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THE GENERALIZED BASIS REDUCTION ALGORITHM

by László Lovász and Herbert E. Scarf

June 1990

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László Lovász and Herbert E. Scarf*

I. Introduction

Let C be a compact convex body in \mathbb{R}^n , of positive volume and symmetric about the origin, and let L be the lattice of integer vectors in \mathbb{R}^n . The body can be used to define a distance function $F(x) = \inf(\lambda \geq 0 \mid x/\lambda \in C)$, with the properties:

1. $F(x)$ is convex,
2. $F(-x) = F(x)$,
3. $F(tx) = tF(x)$ for $t > 0$.

The dual body C^* is defined to be $\{y \mid y \cdot x \leq 1 \text{ for all } x \in C\}$, and the dual distance function is $F^*(y) = \max_{x \in C} y \cdot x$.

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In order to determine a smallest non-zero lattice point according to the distance function F , we introduce the concept of a reduced basis with respect to F . Let b^1, b^2, \dots, b^n be a basis for the integer lattice L . For each i we project C , along the vectors b^1, \dots, b^{i-1} , into the affine space $E_i = \langle b^i, \dots, b^n \rangle$ obtaining C_i . In other words, $x = x_i b^i + \dots + x_n b^n \in C_i$ if and only if there are $\alpha_1, \dots, \alpha_{i-1}$ such that $x + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1} \in C$. The lattice L_i , obtained by projecting L along b^1, \dots, b^{i-1} into $\langle b^i, \dots, b^n \rangle$, is the set of integral linear combinations of the vectors b^i, \dots, b^n .

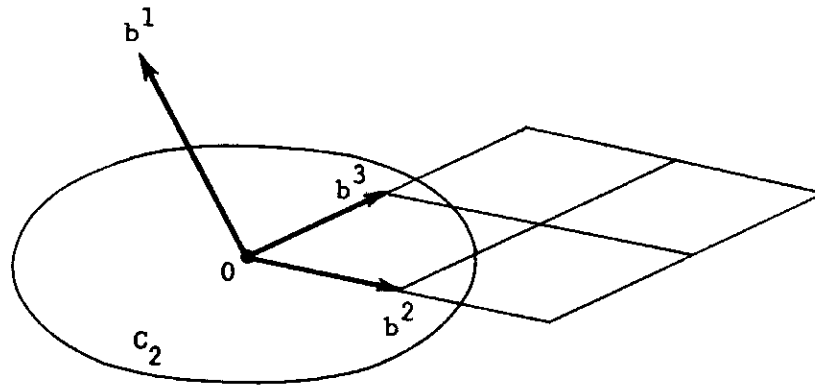


FIGURE 1

The distance function $F_i(x)$, associated with the projected body C_i , is defined for $x \in E_i$ by

$$F_i(x) = \min F(x + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}),$$

with the minimum taken over $\alpha_1, \dots, \alpha_{i-1}$. The function may, of course, be defined for all x in R^n by the same formula; if $x = \sum x_j b^j$, then $F_i(x)$ will be independent of x_1, \dots, x_{i-1} .

Fix $0 < \epsilon < 1/2$. The basis is reduced, for this ϵ , if the following two conditions hold for $i = 1, \dots, n-1$:

1. $F_i(b^{i+1} + \mu b^i) \geq F_i(b^{i+1})$ for integral μ , and
2. $F_i(b^{i+1}) \geq (1-\epsilon)F_i(b^i)$.

If C is an ellipsoid - or, alternatively, if C is the unit ball and the lattice is a general lattice in R^n - this definition of a reduced basis is identical with the definition in A. K. Lenstra, H. W. Lenstra, Jr. and L. Lovász(1982).

In Section II, we discuss the properties of a reduced basis, demonstrating, in particular, that for such a basis b^1 is an approximation to the shortest non-zero lattice point. In addition, b^i is an approximation to a lattice point realizing the i th successive minimum, according to Minkowski. We also provide a polynomial algorithm for fixed n which finds the shortest non-zero lattice point rather than an approximation.

In Section III, the basis reduction algorithm is described and shown to execute in polynomial time, for fixed n . In Section IV, we examine a special basis - the Korkine-Zolotarev basis - associated with a distance function F . Using the Korkine-Zolotarev basis, we provide an alternative demonstration of a theorem to be found in Kannan and Lovász (1988), that a lattice-free body K , in R^n , has associated with it a non-zero lattice point h , such that the width of the body in the direction h satisfies

$$\max_{x \in K} \{h \cdot x\} - \min_{x \in K} \{h \cdot x\} \leq c_0 n(n+1)/2,$$

with c_0 , a universal constant.

Lenstra's polynomial algorithm (H. W. Lenstra, Jr.(1983)) for integer programming with a fixed number of variables makes use of the spherical basis reduction algorithm. He begins by a preliminary reduction to the

problem of determining a lattice point in a convex polyhedron K , in \mathbb{R}^n , defined by a system of linear inequalities $Ax \leq c$. To find such a lattice point, the polyhedron is approximated by an ellipsoid E , and a hyperplane with integer normals h is found so that the width of the ellipsoid in the direction h ,

$$\max_{x \in E} (h \cdot x) - \min_{x \in E} (h \cdot x)$$

is as small as possible, aside from a factor depending only on the number of variables, n . If this width is sufficiently large, the polyhedron is sure to contain a lattice point. In the alternative case, in which the width is not large, we consider the intersections of the polyhedron with the hyperplanes $hx = h_0$, with h_0 assuming all integral values between $\min_{x \in E} h \cdot x$ and $\max_{x \in E} h \cdot x$. The n dimensional problem is thereby reduced to the problem of determining a lattice point in one of a small number of $n-1$ dimensional polyhedra. Each of these polyhedra is then approximated by its own ellipsoid and the algorithm continues.

