

## The Gent–McWilliams Skew Flux

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(Manuscript received 5 May 1997, in final form 6 August 1997)

### ABSTRACT

This paper formulates tracer stirring arising from the Gent–McWilliams (GM) eddy-induced transport in terms of a skew-diffusive flux. A skew-diffusive tracer flux is directed normal to the tracer gradient, which is in contrast to a diffusive tracer flux directed down the tracer gradient. Analysis of the GM skew flux provides an understanding of the physical mechanisms prescribed by GM stirring, which is complementary to the more familiar advective flux perspective. Additionally, it unifies the tracer mixing operators arising from Redi isoneutral diffusion and GM stirring. This perspective allows for a computationally efficient and simple manner in which to implement the GM closure in  $z$ -coordinate models. With this approach, no more computation is necessary than when using isoneutral diffusion alone. Additionally, the numerical realization of the skew flux is significantly smoother than the advective flux. The reason is that to compute the skew flux, no gradient of the diffusivity or isoneutral slope is taken, whereas such a gradient is needed for computing the advective flux. The skew-flux formulation also exposes a striking cancellation of terms that results when the GM diffusion coefficient is identical to the Redi isoneutral diffusion coefficient. For this case, the horizontal components to the tracer flux are aligned down the horizontal tracer gradient, and the resulting computational cost of Redi diffusion plus GM skew diffusion is roughly half that needed for Redi diffusion alone.

### 1. Introduction

Gent and McWilliams (1990, GM hereafter) and Gent et al. (1995, hereafter GWMM) suggested a closure for the tracer equation to be used in ocean models. With this closure, certain adiabatic stirring effects from ocean mesoscale eddies are encapsulated by a divergence-free eddy-induced velocity. The GM velocity incorporates that aspect of baroclinic eddies representing the transfer of available potential energy to eddy kinetic energy. It has been noted in various atmospheric contexts (e.g., Plumb 1979; Plumb and Mahlman 1987) that eddy-induced transport velocities are generally equivalent to antisymmetric components in the tracer mixing tensor. This mixing will not alter any of the tracer moments as long as no-normal flow, or equivalently no-flux, boundary conditions are applied to the corresponding advective or skew-diffusive tracer flux. In this sense, the mixing is nondissipative, reversible, and sometimes referred to as “stirring” (Eckart 1948).

Prior to the work of GM, Redi (1982) (see also Solomon 1971) noted that a symmetric component to the mixing tensor should be present in order to represent irreversible downgradient diffusive effects of various subgrid-scale processes. The orientation of the diffusive

flux is down the tracer gradient as it occurs along the neutral directions. The result is to align the tracer parallel to the neutral direction in the process of dissipating all tracer moments except the mean. Such diffusion will not affect locally referenced potential density. Therefore, isoneutral diffusion will not change the system’s available potential energy. More discussion of isoneutral diffusion, and references, can be found in the companion paper by Griffies et al. (1998, hereafter referred to as GGPLDS).

Gent and McWilliams stirring and Redi diffusion form a framework in which many coarse-resolution ocean models parameterize the mixing of tracers. Currently, there is a great deal of energy focused on understanding the implications and relevance of this framework for simulating ocean circulation. There have been notable improvements in the simulations (e.g., Danabasoglu and McWilliams 1995; Hirst and McDougall 1996) and yet there have also been some rather tentative results (e.g., England 1995; England and Holloway 1996; Duffy et al. 1995). In addition to realistic coarse-model simulations with the GM and Redi parameterizations, there is an increasing number of theoretical and idealized studies aimed at clarifying certain of the conceptual issues (e.g., Held and Larichev 1996; McDougall and McIntosh 1996; Tandon and Garrett 1996; Holloway 1997; Treguier et al. 1997; Visbeck et al. 1997; Greatbatch 1998; Killworth 1998, Gille and Davis 1997, manuscript submitted to *J. Phys. Oceanogr.*; Dukowicz and Smith 1997).

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This paper does not resolve any of the outstanding issues. Rather, it simply endeavors to bring the Redi and GM ideas onto an equal footing so that certain of their mathematical and physical properties can be directly compared and contrasted. The purpose of such an effort is twofold: First, the results presented here are arguably the simplest conceptual framework for thinking about the individual and combined effects of GM stirring and Redi diffusion (see also Holloway 1997). This framework may be useful when examining the effects of these subgrid-scale parameterizations on ocean density and tracer fields. Second, and most pragmatically, these results provide an almost trivial manner for which to implement GM and Redi in  $z$ -coordinate ocean models. The key element in this effort is the skew-diffusive flux (e.g., Plumb 1979; Moffatt 1983; Middleton and Loder 1989) arising from the GM closure. The perspective engendered by the GM skew flux provides some useful insights, which can be considered complementary to the more familiar advective flux formulation of GWMM.

The plan of this paper is the following. General kinematical notions of skew fluxes are presented in section 2. Properties of the GM skew flux are discussed in section 3. The combined effects of GM skew diffusion and Redi diffusion are given in section 4, and numerical considerations are presented in section 5. Summary and conclusions are provided in section 6.

## 2. The advective flux and skew-diffusive flux

The results in this paper depend on the mathematical and physical equivalence of the stirring operator obtained by taking the divergence of either an advective flux or its corresponding skew-diffusive flux. The mathematical details of this equivalence are described in this section. Further discussion of these points in an oceanographic context can be found in Middleton and Loder (1989) and McDougall and McIntosh (1996). The notation used here is consistent with that used by GGPLDS. Most notably, the summation convention is followed in which repeated indexes are summed over the three spatial directions.

Let us begin with the tracer equation written in the form

$$(\partial_t + \mathbf{u} \cdot \nabla)T = R(T), \quad (1)$$

where the tracer mixing operator  $R(T)$  is given in an orthogonal coordinate system by

$$R(T) = \partial_m (J^{mn} \partial_n T). \quad (2)$$

Here  $\mathbf{J}$  is a second-order tracer mixing tensor whose contravariant components are written as  $J^{mn}$ ;  $T$  represents a tracer such as potential temperature, salinity, or a passive tracer; and  $\mathbf{u}$  is a divergence-free Eulerian velocity ( $\nabla \cdot \mathbf{u} = 0$ ) representing the resolved currents in the model.

It is useful to split the mixing tensor  $J^{mn}$  into its symmetric,  $2K^{mn} = J^{mn} + J^{nm}$ , and antisymmetric,  $2A^{mn}$

$= -2A^{nm} = J^{mn} - J^{nm}$ , parts since they parameterize physically distinct mixing processes. For diffusive or dissipative mixing,  $K^{mn}$  is positive semidefinite. For downgradient isoneutral diffusion,  $K^{mn}$  is the Redi (1982) diffusion tensor. Kinematical aspects of isoneutral diffusion are discussed in GGPLDS. The kinematics of the antisymmetric stirring tensor  $A^{mn}$ , and the resulting stirring operator  $R_A(T) = \partial_m (A^{mn} \partial_n T)$ , are central to the development in this paper, and so are established here.

There are two mathematically equivalent ways to construct the stirring operator  $R_A(T)$ . One is through computing the convergence of an advective tracer flux, which is based on the observation that

$$\begin{aligned} R_A(T) &= (\partial_m A^{mn}) \partial_n T \\ &= \partial_n (\partial_m A^{mn} T) \\ &= -\partial_n (U_*^n T), \end{aligned} \quad (3)$$

where  $A^{mn} \partial_m \partial_n T = 0$  and  $\partial_m \partial_n A^{mn} = 0$  were used to establish this result, and these two identities follow solely from the antisymmetry of  $A^{mn}$ . This manipulation allows for the identification of an advective tracer flux

$$F_{\text{adv}}^n = U_*^n T, \quad (4)$$

whose convergence yields the stirring operator  $R_A(T) = -\nabla \cdot \mathbf{F}_{\text{adv}}$ . The corresponding tracer transport velocity

$$U_*^n = -\partial_m A^{mn} \quad (5)$$

has zero divergence since  $\partial_n U_*^n = -\partial_n \partial_m A^{mn} = 0$ . With no-normal flow boundary conditions placed on the transport velocity, the resulting stirring operator will conserve all tracer moments, just as occurs for transport by the Eulerian current  $\mathbf{u}$ .

