

The Generalized Oka–Grauert Principle for 1-Convex Manifolds

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1. Introduction and Main Theorem

Let Y be a complex manifold satisfying the convex approximation property (CAP) and let X be an arbitrary Stein manifold. Then the Oka–Grauert (or homotopic) principle holds for mappings $X \rightarrow Y$ ([F2]; the name *Oka manifold* has recently been suggested for such manifolds Y). This means that each homotopy class of mappings $X \rightarrow Y$ admits a holomorphic representative.

Manifolds satisfying CAP are in some sense “large”. As an example of a manifold failing to satisfy CAP, consider the annulus $Y = \{z \in \mathbb{C}, 1/2 < |z| < 2\}$. Let $X = \{z, 1/3 < |z| < 3\}$. There are plenty of continuous mappings from X to Y but no nontrivial holomorphic ones. The reason is that Y is “too small” for X . If we are free to change the holomorphic structure on X , then we can find for every continuous mapping $f_0: X \rightarrow Y$ another Stein structure J_1 on X that is homotopic to the initial one as well as a holomorphic mapping $f_1: (X, J_1) \rightarrow Y$ in the same homotopy class as f_0 . In general, the change of structure depends on both Y and f_0 . In the simple example just given, the manifold X is homotopically equivalent to the unit circle $S^1 \subset X$ and we change the homotopic structure of X simply by squeezing it diffeotopically into a small neighborhood of the unit circle. For a general Stein manifold X we can proceed analogously, replacing S^1 by a suitably fattened CW complex embedded in X and homotopically equivalent to X to obtain the following.

GENERALIZED OKA–GRAUERT PRINCIPLE. Every continuous mapping $X \rightarrow Y$ from a Stein manifold X to a complex manifold Y is homotopic to a holomorphic one provided that either Y satisfies CAP or we are free to change the complex structure on X [FS1]. In addition, we can also require that the structure is fixed on a neighborhood of an analytic set $X_0 \subset X$ if the initial mapping is holomorphic on a neighborhood of X_0 [FS2].

It has recently been shown that, if X is 1-convex and Y satisfies CAP, then the following version of the Oka–Grauert principle holds.

RELATIVE OKA–GRAUERT PRINCIPLE FOR MAPPINGS. Every continuous mapping $X \rightarrow Y$ from a 1-convex manifold X to a complex manifold Y that satisfies

Received October 14, 2009. Revision received December 21, 2010.

The authors’ work was supported by the research program P1-0291 at the Slovenian Research Agency.

CAP and that is already holomorphic on a neighborhood of the exceptional set is homotopic to a holomorphic map, and the homotopy is fixed on the exceptional set (cf. [LV; P]).

Recall that a complex space X is 1-convex if it possesses a plurisubharmonic exhaustion function that is strictly plurisubharmonic outside a compact set and that the Remmert reductions of 1-convex spaces are Stein. It is therefore possible to combine these two principles as follows.

GENERALIZED OKA–GRAUERT PRINCIPLE FOR 1-CONVEX MANIFOLDS. Every continuous mapping $X \rightarrow Y$, from a 1-convex manifold X to a complex manifold Y , that is already holomorphic on a neighborhood of the exceptional set is homotopic to a holomorphic map provided that either Y satisfies CAP or we are free to change the complex structure on X .

The main theorem of this paper may now be stated as follows.

THEOREM 1.1 (Generalized Oka–Grauert principle for 1-convex manifolds). *Let (X, J_0) be a 1-convex manifold of dimension at least 3, and let S be its exceptional set. Let $K \subset X$ be a holomorphically convex compact subset of X containing S , let Y be a complex manifold, and let $f_0: X \rightarrow Y$ be a continuous mapping that is holomorphic in a neighborhood of K . Then there exist a homotopy $f_t: X \rightarrow Y$ and a homotopy J_t of complex structures on X such that*

- (1) $f_t(x) = f_0(x)$ for $x \in S$,
- (2) $J_t = J_0$ on a neighborhood of K ,
- (3) (X, J_1) is 1-convex with the exceptional set S ,
- (4) the mappings f_t are J_t -holomorphic on a neighborhood of K and approximate f_0 on K as well as desired, and
- (5) f_1 is J_1 -holomorphic on X .

2. Technicalities

2.1. Handle Attaching in Stein Category

Let (X, J) be a complex manifold with a complex structure J . A real immersed submanifold $i: \Sigma \rightarrow X$ is *totally real* or *J -real* in X if at every point $p \in \Sigma$ we have $T_p i(\Sigma) \cap J(T_p i(\Sigma)) = 0$. The condition implies that $\dim_{\mathbb{R}} \Sigma \leq \dim_{\mathbb{C}} X$.

