K^f : B^* \to K$ is single-valued.*

PROOF: (f_1) For any point $\varphi \in B^*$, Theorem 2.1 implies that $\pi_K^f \varphi$ is nonempty. Since f is bounded from below and $G(\varphi, x) \geqslant \big(\|\varphi\| - \|x\|\big)^2 + f(x)$, it is easy to see that $\pi_K^f \varphi$ is bounded. Next we prove that $\pi_K^f \varphi$ is closed. Suppose $\{x_n\} \in \pi_K^f \varphi$ and $x_n \to x_0$ as $n \to \infty$. By property (viii) of the functional G, we have

$$G(\varphi, x_0) \leqslant \liminf_{n\to\infty} G(\varphi, x_n) = \lim_{n\to\infty} G(\varphi, x_n) = \inf_{y\in K} G(\varphi, y).$$

Thus, $x_0 \in \pi_K^f \varphi$ and so $\pi_K^f \varphi$ is closed. Finally, we prove that $\pi_K^f \varphi$ is convex. Suppose $x_1, x_2 \in \pi_K^f \varphi$ and $0 \leqslant t \leqslant 1$. From property (viii) of the functional G, we have

$$\begin{aligned} G(\varphi, tx_1 + (1-t)x_2) &\leqslant tG(\varphi, x_1) + (1-t)G(\varphi, x_2) \\ &= t \inf_{y\in K} G(\varphi, y) + (1-t) \inf_{y\in K} G(\varphi, y) \\ &= \inf_{y\in K} G(\varphi, y) \end{aligned}$$

https://doi.org/10.1017/S0004972700038892 Published online by Cambridge University Press

and so $tx_1 + (1-t)x_2 \in \pi_K^f \varphi$. This implies that $\pi_K^f \varphi$ is convex.

(f_2) For any $\varphi_1, \varphi_2 \in B^*$, $x_1 \in \pi_K^f \varphi_1$ and $x_2 \in \pi_K^f \varphi_2$, from definition π_K^f, we have

$$\|\varphi_1\|^2 - 2\langle \varphi_1, x_1 \rangle + \|x_1\|^2 + f(x_1) \leqslant \|\varphi_1\|^2 - 2\langle \varphi_1, x_2 \rangle + \|x_2\|^2 + f(x_2) \tag{2.4}$$

and

$$\|\varphi_2\|^2 - 2\langle \varphi_2, x_2 \rangle + \|x_2\|^2 + f(x_2) \leqslant \|\varphi_2\|^2 - 2\langle \varphi_2, x_1 \rangle + \|x_1\|^2 + f(x_1). \tag{2.5}$$

It follows from (2.4) and (2.5) that π_K^f is monotone.

(f_3) We first prove that $x \in \pi_K^f \varphi$ implies that

$$2\big\langle \varphi - J(x), x - y \big\rangle + f(y) - f(x) \geqslant 0, \quad \forall y \in K.$$

In fact, for any $y \in K$ and $t \in (0,1]$, it follows from the definition of $\pi_K^f \varphi$ that

$$G(\varphi, x) \leqslant G\big(\varphi, x + t(y-x)\big).$$

Thus,

$$\begin{aligned}\|\varphi\|^2 - 2\langle \varphi, x \rangle + \|x\|^2 + f(x) \leqslant \|\varphi\|^2 &- 2\big\langle \varphi, x + t(y-x) \big\rangle \\ &+ \big\|x + t(y-x)\big\|^2 + f\big(x + t(y-x)\big)\end{aligned}$$

and so

$$\begin{aligned}2\big\langle \varphi, t(y-x) \big\rangle + \|x\|^2 + f(x) &\leqslant \big\|x + t(y-x)\big\|^2 + f\big(x + t(y-x)\big) \\ &\leqslant \big\|x + t(y-x)\big\|^2 + (1-t)f(x) + tf(y).\end{aligned}$$

It follows that

$$2\big\langle \varphi, t(y-x) \big\rangle + \|x\|^2 \leqslant \big\|x + t(y-x)\big\|^2 + t\big(f(y) - f(x)\big). \tag{2.6}$$

Now from the properties of the normalised duality mapping, we have

$$\Big\langle J\big(x + t(y-x)\big), -t(y-x) \Big\rangle \leqslant \frac{1}{2}\|x\|^2 - \frac{1}{2}\big\|x + t(y-x)\big\|^2.$$

By (2.6), we get

$$2\Big\langle J\big(x + t(y-x)\big), y - x \Big\rangle \geqslant f(x) - f(y) + 2\big\langle (\varphi, y - x) \big\rangle.$$