A non-zero lattice point h , which minimizes the width of the ellipsoid E , is a shortest non-zero lattice point for the body $(E-E)^*$, itself an ellipsoid. If this latter ellipsoid is transformed to a sphere by a linear transformation, an approximation to the shortest non-zero lattice point can be found using the spherical basis reduction algorithm for a general lattice.

The arguments of this note can be used to find a short non-zero lattice point for the body $(K - K)^*$ directly, thereby avoiding the series of

ellipsoidal approximations. The basis reduction algorithm is applied to $C = (K - K)^*$, where $K = \{x | Ax \leq c\}$, with the distance functions

$$F_i(\xi) = \min_{\alpha} F_1(\xi + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}),$$

$$\min_{\alpha} \max\{(\xi + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}) \cdot (x-y) | Ax \leq c, Ay \leq c\}$$

$$= \min_{\alpha, t, u} c \cdot (t + u), \text{ subject to } t, u \geq 0,$$

$$tA = \xi + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1},$$

$$uA = -(\xi + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}), \text{ (from the duality}$$

theorem for linear programming.)

$$= \min_{\alpha, t, u} c \cdot (t + u), \text{ subject to } t, u \geq 0,$$

$$tA - \alpha_1 b^1 - \dots - \alpha_{i-1} b^{i-1} = \xi,$$

$$uA + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1} = -\xi,$$

= max $\xi \cdot (x - y)$, subject to

$$Ax \leq c, Ay \leq c, b^1 \cdot (x - y) = 0, \dots, b^{i-1} \cdot (x - y) = 0 \text{ (using the}$$

duality theorem again.).

The general basis reduction algorithm requires the solution of many linear programs, and there are tradeoffs between using an ellipsoidal approximation to K , or working directly with the body, itself, to resolve the question of whether K contains a lattice point. A number of computational experiments are currently being attempted on integer programming problems of moderately large size to evaluate the merits of the two procedures.

II. Properties of a Reduced Basis

Theorem 1. Let b^1, \dots, b^n be a reduced basis. Then

$$F_{i+1}(b^{i+1}) \geq (1/2 - \epsilon) F_i(b^i) \text{ for } i = 1, \dots, n-1.$$

Proof: We have the identity

$$\min F_i(x + \alpha b^i) = F_{i+1}(x)$$

with the minimum taken over all real α . Since we can round α to the nearest integer μ , it follows that

$$(1) \quad \min F_i(x + \mu b^i) \leq F_{i+1}(x) + 1/2 F_i(b^i),$$

with the minimum taken over integer μ . If x is taken to be b^{i+1} then (1), in conjunction with the definition of a reduced basis, tells us that

$$\begin{aligned} (1-\varepsilon)F_i(b^i) &\leq F_i(b^{i+1}) = \min F_i(b^{i+1} + \mu b^i) \\ &\leq F_{i+1}(b^{i+1}) + 1/2 F_i(b^i). \quad \otimes \end{aligned}$$

Theorem 2. Let b^1, \dots, b^n be a reduced basis, and let

$$\lambda_1 = \min F(h), \text{ for all non-zero lattice points } h.$$

$$\text{Then } \lambda_1 \geq F(b^1) \cdot (1/2 - \varepsilon)^{n-1}.$$

Proof: Let $h = l_1 b^1 + \dots + l_k b^k$, with l_1, \dots, l_k integral and l_k different from zero, be a shortest non-zero lattice point according to the distance function F . Then

$$\lambda_1 = F(h) \geq F_k(h) = |l_k| F_k(b^k) \geq F(b^1) \cdot (1/2 - \varepsilon)^{k-1}. \quad \otimes$$

Theorem 2 states that the first vector, b^1 , in a reduced basis is an approximation to the shortest non-zero lattice point. In a similar fashion the other basis vectors approximate the successive minima of the lattice with respect to the distance function.

Definition: $\lambda_1, \dots, \lambda_n$ are the successive minima of the lattice with respect to F if there are lattice points h^1, \dots, h^n , with $\lambda_i = F(h^i)$, such that for each $i = 1, \dots, n$, h^i is the shortest lattice point which is linearly independent of h^1, \dots, h^{i-1} .

An equivalent definition is $\lambda_i = \min \{ \lambda \mid F(x) \leq \lambda \text{ contains } i \text{ linearly independent lattice points} \}$. The successive minima λ_i are uniquely defined by the distance function F , but there may be more than one set of lattice points h^i which realize these values. We have the following generalization of Theorem 2.

Theorem 3. Let b^1, \dots, b^n be a reduced basis.

