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The generalized Marcum Q−function: an orthogonal polynomial approach

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Abstract. A novel power series representation of the generalized Marcum Q-function of positive order involving generalized Laguerre polynomials is presented. The absolute convergence of the proposed power series expansion is showed, together with a convergence speed analysis by means of truncation error. A brief review of related studies and some numerical results are also provided.

1 Introduction

For ν real number let I_ν be denotes the modified Bessel function [49, p. 77] of the first kind of order ν, defined by

$$I_\nu(t) = \sum_{n\geq 0} \frac{(t/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}, \tag{1}$$

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and let $b \mapsto Q_\nu(a,b)$ be the generalized Marcum Q-function, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)dt, \tag{2}$$

where $b \geq 0$ and $a, \nu > 0$. Here Γ stands for the well-known Euler gamma function. When $\nu = 1$, the function

$$b \mapsto Q_1(a,b) = \int_b^\infty te^{-\frac{t^2+a^2}{2}} I_0(at)dt$$

is known in literature as the (first order) Marcum Q-function. The Marcum Q-function and its generalization are frequently used in the detection theories for radar systems [27] and wireless communications [12, 13], and have important applications in error performance analysis of digital communication problems dealing with partially coherent, differentially coherent, and non–coherent detections [38, 40]. Since, the precise computations of the Marcum Q-function and generalized Marcum Q-function are quite difficult, in the last few decades several authors worked on precise and stable numerical calculation algorithms for the functions. See the papers of Dillard [14], Cantrell [7], Cantrell and Ojha [8], Shnidman [34], Helstrom [17], Temme [46] and the references therein. Moreover, many tight lower and upper bounds for the Marcum Q-function and generalized Marcum Q-function were proposed as simpler alternative evaluating methods or intermediate results for further integrations. See, for example, the papers of Simon [35], Chiani [10], Simon and Alouini [37], Annamalai and Tellambura [1], Corazza and Ferrari [11], Li and Kam [22], Baricz [4], Baricz and Sun [5, 6], Kapinas et al. [19], Sun et al. [41], Li et al. [23] and the references therein. In this field, the order ν is usually the number of independent samples of the output of a square–law detector, and hence in most of the papers the authors deduce lower and upper bounds for the generalized Marcum Q-function with order ν integer. On the other hand, based on the papers [8, 27, 34] there are introduced in the Matlab 6.5 software the Marcum Q-function and positive integer order generalized Marcum Q-function[1]: `marcumq(a,b)` computes the value of the first order Marcum Q-function $Q_1(a,b)$ and `marcumq(a,b,m)` computes the value of the mth order generalized Marcum Q-function $Q_m(a,b)$, defined by (2), where m is a positive integer. However, in some important applications, the order $\nu > 0$ of the generalized Marcum Q-function is not necessarily an integer number. The

[1]See `http://www.mathworks.com/access/helpdesk/help/toolbox/signal/marcumq.html` for more details.

generalized Marcum Q-function is the complementary cumulative distribution function or reliability function of the non–central chi distribution with 2ν degrees of freedom [18, 39, 41]. Moreover, real order generalized Marcum Q-function has been used to characterize small–scale channel fading distributions with line–of–sight channel components [24, 50] or cross–channel correlations [2, 3, 19, 20, 38, 44, 45].

In this paper, we present a novel generalized Laguerre polynomial series representation of the generalized Marcum Q-function, which extends the result of the first order Marcum Q-function in Pent's paper [32] to the case of the generalized Marcum Q-function with real order $\nu > 0$. We further show the absolute convergence of the proposed power series expansion, together with a convergence speed analysis by means of truncation error. A brief review of related studies in the literature is provided, which may assist the readers to get a more complete vision of this area. Finally, some numerical results are provided as a complementary of these theoretical analysis.

