Bull. Korean Math. Soc. **46** (2009), No. 3, pp. 409–434
DOI 10.4134/BKMS.2009.46.3.409

THE GENERALIZED RIEMANN PROBLEM FOR FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS I

Shouxin Chen, Decheng Huang, and Xiaosen Han

Abstract. In this paper, we consider a generalized Riemann problem of the first order hyperbolic conservation laws. For the case that excludes the centered wave, we prove that the generalized Riemann problem admits a unique piecewise smooth solution $u = u(t,x)$, and this solution has a structure similar to the similarity solution $u = U\left(\frac{x}{t}\right)$ of the corresponding Riemann problem in the neighborhood of the origin provided that the coefficients of the system and the initial conditions are sufficiently smooth.

1. Introduction

Consider the first order quasilinear hyperbolic systems of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \dots, u_n)^T$ is an unknown vector function of (t,x), $x \in \mathbb{R}, t > 0$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function of u. Assume that the system (1.1) is strictly hyperbolic on the domain under consideration, i.e., $A(u) = \nabla_u f(u)$ has n real distinct eigenvalues:

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ and $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$ be the left eigenvector and right eigenvector corresponding to the eigenvalue $\lambda_i(u)$, $i = 1, \dots, n$, respectively. Without loss of generality, we may assume that

$$l_i(u) \cdot r_j(u) = \delta_{ij}, \quad (i, j = 1, \dots, n), \tag{1.3}$$

where δ_{ij} is the Kronecker's symbol. Obviously, $\lambda_i(u)$, $l_i(u)$ and $r_i(u) (i = 1, \dots, n)$ have the same regularity as $A(u)$.

Received December 17, 2007; Revised November 11, 2008.
2000 *Mathematics Subject Classification.* 35L45, 35A07.
Key words and phrases. quasilinear hyperbolic systems, generalized Riemann problem, local solution.
This work is supported by natural science fund of Henan education office (No.2007110004).

We prescribe the following piecewise constant initial data:

$$t = 0: \quad u = \begin{cases} \widehat{u}_l, & x \le 0, \\ \widehat{u}_r, & x \ge 0, \end{cases} \tag{1.4}$$

where $\widehat{u}_l$ and $\widehat{u}_r$ are constant vectors satisfying:

$$\widehat{u}_l \neq \widehat{u}_r. \tag{1.5}$$

We first give the following hypothesis:

($\mathbf{H_1}$) The Riemann problem (1.1), (1.4) admits a similarity solution $u = U\left(\frac{x}{t}\right)$, which is composed of $n+1$ constant states $\widehat{u}_0 = \widehat{u}_l,\ \widehat{u}_1, \ldots, \widehat{u}_{n-1},\ \widehat{u}_n = \widehat{u}_r$ and n waves through the origin (containing shock wave, rarefaction wave or contact discontinuity), the states $\widehat{u}_{i-1}$ and $\widehat{u}_i$ are connected by the i-th wave $(i = 1, \ldots, n)$.

For a general quasilinear hyperbolic systems of conservation laws, under the assumption that every eigenvalue $\lambda_i(u)$ is either genuinely nonlinear in the sense of P. D. Lax:

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0, \tag{1.6}$$

or linearly degenerate in the sense of P. D. Lax:

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 0. \tag{1.7}$$

P. D. Lax [8] proved that the Riemann problem (1.1), (1.4) admits a unique similarity solution $u = U\left(\frac{x}{t}\right)$ provided $|\widehat{u}_r - \widehat{u}_l|$ is sufficiently small, which is composed of n small amplitude waves. In this paper, we only consider a similarity solution $u = U\left(\frac{x}{t}\right)$ given by (H_1), regardless of its uniqueness, also disregarding whether its n waves having small amplitude or not.

In this paper, we consider the system (1.1) with the following discontinuous initial data:

$$t = 0: \qquad u = \begin{cases} \widehat{u}_l(x), & x \le 0, \\ \widehat{u}_r(x), & x \ge 0, \end{cases} \tag{1.8}$$

where $\widehat{u}_l(x)$ and $\widehat{u}_r(x)$ are given smooth vector functions defined on $x \le 0$ and $x \ge 0$ satisfying

$$\widehat{u}_l(0) = \widehat{u}_l, \quad \widehat{u}_r(0) = \widehat{u}_r,$$

respectively. Since the generalized Riemann problem (1.1), (1.8) may be regarded as a perturbation of the corresponding Riemann problem (1.1), (1.4), we naturally study the following local problem:

In which condition, the generalized Riemann problem (1.1), (1.8) admits a unique piecewise smooth solution $u = u(t, x)$ which possesses a similar structure in a neighborhood of the origin as the solution of the corresponding Riemann problem (1.1), (1.4). Namely, the solution still contains n waves through the origin, for any i $(i = 1, \ldots, n)$, the type of the i-th wave is same as the i-th wave of the similarity solution $u = U(\frac{x}{t})$; the i-th wave coincides with the i-th wave of $u = U\left(\frac{x}{t}\right)$ at the origin. Moreover, the i-th wave links two known states $\widehat{u}_{i-1}$ and $\widehat{u}_i$.

Tikhonov and Samarsky [20] discussed the problem in the case of a single equation ($n = 1$). The earliest studies for the case of systems were as follows: one-dimensional isentropic flow systems ($n = 2$) was discussed in [2], Gu, Li and Hou [3, 4, 5, 6] discussed the general reducible systems ($n = 2$). Furthermore, in [1, 10, 11] one-dimensional gas dynamics systems ($n = 3$) was studied. All the above articles were devoted to investigation of arbitrary discontinuity $|\widehat{u}_r - \widehat{u}_l|$ of the initial data. For the general first order quasilinear hyperbolic systems of conservation laws, Li and Yu [12, 13, 14, 15, 16, 17] have shown that the problem admits a unique local solution when $|\widehat{u}_r - \widehat{u}_l|$ is sufficiently small for the corresponding similarity solution $u = U\left(\frac{x}{t}\right)$ with small amplitude, provided that all the eigenvalues are genuinely nonlinear or linearly degenerate in the sense of P. D. Lax. Li [9] thought the result was still valid for the case where the discontinuity $|\widehat{u}_r - \widehat{u}_l|$ is arbitrary and n waves are composed of shocks and contact discontinuities, while not giving the proof. In this paper, we shall give a complete proof for that case. For the case that includes centered waves, we deal with it in a forthcoming paper. For more related results, see the monographs [7, 19].

2. Main results

Suppose that we prescribe a similarity solution $u = U(\frac{x}{t})$ of the Riemann problem, which is composed of $n+1$ constant states $\widehat{u}_0 = \widehat{u}_l, \widehat{u}_1, \ldots, \widehat{u}_{n-1}, \widehat{u}_n = \widehat{u}_r$ and n waves (see Figure 1), in Figure 1,

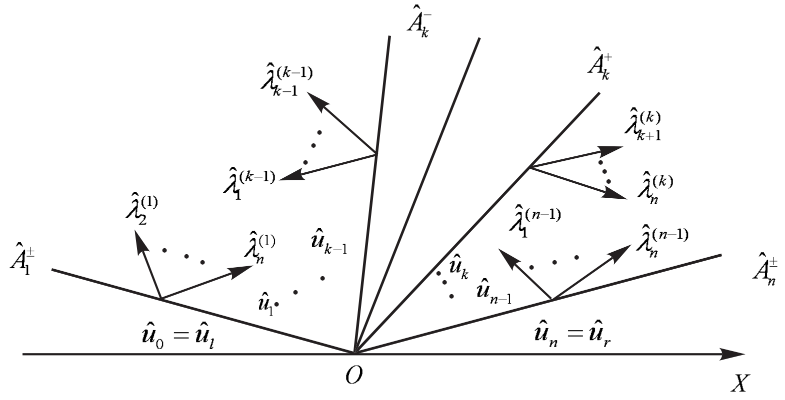

FIGURE 1. Similarity solution of Riemann problem

$$O\widehat{A}_k^{\pm} : x = \widehat{\sigma}_k^{\pm} t, \quad (k = 1, \ldots, n), \tag{2.1}$$

is the right (left) boundary of the k-th wave, $\widehat{u}_k$ is the constant state between $O\widehat{A}_k^+$ and $O\widehat{A}_k^-$; the eigenvalues $\widehat{\lambda}_1^{(k-1)},\ldots,\widehat{\lambda}_{k-1}^{(k-1)}$ and $\widehat{\lambda}_{k+1}^{(k)},\ldots,\widehat{\lambda}_n^{(k)}$ labeled on both sides of $O\widehat{A}_{k+1}^-$ are called "coming characteristics", where

$$\widehat{\lambda}_j^{(i)} = \lambda_j(\widehat{u}_i), \quad (i = 1,\ldots,n-1; j = 1,\ldots,n).$$

Our aim is to investigate in what condition, the generalized Riemann problem (1.1), (1.8) admits a unique piecewise smooth solution that possesses a similar structure (see Figure 2), namely, any wave through the origin

$$OA_k^\pm : x = x_k^\pm(t), (x_k^\pm(0) = 0) \quad (k = 1,\ldots,n)$$

has the same type (shock wave, contact discontinuity or centered wave) as

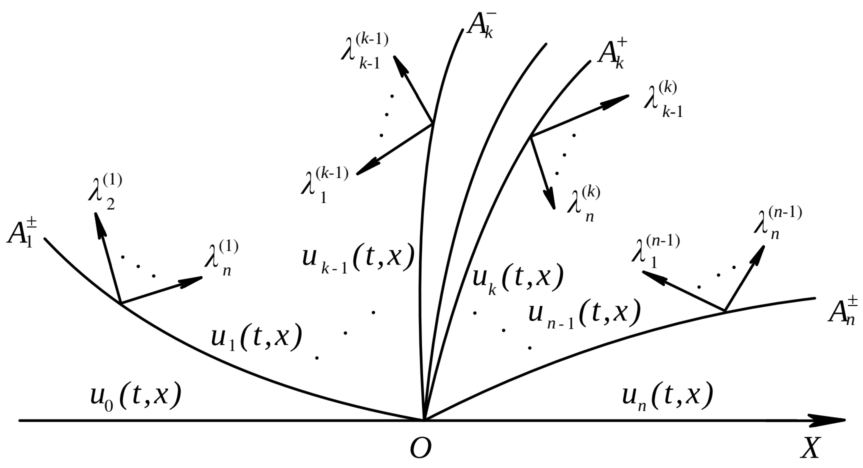

FIGURE 2. Solution of generalized Riemann problem

$O\widehat{A}_k^\pm$ in the solution of the Riemann problem (1.1), (1.4), and

$$x_k^\pm(0) = \widehat{\sigma}_k^\pm, \quad (k = 1,\ldots,n),$$

where $\widehat{\sigma}_k^\pm$ are given by (2.1). $u_0,\ldots,u_n$ satisfy the system (1.1) in the classical sense on their respective domains, and

$$u_k(0,0) = \widehat{u}_k, \quad (k = 0,\ldots,n). \tag{2.2}$$

For the case of the k-th ($1 \le k \le n$) wave being a shock wave or a contact discontinuity, we have

$$\begin{cases} \widehat{\sigma}_k^+ = \widehat{\sigma}_k^-, \\ O\widehat{A}_k^+ = O\widehat{A}_k^-, \end{cases}$$

denoting them $\widehat{\sigma}_k$ and $O\widehat{A}_k$, respectively. On $O\widehat{A}_k$ the following Rankine-Hugoniot condition:

$$(\widehat{u}_k - \widehat{u}_{k-1})\widehat{\sigma}_k = f(\widehat{u}_k) - f(\widehat{u}_{k-1})$$

must be satisfied and since it must satisfy the entropy condition if $O\widehat{A}_k$ is a shock wave, and be the k-th characteristic line if $O\widehat{A}_k$ is a contact discontinuity, combining (1.2) one yields

$$\begin{cases} \lambda_1(\widehat{u}_{k-1}) < \cdots < \lambda_{k-1}(\widehat{u}_{k-1}) < \widehat{\sigma}_k \leq \lambda_k(\widehat{u}_{k-1}), \\ \lambda_k(\widehat{u}_k) \leq \widehat{\sigma}_k < \lambda_{k+1}(\widehat{u}_k) < \cdots < \lambda_n(\widehat{u}_k), \end{cases} \tag{2.3}$$

where "=" corresponds to the contact discontinuity; "<" corresponds to the shock wave.

