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The Generalized Emden-Fowler Equation

L.M. BERKOVICH

Samara State University, 443011, Samara, Russia
E-mail: berk@info.ssu.samara.ru

Abstract

We give the description of nonlinear nonautonomous ordinary differential equations of order n with a so-called *reducible* linear part. The group classification of generalized Emden-Fowler equations of the mentioned class is done. We have found such laws of the variation of $f(x)$ that the equation admits one, two, or tree one-parameter Lie groups.

1. Introduction: the method of autonomization [1, 2]

Nonlinear nonautonomous equations with a reducible linear part form a wide class of ordinary differential equations (ODE) that have both theoretical and applied significance. We can write

$$(NLNA)y \equiv \sum_{k=0}^{n} \binom{n}{k} a_k y^{(n-k)} = \Phi(x, y, y', \ldots, y^{(m)}), \quad a_k \in \mathbf{C^{n-k}}(I), \tag{1.1}$$

$I = \{x|,\ a \le x \le b\}$, where the corresponding linear equation

$$L_n y \equiv \sum_{k=0}^{n} \binom{n}{k} a_k y^{(n-k)} = 0,$$

can be reduced by the Kummer-Liouville (KL) transformation

$$y = v(x)z, \quad dt = u(x)dx, \quad v,\ u \in \mathbf{C^n}(I), \quad uv \ne 0,\ \forall x \in I, \tag{1.2}$$

to the equation with constant coefficients

$$M_n z \equiv \sum_{k=0}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \quad b_k = \text{const}.$$

Theorem 1.1. *For the reduction of (1.1) to the nonlinear autonomous form*

$$(NLA)z \equiv \sum_{k=0}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = aF(z, z'(t), \ldots, z^{(m)}), \quad a = \text{const},$$

by the KL transformation (1.2), it is necessary and sufficient that $L_n y = 0$ is reducible and the nonlinear part Φ can be represented in the form:

$$\Phi(x, y, y', \ldots, y^{(m)}) = au^n vF\left[\frac{y}{v}, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)y, \ldots, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)^m y\right],$$

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where $D = d/dx$, $\left(\frac{1}{u}D - \frac{v'}{vu}\right)^k y$ *is the* k*-th iteration of differential expression* $\left(\frac{1}{u}D - \frac{v'}{vu}\right) y$, *and* $u(x)$ *and* $v(x)$ *satisfy the equations*

$$\begin{aligned} &\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \frac{3}{n+1}B_2u^2 = \frac{3}{n+1}A_2, \\ &v(x) = |u(x)|^{(1-n)/2}\exp\left(-\int a_1 dx\right)\exp\left(b_1\int u dx\right) \end{aligned} \tag{1.3}$$

respectively; $A_2 = a_2 - a_1^2 - a_1'$, $B_2 = b_2 - b_1^2$, *i.e., (1.1) is invariant under a one-parameter group with the generator*

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \quad \xi(x,y) = \frac{1}{u(x)}, \quad \eta(x,y) = \frac{v'}{uv}y. \tag{1.4}$$

In this case, (1.1) assumes partial solutions of the kind

$$y = \rho v(x), \ b_n = aF(\rho, 0, \ldots, 0). \tag{1.5}$$

Theorem 1.2. *1) If the linear part* $L_n y$ *of the equation*

$$N_n(y) \equiv L_n y + \sum_{s=1}^{l} f_s(x) y^{m_s} = F(x), \quad 1 \le m_1 < m_2 < \ldots < m_l, \tag{1.6}$$

can be reduced by the KL transformation and, in addition, the following conditions

$$p_s u^n = f_s(x) v^{m_s - 1}, \quad p_s = \text{const},$$

are fulfilled, then equation (1.6) can be transformed to the equation

$$M_n(z) + \sum_{s=1}^{l} p_s z^{m_s} = v^{-1}(x(t)) u^{-n}(x(t)) F(x(t));$$

2) the equation

$$L_n y + \sum_{s=1}^{l} f_s(x) y^{m_s} = 0,$$

corresponding (1.6) assumes the solutions of the form (1.5), where $v(x)$ *not only satisfies to relation (1.3) but it is also a solution of the linear equation*