2 The generalized Marcum Q-function via Laguerre polynomials

2.1 Novel series representation of the generalized Marcum Q-function

We start with the following well–known formula [43, p. 102]

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!} = \Gamma(\alpha+1)e^z(xz)^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{xz}), \tag{3}$$

where $x, z \in \mathbb{R}$ and $\alpha > -1$. Here J_α stands for the Bessel function of the first kind of order α, $L_n^{(\alpha)}$ is the generalized Laguerre polynomial of degree n and order α, defined explicitly as

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!}\left(e^{-x}x^{n+\alpha}\right)^{(n)} = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)\Gamma(n-k+1)} \frac{(-x)^k}{k!}.$$

Changing in (3) z with $-z$ and taking into account $I_\nu(x) = i^{-\nu}J_\nu(ix)$ we obtain that [26]

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{(-1)^n z^n}{n!} = \Gamma(\alpha+1)e^{-z}(xz)^{-\frac{\alpha}{2}} I_\alpha(2\sqrt{xz}). \tag{4}$$

Now, if we use

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)},$$

and replace x with a and α with $\nu-1$, respectively, (4) can be rewritten as

$$\left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az}) = e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^{n+\nu-1}, \tag{5}$$

which holds for all $a, \nu > 0$ and $z \geq 0$.

Now, consider the following formula [46, 47]

$$\begin{aligned} Q_\nu(\sqrt{2a},\sqrt{2b}) &= e^{-a}\sum_{n\geq 0}\frac{\Gamma(\nu+n,b)}{\Gamma(\nu+n)}\frac{a^n}{n!} \\ &= \int_b^\infty \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az})dz, \end{aligned} \tag{6}$$

where $a, \nu > 0$ and $b \geq 0$. We note that the function $b \mapsto Q_\nu(\sqrt{a},\sqrt{b})$, defined by

$$Q_\nu(\sqrt{a},\sqrt{b}) = \frac{1}{2}\int_b^\infty \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-\frac{z+a}{2}} I_{\nu-1}(\sqrt{az})dz,$$

is in fact the survival function (or the complementary of the cumulative distribution function with respect to unity) of the non–central chi–square distribution with 2ν degrees of freedom and non–centrality parameter a. With other words, for all $a, \nu > 0$ and $b \geq 0$ we have

$$Q_\nu(\sqrt{a},\sqrt{b}) = 1 - \frac{1}{2}\int_0^b \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-\frac{z+a}{2}} I_{\nu-1}(\sqrt{az})dz. \tag{7}$$

See [39] for more details. Combining (5) with (7) we obtain

$$\begin{aligned} Q_\nu(\sqrt{2a},\sqrt{2b}) &= 1 - \int_0^b \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az})dz \\ &= 1 - \int_0^b e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^{n+\nu-1}dz \\ &\overset{(a)}{=} 1 - e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)}\int_0^b z^{n+\nu-1}dz \\ &= 1 - \sum_{n\geq 0}(-1)^n e^{-a}\frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n+1)} b^{n+\nu}, \end{aligned}$$

where in (a) the integration and summation can be interchanged, because the series on the right–hand side of (5) is uniformly convergent for $0 \leq z \leq b$. For more details see the last paragraph of Section 2.2. After some simple manipulation, we obtain a new formula of the generalized Marcum Q-function, i.e.,

$$Q_\nu(a,b) = 1 - \sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}, \tag{8}$$

valid for all $a,\nu > 0$ and $b \geq 0$.

In order to simplify the numerical evaluation of the series (8), we consider the expression

$$P_{\nu,n}(a,b) = \frac{b^n L_n^{(\nu-1)}(a)}{\Gamma(\nu+n+1)},$$

which satisfies the recurrence relation

$$\begin{aligned} P_{\nu,n+1}(a,b) =& \frac{(2n+\nu-a)b}{(n+1)(\nu+n+1)}P_{\nu,n}(a,b) \\ &- \frac{(n+\nu-1)b^2}{(n+1)(\nu+n)(\nu+n+1)}P_{\nu,n-1}(a,b) \end{aligned}$$

for all $a,\nu > 0$, $b \geq 0$ and $n \in \{1,2,3,\dots\}$, with the initial conditions

$$P_{\nu,0}(a,b) = \frac{1}{\Gamma(\nu+1)} \quad \text{and} \quad P_{\nu,1}(a,b) = \frac{(\nu-a)b}{\Gamma(\nu+2)}.$$