For the corresponding generalized Riemann problem, set

$$OA_k = OA_k^+ = OA_k^- : x = x_k(t),$$

then $x_k(t)$ satisfies

$$x_k'(0) = \widehat{\sigma}_k. \tag{2.4}$$

On both sides of OA_k $u_{k-1}(t,x)$ and $u_k(t,x)$ have to satisfy the Rankine-Hugoniot condition

$$(u_k(t,x) - u_{k-1}(t,x))\frac{dx_k(t,x)}{dt} = f(u_k(t,x)) - f(u_{k-1}(t,x)) \text{ on } x = x_k(t), \tag{2.5}$$

and by (2.2), (2.3), noting the continuity and the property of contact discontinuity, at least in a neighborhood of the origin it follows that

$$\begin{cases} \lambda_1(u_{k-1}(t,x)) < \cdots < \lambda_{k-1}(u_{k-1}(t,x)) < x_k'(t) \leq \lambda_k(u_{k-1}(t,x)), \\ \lambda_k(u_k(t,x)) \leq x_k'(t) < \lambda_{k+1}(u_k(t,x)) < \cdots < \lambda_n(u_k(t,x)), \end{cases} \tag{2.6}$$

where "=" corresponds to the contact discontinuity; "<" corresponds to the shock wave. By (2.6) we can label the "coming character" $\lambda_i^{(k-1)} (i = 1, \ldots, k-1)$ and $\lambda_i^{(k)} (i = k+1, \ldots, n)$ on both sides of OA_k, where

$$\begin{cases} \lambda_i^{(k-1)} = \lambda_i(u_{k-1}(t,x)), \quad (i = 1, \ldots, k-1), \\ \lambda_i^{(k)} = \lambda_i(u_k(t,x)), \quad (i = k+1, \ldots, n). \end{cases}$$

Let

$$\begin{cases} u_{k-1} = \sum_{i=1}^n v_i^{k-1} r_i(\widehat{u}_{k-1}), \\ u_k = \sum_{i=1}^n v_i^k r_i(\widehat{u}_k), \quad (i = 1, \ldots, n). \end{cases}$$

Then it follows from (1.3) that

$$\begin{cases} v_i^{k-1} = l_i(\widehat{u}_{k-1})u_{k-1}, \\ v_i^k = l_i(\widehat{u}_k)u_k, \qquad (i = 1, \dots, n). \end{cases} \tag{2.7}$$

We present the following hypothesis:

(**H2**) The Rankine-Hugoniot condition (2.5) can equivalently be written as the explicit form of those variables v corresponding to "coming characteristics". Precisely speaking, the Rankine-Hugoniot condition on OA_k can be written as

$$\frac{dx_k(t,x)}{dt} = F_k(u_{k-1}, u_k), \;\; x_k(0) = 0, \tag{2.8}$$

$$\begin{cases} v_i^{k-1} = g_i^{k-1}(v_k^{k-1}, \dots, v_n^{k-1}, v_1^k, \dots, v_k^k), \quad (i = 1, \dots, k-1), \\ v_j^k = g_j^k(v_k^{k-1}, \dots, v_n^{k-1}, v_1^k, \dots, v_k^k), \quad (j = k+1, \dots, n). \end{cases} \tag{2.9}$$

Remark 2.1. To verify the hypothesis (H2), we only need to use the implicit function theorem.

If OA_k is a shock wave, it is easy to prove the hypothesis (H2) is fulfilled provided that

$$\det(r_1(\widehat{u}_{k-1}), \dots, r_{k-1}(\widehat{u}_{k-1}), \widehat{u}_k - \widehat{u}_{k-1}, r_{k+1}(\widehat{u}_k), \dots, r_n(\widehat{u}_k)) \neq 0$$

If OA_k is a contact discontinuity, assume $\lambda_k(u)$ is linearly degenerate in the sense of P. D. Lax, then the Rankine-Hugoniot condition on OA_k can equivalently be written as

$$\omega_i(u_k) = \omega_i(u_{k-1}), \;\; (i = 1, \dots, k-1, k+1, \dots, n),$$
$$\frac{dx_k(t)}{dt} = \lambda_k(u_{k-1}) (= \lambda_k(u_k)),$$

where $\omega_i(u)$ are $n-1$ independent Riemann invariants corresponding to $\lambda_k(u)$, defined as follows:

$$\nabla\omega_i(u) \cdot r_k(u) = 0.$$

Obviously, if

$$\det\begin{pmatrix} \nabla\omega_i(\widehat{u}_{k-1}) \cdot r_j(\widehat{u}_{k-1}) & \nabla\omega_i(\widehat{u}_k) \cdot r_j(\widehat{u}_k) \\ (j = 1, \dots, k-1) & (j = k+1, \dots, n) \end{pmatrix} \neq 0$$

where $i = 1, \dots, k-1, k+1, \dots, n$, then (H2) is fulfilled.

Remark 2.2. (2.6) implies that $u_0(t,x)$ and $u_n(t,x)$ can be respectively obtained by solving the Cauchy problem with initial data $\bar{u}_l(x)$ and $\bar{u}_r(x)$, hence, if OA_k is a shock wave or a contact discontinuity, then the Rankine-Hugoniot condition can be written as

$$\frac{dx_1(t)}{dt} = F_1(t, x, u_1), \;\; x(0) = 0, \tag{2.10}$$

$$v_i^1 = g_i^1(t, x, v_1^1), \;\; (i = 2, \dots, n). \tag{2.11}$$

Likewise, if OA_n is a shock wave or a contact discontinuity, then the Rankine-Hugoniot condition can be written as

$$\frac{dx_n(t)}{dt} = F_n(t, x, u_{n-1}), \quad x(0) = 0, \tag{2.12}$$

$$v_j^{n-1} = g_j^{n-1}(t, x, v_n^{n-1}), \quad (j = 1, \ldots, n-1). \tag{2.13}$$

In what follows we write two groups of $n(n-1) \times n(n-1)$ matrices $\Theta_j (j = 1, 2, \ldots)$ and $\bar{\Theta}_j (j = 0, 1, \ldots)$, and then obtain the main results.

Let

$$\begin{cases} \tau_i^k = \frac{\widehat{\lambda}_i^k - \widehat{\sigma}_k^+}{\widehat{\lambda}_i^k - \widehat{\sigma}_{k+1}^-}, \ (i = 1, \ldots, k), \\ \tau_i^k = \frac{\widehat{\lambda}_i^k - \widehat{\sigma}_{k+1}^-}{\widehat{\lambda}_i^k - \widehat{\sigma}_k^+}, \ (i = k+1, \ldots, n), \end{cases} \qquad (k = 1, \ldots, n-1), \tag{2.14}$$

where $\widehat{\lambda}_i^k = \lambda_i(\widehat{u}_k)$, $\widehat{\sigma}_i^{\pm}$ are given by (2.1). Obviously,

$$0 \le \tau_i^k < 1 \quad (i = 1, \ldots, n; k = 1, \ldots, n-1).$$

For $k (1 \le k \le n)$ corresponding to the shock wave or the contact discontinuity, let

$$\begin{cases} (\Theta_j)_{n(k-2)+p, n(k-2)+q} = (\bar{\Theta}_j)_{n(k-2)+p, n(k-2)+q} \\ \qquad = \dfrac{\partial g_p^{k-1}}{\partial v_q^{k-1}} (\tau_q^{k-1})^j, \quad (q = k, \ldots, n), \\ (\Theta_j)_{n(k-2)+p, n(k-1)+q} = (\bar{\Theta}_j)_{n(k-2)+p, n(k-1)+q} \\ \qquad = \dfrac{\partial g_p^{k-1}}{\partial v_q^{k}} (\tau_q^{k})^j, \quad (q = 1, \ldots, k), \\ (\Theta_j)_{n(k-2)+p, q} = (\bar{\Theta}_j)_{n(k-2)+p, q} = 0, \\ (q < n(k-2) + k \text{ or } q > n(k-1) + k), (p = 1, \ldots, k-1), \end{cases} \tag{2.15}$$

$$\begin{cases} (\Theta_j)_{n(k-1)+p, n(k-2)+q} = (\bar{\Theta}_j)_{n(k-1)+p, n(k-2)+q} \\ \qquad = \dfrac{\partial g_p^{k}}{\partial v_q^{k-1}} (\tau_q^{k-1})^j, \quad (q = k, \ldots, n), \\ (\Theta_j)_{n(k-1)+p, n(k-1)+q} = (\bar{\Theta}_j)_{n(k-1)+p, n(k-1)+q} \\ \qquad = \dfrac{\partial g_p^{k}}{\partial v_q^{k}} (\tau_q^{k})^j, \quad (q = 1, \ldots, k), \\ (\Theta_j)_{n(k-1)+p, q} = (\bar{\Theta}_j)_{n(k-1)+p, q} = 0, \\ (q < n(k-2) + k \text{ or } q > n(k-1) + k), (p = k+1, \ldots, n), \end{cases} \tag{2.16}$$

where the functions on the right side of (2.15) and (2.16) take values on $t = 0, x = 0$, $v^i = \widehat{v}^i (i = 1, \ldots, n-1)$, (2.11), (2.13) imply that $(\Theta_j)_{pq} (j = 1, 2, \ldots)$ and $(\bar{\Theta}_j)_{pq} (j = 0, 1, \ldots)$ do not have elements not vanishing until $1 \le p \le$

$n(n-1), 1 \le q \le n(n-1)$, thus we define two groups of $n(n-1) \times n(n-1)$ matrices $\Theta_j (j = 1, 2, \ldots)$ and $\bar{\Theta}_j (j = 0, 1, \ldots)$ depending only on the solution of the Riemann problem.

Let the $n(n-1) \times n(n-1)$ diagonal matrix τ be

$$\tau = \text{diag}\{\tau_1^1, \ldots, \tau_n^1, \ldots, \tau_1^{n-1}, \ldots, \tau_n^{n-1}\}. \tag{2.17}$$

For $N \times N$ matrix $A = (a_{ij})$ define the following minimal characterizing number:

$$\|A\|_{\min} = \inf_{\gamma} \|\gamma A \gamma^{-1}\|,$$

where $\gamma = \text{diag}\{\gamma_1, \ldots, \gamma_N\}, \gamma_i \neq 0 \ (i = 1, \ldots, N)$, and

$$\|A\| = \max_{i=1,\ldots,N} \sum_{j=1}^{N} |a_{ij}|.$$

We get the following main theorems:

Theorem 2.1. *Under the hypotheses* (H1),(H2)*, if* $f(u), \hat{u}_l(x), \hat{u}_r(x)$ *are* C^{m+1} *functions, then if*

$$\det|I - \Theta_j| \neq 0 \quad (j = 1, \ldots, n-1),$$

$$\|\bar{\Theta}_m\|_{\min} < 1, \tag{2.18}$$

the generalized Riemann problem (1.1)*,* (1.8) *admits a unique piecewise* C^{m+1} *local solution* $u = u(t, x)$ *except the origin, which possesses a similar structure at least in a neighborhood of the origin with the given similarity solution of the Riemann problem* (1.1)*,* (1.4)*.*

Remark 2.3. As long as one introduces the reversible transformation $\bar{v} = \gamma v$ of the unknown function, where

$$\gamma = \text{diag}\{\gamma_1, \ldots, \gamma_{n(n-1)}\}, \gamma_i \neq 0 \quad (i = 1, \ldots, n(n-1)),$$

$$v = (v_1^1, \ldots, v_n^1, \ldots, v_1^{n-1}, \ldots, v_n^{n-1})^T,$$

then $\bar{\Theta}_j$ is reduced to $\gamma \bar{\Theta}_j \gamma^{-1}$, hence in the proof of Theorem 2.1 we can substitute

$$\|\bar{\Theta}_m\| < 1$$

for (2.18).

