Nonlinear Functional Analysis and Applications
Vol. 19, No. 1 (2014), pp. 61-77

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

THE GENERALIZED CONVOLUTIONS WITH A WEIGHT FUNCTION FOR LAPLACE TRANSFORM

Nguyen Xuan Thao[1], Trinh Tuan[2] and Le Xuan Huy[3]

[1]School of Applied Mathematics and Informatics
Hanoi University of Science and Technology,
1 Dai Co Viet, Hanoi, Vietnam
e-mail: thaonxbmai@yahoo.com

[2]Postgraduate Department, Electric Power University
235 Hoang Quoc Viet, Hanoi, Vietnam
e-mail: tuantrinhpsac@yahoo.com, tuantrinh@epu.edu.vn

[3]Faculty of Basic Science
University of Economic and Technical Industries,
456 Minh Khai, Hanoi, Vietnam
e-mail: lxhuy@uneti.edu.vn

Abstract. In this paper, we introduce several new generalized convolutions with a weight function for the Laplace, Fourier sine and Fourier cosine integral transforms. Convolution properties and their applications for solving a class of integral equations and systems of integral equations are presented.

1. Introduction

Convolutions and generalized convolutions have attracted considerable attention from many researchers in the past decades. The development of this research trend can be found in [1, 3, 6, 8, 9, 10, 11, 13, 16, 19, 20]. It should be noted that the definition of the generalized convolution with a weight function for three arbitrary integral transforms K_1, K_2, K_3 is given in [4]. Accordingly, for the recent years there have been some results on the generalized

[0]Received July 20, 2013. Revised August 7, 2013.
[0]2010 Mathematics Subject Classification: 33C10, 44A35, 45E10, 45J05, 47A30, 47B15.
[0]Keywords: Convolution, Fourier sine, Fourier cosine, Laplace, transform, integral equation.
[0]This research is funded by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2011.05.

convolutions with a weight function for the Fourier transform, Fourier sine transform, Fourier cosine transform, Kontorovich-Lebedev transform and their applications (see [5, 14, 15, 20]). However, the convolution for the Laplace transform has been studied since the early years of the 20th century (see [1, 2, 6, 12, 16, 17, 18]), but so far there have not been any research results about the generalized convolution for the Fourier sine, Fourier cosine and Laplace integral transforms.

In this paper, we introduce generalized convolution with a weight function for the Laplace, Fourier sine and Fourier cosine transforms and we study the algebraic properties, convolution type inequalities and its applications.

The most obvious difference with results stated in [15] is that in this paper we use the Laplace transform to construct a new generalized convolution. Then the respective equations and systems of integral equations are quite difference, so that we can not use the technique used in [15] to solve these problems. Moreover, in this paper, our results are not only in space $L_1(\mathbb{R}_+)$ but also in several weighted spaces $L_p^{\alpha,\beta}(\mathbb{R}_+)$. This paper is organized as follows. In section 2 we recall several convolutions and generalized convolutions related with our research results; In section 3 we introduce the new generalized convolutions for three integral transform and obtain the main result of this paper, Theorem 3.1. Moreover, in this section we also prove some inequalities on different functional spaces and algebraic properties of the convolution operator; In section 4, we apply the new generalized convolution to solve a class of integral equations and systems of two integral equations.

2. Well-known convolutions

The Fourier sine, Fourier cosine and Laplace transforms are of the following form (see [12])

$$\big(F_s f\big)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin xy dx,\ y > 0,$$
$$\big(F_c f\big)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy dx,\ y > 0,$$
$$\big(Lf\big)(y) = \int_0^\infty f(x) e^{-yx} dx,\ y > 0.$$

The convolution of two functions f and g for the Laplace transform is of the form (see [12])

$$(f * g)(x) = \int_0^x f(x-y) g(y) dy,\ x > 0, \tag{2.1}$$

which satisfies the following factorization identity

$$L(f * g)(y) = (Lf)(y)(Lg)(y). \tag{2.2}$$

The convolution of two functions f and g for the Fourier cosine transform is of the following form (see [12])

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[g(x+y) + g(|x-y|)\big]dy,\ x > 0, \tag{2.3}$$

which satisfies the following factorization identity

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)(F_c g)(y),\ \ \forall\, y > 0,\ f, g \in L_1(\mathbb{R}_+). \tag{2.4}$$

The generalized convolution for the Fourier sine and Fourier cosine transforms of f and g is defined as follows (see [12])

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[g(|x-y|) - g(x+y)\big]dy, \quad x > 0, \tag{2.5}$$

or can be defined in the following form

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(y)\big[f(x+y) + \operatorname{sign}(x-y) f(|x-y|)\big]dy, \quad x > 0,$$

which satisfies the following factorization identity

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y),\ f, g \in L_1(\mathbb{R}_+). \tag{2.6}$$

The generalized convolution for the Fourier cosine and Fourier sine transforms is defined by (see [5])

$$\big(f \underset{2}{*} g\big)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\big[g(x+y) + \operatorname{sign}(y-x) g(|y-x|)\big]dy, x > 0, \tag{2.7}$$

which satisfies the following factorization identity

$$F_c\big(f \underset{2}{*} g\big)(y) = \big(F_s f\big)(y)\big(F_s g\big)(y). \tag{2.8}$$

We consider the two parameters weighted function space $L_p(\mathbb{R}_+, x^\alpha e^{-\beta x} dx)$, or $L_p^{\alpha,\beta}(\mathbb{R}_+)$ for convenient, with the norm as follows

$$||f(x)||_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \Big(\int_0^\infty |f(x)|^p x^\alpha e^{-\beta x} dx\Big)^{1/p}, \quad 1 \le p < \infty.$$

3. The Laplace, Fourier sine and Fourier cosine generalized convolutions

Definition 3.1. The generalized convolutions with a weight function $\gamma(y) = e^{-\mu y}\sin y, \mu > 0$ of two functions f, g for the Laplace, Fourier sine and Fourier cosine transforms are defined by

$$\begin{aligned}&\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x) \qquad (3.1)\\ =&\frac{1}{2\pi}\int_0^\infty\int_0^\infty \Big\{\Big[\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}\Big]\\ &-\Big[\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big]\Big\} f(u)g(v)dudv,\ x>0.\end{aligned}$$

The most important property of a generalized convolution is the factorization identity. The below theorem will show that in the factorization identity of the generalized convolution (3.1) contains the Laplace transform while in [15], the respective factorization identity contained the Kontorovich-Lebedev transform.

