THE GENERALIZED WALSH FUNCTIONS

BY

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1. **Introduction.** In a recent paper[1] the author discussed various properties of the Walsh functions $\{\psi_n(x)\}$ and attempted to exhibit the close analogy between them and the exponentials $\{\exp 2\pi i n x\}$. This analogy was seen to stem from the fact that each system is essentially the character group of a certain compact commutative group, and that it is possible to set up a reasonably faithful correspondence between the two groups. It is natural to ask whether the analogy can be extended to the system $\{\exp 2\pi i y x\}$, that is, whether the Walsh functions can be imbedded in a larger class $\{\psi_y(x)\}$ so as to preserve most of the properties of the exponential which are desirable and useful in analysis. This question is answered in the affirmative here, and again group-theoretic considerations play an important role.

In §2 we construct a topological field $\mathfrak{F}$ analogous to the reals, and show that the characters of the additive group F of $\mathfrak{F}$ are generated by means of a single character and the multiplication in $\mathfrak{F}$. If $\chi_1(\bar{x})$ is this character, and χ an arbitrary character, there is a unique $\bar{y}\in F$ such that $\chi(\bar{x})=\chi_1(\bar{y}\bar{x})$. The correspondence $\bar{y}\leftrightarrow\chi$ is indeed an isomorphism between F and its character group $\mathfrak{X}$. It follows that $\mathfrak{X}$ may be made into a field isomorphic with $\mathfrak{F}$. The relation $F\cong\mathfrak{X}$ may also be deduced from $F\cong G\times\text{Char } G$, where G is the dyadic group defined in WF(§2), but the analogy with the reals is somewhat obscured by the direct product decomposition of F, the analogue of which does not exist for the reals. The relevant facts are that there is a homomorphism α of F on G, the kernel of which is isomorphic with Char G, and that $\mathfrak{X}$ contains a subgroup X' isomorphic with Char G, defined by the correspondence $\chi\in\text{Char } G\rightarrow\chi'(\bar{x})\equiv\chi(\alpha(\bar{x}))$.

We should remark here that the group F and its character group $\mathfrak{X}$ have been discussed briefly by Paley and Wiener[2], without, however, any mention of the field or of the connection with the Walsh functions. It is quite likely, in view of Paley's work on the Walsh functions[3], that they were aware of the connection.

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(1) *On the Walsh functions*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 372–414, referred to here as WF.

(2) R.E.A.C. Paley and N. Wiener, *Characters of Abelian groups*, Proc. Nat. Acad. Sci. U.S.A. vol. 19 (1933) pp. 253–257.

(3) *A remarkable series of orthogonal functions*, Proc. London Math. Soc. vol. 34 (1932) pp. 241–279.

In §3 we define a continuous mapping λ of $\mathfrak{F}$ onto the non-negative reals and an "inverse" μ which is made unique by excluding the elements of a set $\mathfrak{E}$ in $\mathfrak{F}$ as images under μ. The set $\mathfrak{E}$ plays a necessarily exceptional part in the whole discussion. The generalized Walsh functions are defined by $\chi(\mu(x))$, $\chi\in\mathfrak{X}$, $0\leqq x<\infty$. Those $\chi\in\mathfrak{X}$ which correspond to $\bar{y}\in\mathfrak{E}$ under the isomorphism $\mathfrak{F}\cong\mathfrak{X}$ are denoted by $\psi_y^*(x)$, $y=\lambda(\bar{y})$; the others are denoted by $\psi_y(x)$, $y=\lambda(\bar{y})$. Several alternate definitions of the generalized Walsh functions are derived, and it is shown that they satisfy a functional equation similar to $f(x+y)=f(x)f(y)$. In §4 we prove that they are the only non-trivial measurable solutions of this functional equation, and in §5 we prove that the only periodic $\psi_y(x)$ are those for which y is a dyadic rational, and that no $\psi_y^*(x)$ is periodic.

