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THE GENERALIZED WAR OF ATTRITION

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ABSTRACT

We generalize the War of Attrition model to allow for $N + K$ firms competing for N prizes. Two special cases are of particular interest. First, if firms continue to pay their full costs after dropping out (as in a standard-setting context), each firm's exit time is independent both of K and of the actions of other players. Second, in the limit in which firms pay no costs after dropping out (as in a natural-oligopoly problem), the field is immediately reduced to $N + 1$ firms. Furthermore, we have perfect sorting, so it is always the $K - 1$ lowest-value players who drop out in zero time, even though each player's value is private information to the player. We apply our model to politics, explaining the length of time it takes to collect a winning coalition to pass a bill.

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In August, 1993 the United States Congress passed the Clinton Administration's budget by the narrowest possible margin; if a single supporter in either the House or Senate had switched their vote, the plan would have been defeated. The last Congresswoman to vote for the budget bill, Marjorie Margolies-Mezvinsky, virtually insured her defeat in the next election by supporting the President.[1] But although many Democrats would have preferred to vote against this unpopular bill, they were unwilling to see the new Democratic President defeated on a measure of such importance so, in the words of the New York Times "one member after another reluctantly fell into line to provide the 218-216 victory" in the House of Representatives.[2,3] Likewise, the bill was approved by the Senate, after an unusually protracted debate, by 50 votes plus the Vice-President's casting vote to 50.[4]

While the budget was passed without a single Republican vote, the North American Free Trade Agreement (NAFTA) was a bipartisan effort. According to the Washington Post, "the interparty understanding was that if the Democrats supplied at least 100 votes, Republicans would provide the balance".[5] It is likely that a large majority of the 258 Democrats would have supported NAFTA in a secret ballot, but the political cost of breaking with trade unions on this issue meant that many Representatives preferred that the bill pass with someone else's support.[6]

[1] The bill was especially unpopular in very affluent districts like hers. The Republicans chanted "Bye-bye Marjorie" as she cast the final vote in favour, and she was indeed comfortably defeated in the 1994 election.

[2] This is the narrowest possible margin, since in the U.S., unlike some other countries, representatives do not wish to abstain on a measure of this importance.

[3] Krauss, Clifford, "The Budget Struggle: The House; Whips Use Soft Touch to Succeed", *The New York Times*, August 6, 1993, section 1 p.7.

[4] It is not known how many additional representatives would have voted for the President if their votes had been required. Representatives' incentives were, of course, to deny any willingness to do this, but even so it was reported that Thornton of Arkansas and, perhaps, Minge of Minnesota were available to vote yes if necessary.

[5] Cooper, Kenneth J. "Backers Claim Momentum to Carry NAFTA in House; Trade Pact Split Parties, Crossed Political Lines", *Washington Post*, November 18, 1993, p.A1.

[6] In the end, the Democrats found 102 votes, allowing for a small safety margin, and the Republicans then provided 134 votes, putting the treaty well above the 218 votes needed for a majority. Thus a naive observer might have concluded that the bill passed by a comfortable margin and that the last 18 Democratic votes were unnecessary, but the real margin among votes cast was just 2.

Thus in both cases many Democrats were fighting a *war of attrition* for the prizes of being among the non-supporters of a successful bill. The costs of holding out included the private costs of enduring pressure from the Administration, and the public costs, borne by all the senators, of delaying passage of the bill.[7]

What makes these games different from existing models of the war of attrition is that there were many losers. The literature on the war of attrition, which spans many fields including biology,[8] strikes,[9] politics,[10] and industrial competition,[11] focuses almost exclusively on games with two players, or on the straightforward generalization to $N+1$ players competing for N prizes. We generalize to $N+K$ players competing for N prizes, so K must exit for the game to end.

There are many cases where a multiplayer analysis seems appropriate. For instance, winner-take-all markets, where only one firm is likely to survive, might originally be contested by more than two firms. Two recent papers[12] argue that the way firms negotiate in industry standard-setting committees is most appropriately modeled as a war of attrition.

These examples highlight an important issue in modeling the generalized war of attrition. In a natural monopoly (or natural oligopoly) setting, once a firm has conceded defeat it leaves the game and it stops paying costs. In a standards-committee case, even after a player admits that its preferred standard will not be chosen, it must still bear the costs of industry confusion until the game ends.

To make the distinction clear, consider the following example: The chairman of the economics department calls a meeting, and says that he needs

[7]Delaying the passage of the bill increases public frustration with the political process, delays the bill's benefits, and increases the probability of the bill failing, perhaps through the President giving up on it. (Increasing the risk of the bill failing is not just equivalent to discounting, because it involves the loss of the bill as well as of the prize of not having to vote for it.) Also, the longer it took the Democrats to pass the budget bill the less time they had to work on the rest of their agenda, such as health-care reform.

[8]See, e.g. Maynard Smith (1974), and Riley (1980).

[9]See, e.g. Kennan and Wilson (1989).

[10]For example, Alesina and Drazen (1991) and Casella and Eichengreen (1996) apply the war of attrition to model the process of agreement to macroeconomic stabilizations.

[11]See, e.g. Fudenberg and Tirole (1986).

[12]See Farrell (1993) and David and Monroe (1994). See also Farrell and Saloner (1988).

five volunteers to serve on a committee. The meeting will not end until the committee is chosen. In the "natural oligopoly" game, a faculty member is allowed to leave the meeting as soon as he agrees to serve on the committee, leaving the others to remain. In the "standards" game, everyone must stay in the meeting until the whole committee is selected. Obviously, there is much less incentive to concede quickly in the second game.

