PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 2, February 1983

THE $\hat{A}$-GENUS OF COMPLEX HYPERSURFACES AND COMPLETE INTERSECTIONS

ROBERT BROOKS[1]

ABSTRACT. In this note, we classify the even-dimensional complex hypersurfaces and complete intersections which carry a metric of positive scalar curvature. This is done by computing the $\hat{A}$-genus of these manifolds to eliminate all cases not known to carry such a metric.

If M is a real $4n$-dimensional manifold, then Hirzebruch **[1]** has defined the $\hat{A}$-genus of M as a certain polynomial in the Pontrjagin classes of M.

By a theorem of Lichnerowicz **[5]**, if M is a spin manifold, then the nonvanishing of the $\hat{A}$-genus is an obstruction to the existence of a metric of positive scalar curvature on M.

In this paper, we consider the case when M is a complex hypersurface, or more generally a complete intersection. If we denote by $M = V^{2n}_{d_1,\ldots,d_r}$ the complete intersection of hypersurfaces defined by homogeneous polynomials of degree $d_1,\ldots,d_r$ in $\mathbf{C}P(2n+r)$, then one knows in principle how to compute the $\hat{A}$-genus of M in terms of $d_1,\ldots,d_r$; M is spin precisely when $2n+r+1-\Sigma_i d_i$ is even. Our main observation here is that for complex hypersurfaces V^{2n}_d, the formula for $\hat{A}(V^{2n}_d)$ is somewhat simpler than one might expect:

THEOREM 1.

$$\hat{A}\left(V^{2n}_d\right) = 2 \cdot \frac{(d/2+n)(d/2+(n-1))\cdots(d/2-n)}{(2n+1)!}.$$

The formula of Theorem 1 was also obtained in **[4**, p. 259**]**. Our proof given below extends readily to give a formula for complete intersections, which is however somewhat more cumbersome. Nonetheless, we are able to determine when $\hat{A}(V^{2n}_{d_1,\ldots,d_r})$ is zero, so that we can show our main result:

THEOREM 2. (i) *If $2n+r+1-\Sigma d_i$ is even, then $V^{2n}_{d_1,\ldots,d_r}$ carries a metric of positive scalar curvature if and only if $\Sigma d_i \leq 2n+r$.*

(ii) *If $2n+r+1-\Sigma d_i$ is odd and $n>1$, then $V^{2n}_{d_1,\ldots,d_r}$ always carries a metric of positive scalar curvature.*

Theorem 2(ii) follows immediately from a theorem of Gromov and Lawson **[3]**—any simply-connected manifold of dimension ≥ 5 which is not spin carries a metric of positive scalar curvature.

Received by the editors June 16, 1982.
1980 *Mathematics Subject Classification.* Primary 57R20.
[1]Partially supported by NSF Grant MCS8102747.

In addition, one direction of Theorem 2(i) is an immediate consequence of the Calabi conjecture, as proved by Yau [6]: if $\Sigma d_i \leq 2n + r$, then the first Chern class of $V^{2n}_{d_1,\ldots,d_r}$ is a positive scalar multiple of the Kähler class in $\mathbf{C}P(2n + r)$, restricted to $V^{2n}_{d_1,\ldots,d_r}$. By Yau's theorem, we may then find a Kähler metric on $V^{2n}_{d_1,\ldots,d_r}$ whose first Chern form is equal to this multiple of the Kähler form. Since the first Chern form is essentially the Ricci tensor, we have constructed a metric of positive Ricci curvature (and hence positive scalar curvature) on $V^{2n}_{d_1,\ldots,d_r}$.

The rest of the paper is devoted to a proof of the other direction of Theorem 2(i). The plan is this: in §1 we prove Theorem 1, along the well-known general lines of Hirzebruch [1]. One must use a little care to obtain the formula of Theorem 1. From this, Theorem 2(i) for hypersurfaces is obvious. In §2, we then turn to the general case of complete intersections, where the number theory involved becomes somewhat more difficult.

1. Proof of Theorem 1. Let η denote the Kähler class of $H^2(\mathbf{C}P(2n + 1))$. If we apply the Whitney product formula to

$$T^*(V^{2n}_d) \oplus N(V^{2n}_d) = T^*(\mathbf{C}P(2n + 1)) \tag{1}$$

restricted to V^{2n}_d where $N(M)$ is the normal bundle, together with the identity

$$c_1(N(V^{2n}_d)) = d \cdot \eta \tag{2}$$

(see, for instance, [2, p. 146]), we get

$$(1 + c_1(V^{2n}_d) + c_2(V^{2n}_d) + \cdots + c_{2n}(V^{2n}_d))(1 + d \cdot \eta) = (1 + \eta)^{2n+2}. \tag{3}$$