The last section contains the derivation of an invariant integral on $(0, \infty)$, corresponding to the formula

$$\int_{-\infty}^{+\infty} f(x+a)dx = \int_{-\infty}^{+\infty} f(x)dx.$$

A similar result for (0, 1) has already been derived in WF (§2). Next we prove the Riemann-Lebesgue Theorem,

$$\lim_{y\to\infty}\int_0^\infty \psi_y(x)f(x)dx = 0$$

for $f(x)\in L(0, \infty)$. This is followed by a proof of the Fourier Integral Theorem and a statement of the Poisson Formula for the generalized Walsh functions.

2. **The field $\mathfrak{F}$ and the characters of F.** Let $\mathfrak{F}$ denote the field of formal power series

$$\bar{x} = \sum_{n\geqq N} x_n\zeta^n, \tag{1}$$

in which N is an integer (positive, negative, or zero) which may vary from one element to another; the coefficients x_n are chosen from the field with two elements 0, 1. We define a neighborhood of zero as the set of $\bar{x}$ for which N has a fixed value. With this definition $\mathfrak{F}$ becomes a topological field which is totally disconnected, locally compact but not compact. We denote the additive group of $\mathfrak{F}$ by F, and its character group by $\mathfrak{X}$. Let G be the subgroup of F consisting of all $\bar{x}$ for which $N=1$. It is easily verified that G is isomorphic with the dyadic group defined in WF (§2), and that the mapping α which carries the element $\bar{x}=\sum_{n\geqq N}x_n\zeta^n$ into $\alpha(\bar{x})=\sum_{n\geqq 1}x_n\zeta^n$ is a homomorphism of F on G. The kernel is a discrete group isomorphic with the character group of G[4], and in fact $F\cong G\times\text{Char } G$. It follows that $\mathfrak{X}\cong\text{Char } G$

[4] To every $\bar{y}=\sum_{n\leqq 0}y_n\zeta^n$ in the kernel there corresponds the integer $\sum_{n\geqq 0}y_{-n}2^n = \sum 2^{n_k}$, $n_k<n_{k+1}$. For $\bar{x}\in G$, define $\chi_n(\bar{x})=(-1)^{x_n}$. Then the correspondence $\bar{y}\to\prod\chi_{n_k}(\bar{x})$ is the required isomorphism.

$\times$Char (Char G)$\cong$Char $G\times G\cong F$. We shall now investigate this isomorphism more closely.

Every $\chi\in\mathfrak{X}$ is completely determined by the sequence $\{\chi(\zeta^n)\}$. Since $\zeta^n\to\bar{0}$ as $n\to\infty$, $\chi(\zeta^n)\to 1$, so $\chi(\zeta^n)=1$ for $n>M(\chi)+1$. Hence, with each χ we can associate the sequence of 0's and 1's $\{y_n\}$, defined by $(-1)^{y_n}=\chi(\zeta^{1-n})$, and for $n<-M(\chi)$, $y_n=0$. Define $\bar{y}\in\mathfrak{F}$ by

$$\bar{y} = \sum_{n\geqq -M(\chi)} y_n\zeta^n.$$

Now, for any element $\bar{x}\in\mathfrak{F}$ we have

$$\begin{aligned}\chi(\bar{x}) &= \chi\left(\sum_{n\geqq N} x_n\zeta^n\right)\\ &= \prod_{n\geqq N}\chi(x_n\zeta^n)\\ &= \prod_{N\leqq n\leqq M(\chi)+1}(\chi(\zeta^n))^{x_n}\\ &= \prod_{N\leqq n\leqq M(\chi)+1}(-1)^{x_ny_{1-n}}\\ &= (-1)^{z_1},\end{aligned}$$

where

$$z_1 = \sum_{N\leqq n\leqq M(\chi)+1} x_n y_{1-n}.$$

But z_1 (mod 2) is the coefficient of ζ in the product $\bar{x}\bar{y}=\bar{z}$, and $(-1)^{z_1}$ is clearly a character $\chi_1(\bar{z})$. Hence