For purposes of comparison with the skew flux to be derived below, it is useful to introduce a vector streamfunction  $\boldsymbol{\psi}$ , whose components are related to the antisymmetric tensor through  $A^{mn} = \epsilon^{mnp} \psi_p$ , with  $\epsilon^{mnp}$  the Levi-Civita or alternating tensor.<sup>1</sup> The vector streamfunction  $\boldsymbol{\psi}$  has only two functional degrees of freedom since the third degree of freedom can always be gauged away (gauge freedom is discussed in the next section). Written in terms of  $\boldsymbol{\psi}$ , the advective flux takes the form

$$\mathbf{F}_{\text{adv}} = T(\nabla \times \boldsymbol{\psi}). \quad (6)$$

The second, completely equivalent, form for the stirring operator is given by the convergence of a skew-diffusive flux (Moffatt 1983; Middleton and Loder 1989)

<sup>1</sup> Here  $\epsilon^{123} = 1$ , as does any even permutation of 1, 2, 3;  $\epsilon^{213} = -1$ , as does any odd permutation of 1, 2, 3;  $\epsilon^{rst}$  vanishes if any two labels are the same.

$$\begin{aligned} R_A(T) &= \partial_m(A^{mn}\partial_n T) \\ &= -\partial_m F_{\text{skew}}^m, \end{aligned} \quad (7)$$

where the skew flux has components defined by

$$F_{\text{skew}}^m = -A^{mn}\partial_n T. \quad (8)$$

The skew flux of a tracer is always oriented perpendicular to the gradient of that tracer since

$$\nabla T \cdot \mathbf{F}_{\text{skew}} = -\partial_m T A^{mn} \partial_n T = 0, \quad (9)$$

which follows from the antisymmetry of  $A^{mn}$ . This orientation of the skew flux motivates its name; that is, the flux is precisely “skewed” relative to the tracer gradient. It is neither upgradient nor downgradient. Note, however, that in any given direction, the skew flux may be up or down the component of the tracer gradient in that particular direction. The most general form for the skew flux consistent with this orientation is given by

$$\mathbf{F}_{\text{skew}} = -\nabla T \times \boldsymbol{\psi}, \quad (10)$$

where  $\boldsymbol{\psi}$  is the same vector streamfunction defined by Eq. (6) for the advective tracer flux.

The skew flux  $\mathbf{F}_{\text{skew}} = -\nabla T \times \boldsymbol{\psi}$  and advective flux  $\mathbf{F}_{\text{adv}} = T\mathbf{U}_* = T(\nabla \times \boldsymbol{\psi})$  have different magnitudes and directions. Indeed, in certain circumstances they have a completely opposite orientation (Middleton and Loder provide examples). However, their convergences are identical, which means that the nondissipative stirring operator  $R_A(T)$  is the same. In other words, the fluxes differ by the curl of some function, as can be seen explicitly through the identity

$$\mathbf{F}_{\text{adv}} = \mathbf{F}_{\text{skew}} + \nabla \times (T\boldsymbol{\psi}). \quad (11)$$

### 3. The GM skew-diffusive flux

The particular form of the transport velocity  $\mathbf{U}_*$  corresponding to GM is given by (GWMM),

$$\mathbf{u}_{h*} = -\partial_z(\kappa\mathbf{S}) \quad (12)$$

$$w_* = \nabla_h \cdot (\kappa\mathbf{S}), \quad (13)$$

where  $\mathbf{U}_* = (\mathbf{u}_{h*}, w_*)$ ,  $\nabla_h = (\partial_x, \partial_y, 0)$  is the horizontal gradient operator,  $\mathbf{S} = -\nabla_h \rho / \partial_z \rho$  is the isoneutral slope vector,  $\rho$  is the locally referenced potential density, and  $\kappa$  is a positive diffusion coefficient, which can depend on space–time. The normal component to the eddy-induced velocity is assumed to vanish on all boundaries (GWMM), which corresponds to a no-flux boundary condition on the skew flux and results in a stirring operator that conserves all tracer moments.

The expression  $U_*^n = -\partial_m A^{mn}$  relating the eddy-induced velocity to the antisymmetric tensor allows for an identification of the tensor components corresponding to the GM parameterization. In general, in order to obtain the antisymmetric tensor corresponding to a divergence-free velocity field, a nontrivial elliptic equa-

tion must be solved. However, the antisymmetric tensor corresponding to the GM velocity is extremely simple. For the purpose of determining  $A^{mn}$ , it is important to note that it is unique only to within a gauge. This redundancy means that the same stirring operator results from  $A^{mn}$  or  $A^{mn} + B^{mn}$ , where  $\partial_m B^{mn} = 0$ ,  $B^{mn} = -B^{nm}$ , and  $B^{mn}$  does not affect the boundary conditions. Since the particular gauge used is not physically relevant, it is sufficient to work with the most convenient gauge. For the zonal velocity  $U_*^1 = -\partial_m A^{m1} = -\partial_y A^{21} - \partial_z A^{31} = -\partial_z(\kappa S_x)$ , the most convenient choice suggests setting  $A^{21} = 0$ , which allows for the identification  $A^{31} = \kappa S_x$ . Likewise,  $A^{32} = \kappa S_y$  is suggested by  $U_*^2 = -\partial_z(\kappa S_y)$ . These choices lead to the antisymmetric stirring tensor corresponding to the GM advective transport velocity:

$$\mathbf{A} = [A^{mn}] = \begin{pmatrix} 0 & 0 & -\kappa S_x \\ 0 & 0 & -\kappa S_y \\ \kappa S_x & \kappa S_y & 0 \end{pmatrix}. \quad (14)$$

This tensor was also written down by Visbeck et al. (1997) for a two-dimensional model, and Greatbatch (1998) for the three dimensions considered here. McDougall and McIntosh (1996) also considered such a tensor for generalizations of GM. The corresponding vector streamfunction is given by  $\boldsymbol{\psi} = \mathbf{k} \times \kappa\mathbf{S} = (-\kappa S_y, \kappa S_x, 0)$ , which yields for the GM advective and skew fluxes:

$$\mathbf{F}_{\text{adv}} = T\nabla \times (\mathbf{k} \times \kappa\mathbf{S}) = T\mathbf{U}_* \quad (15)$$

$$\mathbf{F}_{\text{skew}} = -\nabla T \times (\mathbf{k} \times \kappa\mathbf{S}) = \kappa\mathbf{S}\partial_z T - \mathbf{k}(\kappa\mathbf{S} \cdot \nabla_h T). \quad (16)$$

Both fluxes vanish when the isoneutral slope vanishes. Additionally, the skew flux vanishes when the tracer is uniformly distributed, which is also the case for the Redi diffusive flux.