Here, the recurrence relation for $P_{\nu,n}(a,b)$ were obtained from the recurrence relation [43, p. 101]

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)$$

and the initial conditions from

$$L_0^{(\alpha)}(x) = 1 \quad \text{and} \quad L_1^{(\alpha)}(x) = -x+\alpha+1.$$

With the help of the expression $P_{\nu,n}(a,b)$, (8) can be easily rewritten as

$$Q_\nu(a,b) = 1 - \sum_{n\geq 0} e^{-\frac{a^2}{2}}\left(\frac{b^2}{2}\right)^{\nu} P_{\nu,n}\left(\frac{a^2}{2}, -\frac{b^2}{2}\right). \tag{9}$$

2.2 Convergence analysis of the new series representation

We note that for $a > 0$, $\nu \geq 1$ and $b \geq 0$ the absolute convergence of the series in (8) or (9) can be shown easily by using the following inequalities

$$\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}}\frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\frac{\Gamma(\nu+n)}{n!\Gamma(\nu)}e^{\frac{a^2}{4}} \\
&\leq e^{-\frac{a^2}{4}}\frac{1}{\Gamma(\nu)}\left(\frac{b^2}{2}\right)^{\nu-1}\sum_{n\geq 0}\frac{1}{(n+1)!}\left(\frac{b^2}{2}\right)^{n+1} \\
&= e^{-\frac{a^2}{4}}\frac{1}{\Gamma(\nu)}\left(\frac{b^2}{2}\right)^{\nu-1}\left(e^{\frac{b^2}{2}}-1\right)
\end{aligned}$$

or

$$\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}}\frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\frac{\Gamma(\nu+n)}{n!}e^{\frac{a^2}{4}}\left(\frac{a^2}{4}\right)^{1-\nu} \\
&\leq e^{-\frac{a^2}{4}}\left(\frac{2b^2}{a^2}\right)^{\nu-1}\sum_{n\geq 0}\frac{1}{(n+1)!}\left(\frac{b^2}{2}\right)^{n+1} \\
&= e^{-\frac{a^2}{4}}\left(\frac{2b^2}{a^2}\right)^{\nu-1}\left(e^{\frac{b^2}{2}}-1\right),
\end{aligned}$$

which contain the known inequalities of Szegő [43] for generalized Laguerre polynomials

$$|L_n^{\alpha}(x)| \leq \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}e^{\frac{x}{2}}$$

and of Love [25]

$$|L_n^{\alpha}(x)| \leq \frac{\Gamma(\alpha+n+1)}{n!}\left(\frac{x}{2}\right)^{-\alpha} e^{\frac{x}{2}},$$

where in both of the inequalities $\alpha \geq 0$, $x > 0$ and $n \in \{0, 1, 2, \dots\}$.

Moreover, for $a > 0$, $0 < \nu \leq 1$ and $b \geq 0$ the absolute convergence of the series in (8) or (9) can be shown in a similar manner by using the following inequality

$$\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left(2-\frac{\Gamma(\nu+n)}{n!\Gamma(\nu)}\right) e^{\frac{a^2}{4}} \\
&= e^{-\frac{a^2}{4}} \sum_{n\geq 0} \frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right)\left(\frac{b^2}{2}\right)^{n+\nu} \\
&\leq e^{-\frac{a^2}{4}} \sum_{n\geq 0} \frac{2}{n!}\left(\frac{b^2}{2}\right)^{n+\nu} \\
&= 2e^{-\frac{a^2}{4}}\left(\frac{b^2}{2}\right)^{\nu} e^{\frac{b^2}{2}},
\end{aligned}$$

which contains the classical inequality of Szegő [43] for generalized Laguerre polynomials

$$|L_n^{\alpha}(x)| \leq \left(2-\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right) e^{\frac{x}{2}}$$

where $-1 < \alpha \leq 0$, $x > 0$ and $n \in \{0, 1, 2, \dots\}$. In addition here we used the fact that for all fixed $n \in \{1, 2, 3, \dots\}$ the function

$$\nu \mapsto \frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right),$$

which maps 0 into $2/n!$, is decreasing on $(0, 1]$ and consequently for all $n \in \{0, 1, 2, \dots\}$ and $0 < \nu \leq 1$ we have

$$\frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right) \leq \frac{2}{n!}.$$