$$\chi(\bar{x}) = \chi_1(\bar{x}\bar{y}). \tag{2}$$

Thus, corresponding to each $\chi\in\mathfrak{X}$ there is a $\bar{y}\in\mathfrak{F}$ such that (2) holds for all $\bar{x}\in\mathfrak{F}$. If $\bar{y}$ is given, then $\chi(\bar{x})$ as defined by (2) is a character. The one-to-one correspondence thus established between $\mathfrak{X}$ and F is easily seen to be an isomorphism. $\mathfrak{X}$ can be made into a field isomorphic to $\mathfrak{F}$ by defining the field product $(\chi * \chi')$ $(\bar{x})=\chi_1(\bar{x}\bar{y}\bar{y}')$ if $\chi\leftrightarrow\bar{y}$ and $\chi'\leftrightarrow\bar{y}'$.

3. Definition and properties of the generalized Walsh functions. Given $\bar{x}\in\mathfrak{F}$, we define

$$\lambda(\bar{x}) = \lambda\left(\sum_{n\geqq N} x_n\zeta^n\right) = \sum_{n\geqq N} x_n 2^{-n} = x. \tag{3}$$

For $0\leqq x<\infty$, we define the inverse mapping $\mu(x)$ by (3), choosing the finite expansion if x is a dyadic rational. The mapping μ is *into* $\mathfrak{F}$, omitting only the exceptional set $\mathfrak{E}$ consisting of elements with coefficients 1 from some point on. We have

$$\lambda(\mu(x)) = x \qquad (0 \leqq x < \infty), \tag{4}$$

and

$$\mu(\lambda(\bar{x})) = \bar{x} \qquad (\bar{x} \notin \mathfrak{E}). \tag{5}$$

We define the transforms of the field operations:

$$x \oplus y = \lambda(\mu(x) + \mu(y)), \tag{6}$$

$$x \circ y = \lambda(\mu(x)\mu(y)). \tag{7}$$

The operation $\oplus$ is closely related to the $\dotplus$ defined in WF (§2); in fact,

$$(x \oplus y) - (x \dotplus y) = \textit{integer}. \tag{8}$$

It follows from (8) that

$$\psi_n(x \oplus y) = \psi_n(x \dotplus y), \tag{9}$$

since $\psi_n(x)$ has period 1. We recall [WF; (2.12)] that

$$\psi_n(x \dotplus y) = \psi_n(x)\psi_n(y) \tag{10}$$

unless the dyadic expansions of x and y differ from some point on; in our present notation, unless $\mu(x)+\mu(y)\in\mathfrak{E}$. Also, by definition [WF, §1],

$$\psi_{m\oplus n}(x) = \psi_m(x)\psi_n(x). \tag{11}$$

We shall now define a generalization of the Walsh functions. Since the $\psi_n(x)$ are the transforms of the characters of G, it is natural to consider the transforms of the characters of $\mathfrak{F}$. Let $\chi\in\mathfrak{X}$ be given. By (2), there is a $\bar{y}\in\mathfrak{F}$ such that

$$\chi(\bar{x}) = \chi_1(\bar{x}\bar{y}). \tag{12}$$

Let $y=\lambda(\bar{y})$, and define the functions

$$(13) \qquad \begin{aligned} \psi_y(x) &= \chi(\mu(x)) = \chi_1(\mu(x)\bar{y}) && \text{if } \bar{y} \notin \mathfrak{E}, \\ \psi_y^*(x) &= \chi(\mu(x)) = \chi_1(\mu(x)\bar{y}) && \text{if } \bar{y} \in \mathfrak{E}. \end{aligned}$$

For the present, we restrict ourselves to a consideration of $\psi_y(x)$. In order to justify the use the symbol, we shall now prove that definition (13) yields the Walsh function $\psi_n(x)$ whenever y is an integer n, that is, that

$$\chi_1(\mu(x)\mu(n)) = \psi_n(x). \tag{14}$$

Since (14) is trivial for $n=0$, we may assume that

$$n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\nu} \qquad (k_1 > k_2 > \cdots > k_\nu \geqq 0).$$

Then

$$\mu(n) = \zeta^{-k_1} + \zeta^{-k_2} + \cdots + \zeta^{-k_\nu}$$

and

$$\begin{aligned}\chi_1(\mu(x)\mu(n)) &= \prod_{r=1}^{\nu} \chi_1(\mu(x)\zeta^{-k_r})\\ &= \prod_{r=1}^{\nu} (-1)^{x_{k_r}+1}\\ &= \prod_{r=1}^{\nu} \psi_{2^{k_r}}(x)\\ &= \psi_{2^{k_1}\oplus 2^{k_2}\oplus\cdots\oplus 2^{k_\nu}}(x)\\ &= \psi_n(x),\end{aligned}$$

which proves (14). Thus (13) is a valid generalization of the Walsh functions.

