THE GEOMETRY AND STRUCTURE OF ISOTROPY IRREDUCIBLE HOMOGENEOUS SPACES

BY

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To Albert M. Wolf on his sixtieth birthday

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Introduction

As a first step toward understanding the geometry of a riemannian homogeneous space $M=G/K$ it is natural to consider the case where K acts irreducibly on the tangent space. This approach has been very useful in the study of riemannian symmetric spaces; these spaces are now well understood, and their understanding is based on Cartan's classification and structure theory for the irreducible case. Our Chapter I gives a structure theory and classification for nonsymmetric coset spaces G/K where K acts irreducibly on the tangent space. For K compact, this is done in §§ 1–10, then summarized and put into global form in § 11. For K noncompact, we reduce to the compact case by means of Cartan involutions (§ 12). The results are surprising, for there are a large number of nonsymmetric "isotropy irreducible" coset spaces G/K, and only a few examples had been known before. One of the more interesting classes is $\mathbf{SO}(\dim K)/\mathrm{ad}(K)$ for an arbitrary compact simple Lie group K.

Chapter II concentrates on the study of complex and almost complex structures on isotropy irreducible coset spaces, and § 13 is a definitive treatment of this matter. More general structures are introduced and studied in § 14; the quaternionic structures are needed in § 16, and I believe that the notion of commuting structure will become important in riemannian geometry.

Chapter III is the goal of this paper—the riemannian geometry of isotropy irreducible coset spaces. The riemannian metric is unique up to a constant scalar factor; it is an Einstein metric with sectional curvature of one sign. We determine the holonomy group, the full group of isometries, and (in the almost complex case) the full group of almost hermitian isometries. The chapter ends with an examination of riemannian manifolds in which the local isometry group at a point is irreducible on the tangent space.

The de Rham decomposition shows that a riemannian manifold has parallel Ricci tensor if and only if it is locally a product of Einstein manifolds. Our isotropy irreducible riemannian manifolds have parallel Ricci tensor. Thus the classification results of Chapter I provide new examples of Einstein manifolds, and those examples are neither symmetric nor kaehlerian. I have hopes that those examples, especially the $\mathbf{SO}(\dim K)/\mathrm{ad}(K)$ which show a clear pattern, will contribute toward an understanding of Einstein manifolds.

I wish to thank Lois B. Wolf for checking some of my calculations on $\mathbf{E}_7$ and $\mathbf{E}_8$.

Chapter I. The structure and classification of nonsymmetric isotropy irreducible coset spaces G/K

In this chapter we study and classify the coset spaces $M=G/K$ which satisfy the conditions

(i) *G is a connected Lie group and K is a closed subgroup,*

(ii) *$M = G/K$ is a reductive*[1] *coset space on which G acts effectively*[2], *and*

(iii) *the linear isotropy action (on the tangent space of M) of the identity component K_0 of K is a representation which is irreducible over the real number field.*

Conditions (ii) and (iii) together are equivalent to

[(ii) ∪ (iii)]′ *Let χ be the linear isotropy representation of K on the tangent space of M. Then χ is a faithful representation of K and $\chi|_{K_0}$ is irreducible over the real number field.*

The general case can be transformed or reduced to the case where K is compact, by means of Cartan involutions. This is done in § 12. For the next eleven sections, however, we avoid technical difficulties by generally making the working hypothesis

(iv) *K is compact.*

The euclidean spaces and the irreducible riemannian symmetric spaces are the best known spaces which satisfy (i)–(iv). So we avoid duplication of standard material with the working hypothesis

(v) *(G, K) is not a symmetric pair, i.e. K_0 is not the identity component of the fixed point set of an involutive automorphism of G.*

In this chapter we need a certain amount of notation. A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, and E_8 refer to the Cartan classification types of simple Lie groups and algebras. We use boldface to denote the compact simply connected groups. Thus

$\mathbf{A}_n = \mathbf{SU}(n+1)$, special unitary group,

$\mathbf{B}_n = \mathbf{Spin}(2n+1)$, two sheeted covering of the rotation group $\mathbf{SO}(2n+1)$;

$\mathbf{C}_n = \mathbf{Sp}(n)$, unitary symplectic group;

$\mathbf{D}_n = \mathbf{Spin}(2n)$, double covering of $\mathbf{SO}(2n)$;

$\mathbf{G}_2$ is the automorphism group of the Cayley algebra; and so on. German letters denote Lie algebras; thus $\mathfrak{A}_n$, $\mathfrak{Sp}(n)$ and $\mathfrak{G}$ are the Lie algebras of Lie groups $\mathbf{A}_n$, $\mathbf{Sp}(n)$ and G. If K is a Lie subgroup of G, then $\mathfrak{K}$ denotes the corresponding subalgebra of $\mathfrak{G}$. If $g \in G$, then $\mathrm{ad}(g)$ denotes both the inner automorphism $x \to gxg^{-1}$ of G, and the corresponding automorphism of $\mathfrak{G}$; the latter is a representation which we usually denote ad_G.