Theorem 2.2. *Under hypotheses* (H1), (H2)*, if* $f(u), \hat{u}_l(x), \hat{u}_r(x)$ *are* C^∞ *functions, then*

$$\det|I - \Theta_j| \neq 0, (j = 1, 2, \ldots)$$

if and only if the generalized Riemann problem (1.1)*,* (1.8) *admits a unique piecewise* C^∞ *local solution* $u = u(t, x)$ *except the origin, which possesses a similar structure at least in a neighborhood of the origin with* $u = U\left(\frac{x}{t}\right)$ *.*

Remark 2.4. Theorems 2.1, 2.2 remain valid for more general hyperbolic systems of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(t,x,u)}{\partial x} = g(t,x,u),$$

and the system of corresponding Riemann problem is

$$\frac{\partial u}{\partial t} + \nabla_u f(0,0,u)\frac{\partial u}{\partial x} = 0.$$

3. Proof of main results

We consider the generalized Riemann problem of the following form:

$$\frac{\partial u}{\partial t} + \frac{\partial f(t,x,u)}{\partial x} = g(t,x,u), \tag{3.1}$$

$$t = 0: \quad u = \begin{cases} \hat{u}_l(x), & x \le 0, \\ \hat{u}_r(x), & x \ge 0, \end{cases} \tag{3.2}$$

where f is C^{m+2} with respect to x and u, C^{m+1} with respect to t, and $g, \hat{u}_l, \hat{u}_r$ are C^{m+1} functions of all arguments. Suppose a similarity solution $u = U\left(\frac{x}{t}\right)$ of its corresponding Riemann problem

$$\frac{\partial u}{\partial t} + \nabla_u f(0,0,u)\frac{\partial u}{\partial x} = 0, \tag{3.3}$$

$$t = 0: \quad u = \begin{cases} \widehat{u}_l = \hat{u}_l(0), & x \le 0 \\ \widehat{u}_r = \hat{u}_r(0), & x \ge 0 \end{cases} \tag{3.4}$$

is composed of $n+1$ piecewise constant states $\widehat{u}_0 = \widehat{u}_l,\ \widehat{u}_1, \ldots, \widehat{u}_{n-1},\ \widehat{u}_n = \widehat{u}_r$ and n shocks or contact discontinuities. We shall prove the generalized Riemann problem (3.1), (3.2) admits a unique piecewise C^{m+1} solution which has a similar structure.

Assume the matrix $\zeta(t,x,u)$ is composed of n left eigenvectors $l_1, l_2, \ldots, l_n$ of $\nabla_u f(t,x,u)$, and its every element is a piecewise C^{m+1} function. Moreover, in $A_k O A_{k+1}$ $(k = 1, \ldots, n-1)$ we can always take

$$\zeta_{ij}(0,0,\widehat{u}_k) = \delta_{ij}, \quad (i,j = 1,\ldots,n). \tag{3.5}$$

Multiplying (3.1) by ζ from the left, we obtain the characteristic form

$$\zeta(t,x,u)\frac{\partial u}{\partial t} + \lambda(t,x,u)\frac{\partial u}{\partial x} = \mu(t,x,u), \tag{3.6}$$

where $\zeta, \lambda, \mu \in C^{m+1}$,

$$\lambda(t,x,u) = \mathrm{diag}\{\lambda_1(t,x,u), \ldots, \lambda_n(t,x,u)\},$$

$$\lambda_1(t,x,u) < \lambda_2(t,x,u) < \cdots < \lambda_n(t,x,u)$$

on the domain under consideration. Then (2.3) implies that $u_0(t,x)$ and $u_n(t,x)$ can be respectively obtained by solving the Cauchy problem (3.1) with initial data $\widehat{u}_l(x)$ and $\widehat{u}_r(x)$ in a neighborhood of the origin, set

$$OA_k : x = x_k(t), \qquad (k = 1, \dots, n).$$

To get the solution of the generalized Riemann problem (3.1), (3.2), we only have to solve the free boundary problem on the fan-shaped domain

$$\bigcup_{k=1}^{n-1} D_k(\delta) = \{(t,x) \,|\, 0 \le t \le \delta, x_k(t) \le x \le x_{k+1}(t)\},$$

whose solutions $u_k(t,x)$ satisfy equation (3.1) on $D_k(\delta)$, and

$$u_k(0,0) = \widehat{u}_k, \quad (k = 1, \dots, n-1).$$

Furthermore, free boundaries OA_k $(k = 1, \dots, n)$ satisfy (2.8), (2.10) and (2.12), and u_{k-1}, u_k satisfy the Rankine-Hugoniot conditions (2.9), (2.11) and (2.13) on both sides of OA_k. Noting (2.7) and (3.5), we now have

$$u_k(t,x) = v^k(t,x), \quad (k = 1, \dots, n-1).$$

Let

$$T_k(t) = \frac{x_{k+1}(t) - x_k(t)}{t}, \quad (0 \le t \le \delta), \quad (k = 1, \dots, n-1). \tag{3.7}$$

We introduce the following transformation

$$\begin{cases} \bar{t} = t, \\ \bar{x} = \dfrac{x - x_k(t)}{T_k(t)} \end{cases} \text{on } D_k(\delta), \quad (k = 1, \dots, n-1 \text{ and } k \text{ is odd }),$$

$$\begin{cases} \bar{t} = t, \\ \bar{x} = \dfrac{x - x_{k+1}(t)}{T_k(t)} \end{cases} \text{on } D_k(\delta), \quad (k = 1, \dots, n-1 \text{ and } k \text{ is even}).$$

Thus all $D_k(\delta)$ $(k = 1, \dots, n-1)$ are changed to the domain

$$\bar{D}(\delta) = \{(\bar{t}, \bar{x}) \,|\, 0 \le \bar{t} \le \delta, 0 \le \bar{x} \le \bar{t}\}.$$

Moreover, $OA_k (k = 1, \dots, n)$ are respectively mapped onto $\bar{x} = 0$ and $\bar{x} = \bar{t}$ for odd k and even k. Set

$$\bar{u}^k(\bar{t}, \bar{x}) = u^k(\bar{t}, x_k(\bar{t}, \bar{x})), \qquad (k = 1, \dots, n-1),$$

where

$$x_k(\bar{t}, \bar{x}) = \begin{cases} x_k(\bar{t}) + \bar{x} T_k(\bar{t}) & \text{for odd } k, \\ x_{k+1}(\bar{t}) - \bar{x} T_k(\bar{t}) & \text{for even } k. \end{cases} \tag{3.8}$$

Then $\bar{u}^k (k = 1, \dots, n-1)$ satisfy

$$\sum_{i=1}^{n} \bar{\zeta}_{li}^k(\bar{t}, \bar{x}|\bar{u}^k) \left(\frac{\partial \bar{u}_i^k}{\partial \bar{t}} + \lambda_l^k(\bar{t}, \bar{x}|\bar{u}) \frac{\partial \bar{u}_i^k}{\partial \bar{x}} \right) = \bar{\mu}_l^k(\bar{t}, \bar{x}|\bar{u}^k), \quad (l = 1, \dots, n), \tag{3.9}$$

(3.10) $\bar{u}_r^k = G_r^k(\bar{t}\ |\bar{u}_{k+1}^k, \dots, \bar{u}_n^k, \bar{u}_1^{k+1}, \dots, \bar{u}_{k+1}^{k+1})$ on $\bar{x} = \bar{t}$, $(r = 1, \dots, k)$,

(3.11) $\bar{u}_s^k = G_s^k(\bar{t}\ |\bar{u}_k^{k-1}, \dots, \bar{u}_n^{k-1}, \bar{u}_1^k, \dots, \bar{u}_k^k)$ on $\bar{x} = 0$, $(s = k+1, \dots, n)$

for odd k, and

(3.12) $\bar{u}_r^k = G_r^k(\bar{t}\ |\bar{u}_{k+1}^k, \dots, \bar{u}_n^k, \bar{u}_1^{k+1}, \dots, \bar{u}_{k+1}^{k+1})$ on $\bar{x} = 0$, $(r = 1, \dots, k)$,

(3.13) $\bar{u}_s^k = G_s^k(\bar{t}\ |\bar{u}_k^{k-1}, \dots, \bar{u}_n^{k-1}, \bar{u}_1^k, \dots, \bar{u}_k^k)$ on $\bar{x} = \bar{t}$, $(s = k+1, \dots, n)$

for even k, where

(3.14) $\bar{\zeta}_{li}^k(\bar{t}, \bar{x}|\bar{u}^k) = \zeta_{li}(\bar{t}, x_k(\bar{t}, \bar{x}), \bar{u}^k)$, $(l, i = 1, \dots, n; k = 1, \dots, n-1)$,

(3.15)
$$\lambda_l^k(\bar{t}, \bar{x}\ |\bar{u}) = \left((-1)^{k+1}\lambda_l(\bar{t}, x_k(\bar{t}, \bar{x}), \bar{u}^k) - \frac{\partial x_k(\bar{t}, \bar{x})}{\partial \bar{t}}\right) \Big/ T_k(\bar{t}),$$
$$(l = 1, \dots, n; k = 1, \dots, n-1),$$

(3.16) $\bar{\mu}_l^k(\bar{t}, \bar{x}\ |\bar{u}^k) = \mu_l(\bar{t}, x_k(\bar{t}, \bar{x}), \bar{u}^k)$, $(l = 1, \dots, n; k = 1, \dots, n-1)$,

and $\bar{u} = (\bar{u}^1, \dots, \bar{u}^{n-1})$. As k is odd, we have

(3.17)
$$G_r^k(\bar{t}|\bar{u}_{k+1}^k, \dots, \bar{u}_n^k, \bar{u}_1^{k+1}, \dots, \bar{u}_{k+1}^{k+1})$$
$$= g_r^k(\bar{t}, x_{k+1}(\bar{t}), \bar{u}_{k+1}^k(\bar{t}, \bar{t}), \dots, \bar{u}_n^k(\bar{t}, \bar{t}), \bar{u}_1^{k+1}(\bar{t}, \bar{t}), \dots, \bar{u}_{k+1}^{k+1}(\bar{t}, \bar{t})), (r = 1, \dots, k),$$

(3.18)
$$G_s^k(\bar{t}|\bar{u}_k^{k-1}, \dots, \bar{u}_n^{k-1}, \bar{u}_1^k, \dots, \bar{u}_k^k)$$
$$= g_s^k(\bar{t}, x_k(\bar{t}), \bar{u}_k^{k-1}(\bar{t}, 0), \dots, \bar{u}_n^{k-1}(\bar{t}, 0), \bar{u}_1^k(\bar{t}, 0), \dots, \bar{u}_k^k(\bar{t}, 0)), (s = k+1, \dots, n),$$

in addition

(3.19)
$$\begin{cases} \dfrac{dx_{k+1}(\bar{t})}{d\bar{t}} = F_{k+1}(\bar{t}, x_{k+1}(\bar{t}), \bar{u}_{k+1}^k(\bar{t}, \bar{t}), \dots, \bar{u}_n^k(\bar{t}, \bar{t}), \bar{u}_1^{k+1}(\bar{t}, \bar{t}), \dots, \bar{u}_{k+1}^{k+1}(\bar{t}, \bar{t})), \\ x_{k+1}(0) = 0, \end{cases}$$

(3.20)
$$\begin{cases} \dfrac{dx_k(\bar{t})}{d\bar{t}} = F_k(\bar{t}, x_k(\bar{t}), \bar{u}_k^{k-1}(\bar{t}, 0), \dots, \bar{u}_n^{k-1}(\bar{t}, 0), \bar{u}_1^k(\bar{t}, 0), \dots, \bar{u}_k^k(\bar{t}, 0)), \\ x_k(0) = 0, \end{cases}$$

where g_i^k $(i = 1, \dots, n)$, F_k, F_{k+1} are given by (2.8)-(2.9) and (2.10)-(2.13). Likewise for even k, we can also obtain similar boundary conditions.