Theorem 3.2. *Suppose that $f(x)$ and $g(x)$ are two functions in $L_1(\mathbb{R}_+)$. Then, the generalized convolutions $\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}$ belong to $L_1(\mathbb{R}_+)$, and the following estimaties hold*

$$\|\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}\|_{L_1(\mathbb{R}_+)} \le \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Moreover, the generalized convolutions $\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}$ belong to $C_0(\mathbb{R}_+)$ and satisfy the following factorization identities and the Parseval's type identities

$$F_{\left\{\begin{smallmatrix}s\\c\end{smallmatrix}\right\}}\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(y) = \pm e^{-\mu y}\sin y (F_{\left\{\begin{smallmatrix}c\\s\end{smallmatrix}\right\}}f)(y)(Lg)(y),\ \ \forall\, y>0, \qquad (3.2)$$

$$\big(f \overset{\gamma}{*} g\big)_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x) = \pm\sqrt{\frac{2}{\pi}}\int_0^\infty \big(F_{\left\{\begin{smallmatrix}c\\s\end{smallmatrix}\right\}}f\big)(y)\big(Lg\big)(y)e^{-\mu y}\sin y \begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix} dy. \qquad (3.3)$$

Proof. Set

$$\begin{aligned}\theta_{\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}}(x,u,v) =&\Big[\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}\Big]\\ &-\Big[\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2} \pm \frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big].\end{aligned}$$

For $\mu > 0, v \ge 0$, we have the following estimation

$$\Big|\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2}\Big| \le \frac{1}{v+\mu} \le \frac{1}{\mu},$$

which implies that

$$|\theta_{\{\frac{1}{2}\}}(x,u,v)| \leq \frac{4}{\mu}. \tag{3.4}$$

Thus,

$$\begin{aligned} |(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}}| &\leq \frac{2}{\pi\mu}\Big|\int_0^\infty \int_0^\infty f(u)g(v)dudv\Big| \\ &\leq \frac{2}{\pi\mu}\int_0^\infty |f(u)|du \int_0^\infty |g(v)|dv \\ &= \frac{2}{\pi\mu}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \end{aligned} \tag{3.5}$$

Moreover, we have

$$\begin{aligned} \int_0^\infty |\theta_{\{\frac{1}{2}\}}(x,u,v)|dx \leq &\int_{-1-u}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt + \int_{-1+u}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt \\ &+ \int_{1-u}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt + \int_{1+u}^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt \\ =& 4\int_0^\infty \frac{v+\mu}{(v+\mu)^2+t^2}dt = 2\pi. \end{aligned} \tag{3.6}$$

Therefore,

$$\begin{aligned} \|(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}}\|_{L_1(\mathbb{R}_+)} &= \int_0^\infty |(f \overset{\gamma}{*} g)_{\{\frac{1}{2}\}}(x)|dx \\ &\leq \int_0^\infty |f(u)|du \int_0^\infty |g(v)|dv \\ &= \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)} < \infty. \end{aligned}$$

Hence,

$$\big(f \overset{\gamma}{*} g\big)_{\{\frac{1}{2}\}} \in L_1(\mathbb{R}_+). \tag{3.7}$$

From (3.1) and using the formula $\int_0^\infty e^{-\alpha x}\cos xydx = \frac{\alpha}{\alpha^2+y^2}$ $(\alpha > 0)$, we get

$$\begin{aligned}
&\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}_+^3} f(u)g(v)e^{-(v+\mu)y}\Big\{\big[\cos(x-1-u)y \pm \cos(x-1+u)y\big]\\
&-\big[\cos(x+1-u)y \pm \cos(x+1+u)y\big]\Big\}dudvdy\\
=&\pm\frac{2}{\pi}\int_{\mathbb{R}_+^3} f(u)g(v)e^{-(v+\mu)y}\begin{Bmatrix}\sin xy.\sin y.\cos uy\\ \cos xy.\sin y.\sin uy\end{Bmatrix}dudvdy\\
=&\pm\frac{2}{\pi}\int_0^\infty\Big[\int_0^\infty f(u)\begin{Bmatrix}\cos uy\\ \sin uy\end{Bmatrix}du.\int_0^\infty g(v)e^{-vy}dv\Big]e^{-\mu y}\sin y\begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix}dy\\
=&\pm\sqrt{\frac{2}{\pi}}\int_0^\infty (F_{\left\{\substack{c\\s}\right\}}f)(y)(Lg)(y)e^{-\mu y}\sin y\begin{Bmatrix}\sin xy\\ \cos xy\end{Bmatrix}dy.
\end{aligned}$$

Then the Parseval's type identities (3.3) hold. Combining with (3.7), we obtain the factorization identities (3.2). From (3.3) and Riemann-Lebesgue lemma, we get $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}} \in C_0(\mathbb{R}_+)$. □

Theorem 3.2 shows the existence of convolutions (3.1) in $L_1(\mathbb{R}_+)$ and factorization identities (3.2). In Theorem 3.3 and Theorem 3.4 below, we will study the existence of convolutions (3.1) in $L_r^{\alpha,\beta}(\mathbb{R}_+)$ and the corresponding norm of inequalities.