For constructing the GM stirring operator using the advective flux, the relevant part of the eddy-induced velocity field  $\mathbf{U}_*$  is that part parallel to the tracer gradient since  $\nabla \cdot (T\mathbf{U}_*) = \mathbf{U}_* \cdot \nabla T$ . This result means that for locally referenced potential density  $\rho$ , the only relevant piece of  $\mathbf{U}_*$  is that piece in the dianeutral direction. In contrast, the skew-diffusive flux is always perpendicular to the tracer gradient. Therefore, the GM skew flux of  $\rho$ ,

$$\rho^{-1}\mathbf{F}_{\text{skew}}(\rho) = -(\beta\nabla s - \alpha\nabla\theta) \times (\mathbf{k} \times \kappa\mathbf{S}), \quad (17)$$

is manifestly orthogonal to the dianeutral direction,

$$\hat{\gamma} = \frac{\beta\nabla s - \alpha\nabla\theta}{|\beta\nabla s - \alpha\nabla\theta|}, \quad (18)$$

where  $\alpha = -\partial_\rho \ln \rho$ ,  $\beta = \partial_s \ln \rho$ , and  $s$  is salinity (see GGPLDS for further details). The orthogonality holds even in the small slope limit since for that case,  $\hat{\gamma} \approx (\beta\nabla s - \alpha\nabla\theta)|\rho/\partial_z \rho|$ . Therefore, the GM skew flux locally manifests the adiabatic nature of the GM closure

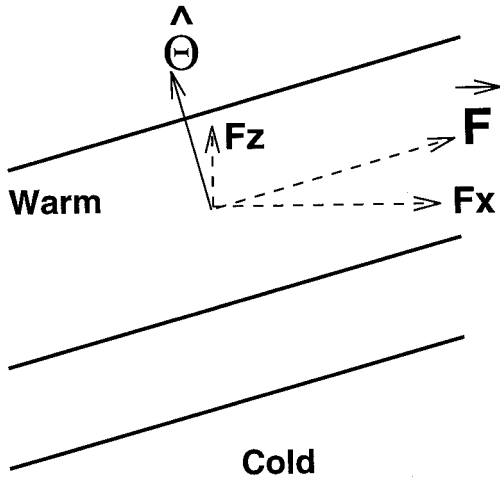


FIG. 1. An example of the GM skew flux arising from a slanted temperature front. The temperature gradient is parallel to the unit vector  $\hat{\theta}$ . The skew flux vector is oriented parallel to the temperature isolines, and the horizontal component is downgradient (to the east), whereas the vertical component is upgradient (upward).

in the sense that the skew flux is precisely aligned parallel to neutral directions.

To help understand some further properties of the GM skew flux, it is useful to write  $\mathbf{F}_{\text{skew}}(\rho)$  more explicitly

$$\mathbf{F}_{\text{skew}h} = -\kappa \nabla_h \rho \quad (19)$$

$$F_{\text{skew}}^z = \kappa |\nabla_h \rho|^2 / \partial_z \rho. \quad (20)$$

It is seen that the horizontal components to this flux are strictly downgradient. This very simple result has been noted previously by McDougall et al. (1996), Visbeck et al. (1997), and Treguier et al. (1997) in various contexts. Note that no quasigeostrophic assumption was used to reach this result; rather, it holds in general for the GM closure. The vertical component in a stably stratified portion of the ocean ( $\partial_z \rho < 0$ ) is always negative. Therefore, the vertical flux component is always directed up the vertical density gradient:  $F_{\text{skew}}^z(\rho) \partial_z \rho \geq 0$ . To reemphasize, the combined effects of the horizontal downgradient components and the vertical upgradient component bring the  $\rho$  skew-flux vector exactly parallel to the neutral directions:  $\mathbf{F}_{\text{skew}}(\rho) \cdot \nabla \rho = 0$ . Hence, there is no net dianeutral GM skew flux of  $\rho$ , and so no ‘‘Veronis effect’’ (Veronis 1975), even though the horizontal flux components are downgradient. This discussion is illustrated in Fig. 1, which shows a temperature field that slopes upward to the east, and temperature is assumed to be the only active tracer. Such a profile has an associated horizontal temperature skew-flux component to the east (down the temperature gradient) and an upward vertical temperature skew-flux component (up the temperature gradient). The full skew-flux vector is parallel to the temperature isolines.

The orientation of the skew-flux components shown in Fig. 1 suggests a general tendency to rotate the temperature profile in a clockwise manner, hence reducing

the available potential energy (APE). Such a rotation is indeed realized in cases where there are nonzero skew-flux divergences. More generally, the upgradient orientation of  $F_{\text{skew}}^z(\rho)$  results in a tendency for the GM closure to reduce gravitational potential energy locally at every point (see also the discussion in GM). In order to understand this property of GM from the skew-flux perspective, it is useful to consider the special case where density is a linear function of potential temperature. In this case, there is a clear definition of APE, and  $\rho$  is materially conserved if there are no diabatic effects. Consider now the gravitational potential energy budget, for which the potential energy density is given by  $P = \rho g z$ . Under the effects of GM, the time tendency for  $P$  is given by

$$\begin{aligned} \partial_t P &= -gz \nabla \cdot \mathbf{F}_{\text{skew}}(\rho) \\ &= -\nabla \cdot (gz \mathbf{F}_{\text{skew}}) + g F_{\text{skew}}^z. \end{aligned} \quad (21)$$

A no-flux boundary condition on the GM skew flux implies that the total derivative term will vanish when integrated over the full domain. Therefore, the potential energy budget takes the form

$$\partial_t \int d\mathbf{x} P = g \int d\mathbf{x} F_{\text{skew}}^z, \quad (22)$$

which expresses the fact that gravitational potential energy is altered by moving parcels in the vertical direction. In a stably stratified column of water, an upgradient vertical component to the density skew flux [ $F^z(\rho) \leq 0$ ] results in a reduction in gravitational potential energy. With nonzero divergences in the skew flux, as the potential energy decreases, the isoneutral slope decreases and likewise the horizontal density gradients decrease. This process brings both the horizontal and vertical components of the skew flux to zero upon reaching a zero slope, for which the available potential energy is zero. Again, this process is adiabatic since  $\mathbf{F}_{\text{skew}}(\rho) \cdot \nabla \rho = 0$ .

#### 4. Mixing tensor for Redi and GM

In models for which there is more than one active tracer or where there are passive tracers, GM stirring is typically combined with Redi isoneutral diffusion. Given the above discussion of the antisymmetric stirring tensor for GM, it is natural to consider the mixing tensor representing the combined effects of GM stirring and Redi diffusion. Such an approach is not novel. For example, Visbeck et al. (1997) employ a unified mixing tensor for their two-dimensional simulations. The full extent of the simplifications resulting from using such a tensor are the subject of the remainder of this paper.

To simplify the diffusion component of the mixing tensor, it is useful to employ the small angle approximation to the Redi diffusion tensor (Cox 1987; GM; GWMM). The differences between the small angle and

full diffusion tensors become on the order of 1% only when the isoneutral slopes reach above 1/10; otherwise, the differences are completely negligible. A slope of 1/10 is a rather large slope realized mostly near convective regions. In such regions, it is unclear how relevant the details of the isoneutral diffusion tensor are for parameterizing tracer mixing. Some discussion of the small versus full tensor is provided by GGPLDS, and further comments are given in the subsequent development. Making the small angle approximation, the combined mixing tensor for GM and small angle Redi diffusion takes the form

$$\mathbf{J} = [J^{mn}] = \begin{pmatrix} A & 0 & (A - \kappa)S_x \\ 0 & A & (A - \kappa)S_y \\ (A + \kappa)S_x & (A + \kappa)S_y & AS^2 \end{pmatrix}. \quad (23)$$

This is the key practical result of this paper. It provides the tracer mixing tensor to be used in  $z$ -level ocean models that implement both Redi isoneutral diffusion and GM skew diffusion. This tensor is quite simple in that it differs from the Redi isoneutral diffusion tensor only by the adjustment of the various diffusivities for the off-diagonal elements; that is, there are no additional terms to compute beyond isoneutral diffusion alone. Hence, the implementation of GM skew diffusion is trivial once a proper implementation of Redi isoneutral diffusion is provided. The special nature of an equal setting for the diffusivities is discussed at length in the following.