Thus, we acquire a functional boundary value problem in terms of $\bar{u}^k$ $(k = 1, \dots, n-1)$ on the angular domain $\bar{D}(\delta)$, which is equivalent to the original problem. We next use the method similar to that used in [18] to extend the systems (3.9)-(3.20).

If $\bar{u}(\bar{t},\bar{x}) \in C^{m+1}$, define operators

$$A = \frac{\partial}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}},\ B = \frac{\partial}{\partial \bar{t}},$$

and set

$$\left\{\begin{array}{l} u^{k,1}(\bar{t},\bar{x}) = \{A\bar{u}_1^k, \ldots, A\bar{u}_k^k, B\bar{u}_{k+1}^k, \ldots, B\bar{u}_n^k\}, \\ v^{k,1}(\bar{t},\bar{x}) = \{B\bar{u}_1^k, \ldots, B\bar{u}_k^k, A\bar{u}_{k+1}^k, \ldots, A\bar{u}_n^k\} \end{array}\right. \tag{3.21}$$

for odd k. Substituting (3.21) into (3.9), we obtain

$$\sum_{r=1}^{k} \bar{\zeta}_{lr}^k(\lambda_l^k u_r^{k,1} + (1-\lambda_l^k) v_r^{k,1}) + \sum_{s=k+1}^{n} \bar{\zeta}_{ls}^k(\lambda_l^k v_s^{k,1} + (1-\lambda_l^k) u_s^{k,1}) = \bar{\mu}_l^k, (l = 1, \ldots, n),$$

from which it yields

$$v_l^{k,1} = \sum_{i=1}^{n} a_{li}^{k,1}(\bar{t}, x_k(\bar{t},\bar{x}), \bar{u}(\bar{t},\bar{x}))u_i^{k,1} + b_l^{k,1}(\bar{t}, x_k(\bar{t},\bar{x}), \bar{u}(\bar{t},\bar{x})), \quad (l = 1, \ldots, n).$$

By (3.5), (3.14), (3.15) we get

$$\bar{\zeta}_{li}^k(0,0|\widehat{u}_k) = \zeta_{li}(0,0,\widehat{u}_k) = \delta_{li}, \quad (l,i = 1, \ldots, n), \tag{3.22}$$

$$\lambda_l^k(0,0|\widehat{u}) = \frac{\lambda_l(0,0,\widehat{u}_k) - F_k(0,0,\widehat{u})}{F_{k+1}(0,0,\widehat{u}) - F_k(0,0,\widehat{u})}, \quad (l = 1, \ldots, n),$$

where $\widehat{u} = \{\widehat{u}_1, \ldots, \widehat{u}_n\}$. Noting (2.4), we have

$$\left\{\begin{array}{l} F_k(0,0,\widehat{u}) = \widehat{\sigma}_k, \\ F_{k+1}(0,0,\widehat{u}) = \widehat{\sigma}_{k+1}. \end{array}\right.$$

By (2.14) we easily calculate

$$a_{li}^{k,1}(0,0,\bar{u}(0,0)) = \tau_l^k \delta_{li}, \quad (l,i = 1, \ldots, n),$$
$$b_l^{k,1}(0,0,\bar{u}(0,0)) = \gamma_l^{k,1}, \quad (l = 1, \ldots, n),$$

where

$$\left\{\begin{array}{l} \gamma_r^{k,1} = \left(\dfrac{F_{k+1} - F_k}{F_{k+1} - \lambda_r}\mu_r\right)(0,0,\hat{u}), \quad (r = 1, \ldots, k), \\ \gamma_s^{k,1} = \left(\dfrac{F_{k+1} - F_k}{\lambda_s - F_k}\mu_s\right)(0,0,\hat{u}), \quad (s = k+1, \ldots, n). \end{array}\right.$$

Consequently, at the origin we have

$$v_l^{k,1} = \tau_l^k u_l^{k,1} + \gamma_l^{k,1}, \quad (l = 1, \ldots, n).$$

Differentiating the system (3.9) with respect to $\bar{t}$ and combining (3.14)-(3.16) yields

$$\begin{aligned} &\sum_{i=1}^{n} \zeta_{li}^{k,1}(\bar{t}, x_k(\bar{t},\bar{x}), u(\bar{t},\bar{x})) \left(\frac{\partial u_i^{k,1}}{\partial \bar{t}} + \lambda_l^k \frac{\partial u_i^{k,1}}{\partial \bar{x}}\right) \\ = \ &\mu_l^{k,1}(\bar{t}, x_k(\bar{t},\bar{x}), u(\bar{t},\bar{x}), u^{k,1}(\bar{t},\bar{x})) \quad (l = 1, \ldots, n). \end{aligned}$$

When ζ, λ, μ in the system (3.6) are C^{m+1} functions, obviously $\zeta^{k,1}, \mu^{k,1}$ are C^m functions, where

$$\begin{cases} \zeta^{k,1}_{rp} = \zeta^k_{rp} + \sum_{q=k+1}^{n} \zeta^k_{rq} a^{k,1}_{qp}, \quad \zeta^{k,1}_{rs} = \sum_{q=k+1}^{n} \zeta^k_{rq} a^{k,1}_{qs}, \\ \zeta^{k,1}_{sr} = \sum_{p=1}^{k} \zeta^k_{sp} a^{k,1}_{pr}, \ \zeta^{k,1}_{sq} = \sum_{p=1}^{k} \zeta^k_{sp} a^{k,1}_{pq} + \zeta^k_{sq}, \ (r,p=1,\dots,k; s,q=k+1,\dots,n). \end{cases}$$

By (3.22) it follows

$$\zeta^{k,1}_{li}(0,0,\widehat{u}) = \delta_{li}, \quad (l,i=1,\dots,n).$$

Repeating the process above m times, we obtain a system in terms of $u^{k,j}(j=0,\dots,m)$, where $u^{k,0} = \bar{u}^k$. On $\bar{D}(\delta)$, $u^{k,j}$ satisfy

(3.23)
$$\sum_{i=1}^{n} \zeta^{k,j}_{li}(\bar{t}, x_k(\bar{t},\bar{x}), u(\bar{t},\bar{x})) \left(\frac{\partial u^{k,j}_i}{\partial \bar{t}} + \lambda^k_l \frac{\partial u^{k,j}_i}{\partial \bar{x}} \right)$$
$$= \mu^{k,j}_l(\bar{t}, x_k(\bar{t},\bar{x}), u^{p,q}(\bar{t},\bar{x})), \quad (p=1,\dots,n-1; q=0,\dots,j), \quad (l=1,\dots,n),$$

where $\zeta^{k,j}, \mu^{k,j}$ $(j=0,\dots,m)$ are at least C^1 functions, and satisfy

$$\zeta^{k,j}_{li}(0,0,\widehat{u}) = \delta_{li}, \quad (l,i=1,\dots,n). \tag{3.24}$$

Likewise, for even k, in (3.21) replacing $u^{k,j}$ by $v^{k,j}$, we can derive similar systems, and (3.24) remains valid.

Next, we shall consider the boundary conditions. As $k=1,\dots,n$ and k is even, $OA_k : \{(t,x) \,|\, 0 \le t \le \delta, x = x_k(t)\}$ is transformed into $\{(\bar{t},\bar{x}) \,|\, 0 \le \bar{t} \le \delta, \bar{x} = \bar{t}\}$, on which we have the boundary condition (3.20) and

$$\begin{aligned} \bar{u}^{k-1}_r &= G^{k-1}_r(\bar{t} \,|\bar{u}^{k-1}_r, \dots, \bar{u}^{k-1}_n, \bar{u}^k_1, \dots, \bar{u}^k_k) \\ &= g^{k-1}_r(\bar{t}, x_k(\bar{t}), \bar{u}^{k-1}_k(\bar{t},\bar{t}), \dots, \bar{u}^{k-1}_n(\bar{t},\bar{t}), \bar{u}^k_1(\bar{t},\bar{t}), \dots, \bar{u}^k_k(\bar{t},\bar{t})), \\ &\qquad (r=1,\dots,k-1), \end{aligned} \tag{3.25}$$

$$\begin{aligned} \bar{u}^k_s &= G^k_s(\bar{t}|\bar{u}^{k-1}_k, \dots, \bar{u}^{k-1}_n, \bar{u}^k_k, \dots, \bar{u}^k_k) \\ &= g^k_s(\bar{t}, x_k(\bar{t}), \bar{u}^{k-1}_k(\bar{t},\bar{t}), \dots, \bar{u}^{k-1}_n(\bar{t},\bar{t}), \bar{u}^k_1(\bar{t},\bar{t}), \dots, \bar{u}^k_k(\bar{t},\bar{t})), \\ &\qquad (r=k+1,\dots,n). \end{aligned} \tag{3.26}$$

Differentiating both sides of (3.25) with respect to $\bar{t}$ yields

$$\begin{aligned} u_r^{k-1,1} &= \sum_{q=k}^{n} \frac{\partial g_r^{k-1}}{\partial \bar{u}_q^{k-1}} v_q^{k-1,1} + \sum_{p=1}^{k} \frac{\partial g_r^{k-1}}{\partial \bar{u}_p^{k}} v_p^{k,1} + \frac{\partial g_r^{k-1}}{\partial t} + \frac{\partial g_r^{k-1}}{\partial x} F_k \\ &= \sum_{q=k}^{n} \frac{\partial g_r^{k-1}}{\partial \bar{u}_q^{k-1}} \left(\sum_{i=1}^{n} a_{qi}^{k-1,1} u_i^{k-1,1} + b_q^{k-1,1} \right) \\ &\quad + \sum_{p=1}^{k} \frac{\partial g_r^{k-1}}{\partial \bar{u}_p^{k}} \left(\sum_{i=1}^{n} a_{pi}^{k,1} u_i^{k,1} + b_p^{k,1} \right) + \frac{\partial g_r^{k-1}}{\partial t} + \frac{\partial g_r^{k-1}}{\partial x} F_k, \\ &\quad (r = 1, \ldots, k-1). \end{aligned}$$

