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The Generalized Distributive Law as Tacit Knowledge in Algebra.

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Bates, Juanita Lavall, Ph.D.

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THE GENERALIZED DISTRIBUTIVE LAW
AS TACIT KNOWLEDGE IN ALGEBRA

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Curriculum and Instruction

by
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August, 1994

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Abstract

The purposes of this study were to investigate theories that explain why common errors of the type

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{occur in algebra}$$

problem solving by novices; and to develop and assess techniques for remediating these errors. The meaning theory of learning (ML), procedural learning theory (PL), and implicit structure learning theory (ISL) are alternative frameworks for the explanation of the errors. The ML theory hypothesizes that experts have rich semantic connections to the procedures and symbols of algebra, but novices lack such connections (Ausubel, Novak & Hanesian, 1978; Brownell, 1947; Wearne & Hiebert, 1985). The PL theory hypothesizes that adept problem solvers have technical proficiency in memorizing and applying mechanical rules (Anderson, 1983; Lewis, Milson, & Anderson, 1987; Matz, 1980). The ISL theory hypothesizes that students enter the classroom with nascent abstract rule structures on which to build a more mature "grammar of algebra" through inductive processes (Bolio, 1989; Drouhard, 1988; Kirshner, 1987).

In order to obtain some measure of the relative efficacy of these theories for remedial purposes, three brief educational treatments have been designed to reflect

the three frameworks for learning. An analysis of variance for repeated measures was used to assess the effectiveness of the treatments in reducing the occurrences of the

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{errors.}$$

Forty students participated in the study. They were enrolled in four intact developmental intermediate algebra classes at Southern University in Baton Rouge. The study used a pretest-posttest-retention test, control group design with three treatments--ML, PL, and ISL--and one control (C) which receives no special instruction concerning the errors. Results indicate that no significant difference was found in the number of errors between the groups on the post and retention tests. However, there was a significant difference between the mean scores of the pretest and the posttest.

These results do not provide support for one theory over another in reducing the error types mentioned above, but do indicate a small decrease in the error rate for distributivity overgeneralization for all treatment groups.

CHAPTER ONE
STATEMENT OF THE PROBLEM

Students' mathematical performance is a major concern for many educators and the focus of much of the current literature and research in mathematics education. This focus is expressed in articles such as "Math and Science: A Nation Still at Risk" (Ashworth, 1990); and "Teaching Mathematics for Tomorrow's World" (Steen, 1989). These authors document that students' performance in mathematics is deficient. Steen (1989) cites reports (e.g. Kirsch & Jungeblut, 1986; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987; Dossey, Mullis, Linguist, & Chambers, 1988; Mullis & Jenkins, 1988; Paulos, 1988; Lapointe, Mead & Phillips, 1989) that indicate serious deficiencies in the mathematical performance of U.S. students. Ashworth cites results from the National Assessment of Educational Progress (NAEP) that indicate "discrepancies between the level of math taught in school and what students can do" (p. 15). NAEP measures the educational attainment of U.S. students and supplies information which can be useful in determining problem areas in education. A review of the NAEP's data on students' performance in mathematics sparked the following statements:

No eighth grader showed the breadth of understanding necessary to begin the study of relatively advanced mathematics (Mullis, Dossey, Owen, & Phillips, 1991, p. 7); and

Approximately half the twelfth graders graduating from today's school appear to have an understanding of mathematics that does not extend much beyond simple problem solving with whole numbers (p. 8).

Based on the NAEP's 1990 trend data in mathematics, Mullis, Dossey, Foertsch, Jones and Gentile (1991) indicate that during the 1980s there was a general pattern of growth in mathematics proficiency; but that discrepancies exist between races/ethnicity (White, Black, Hispanic) and also between genders from 1973 to 1990.

Deficiencies in mathematical performance that the students are displaying suggest that curriculum changes must occur; teaching methods must be innovative; assessment techniques must be varied; and the climate for learning must be different for the student (Ashworth, 1990; Steen, 1989). To help effect these changes, there is a need for more mathematically qualified teachers. However, Fey (1983) indicates that since 1970 there is a sharp decline in the number of U.S. college and university students who choose to major in mathematics. Several reasons have been offered as explanations for such decline. For many students mathematics in general is misunderstood,

feared and avoided (Steen, 1986). Precollege preparation is also offered as a reason for such decline (Fey, 1983). When many students encounter difficulties with mathematical fundamentals, they try to avoid as much mathematics as possible. Dossey (1988) indicates that in 1986 more high school students reported taking advanced courses including Algebra II, Geometry and Calculus, but that the overall percentage of students taking these advanced courses remains low.

Precollege mathematics preparation not only affects the number of mathematics majors but it also affects other disciplines that require advanced mathematics. That is, students who avoid mathematics are generally limited in career choices. They tend to choose careers that require limited mathematics. Deficiencies in high school mathematics preparation act as the "critical filter" barring students, especially women and Blacks, from entry into universities, scientific/technical college majors, and subsequent careers (Sells, 1973; 1976). Other researchers such as Whiteley (1987), Bleyer, Pedersen, and Elmore (1981), Sherman (1982), and Silva and Moses (1990), report similar observations.

On the secondary level, beginning algebra seems to be the course in which difficulty initially is encountered for many students. When difficulty is encountered at this level, many students do not acquire the basic knowledge and

skills that are necessary for higher level mathematics. Algebra is therefore a "gatekeeper" to higher level mathematics as mathematics is a "critical filter" to university entry, scientific/technical majors, and broadened career choices. That is, students who do not have access to algebra or do not have success in beginning algebra tend not to elect higher level mathematics courses. Opening the algebra gate is essential because algebra is a prerequisite for study in nearly every branch of advanced mathematics (Fey, n.d.). Algebra is a basic foundation of higher level mathematics.

The Algebra Project (Silva & Moses, 1990), is one attempt to open the algebra gate: "The conviction of the Algebra Project is that all children can learn algebra" (p. 375). It is an innovative mathematics program that focuses on the students, teachers, and school communities to remove barriers and help students succeed in mastering algebra. In support of the Algebra Project, Kamii (1990) states that this project "has challenged the belief that algebra, currently a 'gatekeeper' course in secondary mathematics education, cannot be grasped by large numbers of inner city minority and poor people" (p. 393). However, she also indicates that if we could start from fundamental knowledge about how all children acquire mathematical concepts then this could validate the adoption of specific innovations in mathematics and science instruction.

Educators generally agree that fundamental knowledge of how students acquire algebraic knowledge is needed to understand the difficulties that students have in the learning of algebra. And theories of algebra need to be applied to problems of curriculum and instruction.

Algebra Instruction

The focus of the standard curriculum is repetitive practice, and algebra pedagogy that attempts to develop algebra competence through drill and practice may be a source of much of the difficulty encountered by students. Such pedagogy "works for routine and repetitive problems but not for the development of free and creative thinking" (Fleming, 1988, p. 19). The logical thinking that is an important aspect of algebra competence is replaced with mindless exercises of manipulating symbols.

Saxon's Incremental Development Model (Saxon, 1982) is an extreme example of such a curriculum. It is based on the idea that algebra is a skill and so practice and repeated review is the major emphasis. Saxon objected to lack of practice time and drill distributed over time as presented in most standard textbooks so he developed a model of instruction and incorporated it into an algebra textbook, whereby the topics are introduced in increments and every topic is practiced in every problem set. Such a curriculum lends itself to mechanical work. It

de-emphasizes the aspect of algebra which encourages, requires, and stimulates thought. A curriculum with such emphasis does not enable students to acquire the algebraic ideas and methods that are required to reason effectively.

In many cases there is little or no attempt to create meaning about the various components of algebraic structural knowledge, even though each of the components is essential in applying algebraic concepts and processes. In the classroom "much time is devoted to the manipulation and simplification of algebraic expressions" (Ernest, 1987, p. 382). Mechanical facility is the focus in the classroom and learning and applying rules is stressed. Therefore, students think of algebra learning as a problem of learning to manipulate symbols according to certain rules (Resnick, Cauzinille-Marmeche, & Mathieu, 1987). "For the most part, students are unaware of or fail to use metacognitive skills" (Schoenfeld, 1989, p. 97). That is, the students do not think about their own thinking. Schoenfeld (1983) suggests that students should be taught to think, to question and to probe; they should be able to employ ideas rather than simply to regurgitate them. However, it is possible to acquire some proficiency through repetitive practice without meaning, but the resultant learning is very fragile as illustrated by common errors.

A Model of Symbolic Algebraic Skill

"The act of encoding natural language and data into a more manageable concise notation is not only advantageous but often virtually essential for the solution of real world problems" (Resnick, 1982, p. 2). This process of encoding natural language and data into a more manageable concise notation can be accomplished utilizing algebra. Algebra then is powerful in that it can be used to represent in concise ways (due to its symbolic system) real world situations. A model of algebraic applications is presented in Figure 1.



Figure 1. A Model of Algebraic Applications

Figure 1 shows that real world situations can be represented by a mathematical model. Formal procedures (algebraic transformations) which involve manipulating and combining symbols in a systematic way, can then be applied to the model so that results can be interpreted algebraically. These algebraic interpretations can then be related back to the context of the real world situation.

To identify the areas of difficulty as well as the specific kinds of difficulty encountered by students in

algebra, a model of algebra (Figure 2), including algebraic knowledge is presented and discussed.

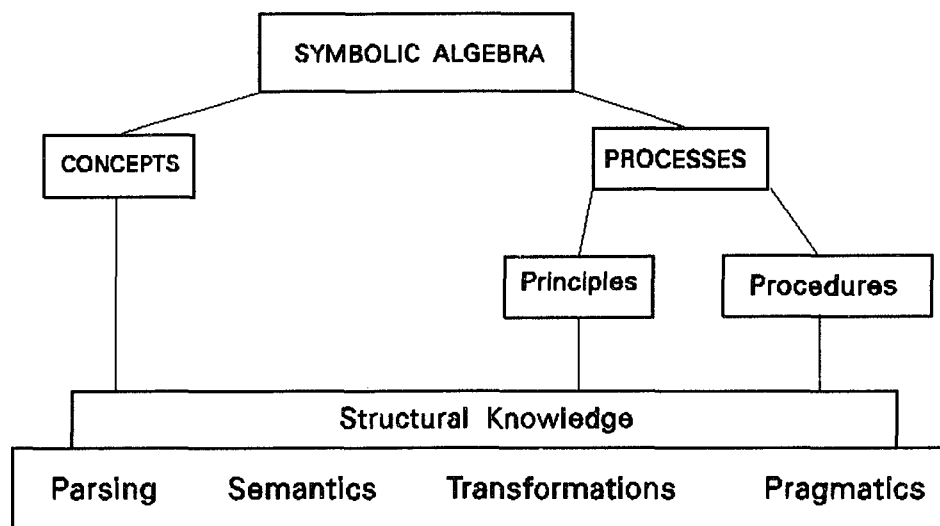


Figure 2. A Model of Symbolic Algebra

"Algebra can be considered as the formulation and manipulation of general statements about numbers" (Kieran, 1989, p. 33). Algebra is composed of concepts and processes. Concepts can be defined as "ordered information about properties of one or more things--objects, events, or processes--that enables any particular thing or class of things to be differentiated from and also related to other things or classes of things" (Sowder, 1980, p. 246). The processes include the principles/laws (such as commutative, associative and distributive) and the procedures (such as simplifying, solving, evaluating, and performing

operations) that apply to the concepts. Knowledge of the structure of algebra is a key element of algebraic competence. Kieran (1989) indicates that the recognition and use of structure is the core of algebra. Additionally, it is the knowledge of the structure of algebra which enables one to apply algebraic concepts and processes.

Yet, certain structural aspects of algebra cause difficulty for students. The kinds of errors that students make in algebra generally indicate a lack of algebraic structural knowledge. The components of algebraic structural knowledge can be identified as knowledge of parsing, knowledge of transformations, semantic knowledge and pragmatic knowledge. These four components of algebraic structural knowledge describe the knowledge necessary to understand and to apply algebraic concepts and processes.

Transformations in Algebra

The components of algebraic structural knowledge are necessary to transform expressions. Parsing involves knowledge of notational conventions that specify grouping rules. Semantic knowledge refers to the referential domain of symbols. Pragmatic knowledge refers to knowledge of symbol manipulations tasks relevant to selecting appropriate transformations. Transformation knowledge refers to knowledge of the rules that take one expression

and derive a new one. A typical example illustrating the transformation of an expression is $x + 4(x + 3) = x + [4(x + 3)] = x + [4 \cdot x + 4 \cdot 3] = x + [4x + 12] = [x + 4x] + 12 = [1x + 4x] + 12 = 5x + 12$. To transform this expression, it is necessary to have the semantic knowledge that the x in this expression represents a number. It is necessary to understand the parse of the expression. That is, addition is the main operator and x and $4(x + 3)$ are subexpressions; multiplication is the main operator of the subexpression $4(x + 3)$; and the operator of $x + 3$ is addition. It is necessary to know the transformations, the distributive property and the associative property. It is necessary to know that x has the explicit representation $1x$. Finally, it is necessary to have pragmatic knowledge. That is, it is necessary to know what to do, what to use and when to use it.