The natural oligopoly case yields a striking result: $K-1$ *firms will exit immediately,* leaving only $N+1$ firms to battle for the N prizes. We call this result *twoness* in that if $N=1$, the game is immediately reduced to the conventional two, regardless of how many firms there are at the beginning.[13] To understand the result, imagine that when $K>1$ exits are still required for the game to end, a player is within ε of his planned dropout time. Then the player's cost of waiting as planned is of order ε, but his benefit is of order ε^K since only when K other players are within ε of giving up will he ultimately win. So for small ε he will prefer to quit now rather then wait, but in this case he should of course have quit ε earlier, and so on. So only when $K=1$ is delay possible.

In the standards version of the game, in which all players pay until the game ends, even if they have already conceded, the result is perhaps equally surprising: *players' strategies are independent of K and of other players' dropout behaviour.* Why does this kind of *strategic independence* arise? Because, as before, when there are still $K>1$ too many firms, a player within ε of his planned exit knows he has no chance (to first order) of winning. So since in this case quitting early does not affect the rate at which the firm pays costs, the firm would quit early if he thereby shortened the expected length of the whole game. Only if the firm's exit decision has no effect on the length of the whole game will it be willing to exit at the "correct" equilibrium time. So no firm can either affect, or be affected by, any other firm's dropout behaviour.

In our general model, which encompasses both the natural oligopoly and

[13]One example of this game is that Avinash Dixit every year offers a $20 prize to the student who continues clapping the longest at the end of his game theory course. Our analysis shows that if they have understood the material, no more than two students should continue applauding after the time when everyone would otherwise have stopped.

the standards versions as special cases, each player's value of surviving in the market is private information to that player. However we always get perfect sorting. Thus even in the limit in which we have twoness, so the field is sorted down to $N + 1$ players immediately, it is the $K - 1$ weakest players that leave in zero time.[14]

Section 1 presents the model. Section 2 analyses the general case. Sections 3 and 4 discuss twoness and strategic independence respectively. Section 5 illustrates the analysis with an example inspired by the 1993 budget battle, and Section 6 concludes.

1 The model

There are $N + K$ risk-neutral firms in a market. As long as a firm continues to compete, it pays a cost that is normalized to 1 per unit time. If it exits, it subsequently pays a cost $c > 0$ per unit time until a total of K firms have quit.[15] If a firm i is one of the N firms which survives, then it wins a prize of v^i which is private information to firm i at the beginning of the game.[16] The values v^i are drawn independently from the distribution $F(v)$,

[14]Other authors have recognised that $K - 1$ players must exit instantaneously, but did not have a formulation that allowed for efficient sorting within the war of attrition. For example, Fudenberg and Kreps (1987) considered a model with identical players who played mixed strategies. They suggested a series of instantaneous randomizations to eliminate at least $K - 1$ bidders prior to the start of the war of attrition. (The analogous mechanism with different "types" of players would have been to "stop the clock" and have bidders participate in a continuous English auction until only $N + 1$ remained, with neither winners nor losers paying anything in the auction but with only the winners participating in a subsequent war of attrition.)

Krishna and Morgan (1994) study second-price all-pay auctions with more than two bidders, but focus on a "sealed bid" game in which firms choose an exit time at the start of the game which cannot be revised based on the play of the game, so "twoness" does not arise even though firms that have exited pay no further costs. Hillman and Riley (1989) considered first-price all-pay auctions. To the best of our knowledge, our paper is the first to consider the case in which firms that exit may continue to lose money, as in our standards story.

[15]Typically we expect c to be no more than 1, but footnote 27 suggets a context in which it might exceed 1. Another example in which c would exceed 1 is a "contributions" game in which N people must each pay for one stage of a building project before it yields any benefits, and discounting means earlier contributions cost more than later ones.

[16]If flow costs differ among firms, we can simply reinterpret v^i as the ratio for player i of the prize value to the flow cost, since this ratio is all that matters to any firm. So all our results will still go through. (Units for measuring costs are of course then different for different firms so that the flow rate of costs is measured as 1 per unit time for each.)

with $F(\underline{V}) = 0, F(\overline{V}) = 1, \underline{V} > 0$ and $\overline{V} < \infty$. We assume $F(\cdot)$ has a strictly positive finite derivative everywhere. It will be convenient to write $h(v)$ for the "hazard rate" $\frac{f(v)}{1-F(v)}$. We also write v_j for the j^{th} highest of the $N + K$ firms' values, and $E(v_j)$ for the expectation of this value. We restrict attention to symmetric equilibria.

At any point of the game let $N + k$ be the remaining number of firms (so k more firms must exit before the game finishes), and let $\underline{v}$ be the lowest possible remaining type conditional on all other firms having thus far followed (symmetric) equilibrium strategies. At any point of time we write $T(v; \underline{v}, k)$ for the additional amount of time a still-surviving firm of type v will wait before exiting if none of the other remaining $N+k$ firms exits beforehand, and $P(v; \underline{v}, k)$ for the firm's probability of being among the N ultimate survivors.

2 The general solution

Lemma 1: *In any equilibrium, $T(v; \underline{v}, k)$ is strictly increasing in v for all $\underline{v}$ and all k, and*

$$P(v; \underline{v}, k) = \sum_{j=k}^{N+k-1} \frac{(N+k-1)!}{(N+k-1-j)!j!} \left(\frac{F(v) - F(\underline{v})}{1 - F(\underline{v})}\right)^j \left(\frac{1 - F(v)}{1 - F(\underline{v})}\right)^{N+k-1-j} \tag{1}$$

Proof: See Appendix □.

Lemma 1 says that because higher-valued firms exit later in a symmetric equilibrium, a firm's probability of winning, $P(v; \underline{v}, k)$, is just the probability that it has one of the N highest values.