$$(L_n - b_n u^n)v = 0,$$

and ρ *satisfies to the algebraic equation*

$$b_n \rho + \sum_{s=1}^{l} p_s \rho^{m_s} = 0.$$

2. The Emden-Fowler equation and the method of autonomization [3]

Let us consider the Emden-Fowler equation

$$y'' + \frac{a}{x}y' + bx^{m-1}y^n = 0, n \neq 0, \qquad n \neq 1,\ m, a, b \text{ are parameters }, \tag{2.1}$$

which is used in mathematical physics, theoretical physics, and chemical physics. Equation (2.1) has interesting mathematical and physical properties, and it has been investigated from various points of view. In this paper, we are interested in it from the point of view of autonomization.

Proposition 2.1. *1) Equation (2.1) can be reduced to the autonomous form*

$$\ddot{z} - \frac{(1-a)(n-1)+2(1+m)}{n-1}\dot{z} + \frac{[(1-a)(n-1)+1+m](1+m)}{(n-1)^2}z + bz^n = 0$$

by the transformation $y = x^{(1+m)/(1-n)}z,\ dt = x^{-1}dx$ *and has the invariant solutions*

$$y = \rho x^{(1+m)/(1-n)}, \qquad \frac{[(1-a)(n-1)+1+m](1+m)}{(n-1)^2}\rho + b\rho^n = 0.$$

2) (2.1) admits the one-parameter group $x_1 = e^{\epsilon}x,\ y_1 = e^{-2\epsilon(1+m)/(n-1)}y,\ \epsilon$ *is a parameter, with the generator*

$$X = x\frac{\partial}{\partial x} + \frac{1+m}{1-n}y\frac{\partial}{\partial y}.$$

3. The generalized Emden-Fowler equation

We consider the group analysis and exact solutions of the equation

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^n = 0, \quad n \neq 0,\ n \neq 1. \tag{3.1}$$

Equation (3.1) can be reduced to the autonomous form

$$\ddot{z} \pm b_1\dot{z} + b_0 z + cz^n = 0 \tag{3.2}$$

by the KL transformation (1.3) under specific laws of variation of $f(x)$.

We have found such laws of variation of $f(x)$ that equation (3.1) admits one, two, or three-parameter Lie groups. It can't admit a larger number of pointwise symmetries.

We call the equation

$$y'' + g(x)y^n = 0, \tag{3.3}$$

a *canonical* generalized Emden-Fowler equation.

Equation (3.1) can always be reduced to the form (3.3) by a KL transformation.

Lemma 3.1. *In order that (3.1) can be reduced to (3.2) by the KL transformation (1.3), it is necessary and sufficient that the following equivalent conditions be satisfied:*

1°*. The kernel* $u(x)$ *of transformation (1.3) satisfies the Kummer-Schwartz equation*

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\delta u^2 = A_0(x),$$

where $\delta = b_1^2 - 4b_0$ *is the discriminant of the characteristic equation* $r^2 \pm b_1 r + b_0 = 0$, *and* $A_0(x) = a_0 - \dfrac{1}{4}a_1^2 - \dfrac{1}{2}a_1'$ *is the semiinvariant of the adjoint linear equation*

$$y'' + a_1(x)y' + a_0(x)y = 0. \tag{3.4}$$

The factor $v(x)$ *of transformation (1.2) has the form*

$$v(x) = |u(x)|^{-1/2} \exp\left(-\frac{1}{2}\int a_1 dx\right) \exp\left(\pm\frac{1}{2}b_1 \int u dx\right). \tag{3.5}$$

Here, the function $f(x)$ *can be represented in the form*

$$f(x) = cu^2(x)v^{1-n}(x), \quad c = \text{const}.$$

2°. *Equation (3.1) admits a one-parameter group Lie group with generator (1.4).*

