THE GENERALIZED FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

A new transform, called the generalized fractional Fourier transform (gFrFT), is proposed. Originally, the eigenfunctions of the fractional Fourier transform (FrFT) are known as the Hermite Gaussian functions (HGFs). Besides, in optics, the HGFs are generalized to be the generalized Hermite Gaussian functions (gHGFs) and their adjoint functions (AgHGFs). Therefore, we can define the gFrFT by the eigenvalues of the FrFT and the eigenfunctions (gHGFs/AgHGFs) in the analysis or synthesis step. Four types of the gFrFT are defined and discussed. The integral forms of the gFrFTs are derived and they are closely related to some popular transforms, such as the Fourier transform (FT), the FrFT, and the complex linear canonical transform (CLCT). We can also extend the FT and the FrFT to the standard and elegant versions. Finally, some properties of the gFrFT are discussed.

Index Terms— Fractional Fourier transforms, Generalized Hermite Gaussian functions

1. INTRODUCTION

The Fourier transform (FT) is a popular signal processing tool [1]. For a given signal, the FT enables us to analyze the frequency contents and has lots of applications. If the signal components are not fully separable in the frequency domain, the fractional Fourier transform (FrFT) [2], [3], which is defined as the fractional order of the FT, might be useful. The signal is transformed to the domain between time and frequency, where different components might be separable. The linear canonical transform (LCT) generalizes the FrFT further. The matrix $\mathbf{M}$ is introduced to generalize the integral kernel. Those transforms correspond to certain operations in the time-frequency plane. The FT or the FrFT rotates the entire plane while the LCT twists the plane. Filtering is viewed as applying a mask in that plane.

One popular definition of the FrFT is based on the Hermite Gaussian functions (HGFs) as the eigenfunctions. In optics, the HGFs are the solutions to the paraxial Helmholtz equation, which the eigenmodes of light propagation satisfy. In addition to the HGFs, there are still other modes, such as the elegant Hermite Gaussian functions [4] and the generalized Hermite Gassian functions [5]. These functions can be used to be eigenfunctions and then we are able to define new transforms.

This paper is organized as follows. Some preliminaries about the FrFT and the LCT, such as the definitions, properties, and the eigenfunctions, are mentioned in Section 2. Different Hermite Gaussian functions are briefly reviewed in Section 3. The definition, integral form, connection to other transforms, and some interesting properties, are investigated in Section 4. Section 5 concludes this paper.

2. PRELIMINARY

The conventional LCT is specified by a 2-by-2 real matrix $\mathbf{M} = [A, B; C, D] \in \mathbb{R}^{2\times 2}$ and defined as

$$F_{\mathbf{M}}(u) = \mathcal{L}_{\mathbf{M}}\left\{f(x)\right\}(u) = K_B \int_{\mathbb{R}} f(x) K_{\mathbf{M}}(x, u) dx, \quad (1)$$

where $K_B = \sqrt{1/(j2\pi B)}$ and

$$K_{\mathbf{M}}(x, u) = e^{j\frac{A}{2B}x^2} e^{-j\frac{1}{B}xu} e^{j\frac{D}{2B}u^2} \quad (2)$$

is the integration kernel with $\det(\mathbf{M}) = 1$. The LCT satisfies the additivity and the reversibility by

$$\mathcal{L}_{\mathbf{M}_2\mathbf{M}_1} = \mathcal{L}_{\mathbf{M}_2}\mathcal{L}_{\mathbf{M}_1}, \quad \mathcal{L}_{\mathbf{M}}^{-1} = \mathcal{L}_{\mathbf{M}^{-1}}. \quad (3)$$

Some basic operations are the special cases of the LCT. For example, the scaling operation $\mathcal{S}_\sigma$ and the chirp multiplication operation $\mathcal{C}_k$ are specified by the matrices $[\sigma, 0; 0, \sigma^{-1}]$ and $[1, 0; k, 1]$, respectively. The well-known FrFT, denoted by the operator $\mathcal{F}_\alpha$, is also a special case of the LCT when

$$\mathbf{M}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \quad (4)$$

The properties of the LCT in (3) are simplified into

$$\mathcal{F}_{\alpha+\beta} = \mathcal{F}_\alpha \mathcal{F}_\beta, \quad \mathcal{F}_\alpha^{-1} = \mathcal{F}_{-\alpha}, \quad (5)$$