While all four components of structural knowledge are necessary for algebra, this study focuses on the transformation component. The transformation component, sometimes called the systemic structure of an expression (Kieran, 1989), is important in the learning of algebra. The specific transformation that was investigated in this study was distributivity. One example is the transformation--distribution of multiplication over addition, $((a + b)c = ac + bc)$ --that is used to transform the expression $(y^2 + 3)4$:

$$(y^2 + 3)4 = y^2(4) + 3(4), \quad (= 4y^2 + 12)$$

(Distributive Law)

Transformation Knowledge

There are several aspects of the transformation component that cause difficulty for students. This component specifies the knowledge that extends the properties of arithmetic operations to establish the properties of algebra, such as the commutative property, the associative property, and the distributive property. That is, the transformation component explains symbol manipulations in algebra. Symbol manipulation is an important aspect of algebraic applications and is a major focus of the present curriculum; therefore, it should remain among the priorities in the algebra curriculum. However, symbol manipulation is also a major stumbling block for many students. New curricula like Fey's (n.d.) Computer Intensive Algebra, recognize a role, though reduced, for traditional symbol skills. Kieran (1989) indicates that even in a modified algebra environment, there would probably still be the need to formalize procedures and symbolize them. Symbol manipulation "can do more than simulate mindless behavior" (Lewis, 1989, p. 164). But for it to be truly useful to students, they must understand the processes they are employing.

Several aspects of the transformation component that cause difficulty for beginning algebra students and the

type of errors that are made by students are discussed here.

Transformation Errors in Algebra

Errors in algebra are varied and wide ranging, with numerous varieties identified by researchers. But there is general agreement that the vast majority of errors are systematic--the result of definite misconceptions or mal-rules--and not chance results of carelessness or inattention (Brown & Van Lehn, 1980).

Some common transformation errors that occur in algebra have been identified as generic deletion operations (also called the cancellation error) and recombination confusion errors (Lewis, 1980); and distribution errors (Kirshner, 1987; Matz, 1980). Lewis (1980) explains that subtraction and division are deletion operations because they have the effect of deleting something. For instance, $x + a \rightarrow x$ if a is subtracted from $x + a$; and $ax \rightarrow x$ if ax is divided by a for $a \neq 0$. However, the students in observing that subtraction and division have similar effects, ignore the difference between these operations. The result is the deletion error, such as

$$\frac{x + ay}{a} = x + y .$$

Lewis also explains that in the

rearrangement and replacement of symbols, some students get

lost in distinguishing between addition and multiplication. The correct rearrangement and replacement of symbols for two expressions $x + x$, and $x \cdot x$ are $2x$ and x^2 respectively. Thus, the recombination confusion results in errors such as $x + x = x^2$ and $y + yt = 2yt$.

Among the most persistent of the systematic errors are errors that overgeneralize distributivity. These are errors that researchers/educators frequently attempt to explain and remediate. These errors are of the type $(a \pm b)^2 = a^2 \pm b^2$ and $\sqrt{a \pm b} = \sqrt{a} \pm \sqrt{b}$. They are among the most persistent and troubling for students (Laursen, 1978; Maron, 1979; Matz, 1980; Schwartzman, 1977). The explanation and remediation of these errors is the focus of this study.

Whereas the education community has agreed that systematic errors are the result of acquiring symbol skills without meaning, there are widely divergent theories as to what constitutes meaning for algebra.

Three Learning Theories

Many authors have discussed the relationship between error analysis and learning. Davis (1979) studied errors made by students and suggested a conceptual framework to interpret the observations of students learning mathematics. Similarly, Davis and McKnight (1979) studied

the mathematical performances of students in order to develop a system for analyzing the performances. The study of errors can contribute to the understanding of how students learn mathematics. But the problem facing educators interested in students' difficulties is that there is not enough knowledge about how students learn mathematics. The difficulties that students have in algebra, particularly with its structure, is a major concern for educators. So researchers should devise studies that will reveal how students come to understand the structure of elementary algebra and algebraic methods. This particular study is an attempt to contribute to the knowledge about how students learn algebra by looking at one rule in the transformational component. This study will focus on errors that are the result of deficient transformational knowledge. A review of three theories which explain the source of certain regular error patterns follows.

The Meaning Theory of Learning

Brownell (1935) proposed the "meaning" theory of arithmetic instruction. He defines meaningful arithmetic as "instruction which is deliberately planned to teach arithmetical meanings and make arithmetic sensible to children through mathematical relationships" (Brownell, 1947). He further indicates that one of the values of meaningful arithmetic is that it safeguards pupils from

answers that are mathematically absurd. This framework is often used by educators to explain why errors occur in mathematics (Greeno 1978; Lewis, 1980). Thus, the meaning theory of learning (ML) hypothesizes that experts have rich semantic connections to the procedures and symbols of algebra but novices lack such connections. According to this theory, these connections serve to constrain students arbitrary mathematics inventions and thus prevent errors (Ausubel, Novak, and Hanesian, 1978; Brownell, 1947; Wearne and Hiebert, 1985).

The Procedural Learning Theory

Procedural learning is often the focus in teaching mathematics. The concern generally is which rules to apply and when. Procedural learning theories generally attempt to explain how students learn rules and/or why students make errors. Thus the procedural learning (PL) theory hypothesizes that adept problem solvers have technical proficiency in memorizing and applying mechanical rules. Such approaches have been extensively studied in cognitive psychology and modeled by production systems (e.g., Anderson, 1983; Lewis, Milson & Anderson, 1987).

Another instance of a procedural learning theory, called Repair Theory, was proposed by Brown and VanLehn (1980) and applied to procedural errors in arithmetic. The arithmetic research of Brown and VanLehn was extended by

Matz (1980) to algebra. Matz analyzes students' errors as unsuccessful attempts to employ extrapolation techniques to adapt previously acquired rules to new situations. The expert solver has learned to constrain extrapolation more successfully.

It is important to note that in these theories, the main sources of relevant knowledge is in the curriculum. The base rules are for the most part given in the textbook. Extrapolation techniques adapt these rules to the task at hand--either successfully (for experts) or unsuccessfully (for novices).

The Implicit Structure Learning Theory

An alternative to the procedure learning theory which assumes that learning is based on students' initial reception of the rules given in the textbook is the implicit structure learning (ISL) theory. The ISL theory hypothesizes that students approach the algebra learning task with nascent rule structures already in place, and that their experience with the symbol system serves in part to constrain and complete these nascent rules (Kirshner, 1987). Additionally it leaves open the possibility that the rules eventually constructed by the successful problem solver may not be the "correct" rules (i.e. the usually accepted rules of the curriculum) whereas, the procedural learning theory holds that the "correct" (i.e. usually

accepted) rules of the curriculum are the basis of eventual mastery.

This study investigated the efficacy of these three hypotheses and explored some possible implications for the teaching of algebra.

Design of the Study

Each of the three theories described above has specific implications for ways in which algebra should be taught in order to overcome the errors, $(a \pm b)^c = a^c \pm b^c$

and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$. These implications are explored in detail in chapter two.

The purposes of the study were (1) to investigate theories that explain why common errors of the type,

$(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ occur in algebra problem solving by novices, and, (2) to develop and assess techniques for remediating these errors.

The general strategy of this study was to provide alternative experiences for groups of novices that may lead to more expert-like performance according to the differing theories of expertise. The study included four groups-- three treatment groups and one control group. Treatment

group 1 was given a rich semantic treatment of the correct rules for expanding squared binomials. This treatment is in accord with the meaning theory of algebraic expertise. Treatment group 2 was taught about the dangers of overgeneralizing distributivity but without a discussion of operation levels. This treatment is in accord with the procedural learning theory. Treatment group 3 was taught the generalized distributive law (GDL) stressing explanations of the error types. This treatment is in accord with the implicit structure learning theory.

Discovering which treatment is most successful in remedying

the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors

would provide indirect evidence as to the nature of the expert's knowledge.

Summary

In this chapter, I presented some current problems in the field of mathematics education. These problems include the following: deficiencies in students' mathematical performance; a need for more mathematically qualified teachers; students' inadequate precollege mathematics preparation; mathematics acts as a critical filter barring students from entering universities and limiting their career choices; algebra is a gatekeeper that prevents students from electing or having success in higher level

mathematics courses; students lack of logical thinking skills; and the need for more knowledge that would enable one to understand the difficulties that students have in algebra.

Next, I discussed algebra instruction and then presented a model of a symbolic algebraic skill, that provides insight about the kind of algebraic knowledge needed to apply algebraic concepts and processes. Also included is an identification and discussion of some types of errors that occur in algebra by students.

Finally, in this chapter I discussed three learning theories that can be applied to avoid or remediate common errors of the type $(a + b)^c = a^c + b^c$ and

$$\sqrt[c]{a + b} = \sqrt[c]{a} + \sqrt[c]{b} .$$

These theories, the meaning theory of

learning, the procedural learning theory and the implicit structure learning provide alternative characterizations of algebra competence.

In the next chapter, a review of the literature is presented which reports the results and findings of research related to the three learning theories and the

errors, $(a + b)^c = a^c + b^c$ and $\sqrt[c]{a + b} = \sqrt[c]{a} + \sqrt[c]{b} .$

Additionally, an explanation of how each of the theories might be applied to the errors is presented.

In chapter three, the method of the study is given which includes the purposes, the general strategy, the experimental design and statistical methods used to analyze the data, the design caveats, the treatments, a description of the subjects who participated in the study and the measures and data analysis. In chapter four, the results of the study are presented. Finally, chapter five contains the discussion of the results of the study, the limitations of the study and the implications for research and practice.

CHAPTER TWO
REVIEW OF THE LITERATURE

This chapter is a review of the literature which focuses on the distributive errors, $(a \pm b)^c = a^c \pm b^c$ and

$\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ and reasons for these errors. Educators

(Brown and VanLehn, 1980; Booth, 1988; Lauren, 1978; Kirshner, 1987; Matz, 1980) identify and research errors that students make in learning elementary algebra as a way of trying to find out what makes algebra difficult for students and in order to contribute to the knowledge of how students learn mathematics. The three theories, meaning theory of learning, (ML), procedural learning, (PL), and implicit structure learning, (ISL), (described in chapter one and that are about the kind of knowledge experts have and how this knowledge is used by experts to prevent errors), attempt to explain the distributive errors listed above. Thus, the findings as reported in the literature, about the distributive property and the three theories, ML, PL, and ISL are presented here. Also presented is an explanation of how these theories might be applied to explain the occurrence of the common errors,

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} .$$

The Meaning Theory of Learning
and the Distributive Property

The widespread acceptance of "meaning" in mathematics education can be traced back to the work of William A. Brownell and to the set of recommendations put forth by the Commission on Mathematics of the College Entrance Examination Board (Begle, 1979). Brownell (1935) formulated the first comprehensive statement of the "Meaning Theory" of arithmetic instruction. He also listed reasons why meanings should be taught in arithmetic. Some of them are: (1) Arithmetic can function in intelligent living only when it is understood; (2) Meanings facilitate learning; (3) Meanings increase the chances of transfer; and (4) Meaningful arithmetic is better retained and is more easily rehabilitated than is mechanically learned arithmetic; (5) Meaningful learning equips pupils with means to rehabilitate quickly, skills that are temporarily weak and (6) Meaningful learning safeguards pupils from answers that are mathematically absurd (1945; 1947). Brownell (1947) also made a distinction between "meaning of" and "meaning for" in order to clarify the term "meaning."

The result of the study "Meaningful vs. Mechanical Learning: A study in Grade III Subtraction (Brownell and Moser, 1949) supported the "meaning theory." This study involved teaching subtraction of whole numbers using two

different methods (decomposition and equal additions) by mechanical instruction and meaningful instruction. As a whole the findings show that meaningful instruction, especially in the case of decomposition, produced results superior to those produced by mechanical instruction.

An overview of the development of meaningful instruction which deals with the meaning theory of arithmetic was discussed by Weaver and Suydam (1972). They reported the results of meaningfully versus non-meaningfully taught content (such as rote, mechanical or rule) and research that explored the effect of teaching various procedures with meaning (compared different procedures but each procedure was taught meaningfully). The result of much of the research reported also supported the "meaning theory." Some of the findings and conclusions of the research of the former type include: greater transfer when content was meaningfully taught; ineffectiveness of premature drill; understanding should precede memorization; high scores on computation and retention was good when a socially meaningful orientation was combined with a mathematically meaningful teaching; practice should occur after understanding; and, increased ability to solve new processes independently when content is meaningfully taught.

Some of the findings and conclusions of the research of the latter type include: no significant difference in

immediate learning, transfer or retention when drill consists of practice through number relationship or drill through repetition, but considerable gain when taught either method meaningfully and followed by drill; computation skill was not an indication of understanding of meanings of procedures; children who demonstrated understanding of operation revealed high computational skill of operation but not vice-versa; higher achievement in computation, problem solving and mathematical concepts; changes in attitude when meaningful methods of teaching arithmetic are used; emphasis upon distributivity led to superior results on transfer ability, retention achievement and retention of transfer when compared to an approach that did not include work with this property; and no significant difference in overall learning of a mathematics principle between pupils who used a meaningful symbolic model and those who used a meaningful concrete model.

Weaver and Suydam conclude from these results that particular advantages will accrue from meaningful mathematics instruction as opposed to rote instruction but that they are less certain about advantages that may accrue from one meaningful approach method to another meaningful one. Other researchers (Baroody and Hume, 1991; Horak and Horak, 1981; Lesgold, 1987; Wearne and Hiebert, 1985; 1988) also affirm and establish the need for meaning in the learning of many mathematical concepts and procedures.

Some of the conclusions from research where the distributive property is the content, support the ML theory. Distributivity is a property that causes difficulty for some students. It is a standard that is found in the curriculum of many grade levels. "As a result of the emphasis on understanding of arithmetic structure in programs of 'new mathematics,' certain ideas have been added to the elementary school curriculum or have been introduced earlier than previously had been" (Schell, 1968, p. 28). The distributive property is one such instance. Schell further indicates that in recently published third grade elementary school mathematics textbooks, the distributive property of multiplication over addition is used when introducing multiplication of one-digit multiplicands by one-digit multipliers. (e.g. $3 \times 7 = (3 \times 4) + (3 \times 3)$). He found that pupils in grade three can learn to use the distributive property of multiplication (using one-digit multiplicands and one-digit multipliers) but that the distributive property items were significantly more difficult for the students than the non-distributive items. Further, he found that while the high scoring pupils performed approximately equal on both types of items, distributive and non-distributive, the low scoring pupils had a more difficult time with the distributive property items. Thus, exposure to the distributive property did not contribute to the

understanding of the structure of arithmetic for the low scoring pupils. Weaver (1973) also found that students perform poorly on items relating to the distributive property.