The corresponding formula in Pontrjagin classes is then

$$(1 + p_1(V^{2n}_d) + \cdots + p_n(V^{2n}_d))(1 + d^2 \cdot \eta^2) = (1 + \eta^2)^{2n+2} \tag{4}$$

or, formally,

$$(1 + p_1(V^{2n}_d) + \cdots + p_n(V^{2n}_d)) = (1 + n^2)^{2n+2}/(1 + d^2 \cdot \eta^2). \tag{4'}$$

We now use the formal factorization of (4′) to evaluate the $\hat{A}$-genus of V^{2n}_d, using the fact that the $\hat{A}$-genus is the multiplicative sequence given by the function $(\sqrt{z}/2)/\sinh(\sqrt{z}/2)$ [1]. We find that

LEMMA 1.

$$\hat{A}(V^{2n}_d) = \left\{ \textit{the coefficient of } z^n \textit{ in } \frac{\left(2\sinh\left(d\sqrt{z}/2\right)\right)}{\left(2\sinh\left(\sqrt{z}/2\right)\right)^{2n+2}} \cdot \left(\sqrt{z}\right)^{2n+1} \right\}.$$

PROOF. By the formula for $\hat{A}$, applied to the factorization (4′),

$$\hat{A}(M) = \left(\frac{\eta}{2\sinh(\eta/2)}\right)^{2n+2} \left(\frac{d\eta}{2\sinh(d\eta/2)}\right)^{-1} \tag{5}$$

and it is easily seen that the coefficient of η^{2n} in the right-hand side of (5) is $1/d$ times the right-hand side of Lemma 1. On the other hand, $\eta^{2n}[V^{2n}_d] = d$, proving Lemma 1.

We now rewrite Lemma 1 in terms of residues:

$$(6)\qquad \hat{A}(V_d^{2n}) = \frac{1}{2\pi i}\int_{\Gamma(0)} \frac{1}{z^{n+1}}\left\{\frac{\left(2\sinh\left(d\sqrt{z}/2\right)\right)}{\left(2\sinh\left(\sqrt{z}/2\right)\right)^{2n+2}}\right\}\left(\sqrt{z}\right)^{2n+1} dz$$

$$= \frac{1}{2\pi i}\int_{\Gamma(0)} \frac{2\sinh\left(d\sqrt{z}/2\right)}{\left(2\sinh\left(\sqrt{z}/2\right)\right)^{2n+2}}\cdot\frac{1}{\sqrt{z}}\,dz$$

where $\Gamma(0)$ is any closed path which winds once about 0.

Using the substitution $u = \exp(\sqrt{z}/2)$, we find

$$\hat{A}(V_d^{2n}) = \frac{1}{2\pi i}\int_{\Gamma(1)} \frac{u^d - u^{-d}}{(u - u^{-1})^{2n+2}}\cdot\frac{2}{u}\,du$$

$$= \frac{1}{2\pi i}\int_{\Gamma(1)} u^{2n+1-d}\frac{[u^{2d} - 1]}{[u^2 - 1]^{2n+2}}\cdot 2\,du$$

where $\Gamma(1)$ is any path winding once about 1, and where we have taken into account that any path which winds once about 1 in the u-plane will wind twice about 0 in the z-plane.

Now substituting $v = u^2$, we find

$$\hat{A}(V_d^{2n}) = \frac{1}{2\pi i}\int_{\Gamma(1)} v^{n-d/2}\frac{[v^d - 1]}{(v - 1)^{2n+2}}\,dv$$

$$= \frac{1}{2\pi i}\int_{\Gamma(0)} \frac{(1+w)^{n+d/2} - (1+w)^{n-d/2}}{w^{2n+2}}\,dw$$

where in the last equation we substituted $w = v - 1$.

This last is then the coefficient of w^{2n+1} in $(1+w)^{n+d/2} - (1+w)^{n-d/2}$, which is then easily seen to be $2\cdot((d/2 + n)\cdots(d/2 - n))/(2n+1)!$, as desired.

2. Complete intersections. We may proceed exactly as in §1 to obtain a formula for $\hat{A}(V_{d_1,\ldots,d_r}^{2n})$, starting from the equation for Chern classes

$$(1)\qquad \left[\prod_i (1 + d_i\cdot\eta)\right]\left[1 + c_1(V) + \cdots + c_n(V)\right] = (1+\eta)^{2n+r+1}.$$

We arrive at the formula

$$(2)\qquad \hat{A}\left(V_{d_1,\ldots,d_r}^{2n}\right) = \int_{\Gamma(0)} (1+w)^{(n-1/2-\Sigma(d_i-1/2))}\frac{\Pi_i\left((1+w)^{d_i} - 1\right)}{w^{2n+r+1}}\,dz.$$

Writing

$$(3)\qquad (1+w)^{d_i} - 1 = (w)\left((1+w)^{d_i-1} + \cdots + 1\right)$$

we can rewrite this as

$$(4)\quad \hat{A}\left(V^{2n}_{d_1,\dots,d_r}\right) = \int_{\Gamma_0} \frac{(1+w)^{n-1/2}}{w^{2n+1}} \left[\prod_i \left[(1+w)^{(d_i-1)/2} + (1+w)^{(d_i-1)/2-1} + \cdots + (1+w)^{(1-d_i)/2} \right] \right] dz.$$

Note that under the assumption that $2n + r + 1 - \Sigma d_i$ is even, we must have an odd number of even d_i's.