Then for $i = 1, \dots, n$,

$$F_i(b^i)(1/2-\epsilon)^{n-i} \leq \lambda_i \leq F_i(b^i)/(1/2-\epsilon)^{i-1}.$$

Proof: We begin by constructing a basis c^1, \dots, c^n for the lattice with

$$(2) \quad F_1(c^i) \leq F_i(b^i)/(1/2-\epsilon)^{i-1},$$

thereby demonstrating the right hand side of the inequality in Theorem 3.

Again we use the inequality

$$\min F_i(x + \mu b^i) \leq F_{i+1}(x) + 1/2 F_i(b^i),$$

with the minimum taken over integer μ . If x is taken to be b^{i+1} then this inequality implies

$$F_i(b^{i+1} + \mu_{i+1,i} b^i) \leq F_{i+1}(b^{i+1}) + 1/2 F_i(b^i)$$

for some integral $\mu_{i+1,i}$. Apply the inequality again with $i+1$ replaced by i and $x = b^{i+1} + \mu_{i+1,i} b^i$, obtaining

$$F_{i-1}(b^{i+1} + \mu_{i+1,i} b^i + \mu_{i+1,i-1} b^{i-1}) \leq \\ F_{i+1}(b^{i+1}) + 1/2 F_i(b^i) + 1/2 F_{i-1}(b^{i-1})$$

for some integral $\mu_{i+1,i}$ and $\mu_{i+1,i-1}$. Continuing, we see that

$$(3) \quad F_1(b^{i+1} + \sum_{j=1}^i \mu_{i+1,j} b^j) \leq F_{i+1}(b^{i+1}) + (1/2) \sum_{j=1}^i F_j(b^j)$$

for some integral $\mu_{i+1,j}$, $j=1, \dots, i$.

We use this construction to define

$$(4) \quad c^{i+1} = b^{i+1} + \sum_{j=1}^i \mu_{i+1,j} b^j.$$

Estimating $\Sigma F_j(b^j)$ by means of Theorem 1, we see that

$$\begin{aligned} F_1(c^{i+1}) &\leq F_{i+1}(b^{i+1}) \cdot (1 + 1/2 \Sigma 1/(1/2-\epsilon)^{i+1-j}) \\ &\leq F_{i+1}(b^{i+1})/(1/2-\epsilon)^i. \end{aligned}$$

To demonstrate the inequalities on the left hand side of Theorem 3, we write

$$\begin{aligned} h^1 &= l_{11}b^1 + l_{12}b^2 + \dots + l_{1n}b^n \\ &\vdots \\ h^i &= l_{i1}b^1 + l_{i2}b^2 + \dots + l_{in}b^n \\ &\vdots \\ h^n &= l_{n1}b^1 + l_{n2}b^2 + \dots + l_{nn}b^n, \end{aligned}$$

with l_{ij} integral and with h^i linearly independent lattice points which realize the successive minima, i.e. $F_1(h^i) = \lambda_i$.

For each index i , there must be a pair of indices j and k with $j \leq i \leq k$ such that $l_{jk} \neq 0$, since otherwise

$$\begin{aligned} h^1 &= l_{11}b^1 + \dots + l_{1,i-1}b^{i-1} \\ &\vdots \\ h^i &= l_{i1}b^1 + \dots + l_{i,i-1}b^{i-1}, \end{aligned}$$

and the vectors h^1, \dots, h^i would be linearly dependent. For each i , therefore, let k be the largest index such that $l_{jk} \neq 0$ for some $j \leq i \leq k$.

But then, since $|l_{jk}| \geq 1$,

$$\begin{aligned} \lambda_i \geq \lambda_j = F_1(h^j) &\geq F_k(h^j) = |l_{jk}| \cdot F_k(b^k) \\ &\geq F_i(b^i) \cdot (1/2-\epsilon)^{k-i} \\ &\geq F_i(b^i) \cdot (1/2-\epsilon)^{n-i}. \end{aligned}$$

This demonstrates Theorem 3. \otimes

According to Minkowski, the successive minima satisfy the inequality

$$\lambda_1 \dots \lambda_n \cdot \text{vol}(C) \leq 2^n.$$

We can show that the basis c^1, \dots, c^n , defined by (4), approximates this result in the following sense:

Theorem 4. Let b^1, \dots, b^n be a reduced basis with respect to F . Then the basis c^1, \dots, c^n satisfies

$$F_1(c^1) \cdot F_1(c^2) \dots F_1(c^n) \cdot \text{vol}(C) \leq 2^n / (1/2 - \epsilon)^{n(n-1)/2}.$$

Proof: The proof depends on the fact that for any basis

$$(5) \quad F_1(b^1) \cdot F_2(b^2) \dots F_n(b^n) \cdot \text{vol}(C) \leq 2^n.$$

To demonstrate (5), let us assume, by induction, that this inequality is satisfied for the $n-1$ dimensional body C_2 obtained by projecting b^1 into the affine space $\langle b^2, \dots, b^n \rangle$, so that

$$F_2(b^2) \dots F_n(b^n) \cdot \text{vol}(C_2) \leq 2^{n-1}.$$

But then $\text{vol}(C) = \int_{C_2} l(x) dx$, with $l(x)$ the length of the intersection of the line $x + \alpha b^1$ with C . From the symmetry and convexity of C , $l(x) \leq l(0) = 2/F_1(b^1)$ so that $\text{vol}(C) \leq 2\text{vol}(C_2)/F_1(b^1)$, thereby demonstrating (5).