Let $W \Subset X$ be a relatively compact domain defined by $W = \{\rho < 0\}$, where ρ is a smooth real function defined in a neighborhood of ∂W and $d\rho \neq 0$ on ∂W . We say that W (or ∂W) is *strongly pseudoconvex* or *J -convex* if $dd^c \rho$ is a positive form in a neighborhood of ∂W , meaning that $dd^c \rho(v, Jv) > 0$ for every $v \in TX|_{\partial W}$. So $dd^c \rho$ defines a metric on $TX|_{\partial W}$ and, in so doing, it also defines normal directions to ∂W . Now let $W = \{\rho < 0\}$ be a J -convex relatively compact domain in (X, J) and let $D = D_k \subset \mathbb{R}^k$ be the closed unit ball with the boundary $S = S^{k-1}$. An *embedding (immersion) of a pair* $G: (D, S) \rightarrow (X \setminus W, \partial W)$ is a smooth embedding (immersion) $G: D \rightarrow X \setminus W$, $G(S) = G(D) \cap \partial W$, with

G transverse to ∂W . We say that G is *normal* to ∂W if dG maps normal vectors from $S = \partial D$ to normal vectors of ∂W and that $G(S)$ is *Legendrian* in ∂W if dG maps vectors tangent to S into the *contact distribution* $T^{\mathbb{C}}W = T\partial W \cap JT\partial W$ along ∂W .

The following lemma, although not given in the manner just stated, is proved in [E]; a complete proof can also be found in [FS1]. The main ingredients are the Legendrization theorem of Gromov [Gro] and Duchamp [D] and the h-principle of Gromov for totally real submanifolds of complex manifolds [Gro]. Lemma 2.1 functions just as well in an almost complex case.

LEMMA 2.1. *Let W be an open, relatively compact J -convex domain with a smooth boundary in a complex manifold (X, J) of the complex dimension $n \geq 3$. For $0 \leq k \leq n$ let (D, S) be the closed unit disc in \mathbb{R}^k with the boundary sphere S and $G_0: (D, S) \rightarrow (X \setminus W, \partial W)$ a smooth embedding. Then there exists a regular homotopy of embeddings $G_t: (D, S) \rightarrow (X \setminus W, \partial W)$, $0 \leq t \leq 1$, such that*

- (1) G_1 is normal to ∂W ,
- (2) $G_1(S)$ is Legendrian in ∂W , and
- (3) $G_1(D)$ is totally real in X .

If ∂W is real analytic in a neighborhood of $G_0(S)$, then G_1 can also be made real analytic.

REMARK 2.2. In the complex dimension 2, Lemma 2.1 is also valid as stated if the attaching disc D is 1-dimensional. If the disc D is 2-dimensional, one cannot get the isotopy of embeddings but only a regular homotopy of immersions, so that the ending map is an embedding near the boundary of D but has special transverse double points in the interior of D . By “special” we mean that the double point is modeled by $\mathbb{R}^2 \cup i\mathbb{R}^2 \subset \mathbb{C}^2$.

Once we have a real-analytic and totally real disc D attached normally (from the outside) along a Legendrian curve to a boundary ∂W of a strictly pseudoconvex domain $W \subset X$, we can use a holomorphic change of coordinates in a neighborhood of ∂D coupled with a C^1 -small, real-analytic deformation of the boundary ∂W near ∂D to get model situations of straight discs attached to a quadratic domain in \mathbb{C}^n . There we can use concrete functions to find strictly pseudoconvex neighborhoods of $W \cup D$ that preserve the topology of $W \cup D$. The construction was first explained in [E] and later also in [FK]. More precisely, we have the following lemma.

LEMMA 2.3. *Let W be an open, relatively compact J -convex domain in a complex manifold (X, J) . For $0 \leq k \leq n$ let (D, S) be the closed unit disc in \mathbb{R}^k with the boundary sphere S and $G_0: (D, S) \rightarrow (X \setminus W, \partial W)$ a smooth totally real embedding with $G_0(S)$ Legendrian. Let d be any smooth Riemannian metric on X and let D_ε be the ε -neighborhood of $G(D)$ in X . Then, for every $\varepsilon > 0$, there exists a relatively compact and strictly pseudoconvex Stein neighborhood \tilde{W} of $W \cup G(D)$ with a smooth boundary completely contained in $W \cup D_\varepsilon$ such that $W \cup G(D)$ is a smooth deformation retract of \tilde{W} and W is Runge in \tilde{W} .*