and for some special α, we have

$$\mathcal{F}_0 = \mathcal{F}_{2\pi} = \mathcal{I}, \quad \mathcal{F}_\pi = \mathcal{P}, \quad \mathcal{F}_{\pi/2} = \mathcal{F}, \tag{6}$$

where $\mathcal{I}$ is the identity operator, $\mathcal{P}$ is the time-reversal operator ($\mathcal{P}f(x) = f(-x)$) and $\mathcal{F}$ is the conventional FT operator. The eigenfunctions of the FrFT are known as the Hermite Gaussian functions (HGFs) $h_n(x)$, which are the solution of the differential equation

$$\left(\mathcal{D}_x^2 - x^2\right) h_n(x) = -(2n+1)h_n(x), \tag{7}$$

where $\mathcal{D}_x = d/dx$ is the differential operator with respect to x. The close form HGFs are

$$h_n(x) = \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(x) e^{-\frac{1}{2}x^2}, \tag{8}$$

where $H_n(x) = (-1)^n e^{x^2} \mathcal{D}_x^n e^{-x^2}$ are the Hermite polynomials. The HGFs are self-orthogonal and complete in $L^2(\mathbb{R})$. In [2], The eigenvalues of the FrFT are known to be $e^{-jn\alpha}$ so that

$$\mathcal{F}_\alpha \left\{h_n(x)\right\}(u) = e^{-jn\alpha} h_n(u). \tag{9}$$

Therefore, we can implement the FrFT by orthogonal expansion in terms of $h_n(x)$.

3. STANDARD/ELEGANT/GENERALIZED HERMITE GAUSSIAN FUNCTIONS

In optics, the HGFs are the solutions of the paraxial equation in Cartesian coordinates [6]. The complete solution is not discussed here due to lots of beam parameters $w(z)$, $q(z)$ and the phase term $P(z)$. We focus ourselves on the solutions at the origin, i.e. $z = 0$, neglect the phase term, and deal with one-dimensional solutions. The simplest solution set is the standard Hermite Gaussian functions (sHGFs), which are

$$\Psi_n^{\text{sHG}}(x) = \left(\frac{\sqrt{2}}{w_0 2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n\left(\frac{\sqrt{2}x}{w_0}\right) e^{-\frac{x^2}{w_0^2}}, \tag{10}$$

where w_0 is the half-width at the beam waist. It is clear that (10) is a scaling version of (8) with scaling factor $\sqrt{2}/w_0$. Because of the scaling relation, the sHG functions are self-orthogonal, $\left\langle \Psi_m^{\text{sHG}}(x), \Psi_n^{\text{sHG}}(x)\right\rangle = \delta_{m,n}$.

In [4], Siegman introduced a symmetrical form, called the elegant Hermite Gaussian functions (eHGFs), still satisfying the paraxial equation. The eHGFs, associated with a complex parameter c, are

$$\Psi_n^{\text{eHG}}(x) = \left(\frac{\sqrt{c}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{c}x) e^{-cx^2}. \tag{11}$$

If we compare (10) with (11), it is observed that the eHGFs seems to be the scaling version of HGFs, however, with different Gaussian weighting functions. As a result, the eHGs are not self-orthogonal but biorthogonal to their adjoint functions, called adjoint elegant Hermite Gaussian functions (AeHGFs), which are

$$\Phi_n^{\text{eHG}}(x) = \left(\frac{\sqrt{c^*}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{c^*}x) \tag{12}$$

so that $\left\langle \Psi_m^{\text{eHG}}(x), \Phi_n^{\text{eHG}}(x)\right\rangle = \delta_{m,n}$. Note that the AeHGFs contain no Gaussian weighting functions, which makes them unbounded.

In [5], Pratesi and Ronchi further defined the generalized Hermite Gaussian functions (gHGFs), which are generalizations between the sHGFs and the eHGFs in complicated beam parameters. We simplify the gHGFs to be [7]

$$\Psi_n^{r,c}(x) = \left(\frac{\sqrt{2rc}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{2rc}x) e^{-cx^2}. \tag{13}$$

The gHGFs are still not self-orthogonal. The adjoint solutions (AgHGFs) are found to be

$$\Phi_n^{r,c}(x) = N_{r,c} H_n(\sqrt{2r^*c^*}x) e^{(c^* - 2r^*c^*)x^2}, \tag{14}$$

with $N_{r,c} = \left(\sqrt{2r^*c^*}/(2^n n! \sqrt{\pi})\right)^{1/2}$. If $r = 1, c = 1/w_0^2 \in \mathbb{R}^+$, the gHGFs and the AgHGFs become the sHGFs. When $r = 1/2$, the gHGFs are the eHGFs and the AgHGFs are identical to the AeHGFs. Therefore, the parameter c can be regarded as the scale of the HGFs while the parameter r controls the the intermediates between the sHGFs and the eHGFs.