Theorem 4 follows from the previously established inequality

$$F_1(c^i) \leq F_i(b^i) / (1/2 - \epsilon)^{i-1}. \otimes$$

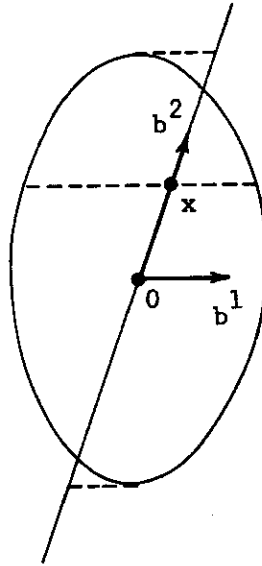


FIGURE 2

If we are given a reduced basis, then a shortest vector h^1 can be calculated in polynomial time for fixed n . We do this by establishing bounds on the coordinates of lattice points satisfying $F_1(h) \leq F_1(b^1)$. Let $h = \sum l_j b^j$ be such a vector. Then $F_1(b^1) \geq F_1(h) \geq F_n(h) = |l_n| \cdot F_n(b^n)$ so that $|l_n| \leq F_1(b^1)/F_n(b^n) \leq 1/(1/2-\epsilon)^{n-1}$.

Now let us suppose that the coordinates l_n, \dots, l_{i+1} have been selected. We find bounds for l_i as follows: Find the real α which minimizes $F_i(l_n b^n + \dots + l_{i+1} b^{i+1} + \alpha b^i)$. If the minimum is greater than $F_1(b^1)$ then there is no h with these final $n-i$ coordinates satisfying $F_1(h) \leq F_1(b^1)$. If, on the other hand, the minimum is less than or equal to $F_1(b^1)$ then since $F_i(l_n b^n + \dots + l_{i+1} b^{i+1} + l_i b^i) = F_i(h) \leq F_1(h) \leq F_1(b^1)$, and F_i is symmetric and convex, we obtain

$$|l_i - \alpha| \cdot F_i(b^i) \leq 2F_1(b^1) \text{ or}$$

$$|l_i - \alpha| \leq 2/(1/2-\epsilon)^{i-1}.$$

This provides us with a tree of depth n and with a "small" number of branches at each node in which to search for the coordinates of the shortest vector. If the tree is used to calculate the shortest non-zero lattice point in actual numerical examples, the estimate $(1/2-\epsilon)^{i-1}$ should be replaced by $F_i(b^i)/F_1(b^1)$, which may be considerably smaller.

If we look for the i th successive minimum by considering those h with $F_1(h) \leq F_i(b^i)/(1/2-\epsilon)^{i-1}$ we obtain precisely the same set of inequalities for l_n, \dots, l_i , but we do not have similar bounds for the first $i-1$ coordinates of h . This yields a "small" number of hyperplanes of dimension $i-1$, one of which contains a lattice point which realizes the i th successive minimum.

III. The Basis Reduction Algorithm

An algorithm for finding a reduced basis may easily be described. We begin with an initial basis a^1, a^2, \dots, a^n for the lattice, and move through a sequence of bases b^1, b^2, \dots, b^n according to the following rules: At each step of the algorithm, we consider the first index i for which one of the conditions

1. $F_i(b^{i+1} + \mu b^i) \geq F_i(b^{i+1})$ for integral μ , and
2. $F_i(b^{i+1}) \geq (1-\varepsilon)F_i(b^i)$.

is not satisfied.

If the first condition is not satisfied, we replace b^{i+1} by $b^{i+1} + \mu b^i$, with μ the integer which minimizes $F_i(b^{i+1} + \mu b^i)$. If, after this replacement, the second condition obtains, we move to level $i+1$. If the second condition is not satisfied, we interchange b^i and b^{i+1} and move to the preceding level $i-1$, unless $i = 1$, in which case we remain at level 1.

In order to demonstrate convergence of the algorithm we consider the vector

$$F_1(b^1), \dots, F_i(b^i), F_{i+1}(b^{i+1}), \dots, F_n(b^n),$$

and remark that the maximum value of the components of the vector does not increase at any step of the basis reduction algorithm. If we replace b^{i+1} by $b^{i+1} + \mu b^i$, none of the terms change; if b^i and b^{i+1} are interchanged, $F_i(b^i)$ becomes $F_i(b^{i+1}) \leq (1-\varepsilon)F_i(b^i)$ and $F_{i+1}(b^{i+1})$ is replaced by

$$\begin{aligned} & \min F(b^i + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1} + \alpha_{i+1} b^{i+1}) \\ & \leq \min F(b^i + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}) = F_i(b^i). \end{aligned}$$

It follows that at any step in the algorithm, $\max F_i(b^i) \leq \max F_i(a^i)$ equal to, say, U .

The basis reduction algorithm is known to converge in polynomial time, including the number of variables, n , for $F(x) = |x|$, and a general lattice given by an integer basis. The argument is based on two observations: first, that an interchange between b^i and b^{i+1} preserves the values of $F_j(b^j)$ for all indices other than i and $i+1$, and secondly, that for $F(x) = |x|$, the product $F_i(b^i)F_{i+1}(b^{i+1})$ is constant when the vectors b^i and b^{i+1} are exchanged. This permits us to deduce that $D(b^1, \dots, b^n) = \Pi(F_i(b^i))^{n-i}$ decreases by a factor of $(1-\epsilon)$ at each interchange. It is easy to show that $\Pi(F_i(b^i))^{n-i} \geq 1$, for any basis, from which the polynomial convergence follows readily.