We note that other uniform bounds for generalized Laguerre polynomials can be found in the papers of Love [25], Lewandowski and Szynal [21], Michalska and Szynal [28], Pogány and Srivastava [33]. See also the references therein.

Finally, note that by using the above uniform bounds for the generalized Laguerre polynomials the uniform convergence of the series on the right-hand side of (5) can be shown easily for $0 \leq z \leq b$. This is important because in order to obtain (8) we have used tacitly that the series on the right–hand side of (5) is uniformly convergent and then we can interchange the integration with summation. For example, if we use the above Szegő's uniform bound, then for all $n \in \{0, 1, 2, \dots\}$, $a > 0$, $\nu \geq 1$ and $0 \leq z \leq b$ we have

$$\left|(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^n\right| \leq \frac{e^{\frac{a}{2}}}{\Gamma(\nu)} \frac{b^n}{n!}.$$

By the ratio test the series $e^b = \sum_{n\geq 0} b^n/n!$ is convergent and thus in view of the Weierstrass M–test the original series on the right–hand side of (5) converges uniformly for all $0 \leq z \leq b$.

2.3 Truncation error analysis

For practical evaluations of our power series expansion, we need to approximate the generalized Marcum Q-function $Q_\nu(a, b)$ by the first $n_0 \in \{1, 2, 3, \dots\}$ terms of (8), i.e.,

$$\hat{Q}_\nu(a, b) = 1 - \sum_{n=0}^{n_0} (-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)} \left(\frac{b^2}{2}\right)^{n+\nu}.$$

We note that the absolute value of the truncation error

$$\varepsilon_t = Q_\nu(a, b) - \hat{Q}_\nu(a, b) = \sum_{n\geq n_0+1} (-1)^{n+1} e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)} \left(\frac{b^2}{2}\right)^{n+\nu}$$

can be upper bounded by using the upper bounds for the generalized Laguerre polynomials as in subsection 2.2. More precisely, by using the same argument as in subsection 2.2 and Sewell's inequality [29, p. 266]

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \leq \frac{xe^x}{n}, \quad n \in \{1, 2, 3, \dots\},\ x \geq 0,$$

we can deduce the following: if $a > 0$, $b \geq 0$ and $\nu \geq 1$, then

$$|\varepsilon_t| \leq e^{-\frac{a^2}{4}} \frac{1}{\Gamma(\nu)} \left(\frac{b^2}{2}\right)^{\nu-1} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0+1} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0+1} \frac{1}{\Gamma(\nu)} \left(\frac{b^2}{2}\right)^{\nu}$$

or

$$|\varepsilon_t| \leq e^{-\frac{a^2}{4}} \left(\frac{2b^2}{a^2}\right)^{\nu-1} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0+1} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0+1} \frac{b^2}{2} \left(\frac{2b^2}{a^2}\right)^{\nu-1}.$$

Similarly, it can be shown that if $a > 0$, $b \geq 0$ and $0 < \nu \leq 1$, then the absolute value of the truncation error is upper bounded as follows

$$|\varepsilon_t| \leq 2e^{-\frac{a^2}{4}} \left(\frac{b^2}{2}\right)^{\nu} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{2e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0} \left(\frac{b^2}{2}\right)^{\nu+1}.$$

Observe that the above upper bounds of the absolute value of the truncation error converge to zero at a speed of $1/n_0$. In practice, we can use these upper bounds to decide the number of terms, i.e. n_0, for achieving a pre–determined accuracy.