It is useful to explicitly note the two different tracer fluxes corresponding to Redi diffusion and GM stirring. The first is given by the sum of the Redi diffusive flux and the GM advective flux

$${}_a\mathbf{F}_h = -A(\nabla_h T + \mathbf{S}\partial_z T) - T\partial_z(\kappa\mathbf{S}) \quad (24)$$

$${}_aF^z = -A(\mathbf{S}\cdot\nabla_h T + S^2\partial_z T) + T\nabla_h\cdot(\kappa\mathbf{S}). \quad (25)$$

The second, which corresponds directly to the mixing tensor in Eq. (23) through  ${}_sF^m = -J^{mn}\partial_n T$ , is given by the sum of the Redi diffusive flux and the GM skew flux:

$${}_s\mathbf{F}_h = -A\nabla_h T - (A - \kappa)\mathbf{S}\partial_z T \quad (26)$$

$${}_sF^z = -(A + \kappa)\mathbf{S}\cdot\nabla_h T - AS^2\partial_z T. \quad (27)$$

The “ $a$ ” and “ $s$ ” prefixes on the fluxes distinguish the tracer flux associated with the GM advective and skew fluxes, respectively. Again, the advective and skew fluxes have the same convergence and, so, correspond to equivalent tracer mixing operators.

The choice of equal diffusivities,  $A = \kappa$ , is commonly used in numerical models, if for no other reason than simplicity. Furthermore, there is some suggestion from the recent work of Dukowicz and Smith (1997) that these coefficients should be equal. However, it should be noted that the Dukowicz and Smith results corre-

spond to the GM closure only in the special case of a constant diffusivity. Additionally, there are currently many issues to be clarified before providing a self-consistent dynamical theory for the diffusivities. Nevertheless, it is intriguing to consider the implications of making such a choice. For the advective flux formulation of GM, there are no significant implications as the form for the flux [Eqs. (24) and (25)] remain structurally the same regardless of the diffusivities. However, with  $A = \kappa$ , the sum of the Redi diffusive and GM skew-diffusive flux takes the very simple form

$${}_s\mathbf{F}_h(T) = -\kappa\nabla_h T \quad (28)$$

$${}_sF^z(T) = -\kappa(2\mathbf{S}\cdot\nabla_h T + S^2\partial_z T). \quad (29)$$

Therefore, the horizontal tracer flux is equivalent to a geopotentially oriented downgradient diffusive flux. This is a kinematical result, which results from an exact cancellation of the off-diagonal term in the horizontal flux occurring in the Redi diffusive flux with the horizontal GM skew flux. Recall that the off-diagonal term arises in the formulation of isoneutral diffusion when representing the horizontal gradient at constant density in the  $z$ -coordinate system, whereas off-diagonal terms in the GM antisymmetric tensor are present in any coordinate representation. Choosing to represent the antisymmetric tensor in  $z$  coordinates using the most obvious gauge exposes the combined Redi diffusion plus GM stirring tensor to this rather striking simplification. Note that this result does not depend on any assumption about the equation of state or the spatiotemporal structure of the diffusivities. It applies to all tracers, active and passive, which are fluxed according to the small angle Redi and GM parameterizations, again using equal diffusivities. Despite the simplification of the horizontal components, the precise form of the vertical flux component is crucial for preserving the underlying physical properties of the parameterizations. In other words, this choice of diffusivities does not yield downgradient diffusion in all three directions, only in the two horizontal directions.

## 5. Numerical considerations

### a. Algorithmic unification

A comparison between the advective and skew-flux forms for the tracer flux [Eqs. (24), (25) and (26), (27)] points out certain simplifications that arise when numerically implementing the skew-flux formulation. Notably, the skew-flux formulation provides for an algorithmic unification of the Redi and GM fluxes. All that is necessary to add GM stirring to Redi diffusion is to alter the mixing coefficients normally used with Redi diffusion alone. No more calculation of the GM eddy-induced velocity or the corresponding advective flux is necessary. Importantly, the efforts of GGPLDS in discretizing the diffusion tensor provide some confidence in implementing tracer mixing tensors in ocean models.

It is therefore natural to exploit that effort for implementing the GM stirring operator. It is worth noting that for diagnostic purposes, it is still useful within the skew-flux approach to compute the eddy-induced velocity in order to construct its streamfunction (see GWMM for examples). Just as for the GM advective flux, the streamfunction of the eddy-induced velocity provides some insight toward how the convergence of the GM skew flux is affecting the tracer.

### b. Reduction in numerical noise

Besides the great algorithmic simplification, the GM skew flux is simpler to compute and is generally more accurate than the GM advective flux. The crucial difference is that, to construct the advective flux, it is necessary to take a spatial derivative on both the slope vector and the diffusivity in order to construct the advection velocity  $\mathbf{U}_*$ . For the skew-flux formulation, the spatial derivative is instead placed on the tracer. The increased numerical accuracy of the skew flux is most easily seen for the case of a single active tracer. For the zonal direction, the downgradient skew flux  $F_{\text{skew}}^x = -\kappa \partial_x \theta$  is trivially computed with second-order accuracy for a numerical scheme using second-order difference operators. The accuracy realized when computing the advective flux  $F_{\text{adv}}^x = \theta \partial_z (\kappa \partial_x \theta / \partial_z \theta)$ , however, is severely compromised due to the extra spatial derivative acting on the ratio of two terms, each of which consist of spatial derivatives. The zonal advective flux is also ill-behaved in regions near steep isoneutral slopes where artificial slope checking schemes are necessary in order to maintain numerical stability (see GGPLDS for a review of such schemes). The vertical skew flux  $F_{\text{skew}}^z = \kappa |\nabla_h \theta|^2 / \partial_z \theta$  is less trivial to compute than the horizontal skew flux. Yet it again involves no derivative on the diffusivity or slope vector, whereas the vertical advective flux  $F_{\text{adv}}^z = -\theta \nabla_h \cdot (\kappa \nabla_h \theta / \partial_z \theta)$  requires a computation of the horizontal divergence of the slope weighted by the diffusivity. The difficulties of numerically maintaining integrity with the advective flux formulation has been recently documented by Weaver and Eby (1997). Their results support the arguments made here.

Weaver and Eby point out additional difficulties arising when positive definiteness of the tracer field is not guaranteed, as occurs when computing GM advective fluxes with centered differencing or many other advection schemes. Coupled to the problems inherent in computing  $\mathbf{U}_*$  near convective regions and its resulting noisy structure, they conclude that it is necessary to employ a positive definite advection scheme such as flux corrected transport (FCT) (see Gerdes et al. 1991) in order to eliminate unphysical tracer extrema. By eliminating the extra slope and diffusivity gradients and by eliminating the computation of advective fluxes, the skew-flux formulation may provide a reasonable alternative to using FCT. In addition, as shown in the subsequent discussion, the skew-flux formulation, when im-

plemented in terms of the density triads of GGPLDS, conserves tracer variance just as a centered difference advective scheme.

For nonconstant diffusivities (e.g., Held and Larichev 1996; Visbeck et al. 1997; Killworth 1998), the spatial derivative of the diffusivity will be nonzero. These coefficients will themselves typically be computed in terms of large-scale Richardson numbers. Consequently, they hold the potential to provide yet another source of noise in the numerical model beyond the calculation of slope derivatives. Hence, it is sensible to eliminate numerical differentiation of these coefficients if possible. Again, there is no differentiation of these coefficients when constructing the GM skew flux.