It is easy to observe from (2) that $\hat{A}(V^{2n}_{d_1,\dots,d_n}) = 0$ when $2n + r + 1 - \Sigma d_i$ is positive and even, since then the numerator of the integrand in (2) is a polynomial in w of degree $n + (r-1)/2 + \Sigma d_i/2 < 2n + r$. It remains to show that if this sum is nonpositive, we get a nonzero number. To accomplish this, we do induction on r. Our main inductive tool is

LEMMA 2. $[1/(1+z)^k] + (1+z)^k$ *is a polynomial in* $(z^2/(1+z))$ *of degree* $\leqslant k$, *with positive coefficients.*

PROOF. It is easily seen that $1/(1+z)^k + (1+z)^k$ is a polynomial in $(z^2/(1+z))$, since it is a symmetric polynomial in $1/(1+z)$ and $(1+z)$, and hence a polynomial in the elementary symmetric functions of $(1+z)$ and $1/(1+z)$, which are 1 and $z^2/(1+z) + 2$. It remains to check that the coefficient of $(z^2/(1+z))^i$ is positive. Let $a_i(k)$ denote the coefficient of $(z^2/(1+z))^i$ in $[1/(1+z)^k] + (1+z)^k$. Then by writing

$$\frac{1}{(1+z)^k} + (1+z)^k = \frac{1}{(1+z)^k}\left(1 + (1+z)^{2k}\right) = \frac{1}{(1+z)^k}\left(2 + 2kz + \cdots + z^{2k}\right)$$

the following properties are evident:

(i) $a_i(k)$ is a polynomial of degree $2i$ in k.
(ii) $a_i(-k) = a_i(k)$.
(iii) $a_i(k) = 0$ for $k < i$.
(iv) $a_i(i) = 1$.

We now have $2i + 1$ values on which to evaluate $a_i(k)$, thus determining it completely. We find

$$a_i(k) = \frac{2k^2(k+1)(k-1)\cdots(k+(i-1))(k-(i-1))}{(2i)!}.$$

It follows that $a_i(k)$ is positive for $k \geqslant i$, and 0 for $0 \leqslant k < i$.

We may now prove Theorem 2 as follows: suppose some d_i is odd, we may assume it is d_r. We may then expand the factor corresponding to r in a polynomial in $z^2/(1+z)$ of degree $d_i - 1$, with positive coefficients. If we multiply through, we obtain a sum of terms with positive coefficients whose value is $\hat{A}(V^{2m}_{d_1,\dots,d_{r-1}})$ where

$2m$ takes on the values $2n-(d_r-1),\dots,2n$. In particular, for at least one of these terms the inequality $2n+(r+1)-\Sigma d_i \leq 0$ continues to hold. By induction, this is a sum of positive numbers, and hence nonzero.

Now suppose all the d_i's are even. If $r=1$, then we are done, by Theorem 1. If $r>1$, then r is odd, and ≥ 3. The product of the terms corresponding to d_r and d_{r-1} is again a polynomial symmetric in the $1/(1+z)^i$'s and the $(1+z)^i$'s, with positive coefficients (the fractional powers of $(1+z)$ cancel out). Again we apply the lemma to obtain a polynomial in $z^2/(1+z)$ of degree $(d_r+d_{r-1})/2-1$. It follows as above that $\hat{A}(V^2_{d_1,\dots,d_{r-1}})$ is expressible as a sum of terms, with positive coefficients, in the $\hat{A}(V^{2m}_{d_1,\dots,d_{r-2}})$, with $2m$ taking on the values $2n-(d_r+d_{r-1}-2),\dots,2n$, so that again this sum must be positive if $2n+r+1-\Sigma d_i \leq 0$. This concludes the proof of Theorem 2.

References

1. F. Hirzebruch, *Topological methods in algebraic geometry*, 3rd ed., Springer-Verlag, Berlin and New York, 1978.

2. P. A. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

3. M. Gromov and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), 423–434.

4. H. B. Lawson and M. L. Michelsohn, *Clifford bundles, immersions of manifolds, and the vector field problem*, J. Differential Geom. **15** (1980), 237–267.

5. A Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7–9.

6. S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation*. I, Comm. Pure Appl. Math. **31** (1978), 339–411.

Department of Mathematics, University of Maryland, College Park, Maryland 20742