Theorem 3.1. *All laws of variation* $f(x)$ *in (3.1), admitting a one-parameter Lie group with generator (1.4), have one of the following forms:*

$$f_1 = F^2(\alpha_1 y_1 + \beta_1 y_2)^{-\frac{n+3}{2} \pm \frac{b_1(1-n)}{2\sqrt{\delta_1}}} (\alpha_2 y_1 + \beta_2 y_2)^{-\frac{n+3}{2} \mp \frac{b_1(1-n)}{2\sqrt{\delta_1}}}, \ \delta_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 0;$$

$$f_2 = F^2(Ay_2^2 + By_2y_1 + Cy_1^2)^{-\frac{n+3}{2}} \exp\left(\pm\frac{1-n}{2}\frac{b_1}{\sqrt{-\delta_2}} \arctan\frac{2Ay_2 + By_1}{\sqrt{-\delta_2}y_1}\right),$$
$$\delta_2 = B^2 - 4AC < 0;$$

$$f_3 = F^2(\alpha y_1 + \beta y_2)^{-(n+3)} \exp\left(\mp\frac{1-n}{2\alpha}\frac{b_1 y_1}{\alpha y_1 + \beta y_2}\right), \quad \delta_3 = 0;$$

$$f_4 = F^2(\alpha y_1 + \beta y_2)^{-\frac{n+3}{2} \pm \frac{b_1(1-n)}{2\alpha}} y_i^{-\frac{n+3}{2} \mp \frac{b_1(1-n)}{2\alpha}}, \quad \delta_4 = \alpha^2 > 0;$$

$$f_5 = F^2 y_i^{-(n+3)} \exp\left(\pm\frac{1-n}{2}b_1\frac{y_2}{y_1}\right), \quad \delta_5 = 0, \ i = 1, 2,$$

where $F = \exp\left(-\int a_1 dx\right)$, *and* $y_1, \ y_2 = y_1 \int F y_1^{-2} dx$ *generate the fundamental system of solutions (FSS) of the linear equation (3.4).*

Here, (3.1) assumes the exact solution

$$y = \rho v(x), \qquad b_0\rho + c\rho^n = 0,$$

where $v(x)$ *satisfies relation (3.5).*

Theorem 3.2. *If* $f(x)$ *is a factor of the nonlinear term of the equation (3.1), admitting symmetry (1.4), then* $f(x)$ *satisfies to one of the following equations:*

$$f'' - \frac{n+4}{n+3}\frac{f'^2}{f} + \frac{n-1}{n+3}a_1 f' - (n+3\left(a_0 - \frac{2(n+1)}{(n+3)^2}a_1^2 - \frac{2}{n+3}a_1'\right) f +$$
$$+(n+3)b_0 \exp\left(\frac{2(1-n)}{3+n}\int a_1 dx\right) f^{\frac{n+7}{n+3}} = 0, \quad b_1 = 0, \ n \neq -3;$$

or the equation

$$f'' - \frac{n+4}{n+3}\frac{f'^2}{f} + \frac{n-1}{n+3}a_1 f' - (n+3\left(a_0 - \frac{2(n+1)}{(n+3)^2}a_1^2 - \frac{2}{n+3}a_1'\right) f +$$

$$+ \frac{[(n+3)b_0 \mp \frac{2(n+1)}{n+3}b_1^2] f^{\frac{n+7}{n+3}} \exp[\frac{2(1-n)}{3+n}\int a_1 dx]}{\left(k \pm \frac{1-n}{n+3}b_1 \int f^{\frac{2}{n+3}} \exp\left(\frac{1-n}{n+3}\int a_1 dx\right) dx\right)^2} = 0, \quad b_1 \neq 0,\ n \neq -3;$$

or the equation

$$2(f' + 2a_1 f)f''' - 3f''^2 - 12(a_1' f + a_1 f')f'' + \left(1 - \frac{\delta}{4b_1^2}\right)\frac{f'^4}{f^2} + 8\left(1 - \frac{\delta}{4b_1^2}\right) a_1 \frac{f'^3}{f} +$$

$$+ \left[\left(1 - 4\frac{\delta}{b_1^2}\right) a_1^2 + 14a_1' - 4a_0\right] f'^2 + 4\left[a_1'' - 4a_0 a_1 + 2a_1 a_1' + \left(1 - 2\frac{\delta}{b_1^2}\right) a_1^3\right] ff' +$$

$$+4\left[2a_0 a_1'' - 3a_1' 2 - 4a_0 a_1^2 + \left(1 - \frac{\delta}{b_1^2}\right) a_1^4 + 2a_1^2 a_1'\right] f^2 = 0, \quad n = -3,\ b_1 \neq 0;$$

or $f(x) = c\exp(-2\int a_1 dx),\ n = -3,\ b_1 = 0,\ a_1 \neq 0$*; or* $f(x) = \text{const},\ n = -3,\ b_1 = 0,\ a_1(x) = 0.$