We shall now derive a somewhat more useful expression for $\psi_y(x)$, depending only on the original set $\{\psi_n(x)\}$. If

$$\mu(x) = \sum x_n\zeta^n, \qquad \mu(y) = \sum y_m\zeta^m,$$

then

$$\chi_1(\mu(x)\mu(y)) = (-1)^{z_1},$$

where

$$z_1 = \sum_{m+n=1} x_n y_m = z + z', \qquad z = \sum_{n\leqq 0} x_n y_m, \qquad z' = \sum_{m\leqq 0} x_n y_m,$$

and $m+n=1$ in both sums. It is easy to see that

$$(-1)^z = \chi_1(\mu([x])\mu(y)), \qquad (-1)^{z'} = \chi_1(\mu(x)\mu([y])),$$

the square brackets denoting the greatest integer function. Hence, by (14),

$$\psi_y(x) = \psi_{[x]}(y)\psi_{[y]}(x). \tag{15}$$

It is interesting to observe that

$$\psi_y(x) = \psi_1(x\circ y) \qquad (\mu(x)\mu(y) \notin \mathfrak{E}). \tag{16}$$

For

$$\begin{aligned}\chi_1(\mu(x)\mu(y)) &= \chi_1\{\mu(\lambda(\mu(x)\mu(y)))\}\\ &= \chi_1\{\mu(x\circ y)\}\\ &= \psi_1(x\circ y).\end{aligned}$$

It is now fairly easy to show that

$$\psi_y(x\oplus x') = \psi_y(x)\psi_y(x') \qquad (\mu(x)+\mu(x') \notin \mathfrak{E}). \tag{17}$$

For, by (15),

$$\psi_y(x\oplus x') = \psi_{[y]}(x\oplus x')\psi_{[x\oplus x']}(y). \tag{18}$$

By (9) and (10), we have

$$\psi_{[y]}(x \oplus x') = \psi_{[y]}(x)\psi_{[y]}(x'); \tag{19}$$

also, since $\mu(x)+\mu(x')\notin\mathfrak{E}$, $[x\oplus x'] = [x]\oplus[x']$, so that, from (11),

$$\psi_{[x\oplus x']}(y) = \psi_{[x]}(y)\psi_{[x']}(y). \tag{20}$$

Equations (18), (19), (20), and (15) now yield the required result (17).

It remains to consider those characters generated by elements of $\mathfrak{E}$, and their transforms $\psi_y^*(x)$ as given by (13). Let

$$\bar{y} = \sum y_n\zeta^n \in \mathfrak{E}, \qquad \lambda(\bar{y}) = y.$$

By the nature of $\bar{y}$, for all sufficiently large N, and for all t such that $y-2^{-N} < t < y$, we have

$$\begin{aligned} t_n &= y_n && (n \leqq N), \\ t_n &\leqq y_n && (n > N), \end{aligned}$$

where the t_n are of course determined by

$$\mu(t) = \sum t_n\zeta^n.$$

Hence

$$\bar{y} + \mu(t) = \sum_{n>N} (y_n - t_n)\zeta^n$$

and

$$z_1 = \sum_{m+n=1} x_m(y_n - t_n)$$

depends only on those x_m with $m<1-N$; indeed $z_1=0$ if $x_m=0$ for $m<1-N$. This is the case for all $x=\sum_{m\geqq 1-N}x_m 2^{-m} < 2^N$. It follows that