Constancy of $F_i(b^i)F_{i+1}(b^{i+1})$ is not valid for a general distance function, and the basis reduction algorithm is not known to execute in polynomial time in the number of variables n . But the algorithm may be shown to be polynomial in the data of the problem for fixed n . We present two arguments for this conclusion, both of which depend on establishing lower bounds for the possible values assumed by $F_i(b^i)$ during the course of the algorithm.

To obtain such a lower bound, assume that $C \subset B(R)$, the ball of radius R . Then $F(x) \geq |x|/R$. Now let b^1, \dots, b^n be any basis for the lattice satisfying $F_i(b^i) \leq U$, and let c^1, \dots, c^n , with $c^{i+1} = b^{i+1} + \sum_{j=1}^i \mu_{i+1,j} b^j$, be the basis constructed in the proof of Theorem 3, which satisfies $F_i(c^i) = F_i(b^i)$ and $F_1(c^i) \leq F_1(b^i) + (1/2)\sum_{j=1}^{i-1} F_j(b^j) \leq nU$. We have, therefore, $|c^i| \leq nUR$.

We estimate $F_i(b^i)$, from below, as follows:

$$\begin{aligned} F_i(b^i) &= \min F(b^i + \alpha_1 b^1 + \dots + \alpha_{i-1} b^{i-1}) \\ &= \min F(b^i + \alpha_1 c^1 + \dots + \alpha_{i-1} c^{i-1}) \\ &\geq \min |(b^i + \alpha_1 c^1 + \dots + \alpha_{i-1} c^{i-1})|/R. \end{aligned}$$

But $\min |(b^i + \alpha_1 c^1 + \dots + \alpha_{i-1} c^{i-1})|$ is the distance between the vector b^i and the space $\langle c^1, \dots, c^{i-1} \rangle$ and is therefore equal to

$$[G(c^1, \dots, c^{i-1}, b^i)/G(c^1, \dots, c^{i-1})]^{1/2},$$

where the Gramian

$$G(x^1, \dots, x^i) = \det[(x^j, x^k)]_{j,k=1}^i.$$

Since c^1, \dots, c^{i-1} and b^i are integral, $G(c^1, \dots, c^{i-1}, b^i) \geq 1$. Moreover,

$$G(c^1, \dots, c^{i-1})^{1/2} \leq |c^1| \dots |c^{i-1}| \leq (nUR)^{i-1}.$$

It follows that

$$F_i(b^i) \geq 1/[R(nRU)^{i-1}] \geq 1/[R(nRU)^{n-1}] = V.$$

We have already shown that each component of

$$F_1(b^1), \dots, F_i(b^i), F_{i+1}(b^{i+1}), \dots, F_n(b^n)$$

is bounded above by $U = \max F_i(a^i)$ throughout the course of the algorithm.

Moreover, the first term in the sequence to change at any iteration decreases by a factor of $(1 - \epsilon)$. Our first argument for polynomial convergence is to observe that the maximal number of interchanges is therefore

$$[\log(U/V)/\log(1/(1-\epsilon))]^n.$$

(Simply record the times at which the first two basis vectors b^1 and b^2 are interchanged. Between any consecutive pair of such times we are faced with an identical problem with $n-1$ variables.) Using our particular lower bound V we see that the number of interchanges of the basis reduction algorithm is

bounded above by

$$(6) \quad [n \log(nUR) / \log(1/(1-\epsilon))]^n.$$

The second argument for polynomiality, which achieves a different bound, depends on the observation that for a general distance function $F(x)$, the product $F_i(b^i)F_{i+1}(b^{i+1})$ increases by a factor less than or equal to 2 after an interchange of b^i and b^{i+1} . The argument makes use of the following Theorem.

Theorem 5. Let S be a compact convex set in R^k , which is symmetric about the origin, and let x and y be two linearly independent vectors on the boundary of S . Define

$$d_x = \max\{\alpha \mid \alpha x + \beta y \in S \text{ for some } \beta\} \text{ and}$$

$$d_y = \max\{\beta \mid \alpha x + \beta y \in S \text{ for some } \alpha\}.$$

$$\text{Then } 1/2 \leq d_x/d_y \leq 2.$$

Proof: If $d_x x + \beta y \in S$, then either $\beta \geq d_x - 1$, or $\beta \leq 1 - d_x$. For if $0 \leq \beta < d_x - 1$, x is a strict convex combination of 0 , $-y$, $d_x x + \beta y$, and is therefore interior to S ; if $1 - d_x < \beta \leq 0$, x is a strict convex combination of 0 , y , $d_x x + \beta y$ and is again interior to S . In the first case, $d_y \geq d_x - 1$. In the second case, since S is symmetric, the vector $-(d_x x + \beta y) \in S$ and again $d_y \geq d_x - 1$. It follows that

$$d_x/d_y \leq 1 + 1/d_y \leq 2,$$

since $d_y \geq 1$. The lower inequality follows from interchanging x and y . \odot