Repeating m times we get that for $j = 1, \ldots, m$

(3.27)

$$\begin{aligned} u_r^{k-1,j} &= \sum_{q=k}^{n} \frac{\partial g_r^{k-1}}{\partial \bar{u}_q^{k-1}} \left(\sum_{i_j=1}^{n} \left(\sum_{i_1,\ldots,i_{j-1}=1}^{n} a_{qi_1}^{k-1,1}, a_{i_1 i_2}^{k-1,2}, \ldots, a_{i_{j-1} i_j}^{k-1,j} \right) u_{i_j}^{k-1,j} \right) \\ &\quad + \sum_{p=1}^{k} \frac{\partial g_r^{k-1}}{\partial \bar{u}_p^{k}} \left(\sum_{i_j=1}^{n} \left(\sum_{i_1,\ldots,i_{j-1}=1}^{n} a_{pi_1}^{k,1}, a_{i_1 i_2}^{k,2}, \ldots, a_{i_{j-1} i_j}^{k,j} \right) u_{i_j}^{k,j} \right) + F_r^{k-1,j} \\ &\triangleq \sum_{q=k}^{n} \frac{\partial g_r^{k-1}}{\partial \bar{u}_q^{k-1}} \left(\sum_{i=1}^{n} \bar{a}_{qi}^{k-1,j} u_i^{k-1,j} \right) + \sum_{p=1}^{k} \frac{\partial g_r^{k-1}}{\partial \bar{u}_p^{k}} \left(\sum_{i=1}^{n} \bar{a}_{pi}^{k,j} u_i^{k,j} \right) \\ &\quad + F_r^{k-1,j}, \quad (r = 1, \ldots, k-1), \end{aligned}$$

here $a^{k-1,j}$ and $a^{k,j}$ are functions of $(t, x, \bar{u})$, $F_r^{k-1,j}$ are functions of $(t, x, u^{p,q})$ $(p = 1, \ldots, n-1;\ q = 0, \ldots, j-1)$, which are at least C^1, and

(3.28) $$a_{li}^{k-1,j}(0, 0, \widehat{u}) = \tau_l^{k-1} \delta_{li}, \quad (l, i = 1, \ldots, n; j = 1, \ldots, m).$$

Therefore we obtain

(3.29) $$\bar{a}_{li}^{k-1,j}(0, 0, \widehat{u}) = (\tau_l^{k-1})^j \delta_{li}, \quad (l, i = 1, \ldots, n; j = 1, \ldots, m),$$

and $a^{k,j}$ also have expressions similar to (3.28). Likewise, for (3.26) and odd k, similar results can be obtained, and (3.28), (3.29) hold.

Lemma 3.1. *In the absence of the centered wave, by equations* (3.9)-(3.20) *the derivatives of the solution* $\bar{u}(\bar{t}, \bar{x})$ *of orders* $\leq m-1$ *at the origin can be determined uniquely if and only if*

$$\det |I - \Theta_j| \neq 0 \ (j = 1, \ldots, m-1),$$

where matrices Θ_j *are defined by* (2.15), (2.16).

Proof. Letting $(\bar{t},\bar{x})=(0,0)$ in (3.27) and noting (3.29), it follows

$$\begin{aligned} u_r^{k-1,j} &= \sum_{q=k}^{n} \frac{\partial g_r^{k-1}}{\partial \bar{u}_q^{k-1}} (\tau_q^{k-1})^j u_q^{k-1,j}(0,0) \\ &\quad + \sum_{p=1}^{k} \frac{\partial g_r^{k-1}}{\partial \bar{u}_p^{k}} (\tau_p^{k})^j u_p^{k,j}(0,0) + F_r^{k-1,j}(0,0). \end{aligned}$$

In view of (2.11) and (2.13) we get an $n(n-1)(m-1)$ system in terms of $u_i^{k,j}(0,0)$ $(k=1,\dots,n-1; j=1,\dots,m-1; i=1,\dots,n)$, whose Jacobi matrix is of the following form

$$\begin{pmatrix} I-\Theta_1 & & & \\ & I-\Theta_2 & & 0 \\ & & \ddots & \\ & * & & I-\Theta_{m-1} \end{pmatrix}.$$

Hence

$$\prod_{j=1}^{m-1} \det |I-\Theta_j| \neq 0$$

if and only if the system has a unique solution, the proof of Lemma 3.1 is complete. □

By Lemma 3.1, we can give the following boundary conditions for the derivatives of $\bar{u}$ of orders $< m$. As $k=1,\dots,n-1$,

$$\text{(3.30)} \quad \begin{cases} u_r^{k,j} = u_r^{k,j}(0,0) + \int_0^{\bar{t}} u_r^{k,j+1}(\bar{t},\bar{t})d\bar{t} \quad \text{on } \bar{x}=\bar{t}, \\ \qquad (r=1,\dots,k; j=0,\dots,m-1) \\ u_s^{k,j} = u_s^{k,j}(0,0) + \int_0^{\bar{t}} u_s^{k,j+1}(\bar{t},0)d\bar{t} \quad \text{on } \bar{x}=0, \\ \qquad (s=k+1,\dots,n; j=0,\dots,m-1) \end{cases}$$

for odd k, and

$$\text{(3.31)} \quad \begin{cases} u_r^{k,j} = u_r^{k,j}(0,0) + \int_0^{\bar{t}} u_r^{k,j+1}(\bar{t},0)d\bar{t} \quad \text{on } \bar{x}=0, \\ \qquad (r=1,\dots,k; j=0,\dots,m-1) \\ u_s^{k,j} = u_s^{k,j}(0,0) + \int_0^{\bar{t}} u_s^{k,j+1}(\bar{t},\bar{t})d\bar{t} \quad \text{on } \bar{x}=\bar{t}, \\ \qquad (s=k+1,\dots,n; j=0,\dots,m-1) \end{cases}$$

for even k. For the m-th order derivatives of $\bar{u}$, letting $j = m$ in (3.27), it follows

(3.32)
$$u_1^{1,m} = \sum_{q=2}^{n} \frac{\partial g_1^1}{\partial \bar{u}_q^1}\Big(\sum_{i=1}^{n} \bar{a}_{qi}^{1,m} u_i^{1,m}\Big) + \sum_{p=1}^{2} \frac{\partial g_1^1}{\partial \bar{u}_p^2}\Big(\sum_{i=1}^{n} \bar{a}_{pi}^{2,m} u_i^{2,m}\Big) + F_1^{1,m} \quad \text{on} \quad \bar{x} = \bar{t},$$

$$u_s^{1,m} = \frac{\partial g_s^1}{\partial \bar{u}_1^1}\Big(\sum_{i=1}^{n} \bar{a}_{1i}^{1,m} u_i^{1,m}\Big) + F_s^{1,m} \qquad \text{on} \quad \bar{x} = 0, \quad (s = 2, \ldots, n). \tag{3.33}$$

As $k = 2, \ldots, n-2$, we have

(3.34)
$$u_r^{k,m} = \sum_{q=k+1}^{n} \frac{\partial g_r^k}{\partial \bar{u}_q^k}\Big(\sum_{i=1}^{n} \bar{a}_{qi}^{k,m} u_i^{k,m}\Big) + \sum_{p=1}^{k+1} \frac{\partial g_r^k}{\partial \bar{u}_p^{k+1}}\Big(\sum_{i=1}^{n} \bar{a}_{pi}^{k+1,m} u_i^{k+1,m}\Big) + F_r^{k,m}$$
$$\text{on} \quad \bar{x} = \bar{t}, \quad (r = 1, \ldots, k),$$

(3.35)
$$u_s^{k,m} = \sum_{p=1}^{k} \frac{\partial g_s^k}{\partial \bar{u}_p^k}\Big(\sum_{i=1}^{n} \bar{a}_{pi}^{k,m} u_i^{k,m}\Big) + \sum_{q=k}^{n} \frac{\partial g_s^k}{\partial \bar{u}_q^{k-1}}\Big(\sum_{i=1}^{n} \bar{a}_{qi}^{k-1,m} u_i^{k-1,m}\Big) + F_s^{k,m}$$
$$\text{on} \quad \bar{x} = 0, (s = k+1, \ldots, n)$$

for odd k.

For even k, we only need to take values of (3.34) on $\bar{x} = 0$, and to take values of (3.35) on $\bar{x} = \bar{t}$. As n is even, we have

$$u_r^{n-1,m} = \frac{\partial g_r^{n-1}}{\partial \bar{u}_n^{n-1}}\Big(\sum_{i=1}^{n} \bar{a}_{ni}^{n-1,m} u_i^{n-1,m}\Big) + F_r^{n-1,m}$$
$$\text{on} \quad \bar{x} = \bar{t}, (r = 1, \ldots, n-1), \tag{3.36}$$

$$\begin{aligned} u_n^{n-1,m} = {} & \sum_{p=1}^{n-1} \frac{\partial g_n^{n-1}}{\partial \bar{u}_p^{n-1}}\Big(\sum_{i=1}^{n} \bar{a}_{pi}^{n-1,m} u_i^{n-1,m}\Big) \\ & + \sum_{q=n-1}^{n} \frac{\partial g_n^{n-1}}{\partial \bar{u}_q^{n-2}}\Big(\sum_{i=1}^{n} \bar{a}_{qi}^{n-2,m} u_i^{n-2,m}\Big) \\ & + F_n^{n-1,m} \qquad \text{on} \quad \bar{x} = 0. \end{aligned} \tag{3.37}$$

Likewise, for odd n, we can obtain the result for odd n by taking values of (3.36) on $\bar{x} = 0$, and taking values of (3.37) on $\bar{x} = \bar{t}$.

Thus, we obtain an $n(n-1)(m+1)$ system (3.23) of the functional form on $\bar{D}(\delta)$ in terms of $u_i^{k,j}$ ($k = 1, \ldots, n-1; i = 1, \ldots, n; j = 0, \ldots, m$) and boundary conditions (3.30)-(3.37) and (3.20). Using Theorem 6.1 of Chapter 2 in [18] yields the following lemma.

Lemma 3.2. *The generalized Riemann problem* (3.1), (3.2) *admits a unique piecewise C^{m+1} solution if and only if the functional boundary value problem,* (3.23)*,* (3.20)*,* (3.30)*-*(3.37)*, admits a unique C^1 solution on $\bar{D}(\delta)$.*

In what follows we shall prove Theorem 2.1, that is to prove if

$$\|\Theta_m\| = \|\bar{\Theta}_m\| < 1,$$

then the problem (3.23), (3.20), (3.30)-(3.37) admits a unique C^1 solution on the angular domain $\bar{D}(\delta)$. To this end, we need to use Theorem 6.1 of Chapter 2 in [18](see the Appendix)

Proof of Theorem 2.1. According to Lemma 3.2, we know that finding the piecewise C^{m+1} solution of the generalized Riemann problem (3.1), (3.2) is equivalent to finding C^1 solution of the functional boundary value problem, (3.23), (3.20), (3.30)-(3.37) on the angular domain $\bar{D}(\delta)$. We first check the conditions (i)-(xi) of Theorem 6.1 in Chapter 2 [18].

Here $u = u_i^{k,j}$ $(k = 1, \dots, n-1;\ j = 0, \dots, m;\ i = 1, \dots, n)$, $\alpha = 0,\ \beta = 0,\ N = n(n-1)(m+1);\ \zeta_{li},\ \lambda_l,\ \mu_l\ (l, i = 1, \dots, N)$ are given by (3.23), $G_l(l = 1, \dots, N)$ are given by (3.30)-(3.37), $u^{k,0}(0,0)(k = 1, \dots, n-1)$ are defined by the solution of the Riemann problem (3.3), (3.4), and $u^{k,j}(0,0)$ $(k = 1, \dots, n-1;\ j = 1, \dots, m)$ are obtained by means of Lemma 3.1. Moreover, in this case, from (2.15) and (2.16) it easily follows

$$\Theta_m = \frac{\partial(g_1^1, \dots, g_n^1, \dots, g_1^{n-1}, \dots, g_n^{n-1})}{\partial(\bar{u}_1^1, \dots, \bar{u}_n^1, \dots, \bar{u}_1^{n-1}, \dots, \bar{u}_n^{n-1})}\Big|_{\bar{t}=\bar{x}=0} \cdot \tau^m,$$

where τ is defined by (2.17). Noting (3.22) and (3.24), we have

$$\zeta_{li}^0 = \delta_{li}, \quad (l, i = 1, \dots, N).$$

We first verify conditions (i)-(vii) for system (3.23).