Since B is smooth, we know that J is demi-continuous. Letting $t \to 0$ in the above inequality, we have

$$2\big\langle J(x) - \varphi, y - x \big\rangle + f(y) - f(x) \geqslant 0.$$

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Conversely, suppose

$$2\langle J(x) - \varphi, y - x\rangle + f(y) - f(x) \geqslant 0 \quad \text{for all} \quad y \in K.$$

Then

$$\|y\|^2 - \|x\|^2 \geqslant 2\langle J(x), y - x\rangle \geqslant 2\langle \varphi, y - x\rangle + f(x) - f(y)$$

which implies that $G(\varphi, x) \leqslant G(\varphi, y)$ for all $y \in K$, that is, $x \in \pi_K^f \varphi$.

(f_4) Assume $x_1, x_2 \in \pi_K^f \varphi$ and $x_1 = \mu x_2$ for some real number $\mu > 0$ with $\mu \neq 1$. Then $G(\varphi, x_1) = G(\varphi, x_2)$ and so

$$2\langle \varphi, x_2 - x_1\rangle = \|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1).$$

Replacing x_1 by μx_2 in above equality, we have

$$2(1 - \mu)\langle \varphi, x_2\rangle = (1 - \mu^2)\|x_2\|^2 + (1 - \mu) f(x_2).$$

Since $\mu \neq 1$, we obtain

$$2\langle \varphi, x_2\rangle = (1 + \mu)\|x_2\|^2 + f(x_2). \tag{2.7}$$

Let

$$x_3 = (x_2 + x_1)/2 = \big((1 + \mu)/2\big)x_2.$$

It follows from (f_1) that $x_3 \in \pi_K^f \varphi$ and so $G(\varphi, x_2) = G(\varphi, x_3)$. Similarly, we can get

$$2\langle \varphi, x_2\rangle = \Big(1 + \frac{1}{2}(1 + \mu)\Big)\|x_2\|^2 + f(x_2). \tag{2.8}$$

Now (2.7) and (2.8) imply that $1+\mu = 1+(1+\mu)/2$ and so $\mu = 1$, which is a contradiction to $\mu \neq 1$. Thus, (f_4) is true.

(f_5) Suppose there exists $\varphi \in B^*$ such that $\pi_K \varphi$ is not a singleton. Then for any $x_1, x_2 \in \pi_K^f \varphi$ and $x_1 \neq x_2$, we have $G(\varphi, x_1) = G(\varphi, x_2)$. This implies

$$2\langle \varphi, x_2 - x_1\rangle = \|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1). \tag{2.9}$$

By property (f_1), for any $t \in [0, 1]$, we know that $x_1 + t(x_2 - x_1) \in \pi_K \varphi$. Since $G\big(\varphi, x_1 + t(x_2 - x_1)\big) = G(\varphi, x_1)$,

$$2t\langle \varphi, x_2 - x_1\rangle = \big\|x_1 + t(x_2 - x_1)\big\|^2 + f\big(x_1 + t(x_2 - x_1)\big) - \|x_1\|^2 - f(x_1). \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$\begin{aligned} t\big(\|x_2\|^2 + f(x_2) - \|x_1\|^2 - f(x_1)\big) &\\ = \big\|x_1 + t(x_2 - x_1)\big\|^2 + f\big(x_1 + t(x_2 - x_1)\big) - \|x_1\|^2 - f(x_1) &\\ \leqslant \big\|x_1 + t(x_2 - x_1)\big\|^2 + t\Big(\big(x_2 - f(x_1)\big)\Big) - \|x_1\|^2. & \end{aligned}$$

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This implies that

$$t\|x_2\|^2 + (1-t)\|x_1\|^2 \leqslant \big\|x_1 + t(x_2 - x_1)\big\|^2 \tag{2.11}$$

and so

$$\begin{aligned}\big\|x_1 + t(x_2 - x_1)\big\|^2 &\leqslant \big(t\|x_2\| + (1-t)\|x_1\|\big)^2 \\ &= t^2\|x_2\|^2 + 2t(1-t)\,\|x_1\|\,\|x_2\| + (1-t)^2\|x_1\|^2 \\ &\leqslant t\|x_2\|^2 + (1-t)\|x_1\|^2 \\ &\leqslant \big\|x_1 + t(x_2 - x_1)\big\|^2.\end{aligned}$$

Thus,

$$t\|x_2\| + (1-t)\|x_1\| = \big\|x_1 + t(x_2 - x_1)\big\|.$$

Taking $t = 1/2$ in the above equation, we get

$$\|x_2\| + \|x_1\| = \|x_1 + x_2\|.$$

From (2.11), we know that if $x_1 = 0$, then $x_2 = 0$. Hence $x_1 \neq 0$ and $x_2 \neq 0$. Since B is strictly convex, according to Lemma 1.1, there exists some $\alpha > 0$ such that $x_1 = \alpha x_2$, which contradicts (f_4). This completes the proof. ∎

From (f_3), it is easy to prove the following result.