$$\chi_1(\mu(x)(\bar{y} + \mu(t))) = 1 \qquad (0 \leqq x < 2^N) \tag{21}$$

But the left member of (21) is equal to

$$\chi_1(\mu(x)\bar{y})\chi_1(\mu(x)\mu(t)) = \psi_y^*(x)\psi_t(x). \tag{22}$$

Therefore $\psi_y^*(x)$ coincides with $\psi_t(x)$ on any given interval $(0, 2^N)$ if t is less than but sufficiently close to y. We have therefore proved that

$$\psi_y^*(x) = \lim_{t\to y-0} \psi_t(x). \tag{23}$$

We have also the result corresponding to (17),

$$\psi_y^*(x \oplus x') = \psi_y^*(x)\psi_y^*(x') \qquad (\mu(x) + \mu(x') \notin \mathfrak{E}). \tag{24}$$

4. Solution of a functional equation. In the preceding section we obtained the solutions $\psi_y(x)$, $\psi_y^*(x)$ for the functional equation

$$f(x \oplus x') = f(x)f(x') \qquad (\mu(x) + \mu(x') \notin \mathfrak{E}). \tag{25}$$

This was done by considering the transforms of all the characters of $\mathfrak{F}$. We shall now show that there are no other nontrivial *measurable* solutions.

Let $f(x)$, not equivalent to zero, be a measurable solution of (25). Taking $x' = x$ in that equation, $f(0) = f^2(x)$, so $f(0) \neq 0$. Taking $x = 0$, $f(0) = 1$; hence $f(x) = \pm 1$ and $f(x)$ is integrable over any finite range. Define $F(x) = f(x)$ in $0 \leqq x < 1$, $F(x+1) = F(x)$ for all x, and consider the Walsh-Fourier coefficients $\{c_n\}$ of $F(x)$. Clearly not all the c_n vanish, for then we would have $F(x) = f(x)$ equivalent to zero in $0 \leqq x < 1$, by the completeness of the Walsh system; since $f(N+x) = f(N \oplus x)$ for all positive integers N and all x in $0 \leqq x < 1$, we would have $f(N+x) = f(N)f(x) = 0$ almost everywhere, which is ruled out by assumption.

Let c_n be a nonzero Fourier coefficient of $F(x)$. Then for every fixed a, $0 \leqq a < 1$,

$$c_n = \int_0^1 \psi_n(x)f(x)dx = \int_0^1 \psi_n(x \dotplus a)f(x \dotplus a)dx$$

by the invariance of the Lebesgue integral [WF, Theorem I]. But for almost all x in (0, 1), we have $\psi_n(x \dotplus a) = \psi_n(x)\psi_n(a)$ by (10), and $f(x \dotplus a) = f(x \oplus a) = f(x)f(a)$ by (25). Hence

$$c_n = \int_0^1 \psi_n(x)\psi_n(a)f(x)f(a)dx = c_n\psi_n(a)f(a).$$

Cancelling c_n, we get

$$f(a) = \psi_n(a) \qquad (0 \leqq a < 1).$$

Hence, for all $x \geqq 0$ with fractional part a,

$$f(x) = f([x] + a) = f([x] \oplus a) = f([x])\psi_n(a) = f([x])\psi_n(x).$$

If $[x] = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_\nu}$, $k_1 > k_2 > \cdots > k_\nu \geqq 0$,

$$f([x]) = f(2^{k_1})f(2^{k_2}) \cdots f(2^{k_\nu}),$$

so $f([x])$ is completely determined by the sequence $\{f(2^n)\}$, $n \geqq 0$. Let us define $\bar{b} \in G$ by the equations $(-1)^{b_{n+1}} = f(2^n)$, $n \geqq 0$. If $\bar{b} \notin \mathfrak{E}$, then $\bar{b} = \mu(b)$, and

$$f([x]) = \psi_{2^{k_1}}(b) \cdots \psi_{2^{k_\nu}}(b) = \psi_{[x]}(b),$$

so that, defining $y = n + b$, we have

$$f(x) = \psi_{[x]}(b)\psi_n(x) = \psi_{[x]}(y)\psi_{[y]}(x) = \psi_y(x).$$

If $\bar{b} \in \mathfrak{E}$ and $b = \lambda(\bar{b})$, then for any given M we can find $t = t(M) < b$ such that $f([x]) = \psi_{[x]}(t)$ for all $x < M$. Since $t \to b$ as $M \to \infty$, we have