4. The case $f(x) = \text{const} = p$

Consideer the equation

$$y'' + a_1(x)y' + a_0(x)y + py^n = 0, \qquad n \neq -3. \tag{4.1}$$

If $b_1 = 0$, we have

$$a_0(x) = \frac{2(n+1)}{(n+3)^2}a_1^2 + \frac{2}{n+3}a_1' + k\exp\left(\frac{2(1-n)}{3+n}\int a_1 dx\right), \quad k = \text{const},$$

or

$$a_0(x) = \frac{2(n+1)}{(n+3)^2}a_1^2 + \frac{2}{n+3}a_1' + q\frac{[(n+3)b_0 \mp \frac{2(n+1)}{n+3}b_1^2]\exp[\frac{2(1-n)}{3+n}\int a_1 dx]}{\left(k \pm \frac{1-n}{n+3}b_1 \int \exp\left(\frac{1-n}{n+3}\int a_1 dx\right) dx\right)^2} = 0,$$

$b_1 \neq 0.\ n \neq -3;\ q = \text{const}.$

Theorem 4.1. *In order that the equation*

$$y'' + a_1 y' + a_0 y + py^n = 0, \quad a_1, a_0 = \text{const} \tag{4.2}$$

have the set of elementary exact solutions depending from one arbitrary constant (besides $a_1 = 0$*), it is sufficient that condition of its factorization,*

$$(n+3)^2 a_0 = 2(n+1)a_1^2 \tag{4.3}$$

hold.

In fact, in this case, equations (4.2), (4.3) admit the factorization:

$$\left(D + \frac{n+1}{n+3}a_1 \mp \frac{n+1}{2}ky^{(n-1)/2}\right)\left(D + \frac{2}{n+3}a_1 \pm ky^{(n-1)/2}\right) y = 0,$$

$k = \sqrt{-2p/(n+1)}.$

In this specific case (at $n = 3$) for some classes of anharmonic oscillators, the exact solutions were obtained in [4] by the Kowalewsky-Painlevé asymptotic method.

Theorem 4.2. *In order that the equation*

$$y'' + a_1(x)y' + py^n = 0, \quad n \neq -3,$$

admit the group with generator (1.4), it is necessary and sufficient that the function $a_1(x)$ satisfy the equation

$$a_1'' + \frac{4n}{n+3}a_1a_1' + \frac{2(n^2-1)}{(n+3)^2}a_1^3 = 0, \tag{4.4}$$

where (4.4) is integrated in elementary functions or quadratures (elliptic integrals). Equation (4.4) can be linearized by the method of the exact linearization (see [5]). Namely, by the substitution $A = a_1^2$, $dt = a_1(x)dx$, it can be reduced to the form

$$\ddot{A} + \frac{4n}{n+3}\dot{A} + \frac{4(n^2-1)}{(n+3)^2}A = 0, \quad (\cdot) = \frac{d}{dt}.$$

It possesses a one-parameter set of solutions

$$a_1(x) = \frac{n+3}{(n-1)(x+c)}, \quad a_1(x) = \frac{n+3}{(n+1)(x+c)} \tag{4.5}$$

and has a general solution of the following parameter kind:

$$a_1 = s^{n-1}(c_1 + s^4)^{1/2}, \quad x = -(n+3)\int s^{-n}(c_1+s^4)^{-1/2}ds + c_2. \tag{4.6}$$

Then it follows from the Chebyshev theorem (see [6])

Corollary 4.1. Equation (4.4), (4.6) (besides $c_1 = 0$, i.e., (4.5)) has elementary solutions for $n = \pm 1 - 4l$, $l \in \mathbf{Z}$.