Blume and Mitchell (1983) indicate that recent textbooks for grades six through nine offer little in the way of applications of the distributive property. They examined some eighth grade students' knowledge of and ability to apply the distributive property using two inventories. The result of the analysis of the students' errors indicates that most students were at a complete loss. According to these authors, situations which can be modeled by the distributive property are not recognized by many students. In teaching distributivity, Blume and Mitchell suggest that "first, there should be increased focus on distributivity as a mathematical model for the sum of two products with a common factor rather than the simple pattern recognition inherent in many textbook approaches. Second, the vehicle for application of the property should be expanded to include a variety of word problems" (p. 221).

Maron (1979) refers to the error, $(x + y)^n = x^n + y^n$, with special cases, (1) $(x + y)^2 = x^2 + y^2$, (n=2),

$$(2) \sqrt{x+y} = \sqrt{x} + \sqrt{y} \text{ , } (n=\frac{1}{2}) \text{ , and } (3) \frac{1}{x+y} = \frac{1}{x} + \frac{1}{y} \text{ (} n=-1 \text{),}$$

as an application of the Student's Universal Distributive Law (SUDL). That is, for many students $f(x+y) = f(x) + f(y)$ for all x and y no matter what the function f . In hopes of providing a weapon against chronic offenders of the distributive property, Maron suggests using the SUDL to show some rather profound results, such as, if

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y} \text{ (where } x=y=2 \text{) then } \sqrt{4} = 2\sqrt{2} \text{ or } 1 = \sqrt{2} \text{ .}$$

Morelli (1992) and Olson (1991) show how models can be used to help students understand mathematical concepts. Morelli (1992) illustrates how the distributive property can be introduced to students and practiced by students using pictures and symbols. She indicates that the connection between pictures and symbols of abstract ideas can be beneficial for students. Olson (1991) advocates teaching algebra from an algorithmic point of view. He indicates that in computing, the tree is one of the most important data structures and that trees are helpful in learning about structural relationships. He illustrates how the distributive law can be represented as the equivalence of two trees to help students understand its application from the structural aspect.

As a whole, advocates of the meaning theory of learning and the findings of many studies that investigate meaningful learning versus non-meaningful learning, support the arguments for meaning set forth by Brownell (1945; 1947).

The Meaning Theory of Learning Applied

The meaning theory of learning would explain the errors $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ as occurring because students lack meanings connected with these expressions. That is, the students have no referents to connect with the expressions $(a \pm b)^c$ and $\sqrt[c]{a \pm b}$ or their constituent symbols that would warn the students that the answer is incorrect.

A number of different sources might be available to ground the formulas meaningfully, such as, numerical referents (use of numbers to replace variables to establish equivalence of expressions), logical axiomatic applications (use of axioms and definitions to establish equivalence of expressions), and geometrical images (use of geometrical figures to establish equivalence of expressions). We can see how some of these sources might apply to the particular problem at hand. For example, the $(a + b)^2 = a^2 + b^2$ error

might be prevented by using $(a + b)^2$ and $a^2 + b^2$ with geometric representations as areas of squares or sums of squares. The $\sqrt{a^2 + b^2} = a + b$ error might be constrained

by the connection of variable symbols with a numerical domain for a and b . According to the meaning theory of learning then, meanings acquired for the expressions

$(a \pm b)^c$ and $\sqrt[c]{a \pm b}$ through mathematical relationships

and connections would prevent the errors $(a \pm b)^c = a^c \pm b^c$

and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$.

The Procedure Learning Theory and the Distributive Property

An aspect of becoming an adept problem solver in elementary algebra is acquiring procedural skills. Procedural learning theories generally attempt to explain how procedural skills are acquired and why students make errors. The ACT* theory (Anderson, 1983) is an instance of a PL theory. Anderson proposes a framework for skill acquisition that includes a declarative stage and a procedural stage. The declarative stage is the stage "in which the facts about the skill domain are interpreted

(Anderson, 1982, p. 369) and the procedural stage is the stage "in which the domain knowledge is directly embodied in procedures for performing the skill" (p. 369). Thus, according to Anderson, the basic progression of a skill acquisition is as follows. It begins as an interpretation of the declarative knowledge where information about the skill is received by the learner; next, the information about the skill is converted into procedural form; and finally the procedural form is refined until the learner is able to speed up the process. Anderson refers to this progression as a stage analysis of human learning.

Another instance of a PL theory is the Repair theory, a generative theory of bugs in procedural skills, proposed by Brown and VanLehn (1980). This theory was developed for algorithms in arithmetic and is motivated by the belief that when a student has unsuccessfully applied a procedure to a given problem, (s)he attempts to repair the procedure by using general knowledge to "patch" the algorithm so that it can be completed. The repair theory predicts the systemic errors (bugs) that students will make in learning a skill.

The PL theory proposed by Matz (1980) is an extension of the research by Brown and VanLehn to algebra. It proposes that "errors are the result of reasonable, although unsuccessful, attempts to adapt previously acquired knowledge to a new situation" (p. 95). The theory

further proposes that problem solving behavior employs two components, base rule (rules of the curriculum/textbook rules) and extrapolation techniques (ways to bridge the gap between known rules and unfamiliar problems). Matz indicates that many common errors are due to (1) using a known rule in an inappropriate situation and (2) incorrectly changing a rule so that it can be applied to a new problem.

Yerushalmy (1991) investigated the effect of a variety of computerized feedback on the student's performance in carrying out algebraic transformations and the student's performance in debugging their own working processes (procedural learning). The research was based on the premise that the source of errors in the simplifying of expressions is students' inability to understand the correct algebraic algorithm and the students' falsely generalizing known rules. There were four groups involved in the study. The control group received no treatment. Each of the other three groups had use of computer software. The software tools were (1) a yes/no error indicator, (2) a manipulator (a tool used to identify the type of legal transformation desired and the terms on which to operate), and (3) a graph. The findings were as follows: (1) when there was no feedback, the students continued the task but did not notice if and where they had made a mistake; (2) when the feedback was yes/no, the

students' working process was longer and they did try to correct their error but often in vain; (3) the manipulator group did achieve higher scores in correct processing than all other groups; and (4) the use of the graph moved the students from debugging of algebraic transformations to identifying and correcting bugs but the use of the graphs did not significantly reduce the number of false steps, nor the number of uncorrected errors.

Other researchers, (Lauren, 1978; Resnick, Cauzinille-Marmeche & Mathieu, 1987) support Matz theory. Lauren identifies several problem types in first year algebra that are particularly difficult for students. She explains the error, $\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2} = a + b$ as an

incorrect application of the principle $\sqrt{a^2 b^2} = \sqrt{a^2} \sqrt{b^2}$, for $a \geq 0, b \geq 0$. Resnick, Cauzinille-Marmeche and Mathieu identify Matz's theory as one of two of the best-developed theories to date of how algebra malrules are invented.

The Procedural Learning Theory Applied

Matz (1980) has sought to explain the

$(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors according

to the procedural learning theory. She identifies these as

part of a class of linear decomposition errors that occur as a result of a reasonable attempt to adapt previous knowledge to a new situation. This means that the errors are the result of the students' attempt to modify a known rule of the curriculum to fit the new situation.

For instance, according to the Procedural Learning theory, this class of errors is the result of linearly decomposing an expression by distributing the top-most operator across its expression parts. However, according to the theory, linear decomposition is sometimes correct and sometimes not. The correct and incorrect examples of linearity applied to various rule patterns are shown below.

Correct Rules

$$a(b + c) = ab + ac$$

$$a(b - c) = ab - ac$$

$$\frac{b + c}{a} = \frac{b}{a} + \frac{c}{a}$$

$$(ab)^c = a^c b^c$$

$$\sqrt[c]{ab} = \sqrt[c]{a} \sqrt[c]{b}$$

Incorrect Rules

$$\sqrt{a + b} = \sqrt{a} + \sqrt{b}$$

$$(a + b)^2 = a^2 + b^2$$

$$a(bc) = ab \cdot ac$$

$$\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$$

$$2^{a+b} = 2^a + 2^b$$

$$2^{ab} = 2^a \cdot 2^b \text{ (Matz, 1980)}$$

Matz further explains that the student having been exposed to the rules $a(b + c) = ab + bc$ and $a(b - c) = ab - ac$, deviates from it to think that the middle operator can be any operator. The student also recalls that other versions of the distributive law worked for some operators other than addition and times such as $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and

$(ab)^2 = a^2b^2$. So the student actually has a wide array of linearity rules to motivate the extrapolation to these invalid instances. When the student encounters $(a \pm b)^c$ or $\sqrt[c]{a \pm b}$ and does not know what rule to apply (s)he adapts

the linearity rules to get the invalid rules

$$(a \pm b)^c = a^c \pm b^c \text{ and } \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} .$$

The Implicit Structure Learning Theory and
the Distributive Property

The implicit structure learning (ISL) theory hypothesizes that students enter the classroom with nascent abstract rule structures on which to build a more mature "grammar of algebra" through inductive processes (Kirshner, 1987). According to this theory, errors like those above evidence students' search for the appropriate constraints under which their nascent rule structures apply.

The ISL theory in algebra is an adaptation of Chomskyan linguistic theory--the system of hypothesis concerning the general features of human language put forth in an attempt to account for a certain range of linguistic phenomena (Chomsky, 1975). Chomsky (1975) indicates that competence in a natural language is developed by each human being for him or herself and can be represented as a system of rules called the "grammar" of the language. "Thus a person who has acquired knowledge of a language has internalized a system of rules that relate sound and meaning in a particular way" (Chomsky, 1972, p.26).

ISL theorists have differed in their reliance on Chomskyan terms and methods. Bolio (1989) and Drouhard (1988) follow Chomsky's (1957; 1965) linguistic conventions closely by using labelled nodes in tree diagrams of expressions to assign structural descriptions to components of expressions. Bolio (1989) develops a generative grammar

for simple expressions and equations of basic math and algebra focusing on the notational language that mathematicians use to communicate the elementary concepts of basic math and algebra to elementary and secondary students. He affirms that there is enough analogous syntactical-linguistic elements in mathematical language to consider mathematics a language in itself.

Drouhard's (1988) grammar is more directed towards providing structural descriptions of students' understanding of expressions. His attempt is to develop an algebraic metalanguage based on structural descriptions and transformations in the grammar that match the structures students develop through their immersion in the algebra class. Kirshner's (1987) theory is like these in using grammatical methods to describe implicit knowledge of algebraic structure; however, he does not rely on labelled nodes for structural descriptions, investing more of his theory in the transformational and transnational components.

In their famous debate, Chomsky and Piaget argue about the processes of language development in the child (Piatelli-Palimarini, 1980). The two theorists agreed that linguistic structure must be induced by the learner based on their experience in a language community. But they disagreed sharply as to the nature of the inductive mechanisms. Piaget believed that general learning

mechanisms were sufficient for inducing linguistic structure. Chomsky believed that syntax is far too complex to be acquired from general learning capacities, and that innate language-specific knowledge of a "universal grammar" must be postulated. The universal grammar thus provides a general starting point for linguistic development, with one's experience in a particular language community serving to specialize the general linguistic endorsement to the local language.

ISL theories subscribe to the notion of abstract, preexisting algebraic structure without necessarily endorsing an innate stance. For instance, Kirshner (1987) proposes that abstract generalized knowledge of distributivity could reflect distributive structures that already have become established in natural language. Sentences like, "I like cake and ice cream," which can be interpreted as "I like cake and I like ice cream" or "I like cake and ice cream" (together), illustrate that English speakers are continually determining whether distributivity applies in some circumstance or another. This natural language experience could be the basis for students' unconscious grappling with distributive structure in algebra. Thus there is no need to take a stance as to whether distributivity or other possible algebraic structures stem from some innate endorsement or from natural language experience.

The notion of operation levels is an important part of the GDL. This idea was introduced by Schwartzman (1977) as he explains the errors

$$(a \pm b)^2 = a^2 \pm b^2, \sqrt{x^2 - y^2} = \sqrt{x^2} - \sqrt{y^2} = x - y, a(xy) = ax \cdot ay$$

as the inability of students to recognize when one operation distributes over another. To offset the problem, Schwartzman defines operation levels and explains that "distributing is applying an operation from a given level to two quantities related by operations of the next lowest level only" (p. 594).

The Implicit Structure Learning Theory Applied

As Matz (1980) notes, the $(a \pm b)^c = a^c \pm b^c$ and

$$\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$$
 errors are instances of an

overgeneralization of distributivity. But whereas Matz (1980) presumes that the basis for overgeneralization is the rules that previously have been learned, Kirshner (1987) hypothesizes that the learner approaches the study of algebra with nascent distributive structures already in place. For instance, in natural language, Kirshner (1987) identifies linguistic processes that are of the same form. The sentences, 'John and Mary went to the store' (meaning

John went to the store and Mary went to the store), and 'The old man and woman came down the stairs' (an ambiguous statement) are examples.

According to the ISL theory, the mastery of algebra consists not so much in learning rules from the curriculum, but in constraining the nascent rule structures that precede algebra instruction. Kirshner (1987) explains the

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{errors in the}$$

following way. Initially, students enter with a very general distributive structure that might be represented by $(a * b) @ c = (a @ c) * (b @ c)$, where $*$ and $@$ represent arbitrary operations. Mastery consists of achieving an abstract (but unconscious) set of constraints on the operations. Using a system of operation levels introduced by Schwartzman (1977), (see Table 1), the maximal appropriate constraints can be symbolized as $|\@| = |*| + 1$

where $|\@|$ is the level.