Lemma 2: *There is at most one equilibrium of the game.*
Proof: See Appendix. □.

The reason for Lemma 2 is that the difference between the expected surpluses of any two types is uniquely determined by standard incentive compatibility arguments.[17] But, since any type's probability of winning a prize

[17]The absolute level of a player's surplus cannot be determined prior to determining

is fixed by Lemma 1, the difference between the two types' waiting costs is therefore also uniquely determined. However, if there were two different equilibria specifying different quitting times $T(v;\underline{v},k)$, these two equilibria would yield different differences between types' waiting costs, for at least one pair of types.[18]

Lemma 3:[19] *The unique symmetric perfect Bayesian equilibrium*[20] *of the subgame in which just one more exit is required to end the game, and the lowest possible remaining type is* $\underline{v}$ *(assuming all players have thus far followed equilibrium strategies) is defined by*

$$T(v;\underline{v},1) = \int_{\underline{v}}^{v} Nxh(x)dx \tag{2}$$

Proof: See Appendix □.

The intuition is straightforward: at each moment the marginal firm with type v faces the prospect of paying an extra $T'(v;\underline{v},1)dv$ to outlast any firms with types between v and $v+dv$, and equates these costs to the value of being a winner, v, times the probability, $\dfrac{Nf(v)dv}{1-F(v)} = Nh(v)dv$, that one of the other N remaining firms will in fact be revealed to have a type below $v+dv$. So $T'(v;\underline{v},1) = Nvh(v)$. Furthermore $T(\underline{v};\underline{v},1) = 0$, since a player of type $\underline{v}$ will never win and so exits immediately. So $T(v;\underline{v},1) = 0 + \int_{\underline{v}}^{v} T'(x;\underline{v},1)dx = \int_{\underline{v}}^{v} Nxh(x)dx$.

We can now state our main result.

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the actual equilibrium because, in contrast to many problems in which the bottom type's surplus is fixed at zero, in our problem the bottom type receives negative surplus for $c > 0$.

[18]This is easy to see if $k = 1$ (i.e. when the game ends after one more quit). So the $k = 1$ subgame is unique. But then if $k = 2$, waiting costs are fixed after one more quit, so two different functions $T(v;\underline{v},2)$ would yield different differences in total waiting costs for some pair of types, so the $k = 2$ subgame is also unique. And so on.

[19]This result can also be found in Bliss and Nalebuff (1984), Nalebuff (1982), and elsewhere.

[20]For this ($k = 1$) case this is also the unique symmetric Nash equilibrium, since each firm knows that its decision to exit is only relevant in the case in which no other firm has previously exited, so the game is strategically equivalent to a static game in which firms simultaneously choose exit times.

Proposition: *In the unique symmetric perfect Bayesian equilibrium of the Generalized War of Attrition in which any player who has quit has flow costs that are c times as large as those of any remaining player, a player with value v will exit after an additional time*

$$T(v; \underline{v}, k) = c^{k-1} \int_{\underline{v}}^{v} N x h(x) dx \tag{3}$$

if no other player exits first, if $N + k$ players remain in contention for N prizes, and if the lowest-possible remaining type has valuation $\underline{v}$ for a prize (assuming all players have thus far followed equilibrium strategies).

Proof: Lemma 3 shows that if $k = 1$ any firm with type $\widetilde{v} < v$ (strictly) prefers to exit than stay. So long as $k > 1$, the firm is indifferent between staying and leaving, since leaving alters its cost per unit time to a fraction c of its pre-exit rate, but also multiplies $T'(\cdot;\cdot,\cdot)$ by $\frac{1}{c}$ so the number of types that are "gone through" per unit time is c times as great. So a firm with $\widetilde{v} < v$ has no incentive to deviate to follow v's strategy. Similarly a firm with $\widetilde{v} > v$ that deviated to mimic v's strategy would not be losing out while the game continued, since both this firm's costs and the rate at which the other firms were exiting would be multiplied by c, but when in the end only N other firms were left, the deviant would regret its deviation if $\widetilde{v}$ still exceeded the marginal v and would be indifferent otherwise. So a firm with $\widetilde{v} > v$ prefers not to deviate. □.

In the general solution types leave at the rate

$$\frac{1}{T'(v; \underline{v}, k)} = \frac{1}{c^{k-1} N v h(v)} \tag{4}$$

per unit time, when k more firms are required to leave and the marginal remaining type is v. Note that this is similar to the standard case in which only a single exit is required ($k = 1$), except that each firm that leaves "slows down" the game to c times its previous speed, so that we "go through" the same number of types in $\frac{1}{c}$ times as much time. For example, if $N = 1, K = 3$ and $c = \frac{1}{2}$, the equilibrium goes through types four times as fast as in the two firm game ($N = K = 1$) until one firm drops out, then twice as fast as the two firm game until a second firm drops out, and then finally at the speed of the two firm game until the final exit.

Notice that a feature of the equilibrium is that the $K-1$ lowest-valued firms are actually indifferent about staying past their equilibrium dropout points; each would be willing to delay until $K-1$ others have quit (assuming each thought the others were following the equilibrium strategies). Of course, if any one of these firms were to delay its departure until $K-1$ others had left, that would speed the game and benefit everyone else.[21]

Note also that, by contrast, the highest-valued losing firm (the only loser in the standard $N+1$ firm model) would hurt everyone else by delaying its exit, so the equilibrium length of the game is non-monotonic in players' valuations. For example, a game with two "tough" players and one "weak" player competing for one prize takes longer than either a game with three tough players or a game with one tough player and two weak players.