By the expressions of $\zeta_{li},\ \lambda_l$ and μ_l $(l, i = 1, \dots, N)$, we know they are C^1 functions, hence (i) is trivial.

For (ii), since $v \in \sum(\delta|\Omega_1)$, obviously we have

$$\|v(\bar{t}, \bar{x}) - v(0,0)\| \le \varepsilon(\delta, \Omega_1). \tag{3.38}$$

Applying (3.7), (3.8), (3.19), (3.20) and the mean value theorem it follows that in A_kOA_{k+1} (taking odd k for an example, for even k the result is similar).

$$\begin{aligned} x_k(\bar{t}, \bar{x}) &= \frac{\bar{x}}{\bar{t}} x_{k+1}(\bar{t}) + \left(1 - \frac{\bar{x}}{\bar{t}}\right) x_k(\bar{t}) \\ &= \frac{\bar{x}}{\bar{t}} (\bar{t} F_{k+1}(\widetilde{t}, x_{k+1}(\widetilde{t}), v(\widetilde{t}, x_{k+1}(\widetilde{t})))) \\ &\quad + \left(1 - \frac{\bar{x}}{\bar{t}}\right) (\bar{t} F_k(\widetilde{\widetilde{t}}, x_k(\widetilde{\widetilde{t}}), v(\widetilde{\widetilde{t}}, x_k(\widetilde{\widetilde{t}})))), \quad (0 \le \widetilde{t}, \widetilde{\widetilde{t}} \le \bar{t}). \end{aligned} \tag{3.39}$$

Since F_k and F_{k+1} are at least C^1 functions, in view of (3.38), we conclude

$$(3.40)\begin{cases} |F_{k+1}(\widetilde{t}, x_{k+1}(\widetilde{t}), v(\widetilde{t}, x_{k+1}(\widetilde{t})))| \le |F_{k+1}(0,0,v(0,0))| + \varepsilon(\delta,\Omega_1), \\ |F_k(\widetilde{\widetilde{t}}, x_k(\widetilde{\widetilde{t}}), v(\widetilde{\widetilde{t}}, x_k(\widetilde{\widetilde{t}})))| \le |F_k(0,0,v(0,0))| + \varepsilon(\delta,\Omega_1). \end{cases}$$

Substituting (3.40) into (3.39), one yields

$$|x_k(\bar{t},\bar{x})| \le \varepsilon(\delta,\Omega_1). \tag{3.41}$$

As a result, since $\mu \in C^1$,

$$\begin{aligned} & |\mu(\bar{t}, x_k(\bar{t},\bar{x}), v(\bar{t},\bar{x}))| \\ \le\ & |\mu(0,0,v(0,0))| + \Big|\bar{t}\frac{\partial \mu}{\partial t}(\eta_1\bar{t},0,v(0,0)) + x_k(\bar{t},\bar{x})\frac{\partial \mu}{\partial x}(0,\eta_2 x_k(\bar{t},\bar{x}),v(0,0)) \\ & + \sum_{i=1}^{N}(v_i(\bar{t},\bar{x}) - v_i(0,0))\frac{\partial \mu}{\partial v_i}(0,0,\eta_3 v(\bar{t},\bar{x}) + (1-\eta_3)v(0,0))\Big| \\ \le\ & |\mu(0,0,v(0,0))| + \varepsilon(\delta,\Omega_1), \end{aligned}$$

where $0 \le \eta_1, \eta_2, \eta_3 \le 1$. Therefore the verification of (ii) is complete.

For (iii), because the functions in $\Gamma[v]$ are continuous, and the continuous function in a closed interval can assume the maximum, hence there exists a constant K_1 depending only on Ω_1 such that

$$\Big\| \Gamma[v] \Big\| \le K_1.$$

For (iv) and (v), by means of checking (ii) it follows that

$$\omega(\eta|v) \le \widetilde{\omega}_0(\eta),$$

$$\omega(\eta|x) \le \widetilde{\omega}_0(\eta), \tag{3.42}$$

where $\widetilde{\omega}_0(\eta)$ is a function depending only on Ω_1, and $\widetilde{\omega}_0(\eta) \to 0$ as $\eta \to 0$. By the expressions of $\zeta_{li},\ \lambda_l,\ \mu_l\ (l,i = 1,\dots,N)$ we know (iv) and (v) hold.

For (vi), $k = 1,\dots,N$, (3.19), (3.20) imply that there exist constants K_2, K_3 such that (in (3.43), take even k for an example, for odd k the result is similar)

$$\begin{aligned} & |x_k(\bar{t}|v') - x_k(\bar{t}|v'')| \\ = & \left| \int_0^{\bar{t}} [F_k(\bar{t}, x_k(\bar{t}|v'), v'(\bar{t},\bar{t})) - F_k(\bar{t}, x_k(\bar{t}|v''), v''(\bar{t},\bar{t}))] d\bar{t} \right| \\ \le & K_2\|v' - v''\| + K_3 \int_0^{\bar{t}} \big|x_k(\bar{t}|v') - x_k(\bar{t}|v'')\big| d\bar{t}. \end{aligned} \tag{3.43}$$

By (3.43) and Gronwall's inequality, it yields that there exists a constant K_4 depending only on δ and Ω_1 such that

$$|x_k(\bar{t}|v') - x_k(\bar{t}|v'')| \le K_4\|v' - v''\|. \tag{3.44}$$

Hence for $k = 1, \ldots, n-1$, there exists a constant K_5 depending only on δ and Ω_1 such that

$$\begin{aligned}
&\big|x_k(\bar t, \bar x|v') - x_k(\bar t, \bar x|v'')\big| \\
&= \Big|\frac{\bar x}{\bar t}\int_0^{\bar t}\Big(F_{k+1}(\bar t, x_{k+1}(\bar t|v'), v') - F_{k+1}(\bar t, x_{k+1}(\bar t|v''), v'')\Big)d\bar t \\
&\quad + \Big(1 - \frac{\bar x}{\bar t}\Big)\int_0^{\bar t}\Big(F_k(\bar t, x_k(\bar t|v'), v') - F_k(\bar t, x_k(\bar t|v''), v'')\Big)d\bar t\Big| \\
&\le K_5\|v' - v''\|
\end{aligned} \tag{3.45}$$

for odd k. Similarly for even k (3.45) holds. Therefore by the expressions of $\zeta_{li},\ \lambda_l,\ \mu_l\ (l, i = 1, \ldots, N)$ we can get (vi).

By the expressions (3.15) of λ we easily obtain (vii) also holds.

So far we have proved the system (3.23) satisfies the conditions (i)-(vii). In the sequel, we shall show the boundary conditions (3.30)-(3.37) satisfy conditions (viii)-(xi). Taking (3.34) for an example, others can be tackled similarly.

(viii) is still trivial.

For (ix), let

$$G_r^{k,m}(\bar t) = u_r^{k,m}(\bar t, \bar t|v(\bar t, \bar x)),$$

differentiating (3.61) with respect to $\bar t$ yields

$$\begin{aligned}
(G_r^{k,m}(\bar t))' \;=\; &\Bigg\{\sum_{q=k+1}^{n}\frac{\partial g_r^k}{\partial \bar u_q^k}\Bigg(\sum_{i=1}^{n}\bar a_{qi}^{k,m}\Bigg(\frac{\partial v_i^{k,m}}{\partial \bar t} + \frac{\partial v_i^{k,m}}{\partial \bar x}\Bigg)\Bigg) \\
&+ \sum_{p=1}^{k+1}\frac{\partial g_r^k}{\partial \bar u_p^{k+1}}\Bigg(\sum_{i=1}^{n}\bar a_{pi}^{k+1,m}\Bigg(\frac{\partial v_i^{k+1,m}}{\partial \bar t} + \frac{\partial v_i^{k+1,m}}{\partial \bar x}\Bigg)\Bigg) \\
&+ \sum_{j=0}^{m-1}\widetilde F_r^{k,m,j}\Bigg(\frac{\partial v^{k,j}}{\partial \bar t} + \frac{\partial v^{k,j}}{\partial \bar x}\Bigg) \\
&+ \sum_{j=0}^{m-1}\widetilde{\widetilde F}_r^{k,m,j}\Bigg(\frac{\partial v^{k+1,j}}{\partial \bar t} + \frac{\partial v^{k+1,j}}{\partial \bar x}\Bigg) \\
&+ \frac{\partial F_r^{k,m}}{\partial t} + \frac{\partial F_r^{k,m}}{\partial x}F_{k+1}\Bigg\}\Bigg|_{\bar x = \bar t},
\end{aligned} \tag{3.46}$$

where $\frac{\partial g_r^k}{\partial \bar u_q^k}$, $\frac{\partial g_r^k}{\partial \bar u_p^{k+1}}$ are C^m functions, $\bar a_{qi}^{k,m}$, $\bar a_{pi}^{k+1,m}$ given by (3.27) are C^1 functions, $\widetilde F_r^{k,m,j}$, $\widetilde{\widetilde F}_r^{k,m,j}$, $\frac{\partial F_r^{k,m}}{\partial t}$, $\frac{\partial F_r^{k,m}}{\partial x}$ are continuous functions of $(\bar t, x_{k+1}(\bar t), v^{p,q})$ $(p = 1, \ldots, n-1; q = 0, \ldots, m-1)$, and F_{k+1} given by (3.19) are C^{m+1} functions.

Since $v(\bar{t},\bar{x}) \in \Sigma(\delta|\Omega_1)$, obviously it holds

$$\|v(\bar{t},\bar{x}) - v(0,0)\| \le \Omega_1\delta.$$

Noticing (3.41), we obtain

$$\begin{cases} \left|\frac{\partial g_r^k}{\partial \bar{u}_q^k} - \frac{\partial g_r^k}{\partial \bar{u}_q^k}|_{\bar{t}=\bar{x}=0}\right| \le \varepsilon(\delta,\Omega_1), \qquad (q = k+1,\dots,n), \\ \left|\frac{\partial g_r^k}{\partial \bar{u}_p^{k+1}} - \frac{\partial g_r^k}{\partial \bar{u}_p^{k+1}}|_{\bar{t}=\bar{x}=0}\right| \le \varepsilon(\delta,\Omega_1), \quad (p = 1,\dots,k+1), \end{cases}$$

$$\begin{cases} \left|\bar{a}_{qi}^{k,m} - \bar{a}_{qi}^{k,m}|_{\bar{t}=\bar{x}=0}\right| \le \varepsilon(\delta,\Omega_1), \qquad (q = k+1,\dots,n; i = 1,\dots,n), \\ \left|\bar{a}_{pi}^{k+1,m} - \bar{a}_{pi}^{k+1,m}|_{\bar{t}=\bar{x}=0}\right| \le \varepsilon(\delta,\Omega_1), \quad (p = 1,\dots,k+1; i = 1,\dots,n), \end{cases}$$

$$\begin{cases} \left|\widetilde{F}_r^{k,m,j}\right| \le \left|\widetilde{F}_r^{k,m,j}|_{\bar{t}=\bar{x}=0}\right| + \varepsilon(\delta,\Omega_1), \\ \left|\widetilde{\widetilde{F}}_r^{k,m,j}\right| \le \left|\widetilde{\widetilde{F}}_r^{k,m,j}|_{\bar{t}=\bar{x}=0}\right| + \varepsilon(\delta,\Omega_1), \end{cases} \quad (j = 0,\dots,m-1),$$

$$\left|\frac{\partial F_r^{k,m}}{\partial t} + \frac{\partial F_r^{k,m}}{\partial x}F_{k+1}\right| \le \left|\left(\frac{\partial F_r^{k,m}}{\partial t} + \frac{\partial F_r^{k,m}}{\partial x}F_{k+1}\right)_{\bar{t}=\bar{x}=0}\right| + \varepsilon(\delta,\Omega_1).$$