Theorem 2.3. *Let A be an arbitrary operator acting from the reflexive and smooth Banach space B to B^*, and $\xi \in B^*$. Then the point $x^* \in K \subset B$ is a solution of the variational inequality*

$$\langle Ax - \xi, y - x\rangle + f(y) - f(x) \geqslant 0, \quad \forall y \in K$$

if and only if x^* is a solution of the following inclusion

$$x \in \pi_K^f\big(J(x) - \frac{1}{2}(Ax - \xi)\big).$$

3. Applications

As an application of our results, in this section, we shall study the existence of solutions to the following variational inequality problem: Find $x^* \in K$ such that

$$\langle Ax^*, y - x\rangle + f(y) - f(x^*) \geqslant 0, \quad \forall y \in K, \tag{3.1}$$

where K is a nonempty, closed and convex subset of the Banach space B, and $A : K \to B^*$ and $f : K \to R \cup \{+\infty\}$ are two mappings.

DEFINITION 3.1: (KKM mapping) Let K be a nonempty subset of a linear space X. A set-valued mapping $G : K \to 2^X$ is said to be a KKM mapping if for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of K, we have

$$co\{y_1, y_2, \cdots, y_n\} \subseteq \bigcup_{i=1}^{n} G(y_i),$$

where $co\{y_1, y_2, \cdots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \cdots, y_n\}$.

LEMMA 3.1. (FanKKM Theorem [20].) *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \to 2^X$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is a compact subset of K, then $\bigcap_{y \in K} G(y) \neq \emptyset$.*

THEOREM 3.1. *Let K be a nonempty, closed and convex subset of a reflexive and smooth Banach space B with dual space B^*. Let $A : K \to B^*$ be a continuous mapping and $f : K \to R \cup \{+\infty\}$ be proper, convex, lower semi-continuous, and bounded from below. Let there exist an element $y_0 \in K$ such that*

$$\left\{x \in K : 2\left\langle Jx - \frac{1}{2}Ax, y_0 - x\right\rangle + \|x\|^2 + f(x) \leqslant \|y_0\| + f(y_0)\right\} \tag{3.2}$$

is a compact subset of K. Then the variational inequality (3.1) has a solution.

PROOF: From Theorem 2.3, we only need to prove that the following inclusion has a solution,

$$x \in \pi_K^f\left(J(x) - \frac{1}{2}Ax\right).$$

Define a set-valued mapping $W : K \to 2^K$ as follows:

$$W(y) = \left\{x \in K : G\left(Jx - \frac{1}{2}Ax, x\right) \leqslant G\left(Jx - \frac{1}{2}Ax, y\right)\right\}.$$

Clearly, for each given $y \in K$, $W(y)$ is nonempty. Next we prove that $W(y)$ is closed for each given $y \in K$. Suppose $\{x_n\} \in W(y)$ and $x_n \to x_0$ as $n \to \infty$. Then,

$$G\left(Jx_n - \frac{1}{2}Ax_n, x_n\right) \leqslant G\left(Jx_n - \frac{1}{2}Ax_n, y\right)$$

and so

$$-2\left\langle Jx_n - \frac{1}{2}Ax_n, x_n\right\rangle + \|x_n\|^2 + f(x_n) \leqslant -2\left\langle Jx_n - \frac{1}{2}Ax_n, y\right\rangle + \|y\|^2 + f(y).$$

Since J and A are continuous and f is lower-semi-continuous,

$$-2\left\langle Jx_0 - \frac{1}{2}Ax_0, x_0\right\rangle + \|x_0\|^2 + f(x_0) \leqslant -2\left\langle Jx_0 - \frac{1}{2}Ax_0, y\right\rangle + \|y\|^2 + f(y).$$

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Hence,

$$G\Big(Jx_0 - \frac{1}{2}Ax_0, x_0\Big) \leqslant G\Big(Jx_0 - \frac{1}{2}Ax_0, y\Big),$$

which implies that $x_0 \in W(y)$.

