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The Geometric Phase in Quantum Systems

Foundations, Mathematical Concepts, and Applications in Molecular and Condensed Matter Physics

With 56 Figures

Springer

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