### c. A comment on steep isoneutral slopes

The cancellation between the off-diagonal terms in the horizontal tracer flux occurring when  $A = \kappa$  occurs only when employing the small angle Redi diffusion tensor. The question therefore arises as to the consistency of using the resulting horizontal diffusion for those steep sloped regions in which the small Redi tensor is not valid. In general, regardless of the relative values of the diffusivities, the issues surrounding steep slopes are quite important since it is for these regions that much of the climatologically crucial middle to high latitude air-sea interaction takes place. In turn, it is the region where the assumptions of adiabaticity tend to break down, so the use of isoneutral diffusion and GM transport may not be completely justified. The details of such boundary regions are still the topic of research. A preliminary discussion of such issues can be found in Treguier et al. (1997). They suggested that horizontal tracer diffusion should be applied in a mixed layer, where their definition of a mixed layer basically equates to regions of steep isoneutral slopes. It is perhaps intuitive that in such regions, eddies will efficiently mix tracers laterally and hence across the mean neutral directions. It should be noted that the arguments for horizontal diffusion in the mixed layer are not universally agreed upon. For example, Large et al. (1997) describe coarse-resolution model results in which all lateral tracer fluxes are eliminated when the isoneutral slopes steepen.

Even though Treguier et al. differ somewhat from GM in their form for a tracer closure, it is interesting to pursue their conclusions regarding horizontal diffusion in the mixed layer within the present context. First, within the framework of the unified mixing tensor given by equation (23), it is simple to prescribe a smooth transfer from interior mixing, using Redi diffusion and GM skew diffusion, to a horizontal-vertical mixed layer diffusion scheme. Second, if choosing to mix in the interior with  $A = \kappa$ , one is led to the conclusion that a horizontal downgradient tracer flux is relevant regardless of the isoneutral slope. In this special case, there is no slope checking for the horizontal flux components since horizontal diffusion is applied everywhere. This approach

brings the onus of the calculation onto the vertical flux component [Eq. (29)]. For this component, a sensible means to scale it to zero when the slopes steepen should be employed (see GGPLDS for a summary of slope checking schemes). In general, a physically based parameterization of boundary layer physics should be implemented in the steep sloped regions [see Large et al. (1994) for a summary].

#### d. Cox isoneutral diffusion and GM advection

Recently, modelers have found that when using the Cox (1987) implemented Redi diffusive flux along with the GM advective flux, there has been a reduction in the need to employ stabilizing horizontal background diffusion (e.g., Danabasoglu and McWilliams 1995; Hirst and McDougall 1996). The question arises as to why such stabilization occurs. As described by GGPLDS, the essential problem with Cox diffusion scheme is that, when the density is a nonlinear function of either the temperature or salinity, the scheme produces an upgradient diffusive flux of locally referenced potential density. This upgradient flux then induces an unbounded growth in tracer variance, hence making the scheme unstable. The upgradient flux in the Cox scheme originates from the off-diagonal term in the horizontal isoneutral diffusion flux components; that is, the term in which the isoneutral slope vector appears [see Eq. (26)].

What apparently occurs is that, even when formulated in terms of advective fluxes, the introduction of GM into the models may alleviate some of the destabilizing effects from the problematical off-diagonal piece of the horizontal diffusive flux. However, this cancellation is incomplete at best since the advective flux is numerically not the same as the skew flux. This incomplete cancellation is consistent with modeling experiences at GFDL in which it has been found that the model stabilization appearing with the GM advective formulation is sensitive to the choice of the GM thickness diffusivity, the momentum dissipation, and the roughness of the bottom topography (R. Toggweiler 1996, personal communication). Indeed, Fig. 2D of GGPLDS shows a case in which the addition of GM advective fluxes to the Cox diffusion scheme results in more unstable numerical behavior than with the Cox scheme alone. However, realistic model tests with the GM skew flux and the unstable Cox isoneutral diffusion scheme indicate complete stabilization of the numerical mixing operator when  $A = \kappa$ , without the addition of horizontal background diffusion (not shown).

#### e. Conservation of tracer variance

One of the advantages of constructing advective fluxes using centered differences is that, with no-flux boundaries, both the tracer mean and variance remain constant in time (Bryan 1969). The GM advective fluxes, when

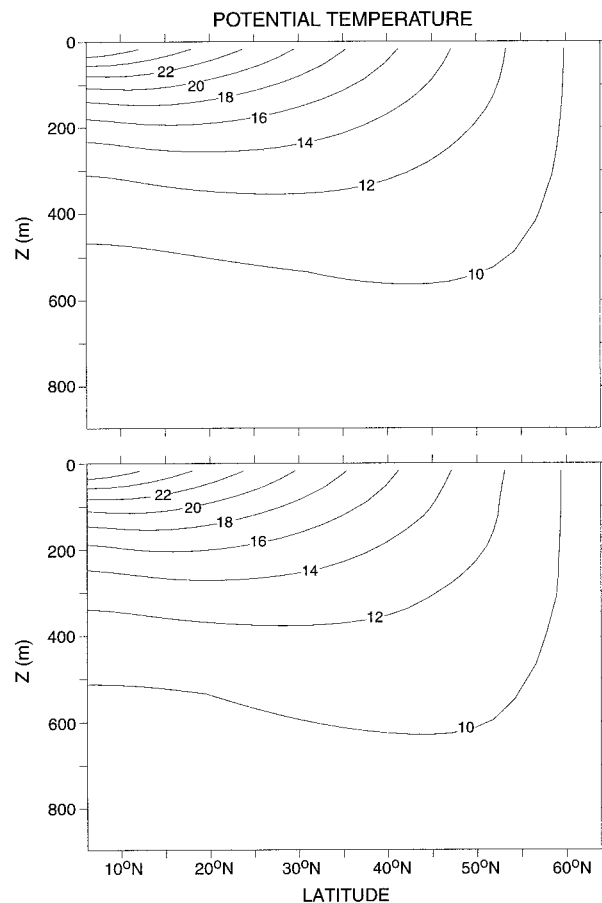


FIG. 2. Zonally averaged meridional-depth snapshot of potential temperature after 4000 years of integration. (a) Upper panel: The result from the advective flux formulation of GM using centered differences to construct the GM stirring operator. (b) Lower panel: The result from the skew-flux formulation of GM. Both solutions employ centered differences for computing the advective flux from the Eulerian velocity.

computed using centered differences, therefore satisfy this property. It is important to test whether the skew-flux formulation, as implemented numerically in terms of the “triad” approach proposed by GGPLDS, will also allow for these properties to be satisfied. First, the tracer mean is trivially conserved because the divergence of the skew flux, summed over the model domain, reduces to the normal component of that flux on the boundaries. A no-flux boundary brings the boundary contribution to zero. In the continuum, the constancy of the tracer variance is directly related to the orthogonal orientation of the skew flux relative to the tracer gradient (i.e.,  $\nabla T \cdot \mathbf{F}_{\text{skew}}(T) = 0$ ). On the lattice, such an orientation will hold within a finite volume (GGPLDS) if implemented in terms of the functional approach described by GGPLDS.