Next we prove that the map $W : K \to 2^K$ is a KKM mapping in K. In fact, suppose $y_1, y_2, \cdots, y_n \in K$ and $0 < \lambda_1, \lambda_2, \cdots \lambda_n \leqslant 1$ with $\sum_{i=1}^{n} \lambda_i = 1$. Let $v = \sum_{i=1}^{n} \lambda_i y_i$. By property (viii) of G, we have

$$\begin{aligned} G(Jv - \frac{1}{2}Av, v) &= G\Big(Jv - \frac{1}{2}Av, \sum_{i=1}^{n} \lambda_i y_i\Big) \\ &\leqslant \sum_{i=1}^{n} \lambda_i G\Big(Jv - \frac{1}{2}Av, y_i\Big). \end{aligned}$$

This implies that

$$G\Big(Jv - \frac{1}{2}Av, v\Big) \leqslant \max_{1\leqslant i\leqslant n} G\Big(Jv - \frac{1}{2}Av, y_i\Big).$$

Hence there exists at least one number $j = 1, 2, \cdots, n$, such that

$$G\Big(Jv - \frac{1}{2}Av, v\Big) \leqslant G\Big(Jv - \frac{1}{2}Av, y_j\Big),$$

that is, $v \in W(y_j)$. Thus, W is a KKM mapping.

If $x \in W(y_0)$, then $G(Jx - (1/2)Ax, x) \leqslant G(Jx - (1/2)Ax, y_0)$. From the definition of G, we have

$$\begin{aligned} &\Big\|Jx - \frac{1}{2}Ax\Big\|^2 - 2\Big\langle Jx - \frac{1}{2}Ax, x\Big\rangle + \|x\|^2 + f(x) \\ &\qquad \leqslant \Big\|Jx - \frac{1}{2}Ax\Big\|^2 - 2\Big\langle Jx - \frac{1}{2}Ax, y_0\Big\rangle + \|y_0\|^2 + f(y_0). \end{aligned}$$

Simplifying the above inequality, we have

$$2\Big\langle Jx - \frac{1}{2}Ax, y_0 - x\Big\rangle + \|x\|^2 + f(x) \leqslant \|y_0\|^2 + f(y_0).$$

We get that

$$W(y_0) = \Big\{x \in K : 2\Big\langle Jx - \frac{1}{2}Ax, y_0 - x\Big\rangle + \|x\|^2 + f(x) \leqslant \|y_0\| + f(x_0)\Big\}.$$

By condition (3.2), we know that $W(y_0)$ is compact. It follows from Lemma 3.1 that $\bigcap_{y\in K} W(y) \neq \emptyset$ and so there exits at least one $x^* \in \bigcap_{y\in K} W(y)$, that is,

$$G\Big(Jx^* - \frac{1}{2}Ax^*, x^*\Big) \leqslant G\Big(Jx^* - \frac{1}{2}Ax^*, y\Big), \quad \forall y \in K.$$

From the definition of the generalised f-projection operator π_K^f, we have

$$x^* \in \pi_K^f\Big(Jx^* - \frac{1}{2}Ax^*\Big).$$

This completes the proof. ∎

THEOREM 3.2. *Let B be a reflexive and smooth Banach space with dual space B^* and K be a nonempty, closed and convex subset that contains the origin θ of B. Let $A : K \to B^*$ be a continuous mapping and $f : K \to R \cup \{+\infty\}$ be proper, convex, lower semi-continuous, and bounded from below. If the set*

$$\{x \in K : \langle Ax, x\rangle + f(x) \leqslant \|x\|^2 + f(\theta)\} \tag{3.3}$$

is compact, then variational inequality (3.1) has a solution.

PROOF: Taking $y_0 = \theta$ in condition (3.2) and noticing that $\langle Jx, x\rangle = \|x\|^2$, it follows from condition (3.3) that all conditions of Theorem 3.1 hold. Thus Theorem 3.1 implies that the conclusion of Theorem 3.2 hold. This completes the proof. ∎

From Theorem 3.2, it is easy to have the following result.

THEOREM 3.3. *Let B be a reflexive and smooth Banach space with dual space B^* and K be a nonempty, closed and convex cone of B. Let $A : K \to B^*$ be a continuous mapping and $f : K \to R \cup \{+\infty\}$ be proper, convex, lower semi-continuous and bounded from below. If*

$$\{x \in K : \langle Ax, x\rangle + f(x) \leqslant \|x\|^2 + f(\theta)\}$$

is compact, then variational inequality (3.1) has a solution.

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https://doi.org/10.1017/S0004972700038892 Published online by Cambridge University Press

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