2.2. Basics for 1-Convex Manifolds

The main problem with 1-convex spaces is the lack of 1-convex neighborhoods of graphs of holomorphic mappings $f: X \rightarrow Y$. However, if we remove from the graph the zero set of a holomorphic function $g: X \rightarrow \mathbb{C}$ that is zero on the exceptional set, we get a Stein space and so its graph has Stein neighborhoods. In addition, we do not want our Stein neighborhoods to be too “thin” in the Y -direction when approaching the graph of f over $g^{-1}(0)$; we prefer the width in the Y -direction to decrease at most polynomially. Such neighborhoods will be called *conic* along the graph of f over $g^{-1}(0)$. The existence of conic neighborhoods is given by our next theorem.

THEOREM 2.4 (Conic neighborhoods [P, Thm. 3.2]). *Let X be a 1-convex complex space with an exceptional set S . Let $A \subset X$ be compact and holomorphically convex with $A \supset S$ and let Y be a complex manifold. Let $f: X \rightarrow Y$ be a continuous mapping that is holomorphic on a 1-convex neighborhood U of A , and let $g: X \rightarrow \mathbb{C}$ be a holomorphic function satisfying $g(S) = 0$. Then, for each 1-convex open set $U' \Subset U$ containing A , there exists a Stein neighborhood of the graph Γf of f over the set $U' \setminus g^{-1}(0)$ in $X \times Y$ that is conic along the graph Γf over $g^{-1}(0)$.*

In the theory of 1-convex spaces there is a version of Cartan’s theorem B for relatively compact, strictly pseudoconvex sets; there is also a version of Cartan’s theorem A.

THEOREM 2.5 (Theorem A for 1-convex spaces [P]). *Let X be a 1-convex space with an exceptional set S , let $U \Subset X$, and let U be an open strictly pseudoconvex set containing S . Let $\mathcal{I} = \mathcal{I}(S)$ be the ideal sheaf generated by the set S , and let \mathcal{Q} be a coherent sheaf on X . Then there exists an $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, the sheaf $\mathcal{Q}\mathcal{I}^n$ is locally generated by $\Gamma(U, \mathcal{Q}\mathcal{I}^n)$ on U .*

When Theorem 2.4 and Theorem 2.5 are combined, there are many important consequences.

COROLLARY 2.6 (Existence of local sprays [P, Sec. 4]). *Let X be a 1-convex space, let $U' \Subset U \Subset X$ be strictly pseudoconvex open sets containing the exceptional set S of X , and let $f: X \rightarrow Y$ be a holomorphic mapping. Then there exists a local spray on U' fixing S that dominates on $U' \setminus S$; in other words, there exists a holomorphic map $F: U' \times B_n(0, \delta) \rightarrow Y$ such that (a) $F(x, \cdot): B_n(0, \delta) \rightarrow Y$, $D_t F(x, t)$ is surjective for $t = 0$ and $x \in U' \setminus S$ and (b) $F(x, 0) = f(x)$ and $F(x, t) = F(x, 0)$ for all $x \in S$.*

Proof. Denote by $VT(X \times Y)$ the kernel of the derivative of the projection $X \times Y \rightarrow Y$, and let V be a conic Stein neighborhood of the graph of f over $U \setminus g^{-1}(0)$ denoted by $\Gamma f|_{U \setminus g^{-1}(0)}$ for some holomorphic function $g: X \rightarrow \mathbb{C}$ with zeros on S . By Cartan’s theorem A for 1-convex spaces (Theorem 2.5), for each sufficiently large $k \in \mathbb{N}$ there exist finitely many vector fields h_1, \dots, h_n of

the bundle $VT(X \times Y)|_{\Gamma f|_U}$ with zeros of order (at least) k on $\Gamma f|_{(g^{-1}(0))}$ generating $VT|_{\Gamma f|_{(U \setminus g^{-1}(0))}}$. We extend these vector fields on V and integrate them. Since V is conic, the fields can be integrated for sufficiently small times $t \leq t_0$ for all $x \in U' \setminus S$ (provided k is big enough). Because of the zeros on $\Gamma f|_{(g^{-1}(0))}$, we can extend the flows of the fields on the graph of f over $(g^{-1}(0))$; this yields a map $F: U' \times B_n(0, \delta) \rightarrow Z$ that fulfills all the requirements. \square

2.3. Special Pseudoconvex Bumps

In this section we construct special pseudoconvex bumps similar to those constructed in [HL]. The main difference is that we have a strictly pseudoconvex open set and a large disc attached to it.