Theorem 3.3. *Let $p > 1, r \geq 1, 0 < \beta \leq 1, f(x) \in L_p(\mathbb{R}_+), g(x) \in L_1(\mathbb{R}_+)$. Then the generalized convolutions $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}$ are well-defined, continuous and bounded in $L_r^{\alpha,\beta}(\mathbb{R}_+)$. Moreover, these generalized convolutions satisfy the following inequalities*

$$\|\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq C\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}, \tag{3.8}$$

where $C = (\frac{2}{\pi\mu})^{1/p}.\beta^{-\frac{\alpha+1}{r}}.\Gamma^{1/r}(\alpha+1)$ and $\Gamma(x)$ is Gamma Euler function.
If we suppose in addition that $f(x) \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$, then the generalized convolutions $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}$ belong to $C_0(\mathbb{R}_+)$, satisfy the factorization identities (3.2), *and satisfy Parseval's type identities* (3.3).

Proof. By applying Hölder's inequality for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and (3.4), (3.6), we get

$$\begin{aligned}
|\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}| \leq & \frac{1}{2\pi}\Big[\int_{\mathbb{R}_+^2} |f(u)|^p |\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)||g(v)|dudv\Big]^{1/p} \\
& \times \Big[\int_{\mathbb{R}_+^2} |g(v)||\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)|dudv\Big]^{1/q} \\
\leq & \frac{1}{2\pi}\Big[\int_{\mathbb{R}_+^2} |f(u)|^p |g(v)| \frac{4}{\mu} dudv\Big]^{1/p} \Big[\int_0^\infty |g(v)| 2\pi dv\Big]^{1/q} \\
= & \Big(\frac{2}{\pi\mu}\Big)^{1/p} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_1(\mathbb{R}_+)}.
\end{aligned}$$

Thus, convolutions (3.1) exist and continuous. By applying formula (3.225.3) in [8] (p.115), we get

$$\int_0^\infty x^\alpha e^{-\beta x} |\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x)|^r dx \leq C^r \|f\|^r_{L_p(\mathbb{R}_+)} \|g\|^r_{L_1(\mathbb{R}_+)}.$$

Hence, convolutions (3.1) are in $L_r^{\alpha,\beta}(\mathbb{R}_+)$ and the identities (3.8) hold.

From the hypothesis of Theorem 3.3 and by similar argument as Theorem 3.2, we get Parseval's type identities (3.3), therefore the factorization identities (3.2) hold. Combining with the Riemann-Lebesgue Lemma, we show that $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x) \in C_0(\mathbb{R}_+)$. □

Theorem 3.4. *Let $\alpha > -1, 0 < \beta \leq 1, p > 1, q > 1, r \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $f(x) \in L_p(\mathbb{R}_+)$ and $g(x) \in L_q(\mathbb{R}_+, e^{(q-1)x})$, the generalized convolutions $(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}$ are well-defined, continuous, bounded in $L_r^{\alpha,\beta}(\mathbb{R}_+)$ and*

$$\|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq C\|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+, e^{(q-1)x})}, \tag{3.9}$$

where $C = (\frac{2}{\pi\mu})^{1/q}.\beta^{-\frac{\alpha+1}{r}}.\Gamma^{1/r}(\alpha+1)$.

Moreover, if $f(x) \in L_1(\mathbb{R}_+) \cap L_p(\mathbb{R}_+)$ and $g(x) \in L_1(\mathbb{R}_+) \cap L_q(\mathbb{R}_+, e^{(q-1)x})$ then the convolutions $(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}} \in C_0(\mathbb{R}_+)$ satisfy the factorization identities (3.2) *and Parseval's type identities* (3.3).

Proof. By applying Hölder's inequality for $p, q > 1$ satisfies and from (3.4), (3.6), we have

$$\begin{aligned}|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}| \leq & \frac{1}{2\pi}\Big[\int_{\mathbb{R}_+^2} |f(u)|^p |\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)| e^{-v} du dv\Big]^{1/p} \\ & \times \Big[\int_{\mathbb{R}_+^2} |g(v)|^q |\theta_{\left\{\substack{1\\2}\right\}}(x,u,v)| e^{(q-1)v} du dv\Big]^{1/q} \\ \leq & \frac{1}{2\pi}\Big[\int_0^\infty |f(u)|^p du \int_0^\infty \frac{4}{\mu} e^{-v} dv\Big]^{1/p}\Big[\int_0^\infty |g(v)|^q e^{(q-1)v} 2\pi dv\Big]^{1/q} \\ = & \Big(\frac{2}{\pi\mu}\Big)^{1/p} \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+, e^{(q-1)x})}.\end{aligned}$$

Therefore, convolutions (3.1) exist and continuous. From which and applying formula (3.225.3) (p.115, in [8]), we obtain

$$\int_0^\infty x^\alpha e^{-\beta x} |(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)|^r dx \leq C^r \|f\|^r_{L_p(\mathbb{R}_+)} \|g\|^r_{L_q(\mathbb{R}_+, e^{(q-1)x})}.$$

It implies the existence of convolutions (3.1) in $L_r^{\alpha,\beta}(\mathbb{R}_+)$ and (3.9) hold.