$$f([x]) = \lim_{t \to b-0} \psi_{[x]}(t).$$

Again defining $y = n + b$, $u = n + t$, we have

$$\begin{aligned} f(x) &= \psi_n(x) \lim_{t \to b-0} \psi_{[x]}(t) \\ &= \lim_{t \to b-0} \psi_{[u]}(x)\psi_{[x]}(u) \\ &= \lim_{u \to y-0} \psi_u(x) \\ &= \psi_y^*(x). \end{aligned}$$

This proves the stated result and shows that there is a one-to-one correspondence between the characters of F and the nontrivial measurable solutions of (25).

5. **Periodicity.** In this section we prove the following theorem:

(i) *If $y = m \cdot 2^{-n}$, where m is odd, n an arbitrary integer, then $\psi_y(x)$ has the exact period 2^n.*

(ii) *If y is not a dyadic rational, $\psi_y(x)$ is not periodic; $\psi_y^*(x)$ is never periodic.*

To prove (i), we observe that for all x, y,

$$\psi_{2^n y}(x) = \psi_y(2^n x). \tag{26}$$

This follows from

$$\psi_{2^n y}(x) = \chi_1(\mu(2^n y)\mu(x)) = \chi_1(\mu(y)\mu(2^n x)) = \psi_y(2^n x).$$

If we set $y = m \cdot 2^{-n}$, $x = 2^{-n}u$, (26) becomes

$$\psi_y(u) = \psi_m(2^{-n}u).$$

Since the exact period of $\psi_m(x) = \psi_1(x)\psi_{2k}(x)$ is 1, the exact period of $\psi_y(x)$ is 2^n.

To prove (ii), we write $\psi_y(x) = \phi(x)f(x)$, where $\phi(x)$ has period 1 and $f(x) = (f[x])$ satisfies

$$f(k \oplus l) = f(k)f(l)$$

for all integers k, $l \geqq 0$, and $f(0) = 1$. If $\psi_y(x)$ is periodic, the period must be rational, so that $f(k)$ has an integral period P. A similar conclusion holds for $\psi_y^*(x)$. In both cases we have $f(2^N) = -1$ for all N in a certain infinite set $\mathfrak{B}$. Hence we can find a residue c (mod P) which is assumed by P distinct values $2^{N_1}, \cdots, 2^{N_P}$ ($N_i \in \mathfrak{B}$). If $s = 2^{N_1} + \cdots + 2^{N_P}$, then $s \equiv 0$ (mod P), so $f(s) = 1$. On the other hand, $f(s) = f(2^{N_1}) \cdots f(2^{N_P}) = (-1)^P$, so P is even, say $P=2P'$. Now define $g(k)=f(2k)$. Clearly g has the period P' and

$$g(k \oplus l) = f(2(k \oplus l)) = f(2k \oplus 2l) = f(2k)f(2l) = g(k)g(l).$$

Finally, $g(2^{N-1})=f(2^N)=-1$ for all $N\geqq 1$ in $\mathfrak{B}$. By the same argument as was used above, we see that P' is even, say $P'=2P''$. In this way we obtain an infinite sequence of positive integers $P>P'>P''> \cdots$. This contradiction proves (ii).

6. **Analytic results.** In this section we shall prove several results which have direct analogues in the trigonometric theory.

We begin with the invariance of the integral:

$$\int_0^\infty f(x \oplus a)dx = \int_0^\infty f(x)dx. \tag{27}$$

This follows directly from the fact that $T_a(x)=x\oplus a$ is a measure-preserving transformation on every interval $(0, 2^n)$, $2^n>a$(5).