2.4 A brief review of related studies

As far as we know the formula (8), or its equivalent form (9), is new. However, if we choose $\nu = 1$ in (9), then we reobtain the main result of Pent [32]

$$Q_1(a, b) = 1 - \frac{b^2}{2} \sum_{n \geq 0} e^{-\frac{a^2}{2}} P_n\left(\frac{a^2}{2}, -\frac{b^2}{2}\right),$$

where

$$P_n(a, b) = P_{1,n}(a, b) = \frac{b^n L_n(a)}{(n+1)!},$$

which for all $a > 0$, $b \geq 0$ and $n \in \{1, 2, 3, \dots\}$ satisfies the recurrence relation

$$P_{n+1}(a, b) = \frac{(2n+1-a)b}{(n+1)(n+2)} P_n(a, b) - \frac{nb^2}{(n+1)^2(n+2)} P_{n-1}(a, b)$$

with the initial conditions

$$P_0(a, b) = 1 \quad \text{and} \quad P_1(a, b) = \frac{(1-a)b}{2}.$$

Here $L_n = L_n^{(0)}$ is the classical Laguerre polynomial of degree n.

It should be mentioned here that another type of Laguerre expansions for the Marcum Q-function was proposed in 1977 by Gideon and Gurland [16], which involves the lower incomplete gamma function. This type of Laguerre expansions requires to use a complementary result of (4), i.e.

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(0)} \frac{(-1)^n x^n}{n!} = \Gamma(\alpha+1)e^{-x}(xz)^{-\frac{\alpha}{2}} I_\alpha(2\sqrt{xz}). \tag{10}$$

Now by some simple manipulation we obtain

$$\left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az}) = z^{\nu-1}e^{-z} \sum_{n\geq 0} \frac{(-a)^n}{\Gamma(\nu+n)} L_n^{(\nu-1)}(z), \tag{11}$$

which is equivalent to Tiku's result [48], available also as equation (29.11) in the book [18]. By integrating (11) in z and by using the differentiation formula [26]

$$\frac{d}{dz}\left[z^{\alpha+1}e^{-z}L_{n-1}^{(\alpha+1)}(z)\right] = nz^\alpha e^{-z} L_n^{(\alpha)}(z),$$

where $n \in \{1, 2, 3, \dots\}$, $\alpha > -1$ and $z \in \mathbb{R}$, we can obtain another generalized Laguerre polynomial series expansion of the generalized Marcum Q-function

$$Q_\nu(\sqrt{2a}, \sqrt{2b}) = 1 - \frac{1}{\Gamma(\nu)}\gamma(\nu, b) - \sum_{n\geq 1} (-1)^n e^{-b} \frac{b^\nu L_{n-1}^{(\nu)}(b)}{n\Gamma(\nu+n)} a^n,$$

which in turn implies that

$$\begin{aligned}
Q_\nu(a,b) &= 1 - \frac{1}{\Gamma(\nu)}\gamma\left(\nu, \frac{b^2}{2}\right) - \sum_{n\geq 1}(-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n \\
&= \frac{1}{\Gamma(\nu)}\Gamma\left(\nu, \frac{b^2}{2}\right) - \sum_{n\geq 1}(-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n \\
&= \lim_{a\to 0} Q_\nu(a,b) - \sum_{n\geq 1}(-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n,
\end{aligned} \tag{12}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function, defined by

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt.$$

Here we used that

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x), \tag{13}$$

and

$$\lim_{a \to 0} Q_\nu(a, b) = \frac{1}{\Gamma(\nu)} \Gamma\left(\nu, \frac{b^2}{2}\right).$$