From the hypothesis of Theorem 3.4 and by similar argument as Theorem 3.3, we get Parseval's type identities (3.3), therefore the factorization identities (3.2) hold. The Riemann-Lebesgue Lemma implies $(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x) \in C_0(\mathbb{R}_+)$. □

Corollary 3.5. *Under the same conditions stated in Theorem* 3.2*, the generalized convolutions* (3.1) *exist, continuous in* $L_p(\mathbb{R}_+)$ *and satisfy the following estimaties*

$$\|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}\|_{L_p(\mathbb{R}_+)} \leq \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+, e^{(q-1)x})}. \tag{3.10}$$

In case $p = 2$*, the following Parseval's type identity hold*

$$\int_0^\infty |(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)|^2 dx = \int_0^\infty |e^{-\mu y} \sin y (F_{\left\{\substack{c\\s}\right\}} f)(y)(Lg)(y)|^2 dy. \tag{3.11}$$

Proof. By applying Hölder's inequality and (3.6), we get

$$\begin{aligned}
&\int_0^\infty |(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)|^p dx \\
\leq&\frac{1}{(2\pi)^p}\int_0^\infty \Big\{\Big[\int_{\mathbb{R}_+^2} e^{-v}|f(u)|^p|\theta(x,u,v)|dudv\Big]^{1/p} \\
&\times\Big[\int_{\mathbb{R}_+^2} e^{(q-1)v}|g(v)|^q|\theta(x,u,v)|dudv\Big]^{1/q}\Big\}^p dx \\
\leq&\frac{1}{(2\pi)^p}\Big[\int_{\mathbb{R}_+^2} e^{-v}|f(u)|^p 2\pi dudv\Big]\Big[\int_0^\infty e^{(q-1)v}|g(v)|^q 2\pi dv\Big]^{p/q} \\
=&\Big[\int_0^\infty |f(u)|^p du.\int_0^\infty e^{-v}dv\Big]\Big[\int_0^\infty e^{(q-1)v}|g(v)|^q dv\Big]^{p/q} \\
=&\|f\|^p_{L_p(\mathbb{R}_+)}\|g\|^p_{L_q(\mathbb{R}_+,e^{(q-1)x})}.
\end{aligned}$$

Therefore, the convolutions $(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)$ are cotinuous in $L_p(\mathbb{R}_+)$ and (3.10) hold.

On the other hand, we get the following Parseval's equalities in $L_2(\mathbb{R}_+)$

$$\|F_{\left\{\substack{s\\c}\right\}}f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

Combining with factorization identities (3.2), we get the Fourier-type Parseval identities (3.11). □

Corollary 3.6. (a) *Let $f(x) \in L_2(\mathbb{R}_+)$, $g(x) \in L_1(\mathbb{R}_+)$. Then the generalized convolutions* (3.1) *exist and belong to $L_r^{\alpha,\beta}(\mathbb{R}_+)$ $(r \geq 1, \beta \geq 0, \alpha > -1)$. Moreover, the following estimaties hold*

$$\|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi\mu}}.\beta^{-\frac{\alpha+1}{r}}.\Gamma^{1/r}(\alpha+1)\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \quad (3.12)$$

(b) *If $f(x), g(x) \in L_1(\mathbb{R}_+)$ then the generalized convolutions* (3.1) *exist and belong to $L_r^{\alpha,\beta}(\mathbb{R}_+)$ $(r \geq 1, \beta \geq 0, \alpha > -1)$, and*

$$\|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \frac{2}{\pi\mu}.\beta^{-\frac{\alpha+1}{r}}.\Gamma^{1/r}(\alpha+1)\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}. \quad (3.13)$$

Proof. (a) By applying Schwartz's inequality and (3.4), (3.6) we get

$$\begin{aligned}
|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}(x)|\leq&\frac{1}{2\pi}\Big[\int_0^\infty 2\pi|g(v)|dv\Big]^{1/2}\Big[\int_{\mathbb{R}_+^2}|f(u)|^2|g(v)|\frac{4}{\mu}dudv\Big]^{1/2} \\
=&\sqrt{\frac{2}{\pi\mu}}\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.
\end{aligned}$$

Combining with formula (3.225.3) (p.115, in [8]), we get

$$\|(f \overset{\gamma}{*} g)_{\left\{\substack{1\\2}\right\}}\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi\mu}}.\beta^{-\frac{\alpha+1}{r}}.\Gamma^{1/r}(\alpha+1)\|f\|_{L_2(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.$$

Thus, (3.12) is proved.
(b) By applying Schwartz's inequality and (3.4), we get

$$\begin{aligned}|\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x)| \leq & \frac{1}{2\pi}\Big[\int_{\mathbb{R}_+^2}|f(u)||g(v)|\frac{4}{\mu}dudv\Big]^{1/2}\Big[\int_{\mathbb{R}_+^2}|f(u)||g(v)|\frac{4}{\mu}dudv\Big]^{1/2}\\ = & \frac{2}{\pi\mu}\|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}.\end{aligned}$$

Combining with formula (3.225.3) (p.115, in [8]), we get (3.13). □

Theorem 3.7. (Titchmarch's type theorem) *Given continuous functions* $g \in L_1(\mathbb{R}_+)$, $f \in L_1(\mathbb{R}_+, e^{\gamma x})$, $\gamma > 0$. *If* $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x) = 0, \ \ \forall x > 0$ *then either* $f(x) = 0, \ \ \forall x > 0$ *or* $g(x) = 0, \ \ \forall x > 0$.