Since, from (3), the expected time between the exits of the $(j+1)^{st}$ and j^{th} highest-value firms (who have actual values v_{j+1} and v_j) is

$$E\left\{c^{j-(N+1)}\int_{v_{j+1}}^{v_j} Nxh(x)dx\right\}, \tag{5}$$

it follows that

Corollary: *The expected time taken to reduce from $N+k$ firms to $N+k-1$ firms is*

$$Nc^{k-1}\frac{E(v_{N+k})}{N+k}$$

and the expected length of the Generalized War of Attrition is

$$N\sum_{j=N+1}^{N+K} c^{j-(N+1)}\frac{E(v_j)}{j} \tag{6}$$

in which $E(v_j)$ is the expected j^{th} highest value of the $N+K$ firms' values.

Note that the expected time between successive departures increases in later stages for three separate reasons. First there are fewer players who

[21] Of course, if $c>1$ each firm that leaves speeds up the game, so leaving late hurts others.

might leave ($N+k$ falls), second the remaining players are stronger ($E(v_{N+k})$ rises as k falls), and third each exit must slow the game (c^{k-1} rises as k falls) in order to make the next firm which drops out indifferent between paying the full costs of remaining in the game a little longer or paying the lower costs per period of being out.

The Appendix offers a purely algebraic proof of the Corollary. However, an approach that is more economic (and economical) is to consider a game in which, after all but j players have been revealed as in our problem, the remaining players fight a standard one-stage war of attrition for $j-1$ prizes. Since this game requires just one more exit, lemma 3 tells us that the time until the lowest of the j remaining firms (with value v_j) quits is $\int_{v_{j+1}}^{v_j}(j-1)xh(x)dx$. But by the Revenue Equivalence Theorem the expected costs per player must be the same as in an English auction, in which $(j-1)$ players win at price v_j, that is, $E\{(j-1)v_j/j\}$.[22] So, $E\left\{\int_{v_{j+1}}^{v_j} xh(x)dx\right\} = \frac{E(v_j)}{j}$, and substitution in (5) and simple summation yields the corollary.

It is easy to extend the model to allow firms' costs to be a function of k.[23] If costs are ℓ_k times as great as in our model when k more firms are required to exit, then types leave ℓ_k times as fast as in (4) at any point of time. Thus the total costs firms incur in the war of attrition are independent of ℓ_k.

It is also easy to see that discounting would have no effect on how firms play the game at any moment of time, since discounting is just equivalent to there being some exogenous flow of probability that the game will end and firms will stop accruing further costs or benefits. So our results and our formulae for $T(v;\underline{v},k)$ are unchanged by discounting, but discounting makes the costs of the war of attrition even greater relative to the discounted value of the prizes.

[22]We can use the Revenue Equivalence Theorem because in both a one-stage war of attrition and an English auction a player who quits immediately receives an expected surplus of zero.

[23]For example, in an oligopoly context losses are probably increasing in k.

3 The special case $c = 0$: " Twoness"

In the limit as c approaches 0, firms drop out arbitrarily fast until only $N+1$ remain. That is, if N firms can be profitable in a market and dropouts pay no costs after exiting, then competition in the symmetric equilibrium will immediately shake out to $N+1$ firms. We call this property "twoness" because when there is just one winner, competition effectively reveals the third-highest value, that is, v_3, immediately, and then yields a conventional two firm game beginning with $\underline{v} = v_3$.

An alternative way of deriving this result that should appeal to auction theorists is to consider the expected total costs paid by the remaining two firms after the buyer with the third-highest value drops out. The Revenue Equivalence Theorem tells us that these costs must be the same as the expected costs in a second-price auction between these firms, namely the expectation of the second-highest value, v_2.[24] Compare this with the expected total costs paid by all the firms in the initial game with K eventual losers. Again by Revenue Equivalence with the second-price auction, total expected costs must be the expectation of v_2 in the initial game. So the expected costs paid to get from the initial game to the subgame with two firms remaining must be zero.[25]

To understand the result observe that if there were positive delay while $K > 1$ exits were still required, then a firm that quit ε earlier than it had originally planned would save ε in waiting costs but would reduce its probability of winning by an amount of order ε^K. So all firms would drop out at least a little earlier than planned, so firms must in fact quit without delay

[24] See Myerson (1981) and Riley and Samuelson (1981) for the earliest statements of the Revenue Equivalence Theorem. Our 1994 paper, Bulow and Klemperer (1994), states and applies the theorem for a setting which, like this one, has multiple objects that are sold in a dynamic game.

[25] The reason this logic only holds when $c = 0$ is that this assures that the expected surplus of a firm with type $\underline{v}$ is zero, since the firm can exit immediately at no further cost. This in turn guarantees that the expected surplus of a firm with type v equals $\int_{\underline{v}}^{v} P(x; \underline{v}, k)dx$ (see equation (10) in the appendix), and hence that the expected total costs paid by all firms (which must equal the sum of the expected gross income to the survivors less the sum of firms' expected surpluses) are the same regardless of whether there is a second-price auction or a symmetric war of attrition in any subgame. If $c > 0$, then the expected surplus of a firm with the lowest possible type is negative, so the war of attrition will be more costly to the firms, in expectation, than a second-price auction.

until only a single firm remains in excess of the number who can ultimately survive.

Another way to understand the result is to consider the limiting case of our problem in which all firms' values are equal (i.e., every firm i has value $v^i = \underline{V} = \overline{V}$), so there is no private information. Then the symmetric equilibrium (again in continuous time) is in mixed strategies, and all firms mix across all possible dropout times.[26] In this case, if a firm survives to be one of the final $N+1$ firms in the market, its expected future payoff is zero (since it is indifferent to dropping out immediately). But it can also earn zero by dropping out at the beginning of the entire game. Therefore, firms will only be willing to wait to become one of the final $N+1$ firms if the cost of waiting is zero - that is, the time that it takes to reduce the field to $N+1$ must be zero regardless of K.