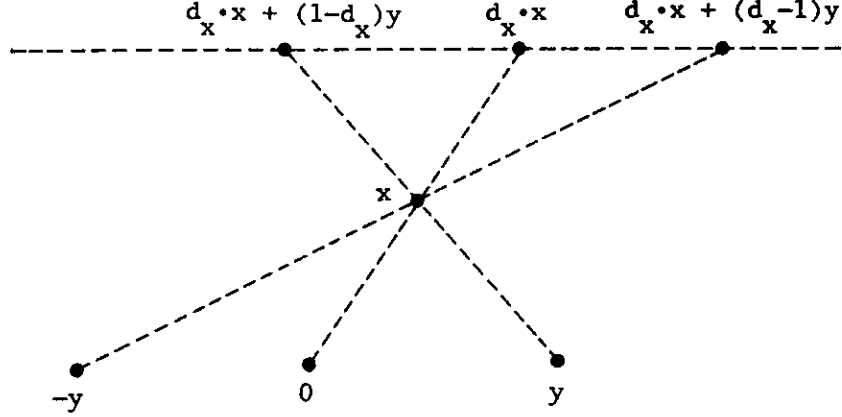


FIGURE 3

Theorem 5 may be used to show that the product

$$F_i(b^i)F_{i+1}(b^{i+1})$$

increases by a factor not larger than 2 at any step of the basis reduction algorithm in which b^i and b^{i+1} are interchanged. Let $S = C_i \subset E_i = \langle b^i, \dots, b^n \rangle$. Assume, without loss of generality, that $F_i(b^i) = 1$, and take $y = b^i$ and $x = b^{i+1}/F_i(b^{i+1})$, both of which are on the boundary of C_i . But then $F_{i+1}(x) = 1/d_x$, and $F_{i+1}^\#(b^i) = 1/d_y$, with $F_{i+1}^\#$ the distance function associated with the projection of C into $\langle b^i, b^{i+2}, \dots, b^n \rangle$. It follows that

$$(7) \quad \begin{aligned} & F_i(b^{i+1})F_{i+1}^\#(b^i)/F_i(b^i)F_{i+1}(b^{i+1}) \\ &= [F_i(b^{i+1})/d_y]/[F_i(b^{i+1})/d_x] = d_x/d_y \leq 2. \end{aligned}$$

Now consider

$$D(b^1, \dots, b^n) = \Pi(F_i(b^i))^{\gamma^{n-i}},$$

with $\gamma = 2 + 1/\log(1/(1-\varepsilon))$. It is a straightforward computation to show that our estimate (7) implies that $D(b^1, \dots, b^n)$ decreases by a factor of at least $(1-\varepsilon)$ at each interchange required by the basis reduction algorithm.

Since $V \leq F_i(b^i) \leq U$ at each step of the algorithm, we see that the number of interchanges is bounded above by

$$\begin{aligned} & [(\gamma^n - 1)/(\gamma - 1)] \log(U/V) / \log(1/(1 - \epsilon)) \\ & \leq [(\gamma^n - 1)/(\gamma - 1)] n \log(nUR) / \log(1/(1 - \epsilon)), \end{aligned}$$

an estimate which is much better than our previous estimate (6) in terms of its dependence on UR . The preceding discussion has established the following theorem:

Theorem 6. The basis reduction algorithm terminates in a polynomial number of steps, for fixed n .

Since the number of possible values of the vector

$$F_1(b^1), \dots, F_i(b^i), F_{i+1}(b^{i+1}), \dots, F_n(b^n)$$

is finite, the basis reduction algorithm executes in finite time even when $\epsilon = 0$. Bárány has recently demonstrated geometric convergence when $\epsilon = 0$ for the case of two variables. Consider two successive steps of the algorithm. Assume that the initial basis is (b^1, b^2) , with b^2 the smallest lattice point on the line $b^2 + \alpha b^1$, and that $\delta_1 F(b^1) < F(b^2) < F(b^1)$. After the first interchange the basis is given by (b^2, b^1) . Let μ^* minimize $F(b^1 + \mu b^2)$ for integral μ and assume that $\delta_2 F(b^2) < F(b^1 + \mu^* b^2) < F(b^2)$ so that another interchange is required leading to the basis $(b^1 + \mu^* b^2, b^2)$. Finally, let μ be the integer which minimizes $F(b^2 + \mu(b^1 + \mu^* b^2))$.

Theorem 7 (Bárány). If $\delta_1 \delta_2 > 1/2$ then $\mu = 0$ or 1 . In either case the basis $(b^1 + \mu^* b^2, b^2 + \mu(b^1 + \mu^* b^2))$ is reduced.

Proof: We argue, first of all, that $|\mu^*| > 1$. If $\mu^* = 0$, there is a contradiction between $F(b^1 + \mu^* b^2) < F(b^2)$ and $F(b^2) < F(b^1)$. If $\mu^* = 1$, then b^2 is not the shortest integral vector on the line $b^2 + \alpha b^1$, and similarly for $\mu^* = -1$. To be specific, let us now assume that $\mu^* \leq -2$.