Proof. We have

$$\begin{aligned}\Big|\frac{d^n}{dy^n}\Big(\begin{Bmatrix}\cos yx\\ \sin yx\end{Bmatrix}f(x)\Big)\Big| = & \Big|f(x)x^n\begin{Bmatrix}\cos(yx+n\frac{\pi}{2})\\ \sin(yx+n\frac{\pi}{2})\end{Bmatrix}\Big| \qquad (3.14)\\ \leq & |e^{-\gamma x}x^n||e^{\gamma x}f(x)| \leq \frac{n!}{\gamma^n}|e^{\gamma x}f(x)|.\end{aligned}$$

Here, we used the following estimation

$$0 \leq e^{-\gamma x}x^n = e^{-\gamma x}\frac{(\gamma x)^n}{n!}\frac{n!}{\gamma^n} \leq e^{-\gamma x}e^{\gamma x}\frac{n!}{\gamma^n} = \frac{n!}{\gamma^n},$$

and $f \in L_1(\mathbb{R}_+, e^{\gamma x})$. Combining with (3.14) we get $\frac{d^n}{dy^n}\Big(\begin{Bmatrix}\cos yx\\ \sin yx\end{Bmatrix}f(x)\Big) \in L_1(\mathbb{R}_+)$.
Since $L_1(\mathbb{R}_+, e^{\gamma x}) \subset L_1(\mathbb{R}_+)$, $(F_{\left\{\substack{c\\s}\right\}}f)(y)$ are analytic in $\mathbb{R}_+$. On the other hand, we get that $(Lg)(y)$ is analytic in $\mathbb{R}_+$. By using the factorization properties (3.2) for $\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x) = 0$ we have $(F_{\left\{\substack{c\\s}\right\}}f)(y)(Lg)(y) = 0$, $\forall y > 0$. It implies that, either $f(x) = 0$, $\forall x > 0$ or $g(x) = 0$, $\forall x > 0$. The theorem is proved. □

Proposition 3.8. *The generalized convolutions* (3.1) *are non-commutative, non-associative but satisfy the following equalities*

$$\big(f \overset{\gamma}{*} g\big)_{\left\{\substack{1\\2}\right\}}(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty g(v)\Big[\big(f(u) \underset{\left\{\substack{1\\F_c}\right\}}{*} k_{\left\{\substack{1\\2}\right\}}(v,\mu,u)\big)(x)\Big]dv,$$

here, $k_{\{ {1 \atop 2} \}}(v,\mu,u) = \dfrac{v+\mu}{(v+\mu)^2+(1+u)^2} \pm \dfrac{v+\mu}{(v+\mu)^2+(1-u)^2}$, $f(x), g(x)$ *are functions in* $L_1(\mathbb{R}_+)$ *and the convolutions* $(.\underset{1}{*}.)$, $(.\underset{F_c}{*}.)$ *are respectively defined by* (2.5) *and* (2.3).

Proof. From (3.1) and by applying (2.5), we get

$$\begin{aligned}
&\big(f \overset{\gamma}{*} g\big)_1(x)\\
&= \frac{1}{2\pi}\int_0^\infty\int_0^\infty \Big[\Big(\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2}+\frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}\Big)\\
&\quad -\Big(\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2}+\frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big)\Big]f(u)g(v)dudv\\
&= \frac{1}{2\pi}\int_0^\infty g(v)\Big[\int_0^\infty f(u)\Big(\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2}-\frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big)du\\
&\quad +\int_0^\infty f(u)\Big(\frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}-\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2}\Big)du\Big]dv\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty g(v)\Big[\Big(f(u)\underset{1}{*}\frac{v+\mu}{(v+\mu)^2+(1+u)^2}\Big)(x)\\
&\quad +\Big(f(u)\underset{1}{*}\frac{v+\mu}{(v+\mu)^2+(1-u)^2}\Big)(x)\Big]dv\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty g(v)\Big[\Big(f(u)\underset{1}{*}\big(\frac{v+\mu}{(v+\mu)^2+(1+u)^2}+\frac{v+\mu}{(v+\mu)^2+(1-u)^2}\big)\Big)(x)\Big]dv.
\end{aligned}$$

The second part can be proved by similar way. □

4. Integral equations and Systems of Integral Equations

Not many integral equations and systems of integral equations can be solved in closed form. The generalized convolutions (3.1) introduced in this paper allow us to get the solutions in closed form for integral equations and system of integral equations.

4.1. Consider the following integral equations

$$f(x) + \int_0^\infty \big(\varphi \underset{\{ {2 \atop 1} \}}{*} f\big)(u)\theta_{\{ {1 \atop 2} \}}(x,u)du = g(x), \tag{4.1}$$

where

$$\begin{aligned}
\theta_{\{ {1 \atop 2} \}}(x,u) =& \frac{1}{2\pi}\int_0^\infty \Big\{\Big[\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2}\pm\frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}\Big]\\
&-\Big[\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2}\pm\frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big]\Big\}\psi(v)dv,
\end{aligned}$$

here, the convolutions $(.\underset{2}{*}.)$, $(.\underset{1}{*}.)$ are respectively defined by (2.7) and (2.5).

Theorem 4.1. *Let $g(x), \varphi(x), \psi(x) \in L_1(\mathbb{R}_+)$. Then, the necessary and sufficient condition for the existence of the unique solutions of equations* (4.1) *in $L_1(\mathbb{R}_+)$ is that $1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y) \neq 0,\ \forall\ y > 0$. Moreover, the solutions can be presented in closed form as*

$$f(x) = g(x) \mp \big(g \underset{\left\{\substack{1\\F_c}\right\}}{*} q\big)(x),$$

where $q \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c q)(y) = \frac{F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}{1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)},$$

and the convolution $(.\underset{F_c}{*}.)$ is defined by (2.3).