Next we prove the Riemann-Lebesgue Theorem: *If* $f(x)$ *is integrable on* $(0, \infty)$, *then*

$$\lim_{y\to\infty} \int_0^\infty \psi_y(x)f(x)dx = 0. \tag{28}$$

Write

$$\int_0^\infty \psi_y(x)f(x)dx = \int_0^n \psi_y(x)f(x)dx + \int_n^\infty \psi_y(x)f(x)dx = I_n + J_n. \tag{29}$$

We may choose n so that for all y,

$$|J_n| \leqq \int_n^\infty |f(x)|\,dx < \epsilon/2.$$

Now

$$I_n = \sum_{k=0}^{n-1}\int_k^{k+1} \psi_y(x)f(x)dx = \sum_{k=0}^{n-1}\psi_k(y)\int_k^{k+1}\psi_{[y]}(x)f(x)dx,$$

$$|I_n| \leqq \sum_{k=0}^{n-1}\left|\int_k^{k+1}\psi_{[y]}(x)f(x)dx\right|. \tag{30}$$

On the right we have a sum of a fixed number of Fourier coefficients of order $[y]$. We may choose y so large that this sum is less than $\epsilon/2$, so that

$$|I_n| + |J_n| < \epsilon,$$

which proves (28).

(5) For a detailed proof of a similar result for the transformation $x\dotplus a$ on the unit interval, see WF(§2).

Finally we prove one form of the Fourier Integral Theorem. For the corresponding trigonometric theorem we refer the reader to Zygmund's *Trigonometrical series*, p. 306. It will be observed that a number of simplifications occur, so that the following proof differs from that of the trigonometric theorem. In fact, we obtain a slightly stronger result.

Let us define

$$J(u;\omega) = \int_0^\omega \psi_y(u)dy, \tag{31}$$

$$S_\omega(x) = \int_0^\infty f(t)J(x\oplus t;\omega)dt. \tag{32}$$

Let I_q denote the interval $q \leqq x < q+1$, where q is a non-negative integer. Assume that $f(t)/(1+t) \in L(0, \infty)$. Let $f_q(t)$ be the function of period 1 which coincides with $f(t)$ on I_q, and let $s_n(x)$ be the nth partial sum of the Walsh-Fourier series for $f_q(t)$. Then as $\omega\to\infty$, $S_\omega(x)-s_{[\omega]}(x)$ tends to zero uniformly for $x\in I_q$.

We require several lemmas.

Lemma 1. *If $\delta(u)$ is the characteristic function of $0\leqq u<1$, then*

$$J(u;\omega) = \delta(u)\sum_{k=0}^{[\omega]-1}\psi_k(u) + \psi_{[\omega]}(u)J([u];\omega-[\omega]).$$

Proof.

$$\begin{aligned} J(u;\omega) &= \int_0^\omega \psi_{[y]}(u)\psi_{[u]}(y)dy \\ &= \sum_{k=0}^{[\omega]-1}\psi_k(u)\int_k^{k+1}\psi_{[u]}(y)dy + \int_{[\omega]}^\omega \psi_{[\omega]}(u)\psi_{[u]}(y)dy \\ &= \delta(u)\sum_{k=0}^{[\omega]-1}\psi_k(u) + \psi_{[\omega]}(u)\int_0^{\omega-[\omega]}\psi_{[u]}(y)dy. \end{aligned}$$

Lemma 2. *For all $n>0$, $|J(n;\omega)|<1/n$.*

Proof. If $2^m \leqq n < 2^{m+1}$, and if γ_m is the number nearest to ω of the form $p/2^m$, then by WF (§3, equation 3.8),

$$J(n;\omega) = |\omega-\gamma_m| \leqq 2^{-(m+1)} < 1/n.$$

Lemma 3. *If $x\in I_q$ and $t\geqq 2^M \geqq 4(q+1)$, then $[x\oplus t]\geqq t/2$.*

Proof. Since $t\oplus x \geqq t-x > t-(q+1)\geqq 3t/4$, $[x\oplus t]\geqq 3t/4-1\geqq t/2$.