Corollary 4.2. The equation

$$y'' + a_0(x)y + py^n = 0, \quad n \neq -3,$$

admits pointwise Lie symmetries only for $a_0(x) = \text{const}$, $(b_1 = 0)$ or $a_0(x) = \dfrac{\nu}{(\lambda + \mu x)^2}$, $(b_1 \neq 0)$.

Corollary 4.3. The Painlevé equation

$$y'' \pm xy = y^3$$

can't be reduced to the autonomous kind by a KL transformation KL (it doesn't admit pointwise Lie symmetries).

Theorem 4.3. *The Ermakov equation (Ermakov V.P., 1880; Pinney, 1951, see, for example, [1, 5])*

$$y'' + a_0(x)y + py^{-3} = 0$$

admits a three-dimensional Lie algebra with the generators

$$X_1 = y_1^2(x)\frac{\partial}{\partial x} + y_1(x)y_1'(x)y\frac{\partial}{\partial y}, \quad X_3 = y_2^2(x)\frac{\partial}{\partial x} + y_2(x)y_2'(x)y\frac{\partial}{\partial y},$$

$$X_2 = y_1y_2\frac{\partial}{\partial x} + \frac{1}{2}(y_1y_2' + y_2y_1')y\frac{\partial}{\partial y},$$

which has the commutators

$$[X_1, X_2] = X_1, \quad [X_2, X_3] = X_3, \quad [X_3, X_1] = -2X_2,$$

and is isomorphic to the algebra $sl(2, R)$ (type G_3 VIII according to the classification of Lie-Bianchi).

5. The special case $n = 2$

Theorem 5.1. *(see [7]). The equation*

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^2 = 0 \tag{5.1}$$

has only point symmetries of the kind

$$X = \xi(x)\frac{\partial}{\partial x} + [\eta_1(x)y + \eta_2(x)]\frac{\partial}{\partial y}, \tag{5.2}$$

where

$$\eta_2'' + a_1\eta_2' + a_0\eta_2 = 0,$$

$$\xi''' - (2a_1' + a_1^2 - 4a_0)\xi' - \left(a_1' + \frac{1}{2}a_1^2 - 2a_0\right)''\xi = 4k\eta_2\xi^{-5/2}\exp\left[\frac{1}{2}\int\left(a_1 \mp \frac{b_1}{\xi}\right)dx\right],$$

$$\eta_1(x) = \frac{1}{2}(\xi' - a_1\xi \pm b_1), \quad f(x) = k\xi^{-5/2}\exp\left[\frac{1}{2}\int\left(a_1 \mp \frac{b_1}{\xi}\right)dx\right], \quad k = \text{const}.$$

Lemma 5.1. *The equation*

$$\xi''' - (2a_1' + a_1^2 - 4a_0)\xi' - \left(a_1' + \frac{1}{2}a_1^2 - 2a_0\right)''\xi = 4k\eta_2\xi^{-5/2}\exp\left(\frac{1}{2}\int a_1dx\right), \quad b_1 = 0,$$

can be reduced to the form

$$\zeta'''(s) = 4k\zeta^{-5/2} \tag{5.3}$$

by the transformation $\xi = u^{-1}\zeta$, $ds = udx$, where

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = A_0(x).$$

Lemma 5.2. *Equation (5.3) assumes an exact linearization by the transformation $Z = \zeta^{-1}$, $dt = \zeta^{-3/2}ds$, namely, $Z'''(t) + 4k = 0$.*

Theorem 5.2. *Equation (5.1) can be reduced to the autonomous form*

$$\ddot{z} \pm b_1\dot{z} + b_0z + c + kz^2 = 0, \quad c = \frac{1}{4k}\left(b_0^2 - \frac{36}{625}b_1^4\right)$$

by the substitution $y = v(x)z + w(x)$, $dt = u(x)dx$,

$$u(x) = \frac{1}{\xi}, \quad v(x) = \exp\left(\int \frac{\eta_1}{\xi} dx\right), \quad w = k\exp\left(\int \frac{\eta_1}{\xi} dx\right)\int \frac{\eta_2}{\xi}\exp\left(-\int \frac{\eta_1}{\xi} dx\right),$$

and has the exact solutions

$$y = \rho v(x) + w(x), \quad \rho = \frac{1}{2}\left(\frac{b_0}{2} \pm \frac{3b_1^2}{25}\right).$$