4. THE GENERALIZED FRACTIONAL FOURIER TRANSFORM

4.1. The definition

In [2], the FrFT is first defined by the eigenvalues $e^{-jn\alpha}$ and the eigenfunctions of the FT. By the above definition, implementing the FrFT involves three steps: analysis, eigenvalue multiplication, and synthesis. The generalized FrFT can be defined in the similar way. The gHGFs/AgHGFs are used for eigenfunctions while the eigenvalues are unchanged.

There are two choices (gHG/AgHG) of the analysis kernel and the synthesis kernel. We can make linear combinations of the kernels and have four types of the gFrFT. The definitions of the four types of the gFrFT are listed in Table 1.

For example, the gFrFT Type-I, denoted by the operator $\mathfrak{F}_{\alpha,r,c}^{(\mathrm{I})}$, is decomposed into three steps:

1. *Analysis*: Decompose the input signal $f(x)$, $x \in \mathbb{R}$ into the coefficients $a_n^{r,c}$ by taking the inner product of $f(x)$ and the analysis kernel $\Phi_n^{r,c}(x)$.

$$a_n^{r,c} = \int_{\mathbb{R}} f(x) \left(\Phi_n^{r,c}(x)\right)^* dx. \tag{15}$$

gFrFT Type	I	II	III	IV
Anal. kernels	$\Phi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$
Eigenvalues	$e^{-jn\alpha}$	$e^{-jn\alpha}$	$e^{-jn\alpha}$	$e^{-jn\alpha}$
Syn. kernels	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$

Table 1. The definitions of the four types of the generalized FrFT with different analysis kernels and synthesis kernels.

2. *Eigenvalue multiplication*: Multiply the coefficients with the eigenvalues, $e^{-jn\alpha}$, which are set to be the same as the eigenvalues of the conventional FrFT.

$$b_n^{r,c} = e^{-jn\alpha} a_n^{r,c}. \quad (16)$$

3. *Synthesis*: Use the modified coefficients to synthesis $F_{\alpha,r,c}^{(\mathrm{I})}(u)$ with respect to the synthesis kernel $\Psi_n^{r,c}(u)$, where $u \in \mathbb{R}$. That is,

$$F_{\alpha,r,c}^{(\mathrm{I})}(u) = \mathfrak{F}_{\alpha,r,c}^{(\mathrm{I})}\{f\}(u) = \sum_{n=0}^{\infty} b_n^{r,c} \Psi_n^{r,c}(u). \quad (17)$$

Other types of the gFrFT are the same as the steps above with some change of the analysis kernels in (15) and the synthesis kernels in (17).

4.2. The integral form

From the definition of the gFrFT, an integral form of the gFrFT can be derived. Substituting (15) and (16) into (17) and using the Mehler's formula,

$$\sum_{n=0}^{\infty} \frac{t^n}{2^n n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-t^2}} e^{\frac{2xyt-\left(x^2+y^2\right)t^2}{1-t^2}}, \quad (18)$$

yield the integral form of the gFrFT Type-I:

$$F_{\alpha,r,c}^{(\mathrm{I})}(u) = \sqrt{\frac{rc(1-j\cot\alpha)}{\pi}} e^{c(-1+r(1+j\cot\alpha))u^2} \times \int_{\mathbb{R}} f(x) e^{c(1-r(1-j\cot\alpha))x^2} e^{-j2rcxu\csc\alpha} dx. \quad (19)$$