LEMMA 2.7. *Let X be a 1-convex manifold with the exceptional set S , and let $\rho: X \rightarrow \mathbb{R}$ be a plurisubharmonic exhaustion function that is strictly plurisubharmonic outside a holomorphically convex compact set $K \supset S$. Let zero be a regular value for ρ . Let $A := \{\rho \leq 0\} \Subset X$ be a strictly pseudoconvex set containing K , and let $D \subset X$ be a Legendrian disc attached to the boundary of A . We shall construct a compact set $C \subset (A \setminus K)$ and a compact set $B \supset D$ such that the following statements hold.*

- (1) $A \cup B$ is strictly pseudoconvex.
- (2) A and B have bases of strictly pseudoconvex open neighborhoods $\{U_A\}$ and $\{U_B\}$ (respectively) such that $\{U_A \cup U_B\}$ is a basis of strictly pseudoconvex open neighborhoods of $A \cup B$; moreover, the sets U_B are Stein.
- (3) $A \cap B = C$, and C is Runge in any of the neighborhoods U_B .
- (4) Separation property: $(A \setminus B) \cup (B \setminus A) = \emptyset$ and $(\overline{U_A} \setminus \overline{U_B}) \cup (\overline{U_B} \setminus \overline{U_A}) = \emptyset$.

Proof. The assertions derive from [HL, Lemma 2.3] and from our ability to reduce this situation to the model one with a straight disc attached to the quadratic domain in \mathbb{C}^n . For the set C we may take a (full and closed) ε -torus T_ε around ∂D intersected by A . Using the same methods as in [HL], we can smooth the edges of $T_\varepsilon \cap A$ to obtain a strictly pseudoconvex set C with smooth boundary. According to Lemma 2.3, there is a 1-convex neighborhood U of $(\text{Int } C) \cup D$ containing $\text{Int } C$. Let $B := \overline{U}$. By definition, the set C shares a piece of boundary with A so the set $A \cup B$ is strictly pseudoconvex and the separation property (4) holds. Since zero is a regular value for ρ , we can produce a basis of neighborhoods $\{U_A\}$ and $\{U_B\}$ with the listed properties simply by taking a family of sublevel sets $\{\rho \leq \varepsilon_n\}$ for some sequence $\varepsilon_n \rightarrow 0$. \square

3. Proof of the Main Theorem

The proof follows the one presented in [FS2] for the Stein case. The key idea in [FS2] is the following. A suitably chosen strictly plurisubharmonic exhaustion function for a Stein manifold X gives an increasing sequence of strictly pseudoconvex sets $\{X_j\}$ such that there is exactly one critical point in each of them. We

move from X_j to X_{j+1} in two steps: first we have to cross the critical point by attaching a suitable disc D to X_j and then approximate our initial function with the one that is holomorphic on a neighborhood of $D \cup X_j$. Second, we must “fatten” the union $D \cup X_j$ in order to get to X_{j+1} . In the limit we get the desired structure on X and a mapping homotopic to the initial one.

Before we start to explain the differences in our approach, let us mention that in practice we construct something that looks like a fattened CW complex embedded in X that is diffeotopically equivalent to X . Following the procedure just described in the case when X is 1-convex, we try to cross the critical point of the suitable exhaustion function by attaching a disc D to X_{j-1} . Recall that the disc is “large” because it’s obtained by pushing down the small disc given by Morse theory so that its boundary hits the boundary of X_{j-1} in the proper way. If X were Stein then at this point we could use [F1, Thm. 3.2] and get the desired neighborhood of $X_{j-1} \cup D$ and the approximation. However, one of the essential ingredients of the proof of [F1, Thm. 3.2] is that X_j is relatively compact and Stein. There seems to be (at least for the authors) no obvious way to replace this with 1-convexity; the Remmert reduction does not help. We must therefore effect the crossing via gluing, following the idea of Henkin and Leiterer. In their notation, we would like to attach to the set $A = X_j$ a “bone” $B \supset D$ such that $[A, B, A \cup B]$ is a (version of a) special pseudoconvex bump. In [HL] the set B is a Stein neighborhood of the disc such that it shares a piece of boundary with A , and then C is the intersection $A \cap B$. In our case the situation is not local, so the only way to get Stein neighborhoods and approximations is by using [F1, Thm. 3.2] and Lemma 2.3. Let A, B, C be as in Lemma 2.7, and let f be holomorphic on \bar{A} . By [F1, Thm. 3.2], for the set C we obtain a thin bone B' together with the desired holomorphic approximation f' for f ; the better the approximation, the thinner the bone B' . We may assume that $A \cap B' = C$. The gluing procedure for f and f' is performed by solving the $\bar{\partial}$ -equation on the union $A \cup B$ so that the resulting function is a small perturbation of the initial ones. The set on which we solve the $\bar{\partial}$ -equation must be prescribed in advance because the norm of the operator solving this equation depends on the set.