The proof that tracer variance remains unchanged with the skew-flux formulation can be seen in its most streamlined form by employing the GGPLDS functional

formalism. The functional corresponding to the skew flux vanishes since  $\nabla T \cdot \mathbf{F}_{\text{skew}} = 0$ . Therefore, the functional<sup>2</sup> for Redi diffusion plus GM skew diffusion is given by

$$\begin{aligned} & \int d\mathbf{x} \nabla T \cdot (\mathbf{F}_{\text{Redi}} + \mathbf{F}_{\text{skew}}) \\ &= - \int d\mathbf{x} \partial_x T [A \partial_x T + S_x (A - \kappa) \partial_z T] \\ & \quad - \int d\mathbf{x} \partial_x T [A \partial_z T S^2 + S_x (A + \kappa) \partial_x T], \quad (30) \end{aligned}$$

where  $y$  variation has been ignored for brevity. For the same reason that the discretized Redi diffusive flux is constrained to reduce the tracer variance (GGPLDS), the discretized skew flux will not change the tracer variance. Namely, all that has been done to construct the Redi plus GM functional is to add zero in a nontrivial manner to the negative semidefinite Redi functional. Such an addition cannot change the global properties of the resulting functional, so long as it is discretized in a consistent fashion.

It is useful to be a bit more explicit in showing that the tracer variance is conserved with the proposed discretization of the skew flux. As shown in GGPLDS, when starting from the functional in Eq. (30) with  $\kappa = 0$ , the deduced grid stencil arising from second-order differencing operators consists of triads of density points. In turn, the triads are used to construct the discretization of the isoneutral slopes, and they weight the gradients of the tracers in the construction of the diffusive flux. The result is a variance-reducing numerical diffusion scheme. When allowing for  $\kappa > 0$ , the functional approach leads to the identical grid stencil in terms of triads. In particular, the resulting zonal and vertical components to the skew flux can be written

$$F^x = \langle \kappa S \partial_x T \rangle_x = - \langle \kappa \partial_z T / \partial_z \rho \rangle_x \partial_x \rho \quad (31)$$

$$F^z = - \langle \kappa S \partial_x T \rangle_z = \langle \kappa \partial_x \rho \partial_x T \rangle_z (\partial_z \rho)^{-1}, \quad (32)$$

where the angled brackets symbolize an average to be taken over the four density triads as described by GGPLDS (see their section 5). For brevity, the lattice labels are absent in these equations. Taking the scalar product of this skew flux with the tracer gradient yields

$$\partial_x T F^x = - (\partial_x T \partial_x \rho) \langle \kappa \partial_x T / \partial_z \rho \rangle_x \quad (33)$$

$$\partial_z T F^z = (\partial_z T / \partial_z \rho) \langle \kappa \partial_x \rho \partial_x T \rangle_z. \quad (34)$$

Summing these terms over the extent of the model lattice will provide for exact cancellation of terms, hence yielding the lattice equivalent of  $\int d\mathbf{x} \nabla T \cdot \mathbf{F}_{\text{skew}} = 0$ . The conservation of tracer variance follows, and this result holds for arbitrary diffusivity.

<sup>2</sup> Actually, twice the functional.

#### f. A numerical example

For the purpose of illustrating the numerical solutions arising from the skew-flux formulation of GM, a single active tracer is employed. This case is sufficient to address the numerical issues raised by Weaver and Eby (1997). With a single active tracer, no isoneutral diffusion will act on this tracer regardless of the equation of state (GGPLDS).<sup>3</sup> As seen in the discussion of section 3, the horizontal temperature skew flux is directed down the horizontal temperature gradient, whereas the vertical skew-flux component is up the vertical gradient. Again, the sum of these two flux components provides for a skew-flux vector that is orthogonal to the temperature gradient, resulting in a zero cross isothermal temperature skew flux.

For the numerical test, the idealized sector model used in GGPLDS, employing the MOM 2 ocean model documented by Pacanowski (1996), is integrated using the two GM formulations. This model has 18 unevenly spaced vertical levels, and the temperature field is restored to a linear profile with a 50-day restoring time over the top model layer 35 m deep. The steady-state solution has convection occurring in the far north due to the cooling and a strong amount of downwelling in the northeast (e.g., see Bryan 1975). Weaver and Eby (1997) employed a similar model for performing their numerical tests of the GM advective flux formulation. The most notable difference between their model and the present one is their use of increased vertical resolution: they used 80 evenly spaced vertical levels reaching to 4000 m in depth.

The triad scheme of GGPLDS (see their section 5) is used to compute the skew flux. The method of Danabasoglu and McWilliams (1995) is used to compute the eddy-induced velocity  $\mathbf{U}_*$ , and centered differences are used to compute the corresponding GM advective flux. Both experiments compute the advective fluxes from the Eulerian current  $\mathbf{u}$  with centered differences. Figures 2a and 2b show a zonally averaged meridional–depth snapshot in the upper portion of the model obtained after 4000 years of integration.<sup>4</sup> Both the advective flux (Fig. 2a) and skew-flux (Fig. 2b) solutions show similar profiles, with the advective flux solution slightly cooler. The colder advective flux solution might be related to the presence of increased dispersion errors associated with problems in the steeply sloped regions.

<sup>3</sup> A point of clarification is warranted. Weaver and Eby employed a linear equation of state and the Cox (1987) diffusion scheme. As shown by GGPLDS, the Cox scheme is stable in this case since it correctly provides an identically zero isoneutral diffusive flux of temperature when using a linear equation of state. Isonneutral diffusion of temperature, therefore, is completely absent in the study of Weaver and Eby.

<sup>4</sup> For the vertical diffusivity  $\kappa_v = 0.5 \text{ cm}^2 \text{ s}^{-1}$  was used. Therefore, the diffusive spinup time is  $D^2/\kappa_v \approx 10^4 \text{ yr}$ , where  $D = 4000 \text{ m}$  is the depth of the model.



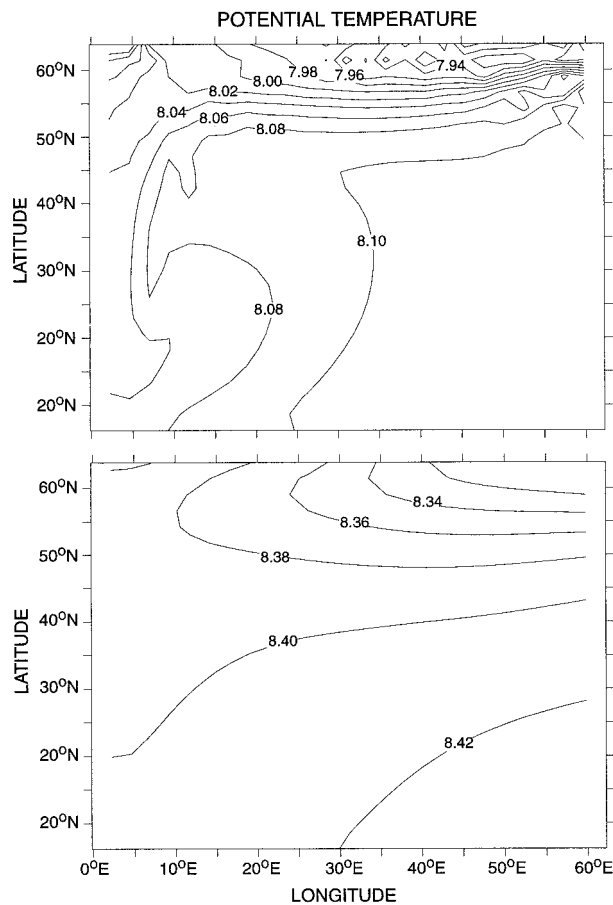


FIG. 3. Snapshot of potential temperature at the bottom level after 4000 years of integration. (a) Upper panel: The result from the advective flux formulation of GM using centered differences to construct the GM stirring operator. (b) Lower panel: The result from the skew-flux formulation of GM. Both solutions employ centered differences for computing the advective flux from the Eulerian velocity. Note the use of identical contour intervals ( $0.02^{\circ}\text{C}$ ), yet different ranges.