4 The special case $c = 1$: " Strategic Independence"

Now consider the special case in which all firms pay full costs until the game is resolved. This is the polar opposite of "twoness", and can be thought of as the standards case, where all firms lose until a standard is established, with losses independent of whether a firm is one of the remaining competitors for establishing the standard.

When $c = 1$, the Proposition yields that types leave at the rate

$$\frac{1}{T'(v; \underline{v}, k)} = \frac{1}{Nvh(v)}$$

when the marginal remaining type is v, so we have "strategic independence". That is, each firm chooses the same dropout time as if it were in a game with just $N+1$ firms and N prizes, independent of the actual number of remaining firms. Having chosen its dropout time at the beginning of the game, the firm then sticks with it.

[26]See Fudenberg and Tirole (1991, p230-232) for discussion of how the mixed strategy equilibrium of the two-player war of attrition with complete information corresponds to the equilibria of wars of attrition with incomplete information in which every type plays a (different) pure strategy.

The intuition is that because a firm's flow costs are unaffected by dropping out before the end of the game, and its probability of winning is also unaffected (to first order) by small changes in its exit time when $K > 1$ exits are still required to end the game, the firm's exit decision cannot affect the length of the game. So other firms' decisions are unaffected by this firm's actions.

As for "twoness", it may help in understanding this result to consider the mixed strategy equilibrium of the limiting case of our game in which all firms' values are known to be equal. In this case firms must be indifferent to dropping out at any time prior to the end of the game. Thus a firm must be indifferent between dropping out now, when more than $N + 1$ firms remain, or waiting until exactly $N + 1$ remain and then dropping out immediately, or dropping out at any time in between. Since in any of these cases the firm does not win and pays costs proportional to the length of the game, the length of the game must be independent of the firm's choice. So the dropout decisions of the first $K - 1$ firms to exit do not affect the decisions of the remaining firms.

The length of the "strategic independence" game is strictly a function of the $N + 1^{st}$ highest value, but the larger K is the longer the game will take in expectation, because the expected value of the $N + 1^{st}$ highest value rises as $N + K$ rises.

Both "twoness" and "strategic independence", take equally long to reduce from $N+1$ firms down to N. The difference is that in "twoness" we get down to $N + 1$ firms immediately, and so only have to incur costs running through the types between the $N + 2^{nd}$ highest value and $N + 1^{st}$ highest value. However, with "strategic independence" all types must be run through in real time, and the amount of time required for the industry to shake down from $N + K$ firms to $N + 1$ may far exceed the time needed to get from $N + 1$ to N.

5 An example

We illustrate our model with an example inspired by the battle over the 1993 budget bill.

Assume for simplicity that there are 51 senators who would each like to see a bill passed but would prefer not to have to vote for it. Each senator has a value independently drawn from a uniform distribution on $[0,\overline{V}]$ of being the one person who need not vote for the bill. We normalise units of time so that the costs of holding out are 1 per unit time for those who have not yet pledged support.

We begin with the "strategic independence" case in which there are no private costs, so all 51 senators suffer political costs equally until the impasse is resolved, whether or not they have themselves given in and agreed to support the President. In this case $c = 1$, so since for the uniform distribution $E(v_j) = (1 - \frac{j}{N+K+1})\,\overline{V}$, the expected delay before passage of the bill is, by our Corollary,

$$N\overline{V}\sum_{N+1}^{N+K}\left(\frac{1}{j} - \frac{1}{N+K+1}\right) \tag{7}$$

which with $N = 1$ and $K = 50$ yields

$$\overline{V}\left(-\frac{50}{52} + \sum_{j=2}^{51}\frac{1}{j}\right) \approx \overline{V}\left(-2 + \sum_{j=1}^{53}\frac{1}{j}\right) \approx \overline{V}(-2 + \log 53 + \gamma)$$

in which γ ($\approx .58$) is the Euler number. Thus the total expected time to completion in the "strategic independence" case is $2.55\,\overline{V}$, so on average even the "winner" suffers costs that are more than two and one half times as great as his prize.[27]

This is an indication that players should do everything they possibly can to change such games.[28] For example, even a senator with a value of $\overline{V}$

[27] Of course, this does not mean that a player should refuse to play. It is common knowledge that each player anticipates negative surplus relative to the bill passing immediately with his vote, but legislators may still obtain positive surplus relative to the bill not passing at all.

In fact, the legislators would be in an even worse game than this if $c > 1$. For example, if pledging one's vote increases the lobbying pressure and hostility one faces from those opposed to the bill, then costs are largest for those who have already pledged.

[28] This may help explain the institution of whips, whose job it is to determine the allocation of prizes (that is, who should be permitted not to vote for a particular measure)

would be better off if he could somehow persuade a random group of 34 other senators to concede with him.[29]

At the other extreme, consider the "twoness" case in which a senator's only costs are his personal costs of withstanding administration pressure, and these costs stop as soon as he knuckles under. In this case, as we have seen, 49 senators give in to the administration immediately, while the two with the highest values hold out in a standard two-player war of attrition. Here $c = 0$, so using (6) the total expected time is just half of the expected value of the lower of the holdouts, $\frac{1}{2}E(v_2)$, that is, since $F(\cdot)$ is uniform, $\frac{25}{52}\overline{V}$.[30]

Table 1 shows the expected length of the game for $N = 1$ and $N = 5$, with values drawn uniformly on $[0, 1]$, for $c = 0$, $\frac{1}{2}$, and 1, and $K = 1$, 2, 3, 10, 50, 100 and 500.[31]

In the "twoness" case, of course, all the time is spent waiting for the last vote, but observe that even in the "strategic independence" case, the first few votes come in relatively quickly (the first 10 votes take less than 1% of the total time on average) while the last few votes take much longer (the last 4 votes take almost half the total time in expectation). The typical case will lie between these extremes. Thus our model both explains why political decision making can sometimes take so long, and why, even when agreements seem close to complete, the hunt for the last few votes can often

without recourse to a war of attrition. Whips will not necessarily select the highest valuers of the prizes but even uninformed randomisation is highly desirable relative to these games. Whips seem to be more effective in resolving smaller issues than larger ones, such as the budget bill, for which it is harder to persuade losers that they will be compensated on future issues.