In the gluing procedure we solve the $\bar{\partial}$ -equation on the union $A \cup B$ of the following type of forms. To explain the idea, assume that f and f' are functions. Consider the form

$$\omega = \bar{\partial}(f + \chi(f' - f))$$

using a suitable cutoff function χ with support of $d\chi$ on C (see e.g. [FPI] for details). Formally, this form is defined on $A \cup B'$. But since its support is contained in C , we can trivially extend it to $A \cup B$. Now we solve the $\bar{\partial}$ -equation for the form ω on $A \cup B$ and get a function that is holomorphic over $A \cup B'$. The sets $A \cup B$ and $A \cup B'$ are homotopically equivalent, and from here we can continue with the fattening procedure. The details are explained in what follows.

Using the Remmert reduction, choose a plurisubharmonic exhaustion function $\rho: X \rightarrow [0, \infty)$ such that $\rho^{-1}(0) = K$ and ρ is strictly plurisubharmonic

outside K . Choose regular values $c_0 < c_1 < c_2 \cdots$ for every j such that there is exactly one critical point p_j of ρ contained in $\rho^{-1}(c_{j-1}, c_j)$, and let k_j denote the Morse index of p_j (recall that $k_j \leq n$ for all j). The sublevel sets $X_j = \rho^{-1}[0, c_j]$ are topologically obtained by attaching a handle of index k_j to the domain X_{j-1} . The handle in question is the thickening in X of the stable disc of p_j of the gradient flow of ρ . Assume that f is already holomorphic in a neighborhood of \bar{X}_{c_0} . We shall construct a sequence of complex structures J_j on X and a sequence of maps $f_j: X \rightarrow Y$ such that:

- (1) for every j , the manifold (X_j, J_j) is 1-convex with the exceptional set S ;
- (2) $J_j = J_{j-1}$ on a neighborhood of \bar{X}_{j-1} ;
- (3) \bar{X}_{j-1} is Runge in (X_j, J_j) ;
- (4) f_j is J_j holomorphic on a neighborhood of \bar{X}_j ;
- (5) f_j and f_{j-1} differ by less than $\varepsilon 2^{-j-1}$ on X_{j-1} and are equal on S ; and
- (6) f_j is homotopic to f_{j-1} by a homotopy that is $(\varepsilon 2^{-j-1})$ -close to f_{j-1} on \bar{X}_{j-1} .

At the $j = 0$ step we take J_0 to be the original complex structure on X and f_0 to be the original function. By assumption, f_0 is holomorphic in a neighborhood of X_0 .

Let us now assume that conditions (1)–(6) are met at the $j - 1$ level. We set $A = X_{j-1}$ and $f_0 = f_{j-1}$. Let M be the stable manifold for the critical point p_j of the gradient flow of ρ by using the metric associated with the positive Levi form $dd^c \rho$. The disc $D = M \cap (X \setminus A)$ is a smooth disc attached transversely to the boundary of A . Using a small perturbation of our domain (or the function ρ), we can assume that ∂A is real analytic in a neighborhood of ∂D . We use Lemma 2.1 with $W = A$ and $J = J_{j-1}$ to deform the disc $(D, \partial D)$ with a small isotopy $(D_t, \partial D_t) \hookrightarrow (X \setminus A, \partial A)$ of pairs to a real-analytic, totally real disc, which we again call D ($D \subset X \setminus A$), attached to ∂A along a real-analytic Legendrian curve. Let B and C be as in Lemma 2.7. Recall that $C = A \cap B$ and that the boundary ∂C agrees with ∂A near ∂D (see Figure 1).