To address this point, it is useful to look at the bottom level since it is for this level that many of the problems occurring in the upper regions accumulate over long integrations. In particular, undershoots creating anomalously cold water parcels will eventually find their way to the bottom due to convective adjustment. Figures 3a and 3b show the bottom-level temperature. As suggested, the advective formulation results in somewhat colder water than the skew-flux formulation. Most importantly, note that the temperature profile arising from the advective formulation is afflicted with unphysical extrema. These extrema are thought to be associated with dispersion errors with the centered difference scheme acting on the noisy  $U_*$  field. In contrast, the skew-flux solution is completely smooth.

It is important to emphasize that the tracer transport obtained with the GM skew flux and centered differences for the Eulerian advective flux conserves tracer variance. Therefore, the smoothness in the solution

shown in Fig. 3b is not achieved with enhanced dissipation coming in “through the back-door.” Rather, it arises from a cleaner formulation of the GM mixing operator, which sidesteps the problems inherent in computing the  $U_*$  velocity and the corresponding problems of using this velocity within the centered difference advection scheme.

Besides problems with the flat bottom experiments described by Weaver and Eby (1997), the numerical integrity of models run with centered difference advective fluxes is sometimes compromised when running with rectangular stepped topography. What can occur is an excessive amount of dispersion error recurring near the topography and producing tracer values far from those that are physically realistic. This “digging” is currently a reason some modelers choose to employ FCT and similar advection schemes for computing advective fluxes corresponding to the Eulerian currents. Due to the presence of the horizontal downgradient fluxes of  $\rho$  in the skew-flux formulation of GM and its inherently large horizontal flux divergences next to the no-flux boundaries, it might be that it could alleviate digging. However, preliminary tests have indicated that the digging near topography is not removed. It seems that such problems, if they are found in a particular model, either require some form of dissipative advection or, more physically, some form of topographic sculpting such as done by Adcroft et al. (1997).

## 6. Summary and conclusions

A closure of the tracer equation in terms of a divergence-free eddy-induced velocity necessarily implies the relevance of both an advective tracer flux and its corresponding skew-diffusive flux. These two fluxes differ by a curl, which means that they lead to the same tracer stirring operator. So far, thinking regarding GM has mostly focused on its advective form (GWMM), for which GM stirring arises from the addition of an eddy-induced velocity to the usual Eulerian velocity. This paper emphasized the skew-diffusive form of GM, and it was argued that it provides a tidy summary of the physics incorporated into the GM closure. In general, the skew flux of *any* tracer is directed parallel to the isolines of that tracer. In particular, the horizontal components of the GM skew flux for locally referenced potential density are directed downgradient. This downgradient flux is combined with an upgradient vertical component rendering the skew-flux vector parallel to the neutral directions. Hence, there is a zero dianeutral component to the GM skew flux of locally referenced potential density: a result indicative of the adiabatic nature of the GM closure. Additionally, the upgradient vertical component is directly associated with the reduction of APE resulting from GM closure.

The skew-flux perspective provides a very useful and efficient means to implement GM stirring in  $z$ -coordinate ocean models. Currently, GM is typically imple-

mented in its advective form, which involves a calculation of the eddy-induced velocity  $\mathbf{U}_*$  and the divergence of the advective tracer flux  $\nabla \cdot \mathbf{TU}_*$ . The calculation of the advective flux suffers from a number of problems. Most notably, 1) it requires taking a spatial derivative of both the isoneutral slope vector  $\mathbf{S} = -\nabla_{\rho} \rho / \partial_{\rho} \rho$  and the diffusivity  $\kappa$  in order to compute  $\mathbf{U}_*$ . This derivative is most difficult to compute numerically in regions where either the slope or the diffusivity are changing rapidly. Furthermore, if  $\kappa$  is proportional to the Richardson number, then these regions may coincide, thus exacerbating the problem. 2) It requires the construction of advective fluxes. If these fluxes conserve tracer variance, they are not monotonic. The numerical implications of both issues were described by Weaver and Eby (1997). They concluded that in order to implement the advective form of GM, even when using constant diffusivities in a flat bottom model, it is necessary to employ a dissipative advection scheme such as FCT when computing the GM advective flux. Otherwise, the numerical integrity of the solution will be greatly compromised. Besides being computationally expensive, the use of such advection schemes in the context of non-constant tracer diffusivities is cumbersome. The reason is that both the physically based closure and the numerically based dissipative advection potentially will be active in the interesting dynamical regions associated with strong currents and convection. This confusion of effects may make it difficult to assess the relevance of various subgrid-scale closures for the diffusivities.

The skew-flux formulation avoids the two problems with the advective formulation. First, to compute the skew flux requires taking a spatial derivative on the tracer rather than on the product of the diffusivity and isoneutral slope. As a result, the skew flux involves the same differentiation operations needed to compute the Redi diffusive flux. The result is an inherently smoother GM skew flux than GM advective flux. Second, formulating GM in terms of its skew flux allows for a clean unification of the symmetric Redi diffusion and anti-symmetric GM stirring tensors. Within this framework of a general mixing tensor [Eq. (23)], there is no need to compute either the GM eddy-induced velocity  $\mathbf{U}_*$ , or its corresponding advective flux. Furthermore, by implementing the mixing tensor using the algorithm of GGPLDS, the discretized skew flux conserves both tracer mean and tracer variance. This approach requires no more computation than required for Redi diffusion alone since there is no longer a separate computation of the GM advective flux and the Redi diffusive flux. The relative savings in computational load increase in proportion to the number of tracers used in the model. For example, many biogeochemical models now employ tens of tracers, so the total cost of those models using Redi diffusion and GM advective transport could be substantially reduced with the skew-flux approach to GM. To support this analysis, an idealized model test was run, where the focus was on the problems with the

advective flux formulation pointed out by Weaver and Eby (1997). The results of this test indicate that the skew-flux formulation resolves the problems with the advective formulation, again, while conserving tracer variance.

Formulating GM in terms of the skew flux exposes the potential to realize a rather striking cancellation between the horizontal Redi diffusive flux and horizontal GM skew flux. Namely, setting the GM diffusivity equal to the Redi isoneutral diffusivity ( $\kappa = A$ ) yields a strictly downgradient horizontal tracer flux for the sum of Redi diffusion plus GM skew diffusion. The vertical flux component, whose precise form is crucial in order to preserve the physics of the closure, is less trivial yet no more difficult to compute than the vertical Redi diffusive flux. The cancellation occurs only when formulating small angle Redi diffusion with GM skew diffusion in a  $z$ -level model. Currently, the justification for setting the diffusivities equal is mostly based on simplicity, and is currently employed by most ocean modelers (e.g., Danabasoglu and McWilliams 1995; Hirst and McDougall 1996).

Regardless of the diffusivities, for those wishing to test the GM and Redi schemes in  $z$ -level models, it is recommended that the GM skew flux be implemented within the framework of the GGPLDS isoneutral diffusion scheme. By doing so, one gains the assurance that the discretized GM skew flux will conserve tracer mean and variance, and the Redi plus GM tracer flux will preserve the physical properties discussed in this paper and in GGPLDS. However, it is recognized that the new algorithm of GGPLDS represents a major change in the diffusion code, and so may require a non-trivial investment in model restructuring if not employing the latest version of the MOM 2 code (versions subsequent to October 1996). What has been shown in this paper is that in general there is no impediment toward implementing GM stirring within *any* model already containing some form of isoneutral diffusion. The reason is that to do so is trivial when using the most common diffusivity setting of  $A = \kappa$ . The resulting mixing operator will be stabilized relative to the behavior encountered using the unstable Cox (1987) diffusion scheme alone, and the stabilization will be realized without adding horizontal background diffusion. Furthermore, the corresponding GM plus Redi tracer mixing scheme requires roughly half the computational load engendered by isoneutral diffusion alone. Such savings have translated into a 30% reduction in total model run time for a realistic four degree global ocean model carrying two active tracers and one passive tracer.