[29]This would reduce the costs to the player because his own exit from the game saves money, and the possible exit of the player with the second- or third-highest value also saves him money, since it is the value of the second-highest participant that determines the length of the game.

[30]We can also obtain this result directly without need of (6) by using the Revenue Equivalence Theorem which says that the expected total costs incurred (that is, twice the expected time) must be the same as in an English auction, that is, $E(v_2) = \frac{50}{52}\overline{V}$. (Note that the Revenue Equivalence Theorem applies here because a player who quits immediately gets zero surplus in this case.)

[31]For the uniform case with $c = 1$, it is easy to use (7) to check that if $N = 1$ and K is large the expected time $\approx \overline{V}(\log K + \gamma - 2)$ while if both N and K are large the expected time $\approx N\overline{V}\left(\log\left(\frac{N+K}{N}\right) - \frac{K}{N+K}\right)$. With $c = 0$, the expected time (using (6)) is $\overline{V}\left(\frac{NK}{(N+1)(N+K+1)}\right)$. For $c \in (0, 1)$ a weak upper bound on the expected time is $\left(\frac{\overline{V}}{1-c}\right)\left(\frac{K}{N+K}\cdot\right)$

seem so excruciatingly slow.[32]

6 Conclusion

We study wars of attrition in which two or more players must exit. Except in the final stage, a player's departure will not end the game, and a player may continue to incur at least some costs even after he has conceded. Therefore, except in the final stage, by the time a player exits he knows a small delay in conceding will not allow him to win, because it is so unlikely that two or more others will exit before him. So the player becomes solely interested in minimizing his costs, and in equilibrium a small change in exit strategy must have no effect on his expected costs.

If costs are as high for those who have already conceded as for those who continue to fight, then there is no incentive to drop out early. For a player to be satisfied with his equilibrium exit time, his departure must neither speed nor slow the ultimate resolution of the game. So each player's exit behavior is unaffected by the number of other competitors and their actions. We call this "strategic independence." Examples where this may occur are battles over standard setting and building voting majorities.

If a player does not pay any costs after he has conceded, as in a natural monopoly game, the market will immediately and efficiently sort down to the final stage. We refer to this property as "twoness", since in equilibrium we should never observe more than two competitors in such games if only one will win.

Even with strategic independence, departures occur at a faster rate in the early stages, because at the beginning of the game there are more, and weaker, players who might concede. If costs are lower for those who have exited than for those still fighting, so there is an incentive to depart early, this effect becomes even more pronounced. Each exit must slow the game sufficiently to make the next dropout indifferent between paying the full costs of remaining in the game a little longer or paying the lower costs per period of being out. Of course in the limit with twoness all but the last departure is

[32]Multilateral international treaties (e.g. GATT) often seem to illustrate this.

instantaneous. So the model explains why rounding up most of the necessary votes for a bill might take very little time, but gathering the last few votes may be time-consuming and costly.

Appendix

We write $C(v; \underline{v}, k)$ for the firm's expected future delay costs over the whole of the rest of the game (net of delay costs thus far incurred), and $S(v; \underline{v}, k)$ for the firm's expected future surplus over the whole of the rest of the game (so $S(v; \cdot, \cdot) \equiv vP(v; \cdot, \cdot) - C(v; \cdot, \cdot)$).

Proof of Lemma 1: A higher-value type of a firm cannot exit before a lower-value type of the same firm would exit. (If a low type gets the same expected surplus from strategies with two different probabilities of being an ultimate survivor, the high type strictly prefers the high-probability strategy, so the high type cannot choose a strategy with a lower probability of survival than the low type.) Also, at no moment of time does any firm exit with strictly positive probability. (By symmetry, all firms would have strictly positive probability of exit, but then any firm would strictly prefer exiting just after this time to exiting at this time). So $T(\cdot; \cdot, \cdot)$ is strictly increasing in v for all $\underline{v}$ and k, and a firm ultimately survives if and only if k or more of the remaining $N + k - 1$ other current survivors have lower values than it. □

Proof of Lemma 2:[33] The proof is by induction. We assume that there is at most one equilibrium of any subgame in which there are $N + k - 1$ firms left, and show that this implies at most one equilibrium with $N + k$ firms remaining.