Proof. Necessity. Assume that the integral equations (4.1) have solutions in $L_1(\mathbb{R}_+)$, for all g in $L_1(\mathbb{R}_+)$. Therefore, there exists $g \in L_1(\mathbb{R}_+)$ such that

$$(F_{\left\{\substack{s\\c}\right\}} g)(y) \neq 0,\ \ \forall\ y > 0. \tag{4.2}$$

By applying Theorem 3.2, and by using factorization properties (3.2) for (4.1), we get

$$(F_{\left\{\substack{s\\c}\right\}} f)(y) \pm e^{-\mu y} \sin y F_{\left\{\substack{c\\s}\right\}}\big(\varphi \underset{\left\{\substack{2\\1}\right\}}{*} f\big)(y)(L\psi)(y) = (F_{\left\{\substack{s\\c}\right\}} g)(y).$$

Combining with (2.8) and (2.6), we obtain

$$(F_{\left\{\substack{s\\c}\right\}} f)(y) \pm e^{-\mu y} \sin y \big(F_s\varphi\big)(y)\big(F_{\left\{\substack{s\\c}\right\}} f\big)(y)(L\psi)(y) = (F_{\left\{\substack{s\\c}\right\}} g)(y).$$

Combining with (3.2), we get

$$(F_{\left\{\substack{s\\c}\right\}} f)(y)\big[1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)\big] = (F_{\left\{\substack{s\\c}\right\}} g)(y). \tag{4.3}$$

Using feedback evidence, assume that there exist $y_0 > 0$, such that $1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y_0) = 0$. Combining with (4.3), we get

$$(F_{\left\{\substack{s\\c}\right\}} g)(y_0) = 0,\ \ \forall\ g \in L_1(\mathbb{R}_+). \tag{4.4}$$

It is a contradiction to (4.2). Hence, $1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y) \neq 0,\ \ \forall\, y > 0.$
Sufficiency. From (4.2) and the assumption of Theorem 4.1, we have

$$\begin{aligned}(F_{\left\{\substack{s\\c}\right\}} f)(y) =& \frac{(F_{\left\{\substack{s\\c}\right\}} g)(y)}{1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}\\ =&(F_{\left\{\substack{s\\c}\right\}} g)(y)\Big[1 \mp \frac{F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}{1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}\Big]\\ =&(F_{\left\{\substack{s\\c}\right\}} g)(y) \mp (F_{\left\{\substack{s\\c}\right\}} g)(y).\frac{F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}{1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}.\end{aligned} \tag{4.5}$$

We recall the Wiener-Levy Theorem (p.63 in [7]) that if l is the Fourier transform of an $L_1(\mathbb{R}_+)$ function, and η is analytic in a neighborhood of the origin that contains the domain $\{l(y), \forall y \in \mathbb{R}\}$ and $\eta(0) = 0$, then $\eta(l)$ is also the Fourier transform of an $L_1(\mathbb{R}_+)$ function. For the Fourier cosine transform it means that if l is the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function, and η is analytic in a neighborhood of the origin that contains the domain $\{l(y), \forall y \in \mathbb{R}_+\}$, and $\eta(0) = 0$, then $\eta(l)$ is also the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function.
With the given conditions $1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y) \neq 0, \forall y > 0$, the function $\eta(z) = \dfrac{z}{1 \pm z}$ satisfies the conditions of the Wiener-Levy Theorem. Then, there exists a function $q \in L_1(\mathbb{R}_+)$ such that

$$(F_c q)(y) = \frac{F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}{1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y)}. \tag{4.6}$$

From (4.5), (4.6) and $q \in L_1(\mathbb{R}_+)$ we have

$$\begin{aligned}(F_{\left\{\substack{s\\c}\right\}} f)(y) =&(F_{\left\{\substack{s\\c}\right\}} g)(y) \mp (F_{\left\{\substack{s\\c}\right\}} g)(y)(F_c q)(y)\\ =&(F_{\left\{\substack{s\\c}\right\}} g)(y) \mp F_{\left\{\substack{s\\c}\right\}}\big(g \underset{\left\{\substack{1\\F_c}\right\}}{*} q\big)(y).\end{aligned}$$

Therefore, $f(x) = g(x) \mp \big(g \underset{\left\{\substack{1\\F_c}\right\}}{*} q\big)(x),\ f(x) \in L_1(\mathbb{R}_+).$ □

Example 4.2. *We choose the functions φ and ψ as follow*

$$\varphi(x) = e^{-ax}, \quad \psi(x) = e^{-bx}\ (a, b > 0),$$

and $\varphi(x)$, $\psi(x) \in L_1(\mathbb{R}_+)$, *we have*

$$(F_s\varphi)(y) = \sqrt{\frac{2}{\pi}}\frac{y}{a^2+y^2}, \quad (L\psi)(y) = \frac{1}{b+y}. \tag{4.7}$$

From the factorization identities (3.2) *and* (4.7), *we have*

$$\begin{aligned} F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y) &= -\, e^{-\mu y}\sin y (F_s\varphi)(y)(L\psi)(y) \\ &= -\sqrt{\frac{2}{\pi}} e^{-\mu y}\sin y. \frac{y}{(a^2+y^2)(b+y)} \in L_1(\mathbb{R}_+). \end{aligned}$$

Then the convolutions in the Theorem 4.1, *we have* $1 \pm F_c\big(\varphi \overset{\gamma}{*} \psi\big)_2(y) \neq 0$, $\forall\ y > 0$. *Due to Wiener-Levy Theorem, there esixts a function* $q \in L_1(\mathbb{R}_+)$ *such that*

$$(F_c q)(y) = \frac{-\sqrt{\frac{2}{\pi}} e^{-\mu y}\sin y. \frac{y}{(a^2+y^2)(b+y)}}{1 \mp \sqrt{\frac{2}{\pi}} e^{-\mu y}\sin y. \frac{y}{(a^2+y^2)(b+y)}} \in L_1(\mathbb{R}_+). \tag{4.8}$$

Therefore

$$\begin{aligned} (q)(x) =& F_c\Big[\frac{-\sqrt{\frac{2}{\pi}} e^{-\mu y}\sin y. \frac{y}{(a^2+y^2)(b+y)}}{1 \mp \sqrt{\frac{2}{\pi}} e^{-\mu y}\sin y. \frac{y}{(a^2+y^2)(b+y)}}\Big](x) \\ =& -\frac{2}{\pi}\int_0^\infty \frac{y \sin y \cos xy}{(a^2+y^2)(b+y)e^{\mu y} \mp \sqrt{\frac{2}{\pi}} y \sin y} dy, \end{aligned}$$

and $f(x) = g(x) \mp \big(g \underset{\left\{\frac{1}{F_c}\right\}}{*} q\big)(x)$.