Consider the convex function $g(\alpha) = F(b^2 + \alpha(b^1 + \mu^* b^2))$. We have $g(0) = F(b^2)$ and from our assumptions $g(0) > \delta_1 F(b^1) > \delta_2 \delta_1 F(b^1)$. Also $g(1) = F((b^1 + (\mu^* + 1)b^2)) \geq F(b^1 + \mu^* b^2) > \delta_2 F(b^2) > \delta_2 \delta_1 F(b^1)$. But $g(-1/\mu^*) = (1/|\mu^*|)F(b^1) \leq (1/2)F(b^1)$. It follows, from the convexity of $g(\alpha)$, that if $\delta_1 \delta_2 > 1/2$, the integral minimum of $g(\alpha)$ is at $\alpha = 0$ or $\alpha = 1$. In the first case, the basis $(b^1 + \mu^* b^2, b^2)$ is reduced since $F(b^1 + \mu^* b^2) < F(b^2)$; in the second case, the basis $(b^1 + \mu^* b^2, b^1 + (\mu^* + 1)b^2)$ is reduced because $F(b^1 + \mu^* b^2) \leq F(b^1 + (\mu^* + 1)b^2) \otimes$

Theorem 7 implies that in $2p$ steps of the basis reduction algorithm, $F(b^1)$ will decrease by a factor of at least $(1/2)^p$. Since $F(h) \geq 1/R$ for any lattice point h , we have geometric convergence of the algorithm for $n = 2$ and $\epsilon = 0$. No argument is currently available for higher dimensions, and $\epsilon = 0$, unless we revise the order in which the steps of the algorithm are executed. For example, following a suggestion made by Bárány, let us assume that we always select the largest index i for which one of the conditions of a reduced basis is not satisfied. It follows that if we ever return to level 1, the basis b^2, \dots, b^n is reduced with $\epsilon = 0$ for the $n-1$ dimensional problem defined by C_2 . If an interchange of b^1 and b^2 is then required, two possible cases arise:

1. $F_1(b^2) \geq (1-\delta)F_1(b^1)$ for some fixed $1/2 < \delta < 1$. But then the basis b^1, \dots, b^n will be δ -reduced for the original problem. Our previous analysis shows that there are a finite number, $N(n, \delta)$ of lattice points h , such that $F_1(h) < F_1(b^1)$, and, therefore, the algorithm requires an exchange of b^1 and b^2 not more than $N(n, \delta)$ times.

2. At each return to level 1, we have $F_1(b^2) < (1-\delta)F_1(b^1)$, and

therefore the number of returns to level 1 is bounded above by

$$\log(U/V)/\log(1/(1-\delta)).$$

We then use an inductive argument on n to achieve polynomial bounds on the running time of the algorithm for $\epsilon = 0$ and fixed n .

IV. The Korkine-Zolotarev Basis

A special basis for a lattice, the Korkine-Zolotarev basis, has been used very successfully by Lagarias, Lenstra and Schnorr (1990) to improve some classical estimates in the Geometry of Numbers relating the successive minima of a body C and its dual body $C^* = \{y \mid y \cdot x \leq 1 \text{ for all } x \in C\}$. In their analysis they approximate a general body by an ellipsoid, transform the ellipsoid to a sphere by a linear transformation and use specific properties of the spherical norm. We shall illustrate, by means of a few examples, that their arguments can be applied, virtually unchanged, to a general body without the prior step of an ellipsoidal approximation.

Let b^1, b^2, \dots, b^n be defined recursively as follows: given b^1, \dots, b^{i-1} , b^i minimizes $F_i(h)$ over all non-zero lattice points in $\langle b^1, \dots, b^n \rangle$. The vectors b^1, b^2, \dots, b^n clearly form a basis, since otherwise there is an integer vector which can be written as a linear combination of the b^i with some fractional coefficients. But then by adding and subtracting suitable integral multiples of (b^j) , we obtain an integral vector

$$h = \alpha_1 b^1 + \dots + \alpha_i b^i,$$

with α_i a proper fraction; $\alpha_i b^i$ is in the lattice projected into $\langle b^1, \dots, b^n \rangle$ and gives a smaller value of $F_i(h)$ than does b^i .

Using this particular basis, the Korkine-Zolotarev basis is defined by

$$c^{i+1} = b^{i+1} + \sum_{j=1}^i \mu_{i+1,j} b^j,$$

as in (4). The basis satisfies the inequalities

$$(5) \quad F_1(c^i) \leq F_1(b^i) + (1/2)\sum_1^{i-1} F_j(b^j).$$

The Korkine-Zolotarev basis may not be unique; there may be several non-zero integral vectors in $\langle b^1, \dots, b^n \rangle$ which minimize $F_i(h)$, and the integers $\mu_{i,j}$ need not be uniquely defined.

Theorem 8. Let c^1, \dots, c^n be a Korkine-Zolotarev basis. Then

$$F_1(c^i)/((i+1)/2) \leq \lambda_i \leq ((i+1)/2)F_1(c^i).$$

Proof: Let h^1, \dots, h^n realize the successive minima. For each i , at least one of the vectors h^1, \dots, h^i must project to a non-zero lattice point in $\langle b^1, \dots, b^n \rangle$, since otherwise the vectors would all lie in $\langle b^1, \dots, b^{i-1} \rangle$ and be linearly dependent. It follows that $\max_{j \leq i} F_i(h^j) \geq F_1(b^i)$ and therefore $\lambda_i = F_1(h^i) = \max_{j \leq i} F_1(h^j) \geq F_1(b^i)$. But then (5) implies that $F_1(c^i) \leq \lambda_i + (1/2)(\lambda_1 + \dots + \lambda_{i-1}) \leq ((i+1)/2)\lambda_i$. This demonstrates the left hand inequality of Theorem 8.