Thus, letting

$$\overline{M}_r = \max_{j=0,\dots,m-1}\left(\left|\widetilde{F}_r^{k,m,j}|_{\bar{t}=\bar{x}=0}\right|, \left|\widetilde{\widetilde{F}}_r^{k,m,j}|_{\bar{t}=\bar{x}=0}\right|\right),$$

$$R_2 = \left|\left(\frac{\partial F_r^{k,m}}{\partial t} + \frac{\partial F_r^{k,m}}{\partial x}F_{k+1}\right)|_{\bar{t}=\bar{x}=0}\right|,$$

and noting (3.29), we obtain

$$\begin{aligned} &\|G_r^{k,m}(\bar{t})'\| \\ \le\ &\sum_{q=k+1}^{n}\sum_{i=1}^{n}\left(\frac{\partial g_r^k}{\partial \bar{u}_q^k}\Big|_{\bar{t}=\bar{x}=0}\cdot(\tau_q^k)^m\delta_{qi} + \varepsilon(\delta,\Omega_1)\right)\left\|\frac{\partial v_i^{k,m}}{\partial \bar{t}} + \frac{\partial v_i^{k,m}}{\partial \bar{x}}\right\| \\ &+ \sum_{p=1}^{k+1}\sum_{i=1}^{n}\left(\frac{\partial g_r^k}{\partial \bar{u}_p^{k+1}}\Big|_{\bar{t}=\bar{x}=0}\cdot(\tau_p^{k+1})^m\delta_{pi} + \varepsilon(\delta,\Omega_1)\right)\left\|\frac{\partial v_i^{k+1,m}}{\partial \bar{t}} + \frac{\partial v_i^{k+1,m}}{\partial \bar{x}}\right\| \\ &+ \sum_{j=0}^{m-1}(\overline{M}_r + \varepsilon(\delta,\Omega_1))\left(\left\|\frac{\partial v^{k,j}}{\partial \bar{t}} + \frac{\partial v^{k,j}}{\partial \bar{x}}\right\| + \left\|\frac{\partial v^{k+1,j}}{\partial \bar{t}} + \frac{\partial v^{k+1,j}}{\partial \bar{x}}\right\|\right) \\ &+ R_2 + \varepsilon(\delta,\Omega_1). \end{aligned}$$

As for condition (x), when $v \in \Sigma(\delta,\Omega_1)$, it holds

$$\Omega(\eta|v) \le \Omega_1\eta.$$

For continuous functions f, g, we have

$$\Omega(\eta|f\cdot g)\le \|f\|\Omega(\eta|g)+\|g\|\Omega(\eta|f),$$
$$\Omega(\eta|f(g))\le \omega(\Omega(\eta|g)|f).$$

Recalling (3.42), we can directly obtain (x) from (3.46).

As for condition (xi), since $F_r^{k,m}$ in (3.34) are C^1 functions, in view of (3.44), we can get (xi).

Since (3.30) and (3.31) are of integral form, it is easy to verify

$$\begin{cases}\|G_r^{k,j}(\bar t)'\|\le \varepsilon(\delta,\Omega_1)\Big\|\dfrac{\partial v}{\partial \bar t}+\dfrac{\partial v}{\partial \bar x}\Big\|+R_2+\varepsilon(\delta,\Omega_1),\\ \omega(\eta|G_r^{k,j}(\bar t)')\le \varepsilon(\delta,\Omega_1)\Omega\left(\eta\Big|\left(\dfrac{\partial v}{\partial \bar t}+\dfrac{\partial v}{\partial \bar x}\right)\right)+\omega_2(\eta),\quad (j=0,\dots,m-1)\\ \|G_r^{k,j}(\bar t|v')-G_r^{k,j}(\bar t|v'')\|\le \varepsilon(\delta,\Omega_1)\|v'-v''\|.\end{cases}$$

Thus, we obtain the characterizing matrix of the functional boundary value problem as follows

$$A=\begin{pmatrix} & 0 & \\ *\dots\dots* & & \bar\Theta_m\end{pmatrix}.$$

It is easy to see

$$\|A\|_{\min}=\|\bar\Theta_m\|_{\min},$$

this completes the proof of Theorem 2.1. □

To prove Theorem 2.2, we need the following regularity lemma.

Lemma 3.3. *Assume that the functional boundary value problem* (3.9)-(3.20) *admits a unique C^{m+1} solution $\bar u(\bar t,\bar x)$ on $\bar D(\delta_0)$, and $\|\bar\Theta_m\|<1$. If the coefficients of* (3.6) *and initial conditions* (3.2) *are C^{M+m+1} functions* ($M\ge 0$), *and* ($M+m+1$)*-th order derivatives of $\mu(t,x,u)$ are Lipschitz continuous with respect to u, then there exists a positive constant $\delta^*\le\delta_0$ independent of M, such that $\bar u$ is a C^{M+m+1} solution of* (3.9)-(3.20) *on $\bar D(\delta^*)$.*

In [21], the authors showed the following regularity lemma of typical boundary value problem.

Lemma 3.4. *Suppose that the typical boundary value problem*

$$\begin{cases}\displaystyle\sum_{i=1}^N \zeta_{li}(t,x,u)\left(\frac{\partial u_i}{\partial t}+\lambda_l(t,x,u)\frac{\partial u_i}{\partial x}\right)=\mu_l(t,x,u),\quad l=1,\dots,N\\ x=t:\ u_r=G_r(t,u_{k+1},\dots,u_N),\quad r=1,\dots,K\\ x=0:\ u_s=G_s(t,u_1,\dots,u_k),\quad s=K+1,\dots,N\end{cases}\tag{3.47}$$

on the angular domain $\bar D(\delta_0)$ admits a unique C^1solution, whose corresponding $\|\tilde{\bar\Theta}_1\|<1$. If $\zeta,\ \lambda,\ \mu,\ G_r,\ G_s$ are $C^{M+1}(M\ge 0)$ functions, and ($M+1$)*-th order derivatives of μ are Lipschitz continuous with respect to u, then there exists a positive constant $\delta^*\le\delta_0$ independent of M, such that u is a C^{M+1} solution of* (3.47) *on $\bar D(\delta^*)$.*

Proof of Lemma 3.3. If $\bar{u} = \bar{u}(\bar{t}, \bar{x})$ is a C^{m+1} solution of (3.9)-(3.20) on $\bar{D}(\delta_0)$, then by Lemma 3.2, $u_i^{k,j}(k = 1, \ldots, n-1; j = 0, \ldots, m; i = 1, \ldots, n)$ is a C^1 solution of the functional boundary value problem (3.23), (3.20), (3.30)-(3.37). Regarding $x_k(\bar{t})$ obtained $(k = 1, \ldots, n)$ as known functions, then we easily know $u_i^{k,j}$ is a C^1 solution of typical boundary value problem (3.47), $\zeta,\ \lambda,\ \mu,\ G_r,\ G_s$ are at least C^2 functions, second order derivatives of μ with respect to u are Lipschitz continuous, and $\bar{\bar{\Theta}}_1 = \bar{\Theta}_m$. By Lemma 3.4, there exists a positive constant $\delta^* \le \delta_0$ such that $u_i^{k,j}$ is a C^2 solution of (3.47) on $\bar{D}(\delta^*)$. Then it follows from (3.20) that $x_k(\bar{t})(k = 1, \ldots, n)$ are at least C^3 functions, so $\zeta,\ \lambda\ ,\mu,\ G_r,\ G_s$ are at least C^3 functions. Repeated application of Lemma 3.4 implies that $u_i^{k,j}$ is a C^{m+1} of (3.47) on δ^*. In view of Lemma 3.2, we obtain u is a C^{M+m+1} solution of (3.9)-(3.20) on $\bar{D}(\delta^*)$. The proof of Lemma 3.3 is complete. □

Proof of Theorem 2.2. Since any element of $\bar{\Theta}_j$ tends to 0 as $j \to +\infty$, there exist a positive integer $m \ge m_0$ and a positive constant δ_0 such that the functional boundary value problem (3.9)-(3.20) admits a unique C^{m+1} solution $\bar{u} = \bar{u}(\bar{t}, \bar{x})$ on $\bar{D}(\delta_0)$. Owing to Lemma 3.3, we obtain that $\bar{u}$ is a C^∞ solution of (3.9)-(3.20) on $\bar{D}(\delta^*)$. By the equivalence of the generalized Riemann problem and the functional boundary value problem (3.9)-(3.20), one yields that $u = u(t, x)$ is a C^∞ solution of the generalized Riemann problem in a neighborhood of the origin. □

4. Appendix

Let

$$R(\delta) = \{(t, x) | 0 \le t \le \delta, \quad \beta t \le x \le \alpha t\}, \quad (\alpha > \beta)$$

be an angular domain. Consider on this domain the following boundary value problem in functional form:

$$\sum_{j=1}^{n} \zeta_{lj}(t, x, |u)\Big(\frac{\partial u_j}{\partial t} + \lambda_l(t, x|u)\frac{\partial u_j}{\partial x}\Big) = \mu_l(t, x|u), \quad (l = 1, \ldots, n), \tag{4.1}$$

$$\sum_{j=1}^{n} \zeta_{rj}^0 = G_r(t, u) \quad \text{on} \quad x = \alpha t, \quad (r = 1, \ldots, m), \tag{4.2}$$

$$\sum_{j=1}^{n} \zeta_{sj}^0 = G_s(t, u) \quad \text{on} \quad x = \beta t, \quad (s = 1, \ldots, n), \tag{4.3}$$

where the coefficients $\zeta_{lj}, \lambda_l, \mu_l$ and the boundary conditions $G_l(l, j = 1, \ldots, n)$ are assumed to be functionals of the unknown function $u = u(t, x)$, and

$$\zeta_{lj}^0 \triangleq \zeta_{lj}(0, 0|0) = \zeta_{lj}(t, x|v)|_{v \equiv 0, t=x=0}.$$

Let

$$\Sigma(\delta) = \big\{v(t, x) \,|\, v \in C_1[R(\delta)], v(0, 0) = 0\big\}$$

and

$$\Sigma(\delta|\Omega_1) = \{v(t,x) \,|\, v \in \Sigma(\delta), \|q\| \le \Omega_1\},$$

where

$$q = \{q_i\} : q_l = \frac{\partial v_l}{\partial t} + \beta \frac{v_l}{\partial x}, \quad q_{n+l} = \frac{\partial v_l}{\partial t} + \alpha \frac{v_l}{\partial x}, \quad (l = 1, \dots, n),$$

$$q^* = \{q_i^*\} : q_l^* = \sum_{j=1}^{n} \zeta_{lj}^0 q_j, \quad q_{n+l}^* = \sum_{j=1}^{n} \zeta_{lj}^0 q_{n+j}, \quad (l = 1, \dots, n).$$