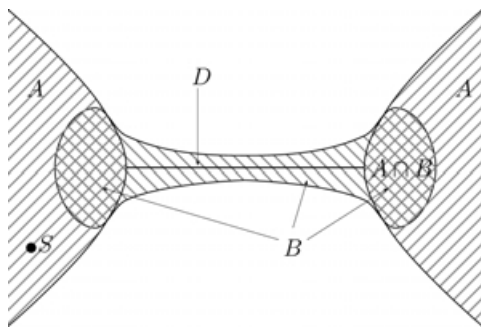


Figure 1 A special pseudoconvex bump

According to Corollary 2.6, there exists a local spray F on a neighborhood U_A of \bar{A} that dominates on $A \setminus S$ and keeps S fixed. According to [F1, Thm. 3.2] there exist an open “thin bone” B' containing $\bar{C} \cup D$, a Stein open set $U_{B'} \supset \bar{B}'$, and a map $G: U_{B'} \times B_n(0, \delta) \rightarrow Y$ such that G approximates F on C as well as desired. There also exists a “gluing” map $\gamma = (\text{id}_x, c): U_C \times B_n(0, \delta') \rightarrow U_C \times B_n(0, \delta')$ such that $F = G \circ \gamma$ on $U_C \times B_n(0, \delta')$. The gluing map may be assumed to be close to the identity map. If it is not, then we can approximate it on C by $\tilde{\gamma} = (\text{id}_x, \tilde{c}): U_B \times \mathbb{C}^n \rightarrow U_B \times \mathbb{C}^n$ well enough so that \tilde{c} is fiberwise invertible over U_C and take $G \circ \tilde{\gamma}$ in the place of G , which gives $(\text{id}_x, \tilde{c}^{-1} \circ c)$ as a gluing map.

Near the zero section, the map γ has a decomposition $\gamma = \beta \circ \alpha^{-1}$, where $\alpha: U_A \times B_n(0, \delta') \rightarrow U_A \times B_n(0, \delta')$ and $\beta: U_B \times B_n(0, \delta') \rightarrow U_B \times B_n(0, \delta')$ are invertible on a neighborhood of the zero section over U_C ; note that U_B is an open neighborhood of \bar{B} . Note also that the decomposition is made over the 1-convex open set $U_A \cup U_B$. The desired decomposition of γ is obtained from an implicit function theorem [FP1, Prop. 5.2]. This theorem employs a solution of a $\bar{\partial}$ -equation with uniform estimates on a neighborhood of strictly pseudoconvex set $A \cup B$ for forms ω such that $\text{supp } \omega \subset A \cap B$, where $A \cup B$ is a subset of a Stein manifold (see Figure 2). Let us mention that this also works for Stein spaces. In the 1-convex case, the same can be done with uniform estimates on the Remmert reduction of a neighborhood of $A \cup B$ because the supports of the forms we deal with do not intersect S . In addition, we can require that the solutions of the $\bar{\partial}$ equation be zero on S without spoiling the uniform estimates (see [FP2] for details). We have $F \circ \alpha(x, 0) = G \circ \beta(x, 0)$ for $x \in U_C$. This defines a holomorphic map f_1 on $A \cup B'$ that is homotopic to f_0 on $A \cup B'$. If B' is thin enough then, according to Lemma 2.3, there exists a strictly pseudoconvex open neighbourhood \tilde{W} of $A \cup D$ that is completely contained in $A \cup B'$. In addition, \tilde{W} is diffeotopically equivalent to X_j . Outside \tilde{W} , the map f_1 can be glued to the map f_0 by a homotopy f_t to yield a continuous map f_1 that is holomorphic on a neighborhood of \tilde{W} , homotopic to f_0 , and approximating f_0 on A . Since $F(x, t) = F(x, 0)$ for all $x \in S$, we also have $f_t(x) = f_0(x)$ on S .

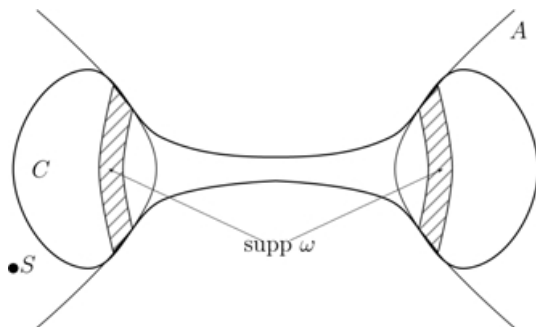


Figure 2 Support of the form ω

All that is needed now is to use a diffeotopy $H_t: X \rightarrow X$ with $H_0 = \text{Id}$, $H_1(X_j) = \bar{W}$, and $H_t = \text{Id}$ on X_{j-1} . Such a diffeotopy exists because X_j is a non-critical extension of \bar{W} . For every $t \in [0, 1]$, let J_t be the pullback of the almost complex structure J_{j-1} by the diffeomorphism H_t and let $f_{1,t}$ be pullbacks of f_1 (as described previously) by H_t . We set $J_j = J_1$ and $f_j = f_{j,t}$.