If we follow the definition in Table 1, we can derive the the integral form of the other three types of the gFrFT. They are

$$F_{\alpha,r,c}^{(\mathrm{II})}(u) = \sqrt{\frac{r^*c^*(1-j\cot\alpha)}{\pi}} e^{c^*(1-r^*(1-j\cot\alpha))u^2} \times \int_{\mathbb{R}} f(x) e^{c^*(-1+r^*(1+j\cot\alpha))x^2} e^{-j2r^*c^*xu\csc\alpha} dx, \quad (20)$$

$$F_{\alpha,r,c}^{(\mathrm{III})}(u) = \sqrt{\frac{|rc|(1-j\cot\alpha)}{\pi}} e^{c(-1+r(1+j\cot\alpha))u^2} \times \int_{\mathbb{R}} f(x) e^{c^*(-1+r^*(1+j\cot\alpha))x^2} e^{-j2|rc|xu\csc\alpha} dx, \quad (21)$$

$$F_{\alpha,r,c}^{(\mathrm{IV})}(u) = \sqrt{\frac{|rc|(1-j\cot\alpha)}{\pi}} e^{c^*(1-r^*(1-j\cot\alpha))u^2} \times \int_{\mathbb{R}} f(x) e^{c(1-r(1-j\cot\alpha))x^2} e^{-j2|rc|xu\csc\alpha} dx. \quad (22)$$

It is observed that all of the kernels are composed of three terms: the x-domain complex Gaussian function, the Fourier kernel involving xu term, and the u-domain complex Gaussian function. Different scaling factors of the three terms yield different kernels. Besides, there are some symmetries between the four kernels. First, if we substitute r and c with their conjugates and exchange the x/u-domain complex Gaussian functions, the type-I kernel becomes the type-II kernel. Then, the type-III kernel combines the x-domain complex Gaussian function of the type-II kernel, the u-domain complex Gaussian function of the type-I kernel, and the scaling factor in the Fourier kernel becomes $|rc|$ together. The type-IV kernel is similar to the type-III kernel with different complex Gaussian functions.

As the FrFT in [2], there is also some ambiguity for the gFrFT when $\alpha = n\pi$. In [8], an extra phase term, specifying the branch of the square-root function, was added to remove the ambiguity. The same method can be applied here.

4.3. Connection to other transforms

4.3.1. The Fourier transforms and their extensions

The conventional Fourier transform can be obtained from the gFrFT. Setting $r = 1$ and $\alpha = \pi/2$ in the type-I kernel yields $\sqrt{c/\pi}e^{-j2cxu}$, which is similar to the kernel of the FT. In addition, if we let $c = 1/2$, $e^{-jxu}/\sqrt{2\pi}$ is the FT kernel associated with the angular frequency, measured in rad/s. When $c = \pi$, $e^{-j2\pi xu}$ is associated with the ordinary frequency, measured in Hertz. Both transforms are unitary. In other words, we can define the new FT with a scaling parameter $c \in \mathbb{R}^+$ and the obtained transform is unitary. The integral form is

$$F_c^{(\mathrm{sFT})}(u) = \sqrt{\frac{c}{\pi}} \int_{\mathbb{R}} f(x) e^{-j2cxu} dx. \quad (23)$$

We call it the "standard Fourier transforms" (sFTs) because in this case, the eigenfunctions are the sHGFs. The sFTs follow the same existence criteria of the FTs [1] because there is only one scaling factor c between the FT and the sFT.

Following the above idea, we can define the "elegant Fourier transforms" (eFTs) whose eigenfunctions are the eHGFs. Setting $r = 1/2$, $c \in \mathbb{R}^+$ and $\alpha = \pi/2$ yields the following integral form,

$$F_c^{(\mathrm{eFT})}(u) = \sqrt{\frac{c}{2\pi}} e^{-cu^2/2} \int_{\mathbb{R}} f(x) e^{cx^2/2} e^{-jcxu} dx. \quad (24)$$

The eFTs might not be stable because of $e^{cx^2/2}$ in its definition. $f(x)$ is elegant-Fourier-transformable if and only if $f(x)e^{cx^2/2}$ is Fourier-transformable.

Finally, we can define the "generalized Fourier transforms" (gFTs) by setting $\alpha = \pi/2$. Its eigenfunctions are the gHGFs.

4.3.2. The fractional Fourier transforms and their extensions

In Section 4.3.1, for some combination of α, r, and c, we can obtain the Fourier transforms and the standard or elegant version. Moreover, if α is not limited to $\pi/2$ but extended to $\mathbb{R}$, we have exactly the FrFT. Similarly, the two cases $c = 1$ and $c = \pi$ lead to two different definitions of FrFT. One is associated with the angular frequency while the other is connected with the ordinary frequency. Therefore, we further define the standard FrFT (sFrFT, $r = 1$), the elegant FrFT (eFrFT, $r = 1/2$) and the generalized FrFT (gFrFT).