Consider the subgame defined by k and $\underline{v}$. (This is well-defined by Lemma 1). There is no finite period of time in which there is zero probability of exit. (If there was, then a type that was due to exit at the end of this period would do better to exit at the beginning of this period; because there is a unique equilibrium after the next exit, its time cannot affect the subsequent development of the game.) So $T(v; \underline{v}, k)$ is continuous, and

$$T(\underline{v}; \underline{v}, k) = 0 \tag{8}$$

so also

$$S(\underline{v}; \underline{v}, k) = -C(\underline{v}; \underline{v}, k). \tag{9}$$

Now note that since in equilibrium no type of firm can gain by following any other type's exit rule,

$$S(v^a; \underline{v}, k) \geq S(v^b; \underline{v}, k) + P(v^b; \underline{v}, k)(v^a - v^b) \qquad \text{for all} \qquad v^a, v^b \in [\underline{v}, \overline{V}].$$

So $S(v; \underline{v}, k)$ has derivative $dS/dv = P(v; \underline{v}, k)$ and therefore

$$S(v; \underline{v}, k) = S(\underline{v}; \underline{v}, k) + \int_{\underline{v}}^{v} P(x; \underline{v}, k)dx \tag{10}$$

[33] This is the most elegant proof we know. An alternative, but here rather cumbersome, approach is to first show the monotonicity and continuity of $T(\cdot; \cdot, \cdot)$ in v and using these show the differentiability of $T(v)$, so that the first-order conditions characterize the equilibrium uniquely. (As in our proof, an inductive argument is required; for arguments along these alternative lines see the Appendix of Gul and Lundholm (1995) or our own working paper (joint with Huang), Bulow et al (1996).)

and, noting (9),

$$C(v;\underline{v},k) = C(\underline{v},\underline{v},k) + vP(v;\underline{v},k) - \int_{\underline{v}}^{v} P(x;\underline{v},k)dx. \tag{11}$$

So (11) and (1) uniquely determine $C(v;\underline{v},k)$, since $C(\underline{v};\underline{v},k)$ equals c times the expected length of the subgame after $\underline{v}$ quits and leaves $(N+k-1)$ firms remaining, and the equilibrium of this subgame is, by assumption, unique.

We can now show that there is at most one equilibrium $T(v;\widehat{v},k)$. Suppose instead that there are two equilibria with $\widetilde{T}(\overline{v};\widehat{v},k) < \widetilde{\widetilde{T}}(\overline{v};\widehat{v},k)$ for some $\overline{v}$. Then by the continuity of $\widetilde{T}(\cdot;\cdot,\cdot)$ and $\widetilde{\widetilde{T}}(\cdot;\cdot,\cdot)$, there exists $\underline{v} \in [\widehat{v},\overline{v})$ such that $\widetilde{T}(\underline{v};\cdot,\cdot) = \widetilde{\widetilde{T}}(\underline{v};\cdot,\cdot) = \tau$ and $\widetilde{T}(v;\cdot,\cdot) < \widetilde{\widetilde{T}}(v;\cdot,\cdot)$ for all $v \in (\underline{v},\overline{v}]$. But if $\widetilde{T}(v;\widehat{v},k)$ and $\widetilde{\widetilde{T}}(v;\widehat{v},k)$ are both equilibria, then $\widetilde{T}(v;\underline{v},k) = \widetilde{T}(v;\widehat{v},k) - \tau$ and $\widetilde{\widetilde{T}}(v;\underline{v},k) = \widetilde{\widetilde{T}}(v;\widehat{v},k) - \tau$ must be equilibria of the subgame defined by $\underline{v}$ and k. But then $\widetilde{T}(v;\underline{v},k) < \widetilde{\widetilde{T}}(v;\underline{v},k)$ for all $v \in (\underline{v},\overline{v}]$, so any $v \in (\underline{v},\overline{v}]$ would expect lower waiting costs under $\widetilde{T}(\cdot;\cdot,\cdot)$ than under $\widetilde{\widetilde{T}}(\cdot;\cdot,\cdot)$ before the next drop out (whether by this firm or another firm) and the same waiting costs thereafter, since by assumption equilibrium is unique after the next dropout. But this contradicts the fact that $C(v;\underline{v},k)$ is the same for both equilibria $\widetilde{T}(v;\underline{v},k)$ and $\widetilde{\widetilde{T}}(v;\underline{v},k)$. This completes the inductive step, and so proves the result, since it holds trivially when just N firms remain. □.

Proof of Lemma 3: Given that all other firms use this exit rule, the expected future surplus of a type v who behaves as a type v^* is

$$U(v,v^*) = -(1-P(v^*;\underline{v},1))T(v^*;\underline{v},1) + \int_{\underline{v}}^{v^*} \frac{\partial P(x;\underline{v},1)}{\partial x}(v - T(x;\underline{v},1))dx$$

in which the first and second terms are v's payoffs from the events that he quits and survives, respectively, $P(v^*;\underline{v},k)$ is defined by (1) and equals $1-\left(\frac{1-F(v^*)}{1-F(\underline{v})}\right)^N$ when $k=1$, and $\frac{\partial P(x;\underline{v},1)}{\partial x} = \frac{Nf(x)}{1-F(x)}\left(\frac{1-F(x)}{1-F(\underline{v})}\right)^N$ is the density with which v wins and the lowest of the other N firms is of type x. Thus (2) satisfies v's first-order condition

$$\begin{aligned} \frac{\partial U}{\partial v^*}(v,v^*) &= 0 \Rightarrow \\ - & (1-P(v^*;\underline{v},1))T'(v^*;\underline{v},1) + \frac{\partial P(v^*;\underline{v},1)}{\partial v^*}v \\ = & \left[-\left(\frac{1-F(v^*)}{1-F(\underline{v})}\right)^N \left\{T'(v^*;\underline{v},1) - \frac{Nvf(v^*)}{1-F(v^*)}\right\}\right] = 0 \end{aligned}$$

at $v^* = v$. It satisfies the second-order conditions since it implies

$$sign\frac{\partial U}{\partial v^*}(v,v^*) = sign(v - v^*).$$

And it also satisfies the boundary condition, (8), $T(\underline{v};\underline{v},1) = 0$. □.