Theorem 5.3. *If equation (5.1) admits a symmetry of the kind (5.2), then the function $f(x)$ satisfies to the system of equations*

$$\varphi'' + a_1\varphi' + a_0\varphi + \frac{1}{2}\varphi^2 = \frac{1}{2}\left(b_0^2 - \frac{36}{625}b_1^4\right)u^4; \tag{5.4}$$

$$\varphi = \frac{1}{5}\frac{f''}{f} - \frac{6}{25}\frac{f'^2}{f^2} + \frac{1}{25}a_1\frac{f'}{f} - \left(a_0 - \frac{6}{25}a_1^2 - \frac{2}{5}a_1'\right) + \left(b_0 - \frac{6}{25}b_1^2\right)u^2; \tag{5.5}$$

$$u = \frac{f^{2/5}\exp(-1/5\int a_1 dx)}{C_1 \mp \frac{1}{5}b_1\int f^{2/5}\exp(-1/5\int a_1 dx)dx}. \tag{5.6}$$

Corollary 5.1. Let $a_1 = 0$, $a_0 = 0$, and $b_0 = \dfrac{6}{25}b_1^2$. Equation (5.4)–(5.6) takes the form

$$f^{iv} - \frac{32}{5}\frac{f'f'''}{f} - \frac{43}{10}\frac{f''^2}{f} + \frac{594}{25}\frac{f'^2}{f^2}f'' - \frac{1782}{125}\frac{f'^4}{f^3} = 0. \tag{5.7}$$

Equation (5.7) admits solutions of the kind $f(x) = \lambda x^\mu$, where μ satisfies to the algebraic equation

$$49\mu^4 + 490\mu^3 + 1525\mu^2 + 1500\mu = 0, \quad \{\mu = -5,\ -20/7,\ -15/7,\ 0\}.$$

Theorem 5.4. *Equation (5.4)–(5.6) in respect of $f(x)$ (at $b_1 = 0$) has the following general solution represented in the parameter form:*

$$f(x)\exp\left(2\int a_1 dx\right)y_1^5 = k\psi^{5/2}, \quad y_2 y_1^{-1} = \int \psi^{-3/2}dt$$

or

$$f(x) = \exp\left(2\int a_1 dx\right)y_2^5 = k\psi^{5/2}, \quad y_2 y_1^{-1} = -\left(\int \psi^{-3/2}dt\right)^{-1},$$

$$\psi = -\frac{2}{3}kt^3 + c_1t^2 + c_2t + c_3,$$

where $F = \exp\left(-\int a_1 dx\right)$, *and* $y_1,\ y_2 = y_1\int F y_1^{-2}dx$ *generate the FSS of the linear equation (3.4).*

Thus, even under the restriction $b_1 = 0$, the function $f(x)$ can be expressed via elliptic integrals. These expressions can be simplified in the case of pseudoelliptic integrals that takes place for the discriminant $\Delta = 0$. Namely,

$$\Delta = c_1^2c_2^2 + \frac{8}{3}kc_2^3 - 4c_1^3c_3 - 12k^2c_3^2 - 12kc_1c_2c_3 = 0.$$

Let, in particular, $c_1 = c_2 = c_3 = 0$. Then $f(x)$ has one of the following forms:

$$f(x) = \lambda \exp\left(-2\int a_1 dx\right) y_1^{-5}\left(\int \exp\left(-\int a_1 dx\right) y_1^{-2} dx\right)^{-15/7},$$

$$f(x) = \lambda \exp\left(-2\int a_1 dx\right) y_1^{-5}\left(\int \exp\left(-\int a_1 dx\right) y_1^{-2} dx\right)^{-20/7},$$

where $y_1(x)$ is a partial solution of equation (3.4).

Example. The equation $y'' + f(x)y^2 = 0$ can be reduced to the autonomous form for $f(x) = \lambda x^{-15/7}$, $f(x) = \lambda x^{-20/7}$, and $f(x) = \lambda x^{-5}$.

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