If there are only finitely many critical points for ρ , then we must make another diffeotopy at the end in order to bring our whole manifold into the last sublevel set. If there are infinitely many critical points, the Runge condition ensures that the limiting complex structure on X does equip X with a structure of 1-convex space with the singular set S . \square

4. The Case of Dimension 2

In the complex dimension 2, the situation is more complicated. The main difference is that Lemma 2.1 is no longer valid in the complex dimension 2 if the attaching disc is also real 2-dimensional. This was already noted by Eliashberg in [E] and was subsequently justified using Seiberg–Witten theory. For example, it is not possible to attach a totally real disc from the outside to the boundary of a unit ball in \mathbb{C}^2 along a Legendrian curve. The obstruction essentially comes from the adjunction inequality for Stein surfaces—which prohibits, for example, any non-null homologous sphere from having self-intersection number larger than -2 (see e.g. [Gol; N]).

If a 2-disc D is attached to the boundary of a strictly pseudoconvex domain in a complex surface X along a Legendrian curve L , then the Lai indices [Lai] $I_{\pm} = e_{\pm} - h_{\pm}$ giving the difference of signed elliptic and hyperbolic complex points on D are invariant under the isotopy of embeddings that keeps the boundary fixed. The indices are calculated from the first Chern class of X evaluated on D and the relative self-intersection number of D . As long as the I_{\pm} both equal zero, one can find an isotopy of the disc D through embeddings to a totally real disc by keeping the boundary fixed. This is a result of Eliashberg and Harlamov [EHa]. The indices I_{\pm} can always be increased by an isotopy of embeddings that also moves the boundary ∂D to a different Legendrian curve. However, it is not possible to arbitrarily decrease the indices I_{\pm} . Using Remark 2.2, we can remedy this by introducing self-intersections (kinks) on the disc D ; this means that, by a regular homotopy of immersions, we can make the disc D be an immersed totally real disc with its boundary circle an embedded Legendrian curve. The immersed disc can be made to have only special double points so that the disc has tubular Stein neighborhood basis.

To prove an analogue of Theorem 1.1 in complex dimension 2, we proceed as follows. As before, we decompose the manifold (X, J) as an increasing union of strictly pseudoconvex domains so as to get to the larger domain: one either adds a handle of index 1 or a handle of index 2. If the critical point we must pass in order to get to the next level has index 1, then there is no difference in the proof because Lemma 2.1 holds in this situation. Let us now explain the difference when

attaching an index-2 critical point. As noted before, the disc D attached to A is in general *not* isotopic to a smooth disc D' ; hence the union $C \cup D'$ has strictly pseudoconvex Stein neighborhoods, which was one of the essential ingredients in the proof in higher dimensions. The idea of how to fix this is Gompf's [Go2] and is explained in more detail (in a similar context) in [FS1]. First we find an isotopy of the disc D (a stable manifold of the critical point) so that it is Legendrian and normal at the boundary. If we can simultaneously also make the disc D totally real, then we can proceed just as in the higher-dimensional case. If not, we add a sufficient number of positive standard kinks to D to get an immersed disc D' such that $C \cup D'$ has thin tubular Stein strictly pseudoconvex neighborhoods. Adding a standard positive kink to D means deleting a small disc in D and, along the boundary circle, gluing back a small disc with exactly one positive transverse double point.

Although the relative homology class of D' is the same as that of D , tubular neighborhoods of D' are not diffeomorphic to the thickening of D because we have introduced extra generators in the π_1 group. To fix this, one adds a trivializing disc for each of the kinks (see Figure 3), whereby the thickening again becomes diffeomorphic to the thickening of D . Unfortunately, each of the trivializing discs also needs exactly one kink to have thin tubular Stein neighborhoods. We therefore repeat the procedure. The limiting procedure (that in this case is necessarily infinite) gives an (nonsmoothly) embedded disc D'' that agrees with D near the boundary and has thin Stein neighborhoods homeomorphic but not necessarily diffeomorphic to the tubular neighborhood of D . These limiting discs are called *Casson handles* and are an essential ingredient in the classification theory of topological 4-manifolds [Fr; FrQ]. Since for each critical point of index 2 we may have to perform infinitely many steps, we use a variant of Cantor's diagonal process: first we make just one step on the new critical point and then go back to make one more step on each of the previous critical points before continuing to the next critical point. The analogue of Theorem 1.1 is as follows.