We begin our proof of the theorem by showing that the integral (32) exists uniformly for $x\in I_q$. Choose M so that $2^M\geqq 4(q+1)$. For $t<2^M$, we have $|J(x\oplus t;\omega)|\leqq\omega$, from (31). For $t\geqq 2^M$, set $u=x\oplus t$ in Lemma 1 and observe that $[u]>0$, so that

$$|J(u;\omega)| = |J([u];\omega-[\omega])| < \frac{1}{[u]} \leqq \frac{2}{t},$$

by virtue of Lemmas 2 and 3. Thus $S_\omega(x)$ exists.

Next we show that $S_{[\omega]}(x)=s_{[\omega]}(x)$. Let $[\omega]=n$. By Lemma 1,

$$J(x \oplus t; n) = \delta(x \oplus t)\sum_{k=0}^{n-1}\psi_k(x \oplus t)$$

and $\delta(x\oplus t)$ is the characteristic function of I_q. Hence

$$\begin{aligned} S_{[\omega]}(x) &= \int_q^{q+1} f(t)\sum_{k=0}^{n-1}\psi_k(x\oplus t)dt \\ &= \sum_{k=0}^{n-1}\psi_k(x)\int_q^{q+1} f_q(t)\psi_k(t)dt \\ &= s_n(x). \end{aligned}$$

Now consider the difference

$$\begin{aligned} S_\omega(x) - s_n(x) &= \int_0^\infty f(t)\int_n^\omega \psi_y(x\oplus t)dydt \\ &= \int_0^{2M} f(t)\int_n^\omega \psi_y(x\oplus t)dydt + \int_{2M}^\infty f(t)\int_n^\omega \psi_y(x\oplus t)dydt \\ &= I_1 + I_2. \end{aligned}$$

Choose M to satisfy the condition of Lemma 3. Since for $t\geqq 2^M$

$$\begin{aligned} \int_n^\omega \psi_y(x\oplus t)dy &= J(x\oplus t;\omega) - J(x\oplus t; n) \\ &= J(x\oplus t;\omega), \end{aligned}$$

we see that I_2 is the tail of the uniformly convergent integral (32). Hence we can choose M so large that $|I_2|<\epsilon$ uniformly for $x\in I_q$. Finally,

$$\begin{aligned} |I_1| &= \left|\int_n^\omega \psi_y(x)\left\{\int_0^{2M} f(t)\psi_y(t)dt\right\}dy\right| \\ &\leqq \sup_{y\geqq n}\left|\int_0^{2M} f(t)\psi_y(t)dt\right|, \end{aligned}$$

which tends to zero as $\omega\to\infty$, by the Riemann-Lebesgue Theorem. This completes our proof.

If $f(t)\in L(0,\infty)$, and the Walsh-Fourier series of $f(t)$ converges to $f(x)$ for

$t=x$, we obtain

$$f(x) = \lim_{\omega\to\infty} \int_0^\infty f(t) \int_0^\omega \psi_y(x \oplus t)dydt$$
$$= \lim_{\omega\to\infty} \int_0^\omega \psi_y(x) \int_0^\infty \psi_y(t)f(t)dtdy,$$

or

$$f(x) = \lim_{\omega\to\infty} \int_0^\omega \psi_y(x)F(y)dy,$$
$$F(y) = \int_0^\infty \psi_y(t)f(t)dt,$$

the usual double-integral form.

It seems clear that a fairly complete Fourier-Transform theory can be developed for the generalized Walsh functions introduced here.

We close with the remark that it is possible to prove the Poisson Formula

$$\sum_{k=0}^\infty g(k) = \sum_{n=0}^\infty \int_0^\infty g(x)\psi_n(x)dx, \tag{33}$$

with the usual assumptions on $g(x)$. We shall omit the proof, which parallels the classical one almost word-for-word.

We take this opportunity to note the following errata in WF:

p. 393, line -2, read (Dini-Lipschitz),

p. 395, equation (7.2), for $D_r(x+u)$ read $D_r(x\dot{+}u)$,

p. 402, line 11, for (8.15) read (8.14).

In the following places, for $+$ read $\dot{+}$: p. 374, display; p. 376, display, line -5, and line -4; p. 379, fifth display; p. 380, line 4, and line 5 throughout; p. 382, line -7; p. 386, line 1; p. 393, display, line -6.

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