Table 1
Levels of Operations

Level I	Level II	Level III
Addition	Multiplication	Exponentiation
Subtraction	Division	Radical

Note that this rule subsumes eight standard curricular rules that Matz (1980) takes to be the fundamental sources of distributive structure.

Level 2 over Level 1

$$\begin{aligned}(a + b)c &= ac + bc \\ (a - b)c &= ac - bc \\ \frac{a + b}{c} &= \frac{a}{c} + \frac{b}{c} \\ \frac{a - b}{c} &= \frac{a}{c} - \frac{b}{c}\end{aligned}$$

Level 3 over Level 2

$$\begin{aligned}(ab)^c &= a^c b^c \\ \left(\frac{a}{b}\right)^c &= \frac{a^c}{b^c} \\ \sqrt[c]{ab} &= \sqrt[c]{a} \sqrt[c]{b} \\ \sqrt[c]{\frac{a}{b}} &= \frac{\sqrt[c]{a}}{\sqrt[c]{b}}\end{aligned}$$

According to Kirshner (1987), the $(a \pm b)^c = a^c \pm b^c$

and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors represent a penultimate state

¹The rules involving the radical operation appears in surface form to be left-distributive; however, Kirshner (1987, p. 93) argues that the deep representation of the radical operation is reversed from its surface form.

of mastery in which the constraints on operation levels are not fully developed: $|@| > |*|$ instead of $|@| = |*| + 1$.

In this case the students' abstract but unconscious grammar permits the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$

errors (level 3 over level 1) as well as the usual correct instances of distributivity.

The three hypotheses and their applications of the errors presented above provided the basis for the development of the techniques used for the remediation of

the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ errors.

CHAPTER THREE

METHOD

Design of the Study

The purposes of this study were (1) to investigate theories that explain why common errors of the type

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{occur in algebra}$$

problem solving by novices, and, (2) to develop and assess techniques for remediating these errors.

General Strategy

The general strategy of the study was to provide alternative experiences for groups of novices intending to lead to more expert-like performance, according to the three differing theories of expertise in algebra described in chapter one. The success of one treatment over another would provide indirect support for the theory--meaning theory of learning, procedural learning theory or implicit structure learning theory--underlying that method. The general character of each of these theories is briefly summarized here, and then the design is described more fully.

The meaning theory of learning hypothesizes that experts have lots of rich meanings connected with procedures and symbols but novices lack these connections.

Thus, according to this theory, the errors

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$$

a lack of connections and references for expressions and symbols involved. Such connections serve to help students remember rules and to constrain students arbitrary mathematics inventions, according to the meaning theory of learning.

The procedural learning theory hypothesizes that adept problem solvers have technical proficiency in memorizing and applying mechanical rules. They are able to match the structure of the problem with the structure of the rule that is to be applied. According to Matz (1980), students errors are the result of unsuccessful attempts to employ extrapolation techniques to adapt previously acquired rules to new situations. The adept problem solver has learned to constrain extrapolations more successfully. Thus according to this theory, the errors $(a \pm b)^c = a^c \pm b^c$

and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ result from the students employing inappropriate extrapolations.

The implicit structure learning theory hypothesizes that people skillful at elementary algebra have developed an unconscious abstract rule system that underlies their

successful performance. According to this theory, students enter the classroom with nascent abstract algebraic rule structures on which to build; and then they begin to sort out the conditions under which the particular structures apply. That is, the students are inductively and unconsciously experimenting to fashion an abstract "grammar" of algebra.

According to this theory, the errors

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$$

of the students (in the process of maturing) experimenting with an abstract distributive rule, $(a * b) @ c = (a@c) * (b@c)$, in search of the maximally permissible context in which it applies in algebra. In the end, the successful ones have learned that addition and subtraction are level one operations; multiplication and division are level two operations; exponentiation and radical are level three operations; and that $|\@| = |*| + 1$ where " $*$ " is an

operation, and " $|*|$ " represents its level. The

constraint to be learned is that the operation being distributed must be one level higher than the other operation.

These three theories give differing explanations about why errors occur and lead to differing predictions about

what kind of remedial strategies ought to be most effective in overcoming the $(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$

errors. Remedial techniques were designed on the basis of the three theories. Implementing these strategies and their effectiveness provided interesting insights into the nature of symbolic competencies in algebra.

The null hypothesis is that there is no significant difference in the number of errors between the four groups on the post and retention tests.

Experimental Design and Statistical Methods

The study was a quasi-experiment conducted at Southern University in Baton Rouge, LA. The study used a pretest-posttest-retention test, control group design with three treatments--meaningful learning (ML), procedural learning (PL), and implicit structure learning (ISL),--and one control (C) which received no special instruction concerning the errors.

An analysis of variance for repeated measures was used to analyze the data. One factor, between subjects, was instructional methods (meaningful learning, procedural learning, implicit structure learning and a control); a second factor, within subjects, was repeated measures (pretest, posttest and retention test). A post hoc test (the Duncan-Range) was used to evaluate the main effects if

there was an overall significant difference.

Design Caveats

The purpose of the study was not to provide definitive proof as to the psychological validity of the three theories; but to carefully frame the theories and to illustrate their applications to curriculum. Whereas the design does have some merits as an empirical test of the theories' validity, the test is very indirect and the data highly interpretable.

The first caveat stems from the indirectness of the link between the theory (as implemented in the treatment) and the learning outcome. For instance, the meaning theory of learning specifies extensive and complex conceptual and semantic connections. A brief instructional treatment cannot make more than modest progress towards this goal. Additionally, the ML treatment is somewhat restricted in that the semantic connections or referents used in this study are all mathematical. That is, no real world referents were used. The instruments provided some specific measures of the students' meaningful understanding; so the effectiveness of the treatment was assessable in the study. But, whereas the general position that an increase in meaningful learning predicts a decrease in error rates seems tenable, knowing the degree of error decrease may depend upon a mere complex analysis of what

aspects of meaningful understanding actually have been acquired.

A second design caveat concerns the nature of the implicit structure learning treatment. According to the theory, expertise is gained through the development of unconscious constraints in the application of rule structures. But the treatment leads to the conscious, explicit elaboration of the constraints. Thus there is an implicit assumption in the design that consciously held structures can function as constraints on behavior analogously to unconscious ones. This is an assumption that may have pedagogical significance, but it is not part of the ISL theory per se. For such reasons as these, the design is not sufficiently robust to serve as a definitive guide to the validity of the three theories. Nevertheless, it provides a basis for a rich contrast and comparison of three very different approaches to a significant education topic.

Treatments

An overview of each treatment is given here. (see Appendices D, E, and F for the detailed instructional sessions for the treatment). The meaningful learning treatment (ML) consisted of a variety of rich semantic experiences: (1) Numerical instances were used to evaluate whether proposed rules are correct or incorrect; (2)

Axiomatic methods were used to establish equivalent expressions; and (3) Geometrical models of expressions were constructed to verify equivalences.

The procedural learning treatment (PL) consisted of reviewing the textbook rules and illustrating these rules with numerical instances. Expressions where these rules can or cannot be applied were identified. Possible new rules were generated by the students for situations in which the given rules are inapplicable, and the validity of these new rules was assessed. The dangers of overgeneralizing given rules without verification were stressed. In addition to usual algebra, a contrived rule system was used to further the students' procedural competence.

The implicit learning treatment consisted of experiences that enabled students to determine the constraints of distributivity: (1) Examples of the distributive structure in natural language were presented and discussed; (2) The students generated correct and incorrect rules of distributivity in algebra to compare and contrast; and (3) The operation levels were presented to provide an additional catalyst that would assist students in determining the constraints of distributivity.

A brief description of the daily activities for the three instructional sessions is given here. See Appendix

D, E and F, for the complete lesson plan for each of the three instructional sessions.

The Meaningful Learning Instructional Sessions

The goal of the meaningful learning sessions was to remediate errors of the type $(a \pm b)^c = a^c \pm b^c$ and

$$\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}, \text{ according to the ML theory.}$$

Day One

The first day's session was designed to help students make the connection between variable symbols and numbers, and to use this connection to evaluate the equivalence of expressions. The first activity involved: determining if expressions were equivalent; establishing how one can determine equivalent expressions; and establishing the meaning of equivalence and non-equivalence. The second activity was an exercise in which the students used numerical instances to determine equivalence or non-equivalence of expressions.

Day Two

The second day's session was designed to introduce a second method to determine equivalence of expressions. Following a brief illustration of the limitations of numerical methods, a review of definitions and axioms was

given. The definitions included squaring and cubing. The axioms included the commutative, associative and distributive axioms. The learning activity was an exercise for the students to establish equivalences using the axiomatic method.

Day Three

The third day's session began with a review of day one and day two's sessions. This session was designed to reinforce some formulas geometrically. $(a + b)^2$ was represented using a square, and $\sqrt{a^2 + b^2}$ was represented as the hypotenuse of a right triangle. The first activity involved the teacher and the students reviewing the concept of area. The area of a rectangle and square was discussed. This was followed by the teacher assisting the students to construct a geometric model of $(a + b)^2$, and to establish its equivalent, $a^2 + 2ab + b^2$. The second activity involved constructing a geometric model of $\sqrt{a^2 + b^2}$. The students were assisted by the teacher. The students established that $\sqrt{a^2 + b^2} \neq a + b$. The students were required to make the constructions at their seats

using rulers as the teacher directed the constructions at the board.

The Procedural Learning Instructional Sessions

The goal of the procedural learning sessions was to remediate errors of the type $(a \pm b)^c = a^c \pm b^c$ and

$$\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} , \text{ according to the PL theory.}$$

Day One

The first day's session was designed to help students identify expressions for which rules can or cannot be applied. Some of these rules were standard textbook rules and others were contrived formal constructions. The first activity involved reviewing and applying rules that the students had previously encountered. After the teacher reviewed the rules, the students studied given expressions and decided whether a rule applied, and if so, they applied that rule. The second activity involved a similar exercise except the rules involved were contrived formal constructions. In both cases, the rules to be applied were visually available to the student.

Day Two

The second day's session was designed to help students identify expressions where rules can and cannot be applied,

to write general rules, and to extend rules where possible. The rules used were not visually available to the students, but needed to be recalled. The first activity involved applying previously encountered rules and then students had to write down a general statement of the rule applied. The second activity involved extending rules when possible and verifying the rules using the axiomatic method.

The Implicit Structure Learning Sessions

The goal of the implicit structure learning sessions was to remediate errors of the type $(a \pm b)^c = a^c \pm b^c$

and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$, according to the ISL theory.

Day One

The first day's session was designed to help students identify the distributive structure in natural language; and to help them identify and generate rules, distributive rules and non-rules in algebra. The first activity involved looking at some English statements, explaining their meaning, formalizing them (using letters) and observing their distributive structure. This was followed by an activity in which the students related the above activity to algebra. They were asked to generate a list of algebra rules of the same form as the statements.

Day Two

The second day's session was designed to help students discover constraints on the application of distributivity in algebra. The first activity involved the teacher identifying the correct rules that were on the list of generated rules from the previous day. The second activity involved the teacher introducing the operation levels. The third activity involved the students comparing the rules (correct and incorrect) in order to formulate expert constraints on distributivity.

Subjects

The subjects in the study were enrolled at Southern University in Baton Rouge. The student population is predominantly Black Americans (96.4 %) and includes Whites (1.9 %), Hispanics (.2 %), Asians or Pacific Islanders (.1 %) and others (1.4 %). Four intact sections of the developmental intermediate algebra (Math 107) were used. Students who were enrolled in this course met one of the following criteria: They obtained less than 11 on the ACT but passed a Developmental Beginning Algebra course with a grade of "C" or better; or obtained an ACT score of 11 - 16. The majority of these students have non-science majors. Their majors include accounting, marketing, business administration, economics, sociology, music and education.

Measures and Data Analysis

The measures that were used in the study were the pretest, the posttest, and the retention test. The pretest, posttest and retention test are matching test constructed by the investigator (see Appendices A, B, and C respectively). The pretest was administered prior to treatment to assess students incoming strengths and knowledge of the distributivity property. The posttest was administered following treatment to assess the immediate effect of the treatment. The retention test was administered approximately three weeks following the post test to assess the long term or retention effect of the treatment.

The data was analyzed using the following procedure. The dependent variable was determined by the number correct responses on the pretest, posttest and the retention test. Each of these tests was divided into four parts. Part 1 consisted of 15 multiple choice items involving the distributive property and required selecting equivalent expressions. Two of these items are,

(1) $(x + y)z =$

(A) $x + yz$ (B) $xz + yz$ (C) $xz + xy$ (D) $x^2 + y^2$

(E) none

$$(2) \quad \sqrt{x + y} =$$

$$(A) \quad \sqrt{x} + \sqrt{y} \quad (B) \quad x + y \quad (C) \quad x^2 + y^2$$

$$(D) \quad \sqrt{xy} \quad (E) \quad \text{none.}$$

A subtest of part 1, identified as error items only, (EIO), was also analyzed. This subtest, EIO, consists only of items from part 1 that involve the error types

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} . \quad \text{The}$$

explanations and remediation of these error types was the focus of this study.

Part 2 was applications of $(a \pm b)^c$ and $\sqrt[c]{a \pm b}$ and consisted of 4 items. Two of these items are,

- (1) Solve for x: $(x + 3)^2 - 9 = 16$. Show your work.
- (2) A flat rectangular packing case is 5" wide by 12" long. What is the length of the longest knife that could be placed in the case? (Hint: use the diagonal). Show your work.

Part 3 consisted of five items based on the ML theory. Two of these items are,

- (1) What do you think is meant by equivalent

expressions.

- (2) Are the expressions $(3x)^3$ and $27x^3$ equivalent?
 (a) If so why? or if not, why not?
 (b) Can you think of another way of proving or demonstrating the equivalence or nonequivalence of $(3x)^3$ and $27x^3$.