Algebraic Proof of Corollary:

Using (5), the expected length of the game equals

$$\sum_{j=N+1}^{N+K} E\left\{c^{j-(N+1)}\int_{v_{j+1}}^{v_j} Nxh(x)dx\right\}$$

(in which $v_{N+K+1} \equiv \underline{V}$). Now,

$$E\int_{\underline{V}}^{v_j} Nxh(x)dx$$

$$= \lim_{\varepsilon\to 0}\int_{\underline{V}}^{\overline{V}-\varepsilon}\binom{N+K-1}{j-1}(F(v))^{N+K-j}(1-F(v))^{j-1}(N+K)f(v)\int_{\underline{V}}^{v} Nxh(x)dxdv$$

which, integrating by parts,

$$=\lim_{\varepsilon\to 0}\left\{\left[\left[\sum_{i=N+K+1-j}^{N+K}\binom{N+K}{i}(F(v))^{i}(1-F(v))^{N+K-i}\right]\int_{\underline{V}}^{v} Nxh(x)dx\right]_{\underline{V}}^{\overline{V}-\varepsilon}\right.$$

$$\left.-\int_{\underline{V}}^{\overline{V}-\varepsilon}\left[\sum_{i=N+K+1-j}^{N+K}\binom{N+K}{i}(F(v))^{i}(1-F(v))^{N+K-i}\right]Nvh(v)dv\right\}.$$ [34]

So

$$E\int_{v_{j+1}}^{v_j} Nxh(x)dx$$

$$= E\int_{\underline{V}}^{v_j} Nxh(x)dx - E\int_{\underline{V}}^{v_{j+1}} Nxh(x)dx$$

$$=\lim_{\varepsilon\to 0}\left\{\int_{\underline{V}}^{\overline{V}-\varepsilon}\binom{N+K}{N+K-j}(F(v))^{N+K-j}(1-F(v))^{j}Nvh(v)dv+\right.$$

$$\left.\left[\binom{N+K}{N+K-j}(F(v))^{N+K-j}(1-F(v))^{j}\int_{\underline{V}}^{v} Nxh(x)dx\right]_{\underline{V}}^{\overline{V}-\varepsilon}\right\}$$

[34] We are being careful to take $\lim_{\varepsilon\to 0}\int^{\overline{V}-\varepsilon}$ since $\lim_{\varepsilon\to 0}\int^{\overline{V}-\varepsilon} Nxh(x)dx$ may be ∞.

which after noting that the second term is zero,[35] and substituting $h(v) \equiv \frac{f(v)}{1-F(v)}$ and $\binom{N+K}{N+K-j} = \frac{N+K}{j}\binom{N+K-1}{j-1}$into the first term,

$$= \int_{\underline{V}}^{\overline{V}} \binom{N+K-1}{j-1} (F(v))^{N+K-j}(1-F(v))^{j-1}(N+K)f(v)\frac{N}{j}dv,$$

$$= \frac{N}{j}E(v_j).$$

So the expected length of the Generalized War of Attrition is therefore

$$N\sum_{j=N+1}^{N+K} c^{j-(N+1)}\frac{E(v_j)}{j}. \quad \square.$$

[35]To show the second term is zero, it suffices to show that

$$\lim_{\varepsilon\to 0}\left[1-F(\overline{V}-\varepsilon)\right]\int_{\underline{V}}^{\overline{V}-\varepsilon} Nxh(x)dx = 0.$$

A careful proof of this is as follows: define

$$g_\varepsilon(x) = \begin{cases} [1-F(\overline{V}-\varepsilon)]Nxh(x) & \text{for } \underline{V} < x < \overline{V}-\varepsilon \\ 0 & \text{for } \overline{V}-\varepsilon < x < \overline{V}. \end{cases}$$

So $[1-F(\overline{V}-\varepsilon)]\int_{\underline{V}}^{\overline{V}-\varepsilon} Nxh(x)dx = \int_{\underline{V}}^{\overline{V}} g_\varepsilon(x)$. Also note that $g_\varepsilon(x) \leq Nxf(x) = G(x)$ for all $x \in [\underline{V},\overline{V}]$, as $\frac{1-F(\overline{V}-\varepsilon)}{1-F(x)} \leq 1$ for 0< $x < \overline{V}-\varepsilon$. Now note that for all $x \in [\underline{V},\overline{V}]$ we have $\lim_{\varepsilon\to 0} g_\varepsilon(x) = 0$. Finally note that $g_\varepsilon \leq g'_\varepsilon$ for $\varepsilon < \varepsilon'$. This implies that for any $\delta > 0$ we can find $\varepsilon_0 > 0$, such that $g_\varepsilon(x) < \delta$ for all $0 < x < \overline{V}-\delta$ and all $\varepsilon < \varepsilon_0$. Therefore we can deduce $0 < \int_{\underline{V}}^{\overline{V}} g_\varepsilon(x) < \delta\overline{V} + \int_{\overline{V}-\delta}^{\overline{V}} G(x)$. As we can get the right-hand side as small as we desire, we obtain finally

$$\lim_{\varepsilon\to 0}\int_{\underline{V}}^{\overline{V}} g_\varepsilon(x) = 0.$$

TABLE 1

Expected Length of War of Attrition

		K = 1	**K = 2**	**K = 3**	**K = 10**	**K = 50**	**K = 100**	**K = 500**
N = 1	**c = 0**	0.1667	0.25	0.3	0.4167	0.4808	0.4902	0.498
	c = $\frac{1}{2}$	0.1667	0.2917	0.3792	0.6059	0.7341	0.753	0.7686
	c = 1	0.1667	0.3333	0.4833	1.1865	2.5573	3.2169	4.7987
N = 5	**c = 0**	0.119	0.2083	0.2778	0.5208	0.744	0.7862	0.8235
	c = $\frac{1}{2}$	0.119	0.253	0.3745	0.8488	1.2951	1.3794	1.454
	c = 1	0.119	0.2976	0.506	2.0495	7.0871	10.046	17.656

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