In the above discussion, we focus ourselves on the gFrFT type-I. Different types of the sFrFT/eFrFT/gFrFT are defined by (20) - (22) using the same parameter set as that in the type-I case.

4.3.3. The complex linear canonical transforms

If we compare (19) with (1) and (2), it is obvious that (19) is a special case of LCT of

$$\mathbf{M}^{(\mathrm{I})}_{\alpha,r,c} = \begin{bmatrix} \cos\alpha + j\frac{r-1}{r}\sin\alpha & \frac{1}{2rc}\sin\alpha \\ 2c(\frac{1}{r}-2)\sin\alpha & \cos\alpha - j\frac{r-1}{r}\sin\alpha \end{bmatrix}. \tag{25}$$

The matrix $\mathbf{M}^{(\mathrm{I})}_{\alpha,r,c}$ is complex in general. The LCT is called the complex LCT (CLCT), which is more general and has lots applications in optics and quantum mechanics. According to the symmetry, $\mathbf{M}^{(\mathrm{I})}_{\alpha,r,c}$ can be decomposed of five steps by

$$\mathbf{M}^{(\mathrm{I})}_{\alpha,r,c} = \begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ k_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1^{-1} \end{bmatrix}, \tag{26}$$

where $\sigma_1 = \sqrt{2rc}$, $k_1 = j(r-1)/r$. In Section 2, $\mathcal{S}_{\sigma_1}$ is the scaling operation associated with σ_1 and $\mathcal{C}_{k_1}$ is the complex-chirp multiplication associated with k_1. Therefore, the gFrFT Type-I operator can be rewritten as

$$\mathfrak{F}^{(\mathrm{I})}_{\alpha,r,c} = \mathcal{S}^{-1}_{\sigma_1}\mathcal{C}^{-1}_{k_1}\mathcal{F}_\alpha\mathcal{C}_{k_1}\mathcal{S}_{\sigma_1}. \tag{27}$$

The other types share the same symmetry. By defining $\sigma_2 = \sigma_1^*$ and $k_2 = k_1^*$, we obtain

$$\mathfrak{F}^{(\mathrm{II})}_{\alpha,r,c} = \mathcal{S}^{-1}_{\sigma_2}\mathcal{C}^{-1}_{k_2}\mathcal{F}_\alpha\mathcal{C}_{k_2}\mathcal{S}_{\sigma_2}, \tag{28}$$
$$\mathfrak{F}^{(\mathrm{III})}_{\alpha,r,c} = \mathcal{S}^{-1}_{\sigma_1}\mathcal{C}^{-1}_{k_1}\mathcal{F}_\alpha\mathcal{C}_{k_2}\mathcal{S}_{\sigma_2}, \tag{29}$$
$$\mathfrak{F}^{(\mathrm{IV})}_{\alpha,r,c} = \mathcal{S}^{-1}_{\sigma_2}\mathcal{C}^{-1}_{k_2}\mathcal{F}_\alpha\mathcal{C}_{k_1}\mathcal{S}_{\sigma_1}. \tag{30}$$

In accordance with the symmetry between the kernels, mentioned in Section 4.2, there are also some symmetries in the above operator decompositions, (27) - (30).

Because the gFrFT is a special case of the CLCT, the properties of the CLCT are applied with given the matrix $\mathbf{M}^{(\mathrm{I})}_{\alpha,r,c}$. Main properties of the CLCT can be found in [9], such as the linear property, time reversal, time shift, modulation, multiplication and differentiation property, etc. From (27) to (30), one can easily verify the additivity and the reversibility among the gFrFTs.

5. CONCLUSION AND FUTURE WORK

In this paper, we introduced the gFrFT, which generalized the conventional FT by replacing the eigenfunctions with the gHGFs. The integral form was derived and it is closely connected with some well-known transforms. In addition, the standard and elegant form of these transforms were defined. Finally, we related the gFrFT with the CLCT closely.

Due to the relation between the gFrFT and the CLCT, it is possible to implement the CLCT by selecting appropriate α, r, and c in the gFrFT. In addition, the discrete implementation of the gFrFT is simple under its definition. As a result, the discrete CLCT is promising with the aid of the gFrFT.

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