Part 4 consisted of four items based on the PL theory. The students were given a set of contrived rules and asked to determine if any of the rules could be applied. If so, they were to write the new expression using the rule selected and indicate the rule used. The rules were,

(A) $(^x)y \rightarrow yx$

(B) $x(^y) \rightarrow y$

(C) $^(xy) \rightarrow x$

(Note: x and y are variables) and two of the items are,

(1) $3(^k) \rightarrow \underline{\hspace{2cm}}$

Rule: (A) (B) (C) (D) none

(2) $(2f)^(g^5ht) \rightarrow \underline{\hspace{2cm}}$

Rule: (A) (B) (C) (D) none.

(For comments on these rules, see the limitation section in chapter five).

The data analyses involved use of a two-way Analysis of Variance for repeated measures where the first factor, which is between subjects, is teaching method and the second factor, which is within subjects, is repeated testing. The repeated measures used to determine the

immediate effect were the pretest and the posttest. The repeated measures used to determine the retention effect were the pretest and the retention test. However, posttest to retention test gains also were examined to determine if there was more consistency between the posttest and retention test than between the pretest and retention test and to help explain some of the unexpected results. The analyses were performed on each of the four parts of the test and on the subtest of part 1, error items only, (EIO).

CHAPTER FOUR

RESULTS

This chapter presents an analyses of the data resulting from this study. Analyses of the data from all 40 subjects is presented below. It should be noted that not all subjects received all parts of the treatment due to absences. A comment will be made on this later.

Tables 2, 3, and 4 show the mean correct answers for the pretest, the posttest and the retention test, respectively for each group.

Analysis of variance (ANOVA) was performed on the pretest to establish whether the groups were initially equivalent. The analysis of the pretest that was taken by all the students showed no significant difference between the groups on any part of the test; thus establishing initial group equivalence. Additionally, the pretest analysis showed that the EIO subtest is below chance for the groups. This is evidence that students have not just learned, they have mis-learned.

The Immediate Effect

The analysis of part 1 (Distributive Subtest) showed no significant difference between the groups, but it showed a significant difference ($p < .005$) between the mean scores of the pretest and the posttest. The mean of the posttest

is significantly higher than the mean of the pretest (see Table 5).

Table 2
Mean Scores of Pretest

		PRETEST				
Group	n	Part 1	EIO	Part 2	Part 3	Part 4
ML	6	4.8	.67	1.7	6.2	2.8
PL	12	5.8	.917	2.9	8.8	3.33
ISL	13	6.0	.923	2.6	7.1	3.31
CONTROL	9	4.7	.67	1.7	6.1	3.2
$\Sigma n = 40$						
Total Possible Points		15.0	5.0	8.0	20.0	8.0

Table 3

Mean Scores of Posttest

		POSTTEST				
Group	n	Part 1	EIO	Part 2	Part 3	Part 4
ML	6	5.3	.33	1.7	7.7	2.8
PL	12	7.2	1.33	2.7	7.6	5.2
ISL	13	7.23	1.38	2.23	6.8	3.2
CONTROL	9	5.8	1.11	2.0	6.4	2.3
$\Sigma n = 40$						
Total Possible Points		15.0	5.0	8.0	20.0	8.0

Table 4
Mean Scores of Retention Test

RETENTION TEST						
Group	n	Part 1	EIO	Part 2	Part 3	Part 4
ML	6	4.7	.83	2.0	6.2	2.5
PL	12	7.2	1.16	3.0	9.3	4.4
ISL	13	6.4	.923	2.4	4.5	3.2
CONTROL	9	5.6	.556	1.8	3.9	1.4
$\Sigma n = 40$						
Total Possible Points		15.0	5.0	8.0	20.0	8.0

Table 5
Immediate Effect for Part 1 (Distributive Subtest)

Source	DF	SS	MS	F-Ratio	p>F
Teaching Method	3	36.71	12.23	1.29	.29
Repeated Testing	1	25.31	25.31	9.62	.0037*
Interaction	3	1.51	.50	.19	.90
Error	36	94.7	2.6		

*Significant

An analyses of the error items only, (EIO), part 2 (Application Subtest), and part 3 (ML Subtest), showed no significant differences between groups.

For part 4 (PL Subtest), the analysis showed a marginally significant interaction between teaching methods and repeated testing. The marginally significant difference was due to the PL group--the mean of the posttest is significantly higher than the mean of the pretest. The procedural group made greater gains than the other groups on this portion of the test directly relevant to the procedural treatment (see Table 6 and Figure 3).

Table 6

Immediate Effect for Part 4 (PL Subtest)

Source	DF	SS	MS	F-Ratio	p>F
Teaching Method	3	28.59	9.53	1.17	.33
Repeated Testing	1	2.11	2.11	.81	.37
Interaction	3	21.65	7.21	2.77	.056**
Error	36	93.7	2.6		

** Marginally Significant

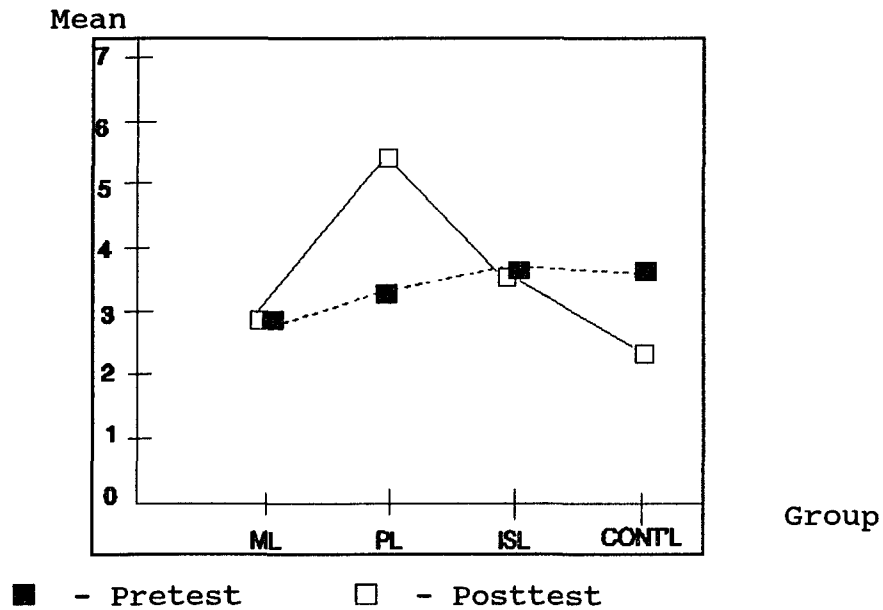


Figure 3: Teaching Method and Repeated Testing Interaction for Part 4 (PL Subtest)- Immediate Effect

The Retention Effect

The analysis of part 1 (Distributive Subtest) showed no significant difference between the groups. However, the analysis showed a marginally significant difference between the mean scores of the pretest and the retention test. The mean of the retention test is marginally significantly higher than the mean of the pretest (see Table 7).

For the error items only, (EIO), and part 2 (Application Subtest), the analyses showed no significant differences between groups.

For part 3 (ML Subtest), the analysis showed a marginally significant difference between the groups with the Procedural Learning group marginally significantly higher than the other three groups. A significant difference was found for repeated testing (the pretest and the retention test) with the pretest significantly higher

Table 7

Retention Effect for Part 1 (Distributive Subtest)

Source	DF	SS	MS	F-Ratio	p>F
Teaching Method	3	37.48	12.49	1.5	.23
Repeated Testing	1	9.79	9.79	3.8	.059**
Interaction	3	5.47	1.82	.71	.55
Error	36	92.7	2.58		

** Marginally Significant

than the retention test. (This was not expected and will be discussed in Chapter 5). There is also a significant interaction between teaching method and repeated testing. The significant interaction was due to the ISL group as well as the control group between the pretest and posttest. In both the ISL group and the control group, the mean score of the pretest was significantly higher than the mean score of the retention test. This was not expected and these

differences will be discussed in Chapter 5 (see Table 8 and Figure 4).

For part 4 (PL Subtest), the analysis showed a marginally significant interaction. The marginally significant interaction was due to the PL group as well as the control group. The mean of the retention test is marginally significantly higher than the mean of the pretest for the PL group. However, the mean of the pretest is marginally significantly higher than the mean of the retention test for the control group. The latter was not expected and will be discussed in Chapter 5 (see Table 9 and Figure 5).

The pretest scores and the retention test scores are the measures that were used in the analyses to determine the retention effect reported above. However, to better understand the unexpected results above, the retention effect was also calculated with the posttest and retention test as measures. Some differences between the analyses (pretest, retention test versus posttest, retention test) were found and are summarized here. On part 1 (Distributive Subtest), no significant difference was found between the mean scores of the posttest and the retention test; previously, the mean score of the retention test was marginally significantly higher than the mean score of the pretest. On part 3 (ML Subtest), no significant difference was found between the groups; previously, a marginally

significantly difference was found between the groups with the mean score of the PL group marginally significantly higher than the mean scores of the other three groups. On part 4 (PL Subtest), a significant difference was found between the groups with the mean score of the PL group higher than the mean score of the other three groups; previously, no significant difference was found between the groups. The results of the analyses for the EIO subtest and part 2 (ML subtest) were the same as the pretest-retention test results. No other differences were found.

Table 8

Retention Effect for Part 3 (ML Subtest)

Source	DF	SS	MS	F-Ratio	p>F
Teaching Method	3	204.46	68.16	2.41	.08**
Repeated Testing	1	28.80	28.80	7.36	.01*
Interaction	3	39.38	13.13	3.36	.03*
Error	36	140.8	3.9		

*Significant **Marginally Significant

Mean

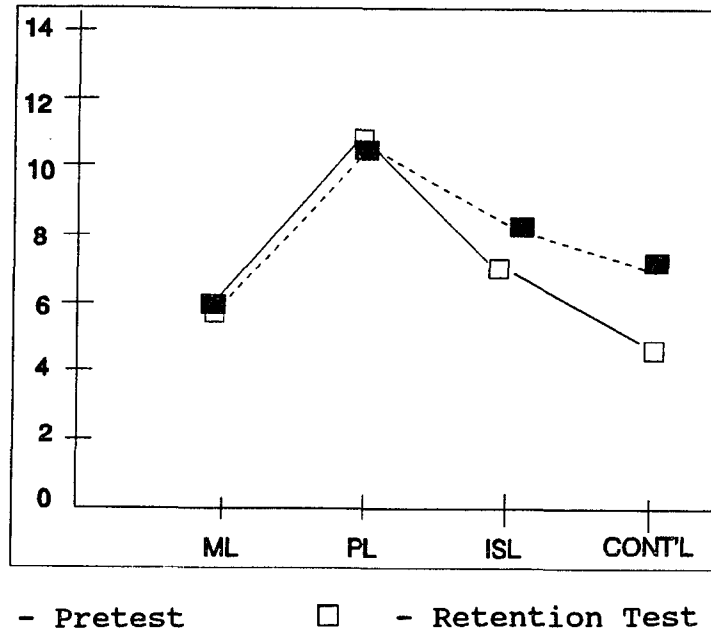


Figure 4: Teaching Method and Repeated Testing Interaction for Part 3 (ML Subtest) - Retention Effect

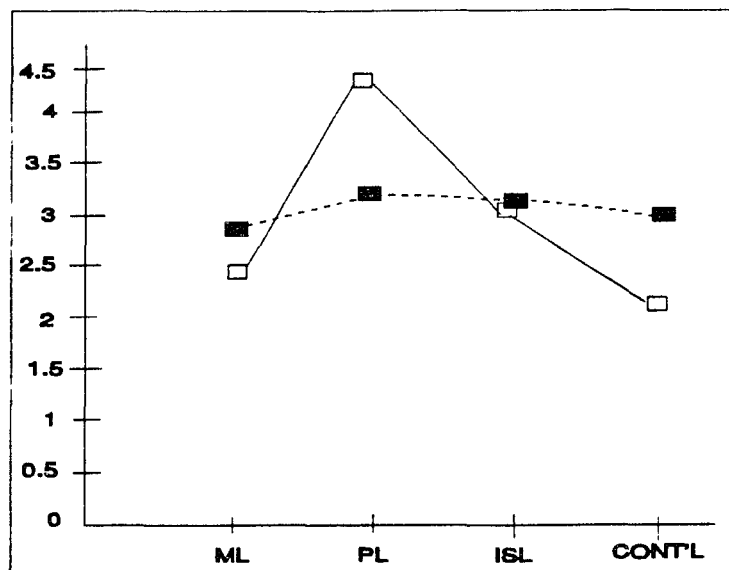
Table 9

Retention for Part 4 (PL Subtest)

Source	DF	SS	MS	F-Ratio	p>F
Teaching Method	3	27.79	9.26	1.29	.29
Repeated Testing	1	.45	.45	.15	.697
Interaction	3	21.19	7.06	2.41	.08**
Error	36	105.36	2.93		

**Marginally Significant

Mean



■ - Pretest □ - Retention Test

Figure 5: Teaching Method and Repeated Testing Interaction for Part 4 (PL Subtest)-Retention Effect

Attendance Filtered Analysis

The reader should note that all of the results above include data of students who did not receive all parts of the treatment. Since attendance is a factor that may affect the result of the treatment, the researcher also analyzed the data collected from the subjects who were in attendance at each session of the treatment. However, a warning is issued that filtering attendance reduces n for the ML group to two, for the PL group to seven, for the ISL group to 11, and for the control group to eight.

The analysis of the immediate effect with attendance filtered was the same except on part 4 (PL Subtest). Previously, a marginally significant interaction was due to the PL group with the mean of the posttest marginally significantly higher than the mean of the pretest. With attendance filtered, no significant interaction occurred.