For $v \in C^1[R(\delta)]$, define

$$\begin{cases} \tilde{\zeta}_{lj} = \zeta_{lj}(t, x|v(x,t)) \\ \tilde{\lambda}_l(t,x) = \lambda_l(t, x|v(t,x)) \\ \tilde{\mu}_l = \mu_l(t, x|v(t,x)) \end{cases}, \quad (l, j = 1, \dots, n)$$

and

$$\Gamma_2[v] = \left\{ \tilde{\zeta}_{lj}, \frac{\partial \tilde{\zeta}_{lj}}{\partial t}, \frac{\partial \tilde{\zeta}_{lj}}{\partial x}, \tilde{\lambda}_l, \frac{\partial \tilde{\lambda}_l}{\partial x}, \tilde{\mu}_l, \frac{\partial \tilde{\mu}_l}{\partial x}, \frac{1}{\det |\tilde{\zeta}_{lj}|}, \frac{1}{\alpha - \tilde{\lambda}_r(t, \alpha t)}, \frac{1}{\tilde{\lambda}_s(t, \beta t) - \beta} \right\}$$
$$(l, j = 1, \dots, n; r = 1, \dots, m; s = m+1, \dots, n).$$

Assume that the functional coefficients of system (4.1) satisfy the following conditions:

(i) For any $v \in C^1[R(\delta)]$, the values of the functions $\tilde{\zeta}_{lj}(t,x),\ \tilde{\lambda}_l(t,x),\ \tilde{\mu}_l(t,x)$ $(l, j = 1, \dots, n)$ on any domain $R(\delta')(0 \le \delta' \le \delta)$ depend only on the values of the function $v(t,x)$ on $R(\delta')$, and all functions in $\Gamma_2[v]$ are continuous on $R(\delta)$;

(ii) On $R(\delta)$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\|\tilde{\mu}\| \le R_1 + \varepsilon(\delta, \Omega_1),$$

where R_1 is independent of δ and Ω_1, and for any fixed Ω_1,

$$\varepsilon(\delta, \Omega_1) \to 0 \quad \text{as} \quad \delta \to 0; \tag{4.4}$$

(iii) On $R(\delta)$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\|\Gamma_2[v]\| \le K_1,$$

where K_1 depends only on Ω_1;

(iv) On $R(\delta)$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\omega(\eta|\tilde{\lambda}_l) + \omega(\eta|\tilde{\mu}_l) \le \omega_0(\eta),$$

where $\omega(\eta|\tilde{\lambda})$, $\omega(\eta|\tilde{\mu})$ are defined by

$$\omega(\eta|w) = \sup_{\substack{i=1,\dots,n \\ (t',x'),(t'',x'') \in R(\delta) \\ |t'-t''| \le \eta, |x'-x''| \le \eta}} \Big| w_i(t',x') - w_i(t'',x'') \Big|,$$

w is an n dimensional vector valued function, and $\omega_0(\eta)$ is a nonnegative function depending only on Ω_1 and $\omega_0(\eta) \to 0$ as $\eta \to 0$;

(v) On $R(\delta)$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\omega(\eta|\Gamma_2[v]) \le K_2\omega(\eta|q) + \omega_1(\eta),$$

where $\omega_1(\eta)$ has the same property as $\omega_0(\eta)$, and K_2 depends only on Ω_1;

(vi) On $R(\delta)$, for any $v', v'' \in \Sigma(\delta|\Omega_1)$,

$$\begin{aligned} \|\zeta_{lj}(t,x|v') - \zeta_{lj}(t,x|v'')\| &\le K_3\|v' - v''\|, \\ \|\lambda_l(t,x|v') - \lambda_l(t,x|v'')\| &\le K_3\|v' - v''\|, \\ \|\mu_l(t,x|v') - \mu_l(t,x|v'')\| &\le K_3\|v' - v''\|, \end{aligned}$$

where K_3 also depends only on Ω_1;

(vii) Let

$$\lambda_l^0 = \lambda_l(0,0|0) = \lambda(t,x|v)\Big|_{t=0,x=0,v\equiv 0} \quad (l = 1,\dots,n).$$

Then for $r = 1,\dots,m$,

$$\lambda_r^0 < \beta$$

or for any $v \in \Sigma(\delta|\Omega_1)$,

$$\lambda_r(t,x|v)|_{x=\beta t} \le \beta.$$

Similarly for $s = m+1,\dots,n$,

$$\lambda_s^0 > \alpha$$

or for any $v \in \Sigma(\delta|\Omega_1)$,

$$\lambda_s(t,x|v)|_{x=\alpha t} \le \alpha.$$

For $v \in C^1[R(\delta)]$, define

$$\begin{cases} \tilde{G}_r(t) = G_r(t|v)|_{x=\alpha t}, & (r = 1,\dots,m), \\ \tilde{G}_s(t) = G_s(t|v)|_{x=\beta t}, & (s = m+1,\dots,n). \end{cases}$$

We suppose that the functional boundary functions in (4.2), (4.3) satisfy the following conditions;

(viii) For any $v \in C^1[R(\delta)]$, $\tilde{G}_l(t)$ $(l = 1,\dots,n)$ are C^1 functions on the interval $0 < t \le \delta$. Moreover, the values of the functions $\tilde{G}_l(t)$ on $0 \le t \le \delta'$ $(0 \le \delta' \le \delta)$ depend only on the values of the functions $v(t,x)$ on $R(\delta')$. In particular, $\tilde{G}_l(0)$ $(l = 1,\dots,n)$ depend only on $v(0,0)$;

(ix) On $0 \le t \le \delta$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\|\tilde{G}_l'(t)\| \le \sum_{k=1}^{n} \big(\theta_{lk} + \varepsilon(\delta,\Omega_1)\big)\mathrm{Max}(\|q_k^*\|, \|q_{n+k}^*\|) + R_2 + \varepsilon(\delta,\Omega_1), (l = 1,\dots,n),$$

where θ_{lk} and R_2 are nonnegative constants independent of δ and Ω_1, $\varepsilon(\delta,\Omega_1)$ satisfies (4.4);

(x) On $0 \le t \le \delta$, for any $v \in \Sigma(\delta|\Omega_1)$,

$$\omega(\eta|\tilde{G}')_l(t) \le \sum_{k=1}^{n} \Big(\theta_{lk} + \varepsilon(\delta,\Omega_1)\Big)\mathrm{Max}\big(\Omega(\eta|q_k^*), \Omega(\eta|q_{n+k}^*)\big) + \omega_2(\eta), (l = 1,\dots,n),$$

where $\Omega(\eta|q_i)$ denotes the modulus of the continuity of q_i on $R(\delta)(i=1,\ldots,2n)$, and $\omega_2(\eta)$ is a nonnegative function depending only on Ω_1 with $\omega_2(\eta)\to 0$ as $\eta\to 0$;

(xi) On $R(\delta)$, for any $v', v''\in\Sigma(\delta|\Omega_1)$,

$$\|G_l(t,x|v')-G_l(t,x|v'')\|\le\sum_{k=1}^{n}\Big(\theta_{lk}+\varepsilon(\delta,\Omega_1)\Big)\|v_k'^*-v_k''^*\|,\ (l=1,\ldots,n),$$

where

$$v_k'^*=\sum_{j=1}^{n}\zeta_{kj}^0 v_j',\quad v_k''^*=\sum_{j=1}^{n}\zeta_{kj}^0 v_j'',\quad (k=1,\ldots,n).$$

Under the preceding assumptions, problem (4.1)-(4.3) is called a typical boundary value problem in functional form and the matrix

$$H=(\theta_{lk})$$

is called the characterizing matrix of this problem. Then the following theorem holds.

Theorem 4.1. *If the minimal characterizing number of H is less than 1, i.e.,*

$$\theta_{\min}=|H|_{\min}<1,$$

then for sufficiently small $\delta>0$, the typical boundary value problem in functional form, (4.1)-(4.3)*, admits a unique solution $u=u(t,x)$ on $R(\delta)$.*

Acknowledgements. The authors would like to express their sincere gratitude to the anonymous referee for his/her valuable comments and useful suggestions on the manuscript of this work.

References

[1] C. Gu, *A boundary value problem for hyperbolic systems and its applications*, Acta Math. Sinica **13** (1963), 32–48.
[2] C. Gu and T. Li et al, *The Cauchy problem of typical hyperbolic systems with discontinuous initial values*, Collections of Mathematical Papers of Fudan University (1960), 1–16.
[3] C. Gu, T. Li, and Z. Hou, *The Cauchy problem of hyperbolic systems with discontinuous initial values*, Collections of Scientific and Technological Papers, Shanghai, Mathematics Chemistry Edition (1960), 55–65.
[4] ———, *Discontinuous initial value problems for systems of quasi-linear hyperbolic equations. I*, Acta Math. Sinica **11**, 314–323.
[5] ———, *Discontinuous initial value problems for systems of quasi-linear hyperbolic equations. II*, Acta Math. Sinica **11** (1961), 324–327.
[6] ———, *Discontinuous initial value problems for systems of quasi-linear hyperbolic equations. III*, Acta Math. Sinica **12** (1962), 132–143.
[7] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag, Berlin, 2000.
[8] P. D. Lax, *Hyperbolic systems of conservation laws. II*, Comm. Pure Appl. Math. **10** (1957), 537–566.

[9] T. Li, *The generalized Riemann problem for quasilinear hyperbolic systems of conservation laws*, Partial differential equations in China, 80–103, Math. Appl., 288, Kluwer Acad. Publ., Dordrecht, 1994.

[10] T. Li and W. Yu, *Local solvability of the boundary value problems for quasilinear hyperbolic systems*, Sci. Sinica **23** (1980), no. 11, 1357–1367.

[11] ———, *Boundary value problems for the first-order quasilinear hyperbolic systems and their applications*, J. Differential Equations **41** (1981), no. 1, 1–26.

[12] ———, *Problèmes d onde centrée pourlas systèmes hyperboliques quasilinéaires et applications*, C. R. Acad. Sci. Paris Ser. I Math. **299** (1984), no. 9, 375–378.

[13] ———, *The discontinuous relations for quasilinear hyperbolic systems*, Journal of Engineering Mathematics **2** (1985), no. 2, 1–10.

[14] ———, *The centered waves for quasilinear hyperbolic systems*, Fudan Journal (Natural Science) **25** (1986), 195–206.

[15] ———, *The centered wave problem for quasilinear hyperbolic systems*, Chin. Ann. Math. **7A** (1986), 423–436.

[16] ———, *A class of mixed problem for quasilinear hyperbolic systems with the initial axis being characteristic*, Fudan Journal (Natural Science) **26** (1987), 1–9.

[17] ———, *The problem for quasilinear hyperbolic systems with discontinuous initial values*, Journal of Engineering Mathematics **4** (1987), no. 2, 1–12.

[18] ———, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Mathematics Series, V. Duke University, Mathematics Department, Durham, NC, 1985.

[19] J. Smoller, *Shock Waves and Reaction-Diffusion Equations: second ed.*, Springer-Verlag, New York, 1994.

[20] A. N. Tikhonov and A. A. Samarsky, *On discontinuous solutions of a quasilinear equation of first order*, Dokl. Akad. Nauk SSSR (N.S.) **99** (1954), 27–30.

[21] J. Yu and Y. Zhao, *The regularity of solution for first order quasilinear hyperbolic systems*, Chin. Ann. Math. **6A** (1985), 595–610.

Shouxin Chen
College of Mathematics and Information Science
Henan University
Kaifeng 475001, China
E-mail address: chensx@henu.edu.cn

Decheng Huang
Department of Mathematics
Xinyang Vocational Technical College
Xinyang 464000, China
E-mail address: huangdecheng123@163.com

Xiaosen Han
College of Mathematics and Information Science
Henan University
Kaifeng 475001, China
E-mail address: xiaosen_han@163.com, xiaosenhan@gmail.com