4.2. Consider the following systems of two integral equations

$$\begin{aligned} f(x) + \int_0^\infty M_{\left\{\frac{1}{2}\right\}}(x,u) g(u) du = p(x), \\ g(x) + \int_0^\infty N_{\left\{\frac{1}{2}\right\}}(x,u) f(u) du = q(x), \end{aligned} \tag{4.9}$$

where

$$
\begin{aligned}
&M_{\left\{\substack{1\\2}\right\}}(x,u)\\
&=\frac{1}{2\pi}\int_0^\infty \varphi(v)\Big\{\Big[\frac{v+\mu}{(v+\mu)^2+(x-1-u)^2}\pm\frac{v+\mu}{(v+\mu)^2+(x-1+u)^2}\Big]\\
&\quad-\Big[\frac{v+\mu}{(v+\mu)^2+(x+1-u)^2}\pm\frac{v+\mu}{(v+\mu)^2+(x+1+u)^2}\Big]\Big\}dv,\\
&N_{\left\{\substack{1\\2}\right\}}(x,u)=\frac{1}{\sqrt{2\pi}}\big[\psi(u+x)\pm\operatorname{sign}(u-x)\psi(|u-x|)\big].
\end{aligned}
$$

Theorem 4.3. *Assume that $\varphi(x),\psi(x),p(x),q(x)\in L_1(\mathbb{R}_+)$ and $1\mp F_c\big(\psi\overset{\gamma}{*}\varphi\big)_2(y)\neq 0$, $\forall y>0$. Systems (4.9) have unique solutions (f,g) in $L_1(\mathbb{R}_+)\times L_1(\mathbb{R}_+)$ given by*

$$
\begin{aligned}
f(x)=&p(x)\mp\big(q\overset{\gamma}{*}\varphi\big)_{\left\{\substack{1\\2}\right\}}(x)\pm\big(p\underset{\left\{\substack{1\\F_c}\right\}}{*}\xi\big)(x)-\Big(\big(q\overset{\gamma}{*}\varphi\big)_{\left\{\substack{1\\2}\right\}}\underset{\left\{\substack{1\\F_c}\right\}}{*}\xi\Big)(x),\\
g(x)=&q(x)-\big(\psi\underset{\left\{\substack{2\\1}\right\}}{*}p\big)(x)\pm\big(q\underset{\left\{\substack{F_c\\1}\right\}}{*}\xi\big)(x)\mp\Big(\big(\psi\underset{\left\{\substack{2\\1}\right\}}{*}p\big)\underset{\left\{\substack{F_c\\1}\right\}}{*}\xi\Big)(x),
\end{aligned}
$$

where $\xi\in L_1(\mathbb{R}_+)$ such that

$$(F_c\xi)(y)=\frac{F_c(\psi\overset{\gamma}{*}\varphi)_2(y)}{1\mp F_c(\psi\overset{\gamma}{*}\varphi)_2(y)}.$$

Here, the convolutions $(.\underset{1}{*}.)$, $(.\underset{F_c}{*}.)$, $(.\underset{2}{*}.)$ are defined by (2.5), (2.3) and (2.7) respectively.

Proof. By using factorization properties (3.2), (2.6) and (2.8) for (4.9), we get

$$
\begin{aligned}
(F_{\left\{\substack{s\\c}\right\}}f)(y)\pm e^{-\mu y}\sin y(F_{\left\{\substack{c\\s}\right\}}g)(y)(L\varphi)(y)&=(F_{\left\{\substack{s\\c}\right\}}p)(y),\\
(F_{\left\{\substack{c\\s}\right\}}g)(y)+(F_{\left\{\substack{s\\s}\right\}}\psi)(y)(F_{\left\{\substack{s\\c}\right\}}f)(y)&=(F_{\left\{\substack{c\\s}\right\}}q)(y).
\end{aligned}\tag{4.10}
$$

Solving the systems of two linear equations (4.10), we get

$$
\begin{aligned}
(F_{\left\{\substack{s\\c}\right\}}f)(y)=&\frac{(F_{\left\{\substack{s\\c}\right\}}p)(y)\mp e^{-\mu y}\sin y(F_{\left\{\substack{c\\s}\right\}}q)(y)(L\varphi)(y)}{1\mp e^{-\mu y}\sin y(F_s\psi)(y)(L\varphi)(y)}\\
=&\Big[(F_{\left\{\substack{s\\c}\right\}}p)(y)\mp F_{\left\{\substack{s\\c}\right\}}\big(q\overset{\gamma}{*}\varphi\big)_{\left\{\substack{1\\2}\right\}}(y)\Big]\Big[\frac{1}{1\mp F_c(\psi\overset{\gamma}{*}\varphi)_2(y)}\Big]\\
=&\Big[(F_{\left\{\substack{s\\c}\right\}}p)(y)\mp F_{\left\{\substack{s\\c}\right\}}\big(q\overset{\gamma}{*}\varphi\big)_{\left\{\substack{1\\2}\right\}}(y)\Big]\Big[1\pm\frac{F_c(\psi\overset{\gamma}{*}\varphi)_2(y)}{1\mp F_c(\psi\overset{\gamma}{*}\varphi)_2(y)}\Big].
\end{aligned}\tag{4.11}
$$

Due to Wiener-Levy Theorem, there exists a function $\xi \in L_1(\mathbb{R}_+)$ such that

$$(F_c\xi)(y) = \frac{F_c(\psi \overset{\gamma}{*} \varphi)_2(y)}{1 \mp F_c(\psi \overset{\gamma}{*} \varphi)_2(y)}. \tag{4.12}$$