The analysis of the retention effect (the difference between the retention test score and the pretest score) with attendance filtered was the same except on part 1 (Distributive Subtest) and part 4 (PL Subtest). Previously, on part 1 (Distributive Subtest), the mean of the retention test was *marginally significantly* higher than the mean of the pretest. Now, when attendance is filtered, the mean of the retention test is *significantly* higher than the mean of the pretest. Previously, on part 4 (PL Subtest), a marginally significant interaction was due to the PL group and in the control group. Now, when attendance is filtered no significant interaction is found. No other differences were found.

Summary

The analyses show no significant differences between the groups on any part of the test (including the subtest, EIO) when analyzing the pretest and the posttest for the immediate effect. The analyses show the mean of the posttest significantly higher than the mean of the pretest on part 1 (Distributive Subtest). Additionally, there is a

marginally significant interaction due to the PL group between teaching method and repeated testing on part 4 (PL Subtest).

The analyses show no significant difference between the groups on parts 1 (Distributive Subtest), part 2 (Application Subtest) and part 4 (PL Subtest) of the test nor the subtest (EIO) when analyzing for the retention effect. There is a marginally significant difference between the groups on part 3 (ML Subtest), with the mean of the PL group higher than the mean of the other three groups. The analyses also show the mean of the retention test marginally significantly higher than the mean of the pretest on part 1 (Distributive Subtest); the mean of the pretest is significantly higher than the mean of the retention test on part 3 (ML Subtest); and there is a significant interaction due to the PL group and the control group between teaching method and repeated testing on part 3 (ML Subtest).

The analyses with attendance as a filter show on part 4, the interaction disappears when analyzing the immediate effect. When analyzing the retention effect, the marginally significant advantage of the retention test over the pretest on part 1 became significant. Further, the interaction found on part 4 disappears. A discussion of these results is found in chapter 5.

CHAPTER FIVE
DISCUSSION AND IMPLICATIONS

Discussion

The purposes of this study were to investigate three theories that explain why common errors of the type

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{occur in algebra}$$

problem solving by novices, and to develop and assess techniques for remediating these errors. The study was a quasi-experiment conducted at Southern University in Baton Rouge, LA. It used a pretest-posttest-retention test, control group design with three treatments--meaningful learning (ML), procedural learning (PL), and implicit structure learning--and one control, which received no special instructions concerning the errors.

The study provided alternative experiences for groups of novices intending to lead to more expert-like performance, according to the three differing theories of expertise. The success of one treatment over the others would provide indirect support for the theory underlying that method.

The meaningful learning treatment consisted of a variety of rich semantic experiences: (1) Numerical substitutions were used to evaluate whether proposed rules

were correct or incorrect; (2) Axiomatic methods were used to establish equivalent expressions; and (3) Geometrical models of expressions were constructed to verify equivalences.

The procedural learning treatment consisted of reviewing the textbook rules and illustrating these rules with numerical instances. Expressions where these rules can or cannot be applied were identified. Possible new rules were generated by the students for situations in which the given rules are inapplicable, and the validity of these new rules was assessed. The dangers of overgeneralizing given rules without verification was stressed. In addition to usual algebra rules, a contrived rule system was used to further the students' procedural competence.

The implicit learning treatment consisted of experiences that enabled students to determine the constraints of distributivity: (1) Examples of the distributive structure in natural language were presented and discussed; (2) The students generated correct and incorrect rules of distributivity in algebra to compare and contrast; and (3) The operation levels were presented to provide an additional catalyst that would assist students in determining the constraints of distributivity.

The null hypothesis was that there would be no significant difference in the number of errors between the

four groups on the posttest and the retention test. The results failed to reject the null hypothesis. That is, no significant difference was found in the number of errors between the four groups on the pretest and posttest. However, the results indicate a significant difference between the mean scores of the pretest and posttest. But this improvement accrued to the control group as well as the treatment groups. Thus the various treatments cannot be individually credited for the gains.

After a period of three weeks, there was still no significant difference between the groups, but the mean score of the retention test was marginally significantly higher than the mean score of the pretest. This suggests that the several improvements noted in the posttest persisted beyond the immediate treatment period.

Parts of the pretest, posttest and retention test were designed to determine if the knowledge components of the ML theory and the PL theory were learned by the subjects in the ML treatment group and the PL treatment group, respectively. The analysis of the knowledge component of the ML theory (part 3 of the tests) for the immediate effect of the treatment, showed no significant difference between the groups. This suggests that the ML group did not learn or internalize the knowledge component of the ML theory as presented during treatment.

The analysis of the knowledge component of the PL theory (part 4 of the tests) for the immediate effect of the treatment, showed a marginally significant interaction due to the PL group between the pretest and posttest. The mean score of the posttest was marginally significantly higher than the pretest. This suggests that the PL group did make some gains on their procedural skills during treatment.

The analysis of the retention effect of the treatment for part 3 (ML subtest) showed a significant difference between the pretest and the retention test (see Table 8) with the mean of the pretest significantly higher than the mean of the retention test; and a significant interaction due to the ISL group as well as the control group between the pretest and retention test (see Figure 5), with the mean score of the pretest higher than the mean score of the retention test for each of these two groups. This result may indicate that since this was the third time a test was given and the questions had not been discussed during treatment, the subjects in both the ISL group and the control group did not put much effort into answering the questions on part 3 (ML Subtest). This is also the explanation offered for the marginal significant interaction due to the control group between the pretest and posttest on part 4 (PL Subtest), where the mean score

of the pretest is higher than the mean score of the retention test.

The analysis of the retention effect also showed a marginally significant difference between the groups with the PL group marginally significantly higher than the other three groups (see Table 8). On part 4 (PL Subtest), the retention effect of the treatment showed a marginally significant interaction due to the PL group between the pretest and the retention test. The mean score of the retention test was marginally significantly higher than the mean score of the pretest. These results suggest that the PL group's superior performance over the other groups was due to the fact that the students learned the knowledge component of the PL theory. In addition, the overall better performance of the PL group on part 3 suggests that they were able to apply or extend this knowledge even to the ML items.

When the retention effect was determined using the posttest and the retention test as the measures, a significant difference was found between the groups on part 4 (PL subtest), with the mean score of the PL group higher than the mean score of the other three groups. This suggests that over a period of time, the improvement due to the PL treatment was significant on the PL items.

Limitations

The first limitation of this study stems from the development of the instructional session for the ML treatment. A pretest was administered but was not used in the development of the instructional sessions for the ML treatment. Confrey (1990) includes preconceptions as one of the conditions for meaningful learning and he indicates that preconceptions should be used to determine the appropriate starting points for instruction. That is, a student's prior knowledge should be understood and should have been used in the development of the instructional sessions for the ML treatment.

A second limitation of the study is that the number of subjects that received all parts of the treatments was small. The number of subjects that received all parts of the treatment was 28 of the 40 subjects.

A third limitation of the study deals with the contrived rule system which was designed to further students procedural competence. The directions for part IV of the pretest, posttest, retention test and exercise 1 (see Appendix E, PL treatment, Day One) lacked a complete clarification about the variables and the symbol representing an operation of the rule system. There needed to be an indication that all variables such as m , n and k are numeric variables and that letters written next to each other does not necessarily mean multiplication.

Additionally, the Rules (A) $(^m)n \rightarrow nm$, (B) $m(^n) \rightarrow n$, and (C) $^(mn) \rightarrow m$ were found to be inconsistent (in the sense that there can be no actual operation that simultaneously satisfy all three rules) and should be replaced with consistent rules.

Implications

This study investigated three theories--ML, PL, and ISL--that are about the kind of knowledge experts have and that attempt to explain why common errors of the type,

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b} \quad \text{occur in algebra}$$

problem solving by novices. Below are the implications for future research and for practice that are based on the results of this study.

Implications for Research

The results of this study do not provide support for one theory over another in terms of reducing the error types mentioned above, so replications of the study are suggested with the following conditions: (1) make sure more subjects receive all parts of the treatment; (2) use intermediate algebra students that are not all developmental; and (3) increase the length of the treatments so that more subjects can internalize the knowledge component of the particular treatment.

The results of this study further verify that students do have difficulty with the structure of algebra, so it is suggested that more research be done that deals with the structural aspect of algebra in order to find out specific aspects causing difficulty.

Implications for Practice

National assessments indicate continuing deficiencies in students' mathematical competence. Educators such as Kieran (1989) have identified algebra, particularly the structure of algebra as an area of great weakness. This is particularly significant in view of the gatekeeper function of algebra to careers in science and technology.

The curriculum is the cause of much of the difficulty students encounter in algebra. Many activities in the current algebra curriculum engage students in mindless repetitive drill and practice. But not all students are performing equally poorly with this current curriculum. A curriculum that is not inherently meaningful has very little intrinsic value or interest to students. Indeed minorities, students with lower SES, and women tend to do less well than other students who have more external support and motivation from their families and general societal expectation. Thus, a mindless curriculum, while optimal for no student does serve to entrench the inequities of our society.

Whereas, the problems with meaningless mathematics are

widely agreed upon by educators, there is less consensus about what constitutes meaningful study of algebra. Achieving consensus on this issue is a necessary step in changing current education practices.

This study examined three theories of meaning in algebra, comparing and contrasting them and investigating their relative success in reducing common errors. This kind of study is designed to help arrive at the consensus needed for change. The results of this study indicated a small decrease in the error rate for distributivity overgeneralization for all treatment groups, including the control group which studied unrelated parts of algebra. This confirms the general observation that students engaged in mathematical activity do make gradual progress towards mastery. Unfortunately the relative ineffectiveness of the treatments, as administered in the study, prevents us from claiming more specific conclusions concerning the relative efficacy of these methods.

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APPENDICES

Appendix A

TEST I (PRETEST)

NAME _____
(Print)

S.S.# _____

SEX _____

CLASS TIME _____

DATE _____

PRETEST

I. Which expression is equivalent to the given expression?

1. $(x + y)z =$

- (A) $x + yz$ (B) $xz + yz$ (C) $xz + xy$ (D) $x^z + y^z$
(E) none

2. $(x + y) + z =$

- (A) $(xy)z$ (B) $(x + z) + (y + z)$ (C) $xy + z$
(D) $xz + yz$ (E) none

3. $(xy)^2 =$

- (A) $xy + xy$ (B) xy^2 (C) $x^2 + y^2$ (D) x^2y^2
(E) none

4. $\sqrt{x + y} =$

- (A) $\sqrt{x} + \sqrt{y}$ (B) $x + y$ (C) $x^2 + y^2$ (D) \sqrt{xy}
(E) none

5. $5(x + y) =$

- (A) $5xy$ (B) $10xy$ (C) $5x + 5y$ (D) $5x + y$
(E) none

6. $\frac{x}{y + z} =$

- (A) $\frac{x}{yz}$ (B) $\frac{x}{y} + \frac{x}{z}$ (C) $xy + xz$ (D) $\frac{y + z}{x}$
(E) none

7. $(x + y)^2 =$

- (A) $xy + xy$ (B) $x^2 + y^2$ (C) $x^2 + 2xy + y^2$
(D) x^2y (E) none

8. $\frac{Fx + Gy}{x + y} =$

- (A) FG (B) $F + G$ (C) $\frac{x + y}{Fx + Gy}$

- (D) $\frac{Fx}{x + y} + \frac{Gy}{x + y}$ (E) none

9. $(x - y)^z =$

- (A) $xz - yz$ (B) $x^z - y^z$ (C) xy^z (D) $-x^zy^z$
(E) none

10. $\sqrt{x^2 - y^2} =$

- (A) $\sqrt{(x - y)^2}$ (B) $x^2 - y^2$ (C) $x - y$

- (D) $\sqrt{x^2} - \sqrt{y^2}$ (E) none

11. $z^{xy} =$

- (A) $z^x z^y$ (B) $zx \cdot zy$ (C) $z^x + z^y$ (D) $(z^x)^y$
(E) none

12. $2^{x+y} =$

- (A) 2^{xy} (B) $2^x 2^y$ (C) $2x \cdot 2y$ (D) $x^2 + y^2$
(E) $2^x + 2^y$

13. $\sqrt{xy} =$

- (A) $\sqrt{x}\sqrt{y}$ (B) $\sqrt{x+y}$ (C) $x^2 + y^2$ (D) xy
(E) none

14. $\frac{z}{5} + \frac{2}{5} =$

- (A) $\frac{z+2}{5}$ (B) $\frac{z+2}{10}$ (C) $\frac{z}{5}$ (D) $\frac{2z}{5}$
(E) none

15. $z^2 + y^2 =$

- (A) $(zy)^2$ (B) $2(z+y)$ (C) $(z+y)^2$ (D) zy^4
(E) none

II.

1. Solve for x: $5(x - 2) = 15$. Show your work.

2. Solve for x: $(x + 3)^2 - 9 = 16$. Show your work.

3. If $\sqrt{a^2 + b^2} = c$, $b = 8$, $c = 10$, find a. Show your work.

4. A flat rectangular packing case is 5" wide by 12" long. What is the length of the longest knife that could be placed in the case? (Hint: Use the diagonal). Show your work.

III. Answer each of the following.

1. Write down two different expressions that you think are equivalent.
2. What do you think is meant by equivalent expressions?
3. Are the expressions $(3x)^3$ and $27x^3$ equivalent?
(a) If so, why? or if not, why not?

(b) Can you think of another way of proving or demonstrating the equivalence or nonequivalence of $(3x)^3$ and $27x^3$.
4. Make up an example of your own of two expressions that are not equivalent.

5. What does $(a + b)^c$ mean?

IV. Imagine a new system of mathematics with the following rules (see below) in which x , y , and z are variables. The parentheses and brackets are used to group symbols (as in algebra).