To obtain the right hand side, notice that for $k \leq i$, $F_k(b^k) \leq F_k(c^i) \leq F_1(c^i)$, since c^i projects into a non-zero lattice point in $\langle b^k, \dots, b^n \rangle$. But

$$\begin{aligned} \lambda_i &\leq \max_{j \leq i} F_1(c^j) \\ &\leq \max_{j \leq i} \{F_j(b^j) + (1/2)\sum_{k \leq j-1} F_k(b^k)\} \\ &\leq F_1(c^i) \max_{j \leq i} \{1 + (1/2)\sum_{k \leq j-1} 1\} \\ &= ((i+1)/2)F_1(c^i) \quad \circledast \end{aligned}$$

We remark that Theorem 8, in conjunction with Minkowski's inequality, implies that a Korkine-Zolotarev basis satisfies

$$F_1(c^1) \cdot F_1(c^2) \cdot \dots \cdot F_1(c^n) \cdot \text{vol}(C) \leq (n+1)!,$$

an improvement over the estimate of Theorem 4.

Let λ_1^* be the length of the shortest non-zero lattice point with respect to the dual body $C^* = \{y \mid y \cdot x \leq 1 \text{ for all } x \in C\}$. Minkowski's first

theorem implies that $\lambda_1 \leq 2/(\text{vol}(C))^{1/n}$ and $\lambda_1^* \leq 2/(\text{vol}(C^*))^{1/n}$, so that an upper bound for $\lambda_1 \lambda_1^*$ may be obtained from a lower bound for the product of the volumes $\text{vol}(C) \cdot \text{vol}(C^*)$. A well-known ellipsoidal approximation to C is sufficient to produce the inequality $\lambda_1 \lambda_1^* \leq n^{3/2}$. A more sophisticated lower bound, quoted by Kannan and Lovász (1988), implies that there exists a universal constant c_0 , such that $\lambda_1 \lambda_1^* \leq c_0 n$. This result is used to demonstrate the following property of a Korkine-Zolotarev basis.

Theorem 9. There is a universal constant c_0 such that for a Korkine-Zolotarev basis, $F_i(b^i) \lambda_1^* \leq c_0(n-i+1)$.

Proof: We assume, without loss of generality, that the Korkine-Zolotarev basis consists of the n unit vectors e^1, \dots, e^n , and let C_i be the projection of C into e^1, \dots, e^n , with associated distance function F_i . The projection of the original lattice is the set of all (x_1, \dots, x_n) with integral coordinates. For this lattice and distance function, $\lambda_1 = F_i(b^i)$.

From the previous discussion, there is a non-zero lattice point $h^i = (h_1, \dots, h_n)$ such that $F_i(b^i) \cdot \max(h^i \cdot x \mid x \in C_i) \leq c_0(n-i+1)$. But this linear function $h^i \cdot x$ may be extended to a linear function $h \cdot x$ in R^n by adding $i-1$ zero coordinates to h^i , so that

$$F_i(b^i) \lambda_1^* \leq F_i(b^i) \cdot \max(h \cdot x \mid x \in C) \leq c_0(n-i+1). \quad \otimes$$

Theorem 9 has an important application to the study of lattice free bodies K which are not symmetric about the origin. As we shall see, any such body has associated with it a non-zero lattice point h such that

$$\max_{x \in K} (h \cdot x) - \min_{x \in K} (h \cdot x) \leq c_0 n(n+1)/2.$$

The argument is based on Theorem 9, and a subsequent result in the paper by Kannan and Lovász which may be described as follows.

Theorem 10. Let $C = (K - K)$, with K a convex body, and let F be the distance function associated with C . For any basis b^1, \dots, b^n , define $\rho = \Sigma F_i(b^i)$. Then the lattice translates of ρK cover R^n .

Proof: We show, by induction on n , that for any $x \in R^n$, there is a lattice point h with $x+h \in \rho K$. Notice that the hypotheses and conclusion of the Theorem are unchanged if we replace K by any translate of itself; we may therefore assume that K has been translated so that both 0 and b^1 are contained in $F_1(b^1)K$. Let K' be the projection of K along the vector b^1 into $\langle b^2, \dots, b^n \rangle$ and x' the corresponding projection of x .

By the induction assumption, there is a lattice point h' in $\langle b^2, \dots, b^n \rangle$ such that $x'+h' \in \Sigma_2^n F_i(b^i)K'$ and therefore $x+\alpha b^1+h' \in \Sigma_2^n F_i(b^i)K$ for some α . It follows that $x+[\alpha]b^1+h' \in \Sigma_2^n F_i(b^i)K + ([\alpha]-\alpha)b^1 \subseteq \Sigma F_i(b^i)K$. \otimes

If the body K is free of lattice points, then its lattice translates do not cover the origin, and therefore $\rho = \Sigma F_i(b^i) > 1$. We see from Theorem 9, that for such a body, $\lambda_1^* < c_0 \Sigma(n-i+1) = c_0 n(n+1)/2$. It should be remarked that the inductive argument provides an algorithm for calculating a lattice point in K , if $\rho \leq 1$.

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