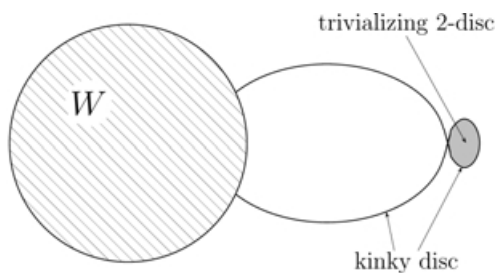


Figure 3 A kinky disc with a trivializing 2-disc

THEOREM 4.1 (Generalized Oka–Grauert principle for 1-convex surfaces). *Let (X, J) be a 1-convex surface and let S be its exceptional set. Let $K \subset X$ be a holomorphically convex compact subset of X containing S , let Y be a complex manifold, and let $f: X \rightarrow Y$ be a continuous mapping that is holomorphic*

in a neighborhood of K . Then there exist a 1-convex surface (X', J') , a holomorphic map $f': X' \rightarrow Y$, and an orientation-preserving homeomorphism $h: X \rightarrow X'$ such that

- (1) h is holomorphic in a neighborhood of K and $h(S)$ is the singular set for X' ,
- (2) $f' \circ h$ is homotopic to f and $f' \circ h|_S = f|_S$, and
- (3) $f' \circ h = f$ approximates f on K as well as desired.

The point of this theorem is that we may need to change the smooth structure on X . New results of Gompf (personal communication) indicate that the same theorem as for higher-dimensional manifolds may hold in this case also.

References

- [D] T. Duchamp, *The classification of Legendre immersions*, preprint, 1984.
- [E] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension > 2* , Internat. J. Math. 1 (1990), 29–46.
- [EHa] Y. Eliashberg and V. Harlamov, *Some remarks on the number of complex points of a real surface in a complex one*, Proc. LITC 82 (1982), 143–148.
- [F1] F. Forstnerič, *Holomorphic submersions from Stein manifolds*, Ann. Inst. Fourier (Grenoble) 54 (2004), 1913–1942.
- [F2] ———, *Runge approximation on convex sets implies the Oka property*, Ann. of Math. (2) 163 (2006), 689–707.
- [FK] F. Forstnerič and J. Kozak, *Strongly pseudoconvex handlebodies*, J. Korean Math. Soc. 40 (2003), 727–745.
- [FP1] F. Forstnerič and J. Prezelj, *Oka’s principle for holomorphic fiber bundles with sprays*, Math. Ann. 317 (2000), 117–154.
- [FP2] ———, *Extending holomorphic sections from complex subvarieties*, Math. Z. 236 (2001), 43–68.
- [FS1] F. Forstnerič and M. Slapar, *Stein structures and holomorphic mappings*, Math. Z. 256 (2007), 615–646.
- [FS2] ———, *Deformations of Stein structures and extensions of holomorphic mappings*, Math. Res. Lett. 14 (2007), 343–357.
- [Fr] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982), 357–453.
- [FrQ] M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton Math. Ser., 39, Princeton Univ. Press, Princeton, NJ, 1990.
- [Gol] R. E. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. (2) 148 (1998), 619–693.
- [Go2] ———, *Stein surfaces as open subsets of \mathbb{C}^2* , J. Symplectic Geom. 3 (2005), 565–587.
- [GR] H. Grauert and R. Remmert, *Theory of Stein spaces*, Grundlehren Math. Wiss., 236, Springer-Verlag, Berlin, 1979.
- [Gro] M. Gromov, *Partial differential relations*, Ergeb. Math. Grenzgeb. (3), 9, Springer-Verlag, Berlin, 1986.
- [HL] G. M. Henkin and J. Leiterer, *The Oka–Grauert principle without induction over the base dimension*, Math. Ann. 311 (1998), 71–93.
- [Lai] H. F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. Amer. Math. Soc. 172 (1972), 1–33.

- [LV] J. Leiterer and V. A. Vâjăitu, *A relative Oka–Grauert principle on 1-convex spaces*, J. Reine Angew. Math. 564 (2003), 85–104.
- [N] S. Nemirovski, *Complex analysis and differential topology of complex surfaces*, Uspekhi Mat. Nauk. 54 (4) (1999), 47–74; English translation in Russian Math. Surveys 54 (1999), 729–752.
- [P] J. Prezelj, *A relative Oka–Grauert principle for holomorphic submersions over 1-convex spaces*, Trans. Amer. Math. Soc. 362 (2010), 4213–4228.

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