RULES: (A) $(^x)y \rightarrow yx$
 (B) $x(^y) \rightarrow y$
 (C) $^(xy) \rightarrow x$

Each rule can be applied to some expressions to get new expressions. For example, if you applied rule (A) to $(^f)(5g)$, you would get $(5g)f$. Look at the expressions (1-4) below and see if any of the rules can be applied. If so, circle the rule and show what new expression you would get. If no rule applies, just circle NONE.

1. $3(^k) \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

2. $^[(2ax)(5y)] \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

3. $(2f)^(g^5ht) \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

4. $(abc)[^(bc)] \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

Appendix B

TEST II (POSTTEST)

NAME _____
(Print)

S.S.# _____

SEX _____

CLASS TIME _____

DATE _____

POSTTEST

I. Which expression is equivalent to the given expression?

1. $(m + n) + k =$

- (A) $(m + k) + (n + k)$ (B) $mn + k$ (C) $mk + nk$
 (D) $(mn)k$ (E) none

2. $\sqrt{m + n} =$

- (A) $m^2 + n^2$ (B) \sqrt{mn} (C) $\sqrt{m} + \sqrt{n}$
 (D) $m + n$ (E) none

3. $\frac{Hm + Kn}{m + n} =$

- (A) HK (B) $\frac{Hm}{m + n} + \frac{Kn}{m + n}$ (C) $\frac{m + n}{Hm + Kn}$
 (D) $H + K$ (E) none

4. $\sqrt{m^2 - n^2} =$

- (A) $\sqrt{m^2} - \sqrt{n^2}$ (B) $m - n$ (C) $m^2 - n^2$
 (D) $\sqrt{(m - n)^2}$ (E) none

5. $(m + n)k =$

- (A) $mk + mn$ (B) $m + nk$ (C) $m^k + n^k$
 (D) $mk + nk$ (E) none

6. $\frac{k}{7} + \frac{3}{7} =$

- (A) $\frac{k}{7}$ (B) $\frac{3k}{7}$ (C) $\frac{k+3}{14}$ (D) $\frac{k+3}{7}$
(E) none

7. $(mn)^2 =$

- (A) mn^2 (B) $mn + mn$ (C) m^2n^2 (D) $m^2 + n^2$
(E) none

8. $k^{mn} =$

- (A) $k^m k^n$ (B) $(k^m)^n$ (C) $km \cdot kn$ (D) $k^m + k^n$
(E) none

9. $\sqrt{mn} =$

- (A) $m^2 + n^2$ (B) mn (C) $\sqrt{m}\sqrt{n}$ (D) $\sqrt{m+n}$
(E) none

10. $\frac{k}{m+n} =$

- (A) $\frac{k}{m} + \frac{k}{n}$ (B) $km + kn$ (C) $\frac{m+n}{k}$
(D) $\frac{k}{mn}$ (E) none

11. $(m+n)^2 =$

- (A) $m^2 + 2mn + n^2$ (B) $m^2 + n^2$ (C) m^2n
(D) $mn + mn$ (E) none

12. $(m - n)^k =$

- (A) $mk - nk$ (B) mn^k (C) $m^k - n^k$ (D) $-m^kn^k$
(E) none

13. $3^{m+n} =$

- (A) 3^{mn} (B) $3^m + 3^n$ (C) $m^3 + n^3$ (D) $3m \cdot 3n$
(E) $3^m 3^n$

14. $3(m + n) =$

- (A) $6mn$ (B) $3m + 3n$ (C) $3m + n$ (D) $3mn$
(E) none

15. $k^2 + n^2 =$

- (A) $(kn)^2$ (B) $(k + n)^2$ (C) kn^4 (D) $2(k + n)$
(E) none

II.

1. Solve for x : $3(x + 5) = 18$. Show your work.

2. Solve for x : $(x + 2)^2 + 16 = 25$. Show your work.

3. If $\sqrt{a^2 + b^2} = c$, $a = 4$, $c = 5$, find b . Show your work.

4. A rectangular screen is 6' wide by 8' long. How many feet of wire was used to support the screen if the wire runs diagonally across the screen? Show your work.

III. Answer each of the following.

1. Write down two different expressions that you think are equivalent.

2. What do you think is meant by equivalent expressions?

3. Are the expressions $(7y)^2$ and $49y^2$ equivalent?
 - (a) If so, why? or if not, why not?

 - (b) Can you think of another way of proving or demonstrating the equivalence or nonequivalence of $(7y)^2$ and $49y^2$.

4. Make up an example of your own of two expressions that are not equivalent.

5. What does $(m + n)^k$ mean?

IV. Imagine a new system of mathematics with the following rules (see below) in which m , n , and k are variables. The parentheses and brackets are used to group symbols (as in algebra).

RULES: (A) $(^m)n \rightarrow nm$
 (B) $m(^n) \rightarrow n$
 (C) $^(mn) \rightarrow m$

Each rule can be applied to some expressions to get new expressions.

For example, rule (A): $(^m)n \rightarrow nm$
 applied to $(^a)(4b) \rightarrow (4b)a$.

Look at the expressions (1 - 4) below and see if any of the rules can be applied. If so, circle the rule and show what new expression you would get. If no rule applies, just circle NONE.

1. $(3a)^{(b^3cd)} \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

2. $(fgh)^{[^(gh)]} \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

3. $^{[(3gf)(7h)]} \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

4. $5(^y) \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

Appendix C

TEST III (RETENTION TEST)

NAME _____
(Print)

S.S.# _____

SEX _____

CLASS TIME _____

DATE _____

RETENTION TEST

I. Which expression is equivalent to the given expression?

1. $r^2 + q^2 =$

- (A) $2(r + q)$ (B) $(rq)^2$ (C) rq^4 (D) $(r + q)^2$
 (E) none

2. $4(p + q) =$

- (A) $4p + q$ (B) $4pq$ (C) $8pq$ (D) $4p + 4q$
 (E) none

3. $\sqrt{p + q} =$

- (A) \sqrt{pq} (B) $p^2 + q^2$ (C) $p + q$
 (D) $\sqrt{p} + \sqrt{q}$ (E) none

4. $(p + q)r =$

- (A) $pr + qr$ (B) $p + qr$ (C) $p^r + q^r$
 (D) $pr + pq$ (E) none

5. $(p + q) + r =$

- (A) $pq + r$ (B) $(pq)r$ (C) $pr + qr$
 (D) $(p + r) + (q + r)$ (E) none

6. $\frac{Px + Qy}{x + y} =$

- (A) $P + Q$ (B) PQ (C) $\frac{Px}{x + y} + \frac{Qy}{x + y}$

- (D) $\frac{x + y}{Px + Qy}$ (E) none

7. $\sqrt{p^2 - q^2} =$

- (A) $\sqrt{p^2} - \sqrt{q^2}$ (B) $\sqrt{(p - q)^2}$ (C) $p^2 - q^2$ (D) $p - q$ (E) none

8. $\frac{3}{p} + \frac{3}{2} =$

- (A) $\frac{3}{p}$ (B) $\frac{3}{p+2}$ (C) $\frac{6}{p+2}$ (D) $\frac{3}{2p}$ (E) none

9. $5p+q =$

- (A) $5p + 5q$ (B) $p^5 + p^5$ (C) $5p \cdot 5q$ (D) $5p \cdot 5q$ (E) none

10. $\frac{p}{d} + \frac{r}{d} =$

- (A) $\frac{d}{p} + \frac{r}{d}$ (B) $\frac{dr}{d}$ (C) $\frac{d}{d+r}$ (D) $pq+pr$ (E) none

11. $\sqrt{pq} =$

- (A) pq (B) $p^2 + q^2$ (C) $\sqrt{p+q}$ (D) $\sqrt{p}\sqrt{q}$ (E) none

12. $r_{pq} =$

- (A) $(r_p)^q$ (B) $r_p \cdot r_q$ (C) $r_p r_q$ (D) $r_p + r_q$ (E) none

13. $(p + q)^2 =$

- (A) p^2q (B) $p^2 + 2pq + q^2$ (C) $pq + pq$
(D) $p^2 + q^2$ (E) none

14. $(pq)^2 =$

- (A) pq^2 (B) $p^2 + q^2$ (C) p^2q^2 (D) $pq + pq$
(E) none

15. $(p - q)^r =$

- (A) $pr - qr$ (B) pq^r (C) $-p^r q^r$ (D) $p^r - q^r$
(E) none

II.

1. Solve for x : $4(x - 3) = 20$. Show your work.

2. Solve for x : $(x + 2)^2 + 9 = 25$. Show your work.

3. If $\sqrt{a^2 + b^2} = c$, $a = 6$, $c = 10$, find b . Show your work.

4. How long is the diagonal of a rectangular sign that is 3' wide by 4' long.

III. Answer each of the following.

1. Write down two different expressions that you think are equivalent.
2. What do you think is meant by equivalent expressions?
3. Are the expressions $(2x)^3$ and $8x^3$ equivalent?
 - (a) If so, why? or if not, why not?
 - (b) Can you think of another way of proving or demonstrating the the equivalence of nonequivalence of $(2x)^3$ and $8x^3$.
4. Make up an example of your own of two expressions that are not equivalent.
5. What does $(p + q)^r$ mean?

IV. Imagine a new system of mathematics with the following rules (see below) in which p and q are variables. The parentheses and brackets are used to group symbols (as in algebra).

RULES: (A) $(^p)q \rightarrow qp$
 (B) $p(^q) \rightarrow q$
 (C) $^(pq) \rightarrow p$

Each rule can be applied to some expressions to get new expressions.

For example, rule (A): $(^p)q \rightarrow qp$
 applied to $(^m)(2n) \rightarrow (2n)m$.

Look at the expressions (1-4) below and see if any of the rules can be applied. If so, circle the rule and show what new expression you would get. If no rule applies, just circle NONE.

1. $(fgh)[^(gh)] \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

2. $5(^y) \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

3. $(3t)^(u^2vw) \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

4. $^[(5mn)(2k)] \rightarrow$ _____

RULE: (A) (B) (C) (D) NONE

Appendix D

**INSTRUCTIONAL SESSION - GROUP 1
MEANINGFUL LEARNING TREATMENT (ML)**

GOAL - TO REMEDIATE ERRORS OF THE TYPE

$(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ using the
meaning theory of learning.

Group 1 - Meaningful Learning Treatment

Day One (ML)

The goal for the first day's session will be to help students make the connection between variable symbols and numbers, and to use this connection to evaluate the equivalence of expressions. Application will include

$$(a + b)^2 \neq a^2 + b^2 ; \quad \sqrt{a + b} \neq \sqrt{a} + \sqrt{b} \quad \text{and}$$

$$(a + b)^2 = a^2 + 2ab + b^2 .$$

The teacher will introduce the day's session by stating a modified form of the day's goal. That is, today's session will involve determining if expressions are equivalent, and how one can go about determining equivalent expressions.

The first activity will involve equivalent and nonequivalent expressions. The teacher will begin by

writing the two expressions $\frac{Ax + x}{x}$ and $A + 1$ on the

board and then ask the following questions. Are these expressions equivalent? How do you know? What does it mean to say that the expressions are equivalent? Can you give an example of two expressions that are not equivalent? How do you know these are not equivalent? What do you think non-equivalence mean? How can we determine whether or not expressions are equivalent?

The second activity will be an exercise (see exercise #1 below) in which students will determine equivalence using numerical instances. The teacher will begin by stating the following. Now that you have decided how to determine whether expressions are equivalent, let's look at a few more expressions. The first four problems will be done as a whole class activity. The teacher will show one problem at a time, giving each student time to work the problem. The teacher will ask the following questions for each of the four problems. Are these equivalent? Can you verify?.

Then the students will be divided into small groups. They will continue to determine whether the expressions in each of the problems #5 - #10 are equivalent. The students will present and explain their findings.

Exercise 1 (ML-Day One)

Determine whether or not the following expressions are equivalent.

1. $(ab)^3 = a^3 b^3$

2. $3(x + y) = 3x + 3y$

3. $\frac{15 - x}{3} = 5 - x$

4. $\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$

5. $(a - b)^3 = a^3 - b^3$

6. $a(bc) = ab \bullet ac$

7. $\sqrt{a^2 - 16} = a - 4$

8. $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

9. $a^{bc} = a^b a^c$

10. $(x + 3)(x - 3) = x^2 - 9$

Day Two (ML)

The goal for the second day's session will be to introduce the axiomatic method and use it to establish equivalent expressions.

The teacher will begin the session by stating that today we will explain another method for determining equivalence. As we indicated during the last session, the method of using numerical instances is good. However, today we want to note that it could have some drawbacks. That is, let's look at the expressions x^3 and x . Are they equivalent? Let's verify. (The students will be given the opportunity to select values to determine equivalence.)

The teacher will then ask, what if we had selected the values, 0, 1, and -1? (The students will be given time to evaluate the expressions for those values.) Does this mean

that the expressions are equivalent? What does this tell us? After comments from the students, the teacher will state that even though the method of using numerical instances is a good method for determining equivalence, it should be used with caution: If you find non-equivalence then you can be sure of it. But with equivalence, you might want to try one or 2 more values.

The method that will be used today is the axiomatic method. This method involves using axioms and definitions. Let's begin by reviewing a few axioms that we may need to use as we establish some equivalent expressions.

The teacher and students will review definitions and the commutative, associative and distributive laws. After this the teacher will initiate using the axiomatic method to establish the equivalence for several expressions. The first expression will be $(ab)^2$. The students will be reminded that the axiomatic method involves using axioms and definitions. The teacher will begin by asking, "What does n^2 mean?" That is, what does the square tell us? Okay, so what does $(ab)^2$ mean? Now what do you think we need to do next? Are there any axioms that we can use at this point. Yes, we can use the associative and commutative laws. So we have $(ab)^2 = (ab)(ab) = a(ba)b = a(ab)b = (aa)(bb) = a^2b^2$.

The teacher and students will then follow the same procedure to establish equivalent expressions for $(a/b)^3$, $(a^2)^3$, and $(a + b)^2$.