From (4.11) and (4.12), we have

$$\begin{aligned}(F_{\left\{\substack{s\\c}\right\}} f)(y) =& \big[\big(F_{\left\{\substack{s\\c}\right\}} p\big)(y) \mp F_{\left\{\substack{s\\c}\right\}}\big(q \overset{\gamma}{*} \varphi\big)_{\left\{\substack{1\\2}\right\}}(y)\big]\big[1 \pm (F_c\xi)(y)\big]\\ =& \big(F_{\left\{\substack{s\\c}\right\}} p\big)(y) \mp F_{\left\{\substack{s\\c}\right\}}\big(q \overset{\gamma}{*} \varphi\big)_{\left\{\substack{1\\2}\right\}}(y) \pm F_{\left\{\substack{s\\c}\right\}}\big(p \underset{\left\{\substack{1\\F_c}\right\}}{*} \xi\big)(x)\\ &- F_{\left\{\substack{s\\c}\right\}}\Big(\big(q \overset{\gamma}{*} \varphi\big)_{\left\{\substack{1\\2}\right\}} \underset{\left\{\substack{1\\F_c}\right\}}{*} \xi\Big)(x).\end{aligned}$$

Therefore,

$$f(x) = p(x) \mp \big(q \overset{\gamma}{*} \varphi\big)_{\left\{\substack{1\\2}\right\}}(x) \pm \big(p \underset{\left\{\substack{1\\F_c}\right\}}{*} \xi\big)(x) - \Big(\big(q \overset{\gamma}{*} \varphi\big)_{\left\{\substack{1\\2}\right\}} \underset{\left\{\substack{1\\F_c}\right\}}{*} \xi\Big)(x).$$

Similarly, we get

$$g(x) = q(x) - \big(\psi \underset{\left\{\substack{2\\1}\right\}}{*} p\big)(x) \pm \big(q \underset{\left\{\substack{F_c\\1}\right\}}{*} \xi\big)(x) \mp \Big(\big(\psi \underset{\left\{\substack{2\\1}\right\}}{*} p\big) \underset{\left\{\substack{F_c\\1}\right\}}{*} \xi\Big)(x).$$

It is easy to see that f, g are functions in $L_1(\mathbb{R}_+)$. The proof is completed. □

Acknowledgments. The authors would like to press their deep thank the referee for his/her constructive comments.

References

[1] L. Biryukov, *Some theorems on integrability of Laplace transforms and their applications*, Int. Trans. and Spec. Func., **18(7)** (2007), 459–469.

[2] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, Chapman and Hall/CRC, Boca Raton (2007).

[3] V.A. Kakichev, *On the convolution for integral transforms*, Izv. Vyssh. Uchebn. Zaved. Mat.(In Russian), **2** (1967), 53–62.

[4] V.A. Kakichev and N.X. Thao, *On the design method for the generalized integral convolutions*, Izv. Vyssh. Uchebn. Zaved. Mat.(In Russian), **1** (1998), 31–40 .

[5] V.A. Kakichev, N.X. Thao and V.K. Tuan, *On the generalized convolutions for Fourier cosine and sine transforms*, East-West Journal of Mathematics, **1(1)** (1998), 85–90.

[6] N.D.V. Nhan and D.T. Duc, *Fundamental inequalities for the iterated Laplace convolution in weighted L_p spaces and their applications*, Int. Trans. and Spec. Func., **19(9)** (2008), 655–664.

[7] R.C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc, NewYork (1949).

[8] I.M. Ryzhik and I.S. Gradshteyn, *Tables of Integrals, Sum, Series and Products*, Moscow (1951).

[9] S. Saitoh, V.K. Tuan and M. Yamamoto, *Conditional stability of a real universe formula for the Laplace transform*, Zeischnift fun Analysis unid ihre Annendungen, **20(1)** (2001), 193–202.

[10] S. Saitoh, V.K. Tuan and M. Yamamoto, *Reverse weighted Lp-norm inequalities in convolutions and stability in inverse problems*, J. of Ineq. in Pure App. Math., **1(1)** (2000), 1–7.

[11] S. Saitoh, V.K. Tuan and M. Yamamoto, *Convolution inequalities and applications*, J. of Ineq. in Pure App. Math., **4(3)** (2003), 1–8.

[12] I.N. Sneddon, *Fourier Transforms*, McGray-Hill, New York (1951).

[13] N.X. Thao and N.T. Hai, *Convolutions for integral transform and their application*, Computer Centre of the Russian Academy(In Russian), Moscow, 44p, (1997).

[14] N.X. Thao and N.M. Khoa, *On the generalized convolution with a weight function for the Fourier sine and cosine transforms*, Int. Trans. and Spec., **9** (2006), 673–685.

[15] T. Tuan, *On the generalized convolution with a weight function for the Furier cosine and the inverse Kontorovich-Lebedev integral transformations*, Nonlinear Funct. Anal. Appl., **12(2)** (2007), 325–341.

[16] V.K. Tuan, *Modified Laplace transforms and a multidimensional H-transform*, Dokl. Akad. Nauk SSSR(in Russian), **313(6)** (1990), 1299–1302.

[17] V.K. Tuan and T. Tuan, *A real-variable inverse formula for the Laplace transform*, Int. Trans. and Spec. Func., **23(8)** (2012), 551–555.

[18] S.B. Yakubovich, *On integral convolutions of Laplace type for G-transforms*, Izv. Akad. Nauk BSSR(in Russian), **6** (1991), 11–16.

[19] S.B. Yakubovich, *Certain isometrics related to the bilaterral Laplace transforms*, Modeling and Analysis, **11(3)** (2006), 331–346.

[20] S.B. Yakubovich and L.E. Britvina, *Convolution related to the Fourier and Kontorovich-Lebedev transforms revisited*, Int. Trans. and Spec. Func., **21(4)** (2010), 259–276.