Some other Laguerre expansions for the Marcum Q-function are provided in Gideon and Gurland's paper [16], available also as equation (29.13) of [18]. Moreover, a new unified Laguerre polynomial–series–based distribution of small–scale fading envelope and power was proposed recently by Chai and Tjhung [9], which covers a wide range of small–scale fading distributions in wireless communications. Many known Laguerre polynomial–series–based probability density functions and cumulative distribution functions of small–scale fading distributions are provided, which include the multiple–waves–plus–diffuse–power fading, non–central chi and chi–square, Nakagami–m, Rician (Nakagami–n), Nakagami–q (Hoyt), Rayleigh, Weibull, Stacy, gamma, Erlang and exponential distributions as special cases. See also [42], which contains some corrections of formulas deduced in [9]. In particular, (12) is a special case of the unified cumulative distribution function given in corrected form in [42]. We note that the expression of (12) and the unified cumulative distribution function in [42] are quite different from our main result (8) or (9). This is because they are based on two different Laguerre polynomial expansions of the modified Bessel function of the first kind I_ν given in (4) and (10). Therefore, these Laguerre polynomial expansions are expanded over different variables of the generalized Marcum Q-function. Finally, we note that since Nakagami's work [30] the Laguerre polynomial series expansions of various probability density functions have been derived. We refer to the papers of Esposito and Wilson [15], Yu et al. [51], Chai and Tjhung [9] and to the references therein.

Finally, by using the infinite series representation of the modified Bessel function of the first kind (1) and the formula

$$\int_\alpha^\infty t^m e^{-\frac{t^2}{2}} dt = 2^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}, \frac{\alpha^2}{2}\right),$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function, defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt,$$

we easily obtain that

$$
\begin{aligned}
Q_\nu(a,b) &= \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} \sum_{n\geq 0} \frac{(at)^{2n+\nu-1}}{2^{2n+\nu-1} n!\Gamma(\nu+n)} dt \\
&= e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{a^{2n}}{2^{2n+\nu-1} n!\Gamma(\nu+n)} \int_b^\infty e^{-\frac{t^2}{2}} t^{2n+2\nu-1} dt \\
&= e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{a^2}{2}\right)^n \frac{\Gamma\left(\nu+n, \frac{b^2}{2}\right)}{\Gamma(\nu+n)} \\
&= 1 - \sum_{n\geq 0} e^{-\frac{a^2}{2}} \left(\frac{a^2}{2}\right)^n \frac{\gamma\left(\nu+n, \frac{b^2}{2}\right)}{n!\Gamma(\nu+n)}. \qquad (14)
\end{aligned}
$$

We note that (14) is usually called the canonical representation of the νth order generalized Marcum Q-function. Recently, Annamalai and Tellambura [2] (see also [3]) claimed that the series representation (14) is new, however it appears already in 1993 in the paper of Temme [46]. See also Temme's book [47] and Patnaik's [31] result from 1949, which can be found also as equation (29.2) in the book [18]. Interestingly, our novel series representation (9) for the generalized Marcum Q-function resembles to the series representation (14).

2.5 Numerical results

We now consider some numerical aspects of our generalized Laguerre polynomial expansions (8) or (9). In practice, we usually need to compute the detection probability for different values of b with fixed a to decide a proper detection threshold. Since the generalized Laguerre polynomial in (8) is determined by only a, we can save computation time by storing the values of the generalized Laguerre polynomials for computing the generalized Marcum Q-function with different values of b.

The next tables contain some values of the generalized Marcum Q-function calculated using (9) and using the Matlab `marcumq` function. For the considered choices of a and b, the numerical value of (9) is exactly the same with that of the Matlab `marcumq` function, if $\nu \in \{1, 5\}$ is integer. When $\nu = 7.7$, the Matlab `marcumq` function does not work, and the numerical value of (9) is provided in the tables. Finally, we note that more accurate intermediate terms are required for larger a and b.

$a = 0.2, b = 0.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.838249985438908	0.999998670306184	0.999999999927717
marcumq	0.838249985438908	0.999998670306184	—

$a = 1.2, b = 1.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.501536568390858	0.994346394491553	0.999944937223540
marcumq	0.501536568390858	0.994346394491553	—

$a = 2.2, b = 2.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.426794627821735	0.929671935077756	0.993735633182201
marcumq	0.426794627821735	0.929671935077756	—

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