The second activity will be an exercise (see exercise #2 below) in which the students will use the axiomatic method to determine equivalence. (These exercises are the same as those in exercise #1)

Exercise 2 (ML-Day Two)

Use the axiomatic method to determine whether the following equations are true. Show all of your work.

1. $(xy)^5 = x^5y^5$

2. $2(x + y) = 2x + 2y$

3. $\frac{15 - x}{3} = 5 - x$

4. $\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$

5. $(a - b)^3 = a^3 - b^3$

6. $5(bc) = 5b \cdot 5c$

7. $\sqrt{a^2 - 36} = a - 6$

8. $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

9. $x^{mn} = x^m x^n$

10. $(x + 4)(x - 4) = x^2 - 16$

Day Three (ML)

The goal for the third session will be to establish and reinforce some formulas geometrically. ($(a + b)^2$ will be represented using a square, and $\sqrt{a^2 + b^2}$ will be represented as the hypotenuse of a right triangle.)

The teacher will begin the day's session by initiating a summary of day one and day two activities and stating the goal for the day. That is, we've seen many expressions and we've looked at methods to decide equivalence. But it helps if we understand formulas from many perspectives. Today we will make geometric models for some of the formulas. (The students will construct models at their seats using rulers as the teacher construct a model on the board).

First let's talk about area. What do we mean by the area of a rectangle or square? Yes, it is the enclosed space of a rectangle and square. Now, how can we find the area of a rectangle and a square?

Next, the teacher will say "let's look at a geometric model of $(a + b)^2$." How can this be done? Well, suppose we let a be one length, _____, and b be another, _____.

Now can anyone think of how to model $(a + b)^2$? Well let's break it down. How can we model just $(a + b)$? The teacher will ask the students to model $a + b$ at their seats. The teacher will continue, now if we want to model $(a + b)^2$, do you think we can relate this to area? If so, how? So if we construct a square so that

each side will have length $(a + b)$, the enclosed space is $(a + b)^2$. Now let's see if we can represent this area another way. Do you think there is a way to divide the area into sections?

The teacher will allow the students to make suggestions and if necessary guide them so that the square will be divided into four sections with areas, a^2 , ab , ab , and b^2 . Then the students will be asked to represent the area of the square using the area of these four sections. i.e. $(a + b)^2 = a^2 + 2ab + b^2$. Thus establishing equivalence.

The next geometric model introduced will be of $\sqrt{a^2 + b^2}$.

The teacher will begin by asking the following questions.

What is $\sqrt{a^2 + b^2}$ equal to? (Students may answer, $a + b$).

The teacher continues, let's make a picture. The teacher draws a rectangle, labeling the width, a and the length, b . and the students will do likewise at their seats. Does anyone know how to find the length of a diagonal of the rectangle? Does anyone remember the Pythagorean theorem?

(Eventually the answer, diagonal = $\sqrt{a^2 + b^2}$) Now, from

the diagram, is $\sqrt{a^2 + b^2} =$

$a + b$? Why or why not?

Appendix E

**INSTRUCTIONAL SESSION - GROUP 2
PROCEDURAL LEARNING TREATMENT (PL)**

GOAL - TO REMEDIATE ERRORS OF THE TYPE

$(a \pm b)^c = a^c \pm b^c$ and $\sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$ using the procedural learning theory.

Group 2 - Procedural Learning Treatment

Day One (PL)

The goal for the first day's session will be to review and give examples of rules (particularly rules of distributivity) that students have previously encountered and will modify (sometimes incorrectly) to solve a new problem; and to apply rules (familiar and new) wherever they can be applied.

The teacher will introduce the day's session by stating that today we want to review, give examples of and apply some rules that you have previously encountered. We will then apply some rules you have never seen.

The first activity will be to present, and illustrate the following rules:

$$1. \quad A(B + C) = AB + AC$$

$$2. \quad A(B - C) = AB - AC$$

$$3. \quad (AB)^c = A^cB^c$$

$$4. \quad \frac{A + B}{C} = \frac{A}{C} + \frac{B}{C}$$

$$5. \quad \frac{A - B}{C} = \frac{A}{C} - \frac{B}{C}$$

$$6. \quad \sqrt{AB} = \sqrt{A}\sqrt{B}$$

The teacher will present each rule, one at a time and illustrate each with one example. One such example is $2(x + y) = 2x + 2y$.

The second activity will be an exercise (see exercise #1 below) in which the students will examine a list of expressions and determine if there is a rule for each of the expressions in the list. If there is a rule for any of the expressions, the students will be asked to apply the rule.

Exercise 1 (PL-Day One)

Do any of the above rules apply to the following? If so, apply them.

$$1. \quad z(3x + y)$$

$$2. \quad (x - y)^c$$

$$3. \quad \frac{2x}{y + z}$$

4. $[(x - y)(z + w^2 - 4)]^5$

5. $a^2[(x + y) + z]$

6. $\sqrt{x^4 - y^4}$

7. $\frac{2x^2 - 3x^5}{z}$

8. $[(2x + 1) - (5y)]^3$

9. $\sqrt{12y^3}$

10. $5y + (z \bullet pq)$

11. $3x(5x^2 + 2x - 6)$

The second activity will involve learning to apply the rules. Suppose we had a new mathematics with odd symbols. All we know is that a , b , and c are variables, parentheses and brackets are used to group symbols (like in algebra), and we have the rules listed below. The teacher will list the following rules:

1. $(;a)b \rightarrow ba$
2. $a!(;b) \rightarrow b!a$
3. $;(a!b) \rightarrow ab$

The teacher will illustrate applying rule #1. Example - $(;x)(yz) \rightarrow (yz)x$.

The students will be given expressions and asked to determine which rule applies and then use it to write an equivalent expression.

Exercise 2 (PL-Day One)

Choose the rule which applies for each of the expressions below and then apply it. If neither rule applies then so state.

1. $(;k)8 \rightarrow$
2. $;(t!(pq)) \rightarrow$
3. $(2x)![(;pt)] \rightarrow$
4. $(2xy);(4k) \rightarrow$
5. $[(;5xy)]!(9yz)$
6. $(5zt)![(;mn)] \rightarrow$
7. $[(;xy)](xy) \rightarrow$
8. $(;x)!(yz) \rightarrow$

Day Two (PL)

The goal for the second day will be to apply previously

encountered rules (but not given); to write down the general rules after they haven been applied; and to extend rules.

The teacher will begin the session by stating that today we want to apply some rules and then write down the general rule for each rule applied. We will then extend rules where necessary.

The first activity will involve applying rules. The teacher will write on the board the expression, $[3(x^2 + 1)]^7$ and ask the students to simplify the expression using a rule (i.e. $[3(x^2 + 1)]^7 = 3^7(x^2 + 1)^7$). Then the students will be asked to write the general rule that was used [i.e. $(ab)^n = a^n b^n$]. The same format will be used for the problems in exercise 3 below.

Exercise 3 (PL-Day Two)

Simplify each of the following where possible. Write the general rule used.

1. $\frac{3y}{2 + z^3}$

2. $x^3(y^3 + z^3)$

3. $(2x \cdot yz)^3$

4. $\sqrt{x^2 - 4y^2}$

5. $\frac{7x^2 - 3y^5}{4z}$

6. $\sqrt{(x - y)(p + q)}$

7. x^{2a+3b}

8. $(4z - 5y)^7$

9. $\frac{p^2 + q^2}{r}$

10. $5x(y^4 - z^4)$

11. $5xyz[2 + x - 7y + 3z]$

12. $(3x + 2y)^4$

13. $\sqrt{4x^2y^6 - 9y^4z^2 + 25}$

14. $[(2x)(5y)(3z)]^5$

15. $\frac{7x^2y - 3yz + 9}{6xyz}$

16.
$$\frac{3x}{2x + 3y + 4z}$$

17.
$$\sqrt{25y^6z^5}$$

The second activity will involve extending rules when necessary. The teacher will state during our last class session and during this class session we concluded that we did not have a rule in our list for expressions such as $3x(5x^2 + 2x - 6)$. What do you think we could do if we wanted an expression that is equivalent to this one? How do you think we can verify that this is correct using our known rules? The students will be allowed to show that $3x(5x^2 + 2x - 6) = 3x[(5x^2 + 2x) - 6] = 3x(5x^2 + 2x) - 3x(6) = 3x \bullet 5x^2 + 3x \bullet 2x - 3x \bullet 6$.

The students will then be asked to review the problems in exercise 3 and simplify expressions, where possible, by extending a rule. They will then be asked to write the general rule that was used and verify each new rule using the axiomatic method.

Appendix F

**INSTRUCTIONAL SESSION - GROUP 3
IMPLICIT STRUCTURE LEARNING TREATMENT (ISL)**

GOAL - TO REMEDIATE ERRORS OF THE TYPE

$$(a \pm b)^c = a^c \pm b^c \quad \text{and} \quad \sqrt[c]{a \pm b} = \sqrt[c]{a} \pm \sqrt[c]{b}$$

using the implicit structure learning theory.

Group 3 - Implicit Structure Learning Treatment

Day One (ISL)

The goal for the first session will be to generate some algebra rules (particularly distributivity) and some non rules.

The teacher will introduce the session by stating we usually think of algebra and language as being separate, but this is not the case. Let's look at some statements (see below). The teacher will write the following statements on the board, one at a time, and ask the following questions. What does the statement mean? Can we put in parentheses? Can we formalize (use letters)? (It will be expected that the students will detect ambiguity in some of the statements).

The teacher and students will analyze each statement below using the following format.

Original Statement: a. John and Mary went to the store.

Another Meaning: b. John went to the store and Mary went to the store.

Symbolic
representation of a: c. (J & M)w

Symbolic
representation of b: d. (Jw) & (Mw)

Statements:

1. John and Mary went to the store.
2. Honesty and integrity are good qualities.
3. The old man and woman came down the stairs.
4. Peanut butter and jelly go well together.
5. I like cake and ice cream.

After the students give the meaning of the statements, put in parenthesis and formalize the statements, they will be asked to generate a list of algebra rules of the same form as the statements. Some of these may be non-rules.

The teacher will extend the list of rules and non-rules to include the ones listed below.

Rules and Non-rules:

1. $(a + b)c = ac + bc$
2. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

3. $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
4. $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$
5. $(a-b)c = ac - bc$
6. $(a-b)^c = a^c - b^c$
7. $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$
8. $x^{a+b} = x^a x^b$
9. $(ab)^c = a^c b^c$
10. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
11. $x^{ab} = x^{a+b}$
12. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Day Two (ISL)

The goal for the second day will be to identify the correct rules, introduce operation levels and formulate a rule (GDL) using operation levels to decide which distributivity rules (of algebra) are true or false.

The teacher will write the list of rules on the board that were generated the previous day. The teacher will state which are correct and which are not.

The teacher will then state, obviously we need some way to keep track of which are true rules and which are non-rules. Obviously, it depends on the operations levels.

The teacher will then introduce the operations levels. The teacher will state, the operations that we encountered in the rules were addition, subtraction, multiplication, division, exponentiation, and root. Addition and subtraction are inverse operations; multiplication and division are inverse operations; and exponentiation and root are inverse operations. So we will rank them by levels. Level 1- addition and subtraction; level 2- multiplication and division; and level 3- exponentiation and root.

The teacher will continue. Now look at the list of correct rules and the list of non-rules. Try to formulate a rule using operation levels to decide which rules are true or

false.

After the rule (GDL) is formulated, the teacher will give examples of good and bad applications (see exercise 1 below). The students will have to decide which are good or bad and why.

Exercise 1 (ISL-Day Two)

Determine which of the following rules are correct and which are incorrect. Explain.

1. $(5b)^3 = 5^3 b^3$

2. $(3x^2 + 5z^2)y = 3x^2y + 5z^2y$

3. $\sqrt{a^2 - 16} = a - 4$

4. $\frac{15 - x}{3} = \frac{15}{3} - \frac{x}{3}$

5. $\frac{a}{b + c} = \frac{a}{b} + \frac{a}{c}$

6. $(2a - b)^3 = (2a)^3 - b^3$

7. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

8. $a(bc) = ab \bullet ac$

9. $(x^4 - y)z = x^4z - yz$

10. $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

11. $a^{bc} = a^b a^c$

12. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

VITA

Juanita LaVall James Bates was born in Hartsville, South Carolina on February 24, 1941 to Robert A. James, Sr. and Laura V. Hughes James. She has two brothers and one sister. Juanita graduated from Southern University Laboratory School in 1958 and from Southern University in 1962 where she received the B.S. degree in mathematics education. In 1967 she received the M.S. degree in mathematics from Atlanta University, Atlanta, Georgia and did further study in mathematics during the academic year 1969-70 at Indiana University, Bloomington, Indiana. In 1972, she married Alton Bates and they are the proud parents of one son, Alton Bates, II.

Juanita's 20 plus years of teaching experience range from the preschool level through the college level. Much of her college teaching involved remedial mathematics. Juanita received the Doctor of Philosophy degree from Louisiana State University, Baton Rouge, Department of Curriculum and Instruction, with an emphasis in Mathematics Education, in August, 1994.

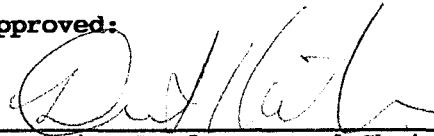
DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Juanita L. James Bates

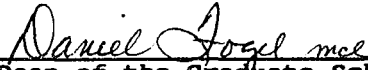
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Title of Dissertation: The Generalized Distributive Law as Tacit Knowledge
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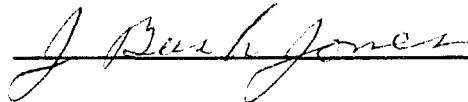


Major Professor and Chairman



Dean of the Graduate School

EXAMINING COMMITTEE:



Date of Examination:

May 18, 1994