*Acknowledgments.* This work grew from many enjoyable discussions about ocean tracer mixing with Kirk Bryan, John Dukowicz, Anand Gnanadesikan, Bob Hallberg, Isaac Held, Vitaly Larichev, Jerry Mahlman, and Ron Pacanowski. Thanks also go to Keith Dixon, Bonnie Samuels, Mike Spelman, Ron Stouffer, and Rob-

bie Toggweiler for their interest in testing the results from this paper in their models.

## REFERENCES

- Adcroft, A., C. Hill, and J. Marshall, 1997: Representation of topography by shaved cells in a height coordinate ocean model. *Mon. Wea. Rev.*, **125**, 2293–2315.
- Bryan, K., 1969: A numerical method for the study of the circulation of the world ocean. *J. Comput. Phys.*, **4**, 347–376.
- , 1975: Three-dimensional numerical models of the ocean circulation. *Numerical Models of Ocean Circulation*, Natl. Acad. Sci., 94–105.
- Cox, M. D., 1987: Isopycnal diffusion in a z-coordinate ocean model. *Ocean Modelling* (unpublished manuscripts), **74**, 1–5.
- Danabasoglu, G., and J. C. McWilliams, 1995: Sensitivity of the global ocean circulation to parameterizations of mesoscale tracer transports. *J. Climate*, **8**, 2967–2987.
- Duffy, P. B., P. Eltgroth, A. J. Bourgeois, and K. Caldeira, 1995: Effect of improved subgrid scale transport of tracers on uptake of bomb radiocarbon in the GFDL ocean general circulation model. *Geophys. Res. Lett.*, **22**, 1065–1068.
- Dukowicz, J. K., and R. D. Smith, 1997: Stochastic theory of compressible turbulent fluid transport. *Phys. Fluids*, in press.
- Eckart, C., 1948: An analysis of the stirring and mixing processes in incompressible fluids. *J. Mar. Res.*, **7**, 265–275.
- England, M. H., 1995: Using chlorofluorocarbons to assess ocean climate models. *Geophys. Res. Lett.*, **22**, 3051–3054.
- , and G. Holloway, 1996: Simulations of CFC-11 outflow and seawater age in the deep North Atlantic. *Applications of Trace Substance Measurements to Oceanographic Problems*, P. Schlosser, W. M. Smethie, and J. R. Toggweiler, Eds., Amer. Geophys. Union.
- Gent, P. R., and J. C. McWilliams, 1990: Isopycnal mixing in ocean circulation models. *J. Phys. Oceanogr.*, **20**, 150–155.
- , J. Willebrand, T. J. McDougall, and J. C. McWilliams, 1995: Parameterizing eddy-induced tracer transports in ocean circulation models. *J. Phys. Oceanogr.*, **25**, 463–474.
- Gerdes, R., C. Köberle, and J. Willebrand, 1991: The influence of numerical advection schemes on the results of ocean general circulation models. *Climate Dyn.*, **5**, 211–226.
- Greatbatch, R. J., 1998: Exploring the relationship between eddy-induced transport velocity, vertical momentum transfer, and the isopycnal flux of potential vorticity. *J. Phys. Oceanogr.*, **28**, 422–432.
- Griffies, S. M., A. Gnanadesikan, R. C. Pacanowski, V. Larichev, J. K. Dukowicz, and R. D. Smith, 1998: Isonutral diffusion in a z-coordinate ocean model. *J. Phys. Oceanogr.*, **28**, 805–830.
- Held, I. M., and V. D. Larichev, 1996: A scaling theory for horizontally homogeneous baroclinically unstable flow on a beta plane. *J. Atmos. Sci.*, **53**, 946–952.
- Hirst, A. C., and T. J. McDougall, 1996: Deep-water properties and surface buoyancy flux as simulated by a z-coordinate model including eddy-induced advection. *J. Phys. Oceanogr.*, **26**, 1320–1343.
- Holloway, G., 1997: Eddy transport of thickness and momentum in layer and level models. *J. Phys. Oceanogr.*, **27**, 1153–1157.
- Killworth, P. D., 1998: On the parameterization of eddy transfer. Part I: Theory. *J. Mar. Res.*, **55**, 1171–1197.
- Large, W. G., J. C. McWilliams, and S. C. Doney, 1994: Oceanic vertical mixing: A review and a model with a nonlocal boundary layer parameterization. *Rev. Geophys.*, **32**, 363–403.
- , G. Danabasoglu, S. C. Doney, and J. C. McWilliams, 1997: Sensitivity to surface forcing and boundary layer mixing in a global ocean model: Annual-mean climatology. *J. Phys. Oceanogr.*, **27**, 2418–2447.
- McDougall, T. J., 1987: Neutral surfaces. *J. Phys. Oceanogr.*, **17**, 1950–1967.
- , and P. C. McIntosh, 1996: The temporal-residual-mean velocity. Part I: Derivation and the scalar conservation equations. *J. Phys. Oceanogr.*, **26**, 2653–2665.
- , A. C. Hirst, M. H. England, and P. C. McIntosh, 1996: Implications of a new eddy parameterization for ocean models. *Geophys. Res. Lett.*, **23**, 2085–2088.
- Middleton, J. F., and J. W. Loder, 1989: Skew fluxes in polarized wave fields. *J. Phys. Oceanogr.*, **19**, 68–76.
- Moffatt, H. K., 1983: Transport effects associated with turbulence with particular attention to the influence of helicity. *Rep. Progress Phys.*, **46**, 621–664.
- Pacanowski, R. C., 1996: MOM 2 documentation, user's guide and reference manual. GFDL Ocean Tech. Rep. 3.1, Geophysical Fluid Dynamics Laboratory/NOAA, 200 pp. [Available from GFDL, Route 1, Box 308, Princeton, NJ 08542.]
- Plumb, R. A., 1979: Eddy fluxes of conserved quantities by small-amplitude waves. *J. Atmos. Sci.*, **36**, 1699–1704.
- , and J. D. Mahlman, 1987: The zonally averaged transport characteristics of the GFDL general circulation/transport model. *J. Atmos. Sci.*, **44**, 298–327.
- Redi, M. H., 1982: Oceanic isopycnal mixing by coordinate rotation. *J. Phys. Oceanogr.*, **12**, 1154–1158.
- Solomon, H., 1971: On the representation of isentropic mixing in ocean models. *J. Phys. Oceanogr.*, **1**, 233–234.
- Tandon, A., and C. Garrett, 1996: On a recent parameterization of mesoscale eddies. *J. Phys. Oceanogr.*, **26**, 406–411.
- Treguier, A. M., I. M. Held, and V. D. Larichev, 1997: On the parameterization of quasi-geostrophic eddies in primitive equation ocean models. *J. Phys. Oceanogr.*, **27**, 567–580.
- Veronis, G., 1975: The role of models in tracer studies. *Numerical Models of Ocean Circulation*, Natl. Acad. Sci., 133–146.
- Visbeck, M., J. Marshall, T. Haine, and M. Spall, 1997: Specification of eddy transfer coefficients in coarse resolution ocean circulation models. *J. Phys. Oceanogr.*, **27**, 381–402.
- Weaver, A. J., and M. Eby, 1997: On the numerical implementation of advection schemes for use in conjunction with various mixing parameterizations in the GFDL ocean model. *J. Phys. Oceanogr.*, **27**, 369–377.