Let $\mathfrak{K}$ be a semisimple Lie algebra. Given a Cartan subalgebra $\mathfrak{H}$ and an ordering of the roots, we have a system $\{\alpha_1, \ldots, \alpha_l\}$ of simple roots. If π is a linear representation of

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(1) This means that the Lie algebra $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ where $\mathfrak{M}$ is a vector space complement of $\mathfrak{K}$ such that $\mathrm{ad}_G(K)\mathfrak{M} = \mathfrak{M}$.

(2) In other words the identity element $1 \in G$ is the only element which acts on M as the identity transformation.

$\mathfrak{K}$ on a complex vector space, then every weight λ satisfies the condition that $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$ are integers; here $\langle\ ,\ \rangle$ denotes the inner product dual to the Killing form. If π is absolutely irreducible and λ is the highest weight, we generally denote π by π_λ because λ specifies π up to equivalence. In turn λ is specified by the nonnegative integers $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle$. Our notation for λ and π_λ is the following: if the integer $2\langle\lambda, \alpha_i\rangle/\langle\alpha_i, \alpha_i\rangle \neq 0$, then we write it next to the vertex of the Dynkin diagram of $\mathfrak{K}^C$ which specifies α_i. For example

$$\overset{1}{\circ}-\cdots-\circ,\quad \overset{1}{\circ}-\cdots-\circ=\bullet,\quad \overset{1}{\bullet}-\cdots-\bullet=\circ,\quad \overset{1}{\circ}-\cdots-\circ\!\!<^{\circ}_{\circ}$$

denote the usual ("vector") representations of $\mathbf{A}_{n-1}$ as $\mathbf{SU}(n)$, $\mathbf{B}_n$ as $\mathbf{SO}(2n+1)$, $\mathbf{C}_n$ as $\mathbf{Sp}(n)$ and $\mathbf{D}_n$ as $\mathbf{SO}(2n)$, respectively. And the adjoint representations are given by

A_1	$\overset{2}{\circ}$	G_2	$\bullet\equiv\overset{1}{\circ}$
$A_n,\ n>1$	$\overset{1}{\circ}-\circ-\cdots-\overset{1}{\circ}$	F_4	$\bullet-\bullet=\circ-\overset{1}{\circ}$
$B_n,\ n>2$	$\circ-\overset{1}{\circ}-\cdots-\circ=\bullet$	E_6	$\circ-\circ-\underset{\substack{\vert \\ \circ\ 1}}{\circ}-\circ-\circ$
$C_n,\ n>1$	$\overset{2}{\bullet}-\bullet-\cdots-\bullet=\circ$	E_7	$1\ \circ-\circ-\underset{\substack{\vert \\ \circ}}{\circ}-\circ-\circ-\circ$
$D_n,\ n>3$	$\circ-\overset{1}{\circ}-\cdots-\circ\!\!<^{\circ}_{\circ}$	E_8	$\circ-\circ-\underset{\substack{\vert \\ \circ}}{\circ}-\circ-\circ-\circ-\underset{1}{\circ}$

Note that we are using the dot convention: if there are two lengths of roots, then the short roots are black in the Dynkin diagram.

1. G is a compact simple Lie group

We will prove:

1.1. THEOREM. *Let $M = G/K$ satisfy conditions* (i) *through* (v) *above. Then G is a compact simple Lie group.*

The proof is divided into several steps, some of which are stated as fairly general lemmas for purposes of reference when we come to the case of noncompact K.

1.2. LEMMA. *Let $M = G/K$ be a reductive coset space of connected real Lie groups such that G acts effectively on M and the linear isotropy representation π of K is $\mathbf{R}$-irreducible. Suppose that G is not semisimple. Then either G is a circle group and $K = \{1\}$, so (G, K) is a* *symmetric pair with symmetry* $g \to g^{-1}$; *or* G *is the semidirect product* $K \times_\pi \mathbf{R}^n$ *with* $n = \dim M$ *and* (G, K) *is a symmetric pair with symmetry* $(k, v) \to (k, v^{-1})$ $[k \in K$ *and* $v \in \mathbf{R}^n]$.

Proof. Let $\mathfrak{G}$ and $\mathfrak{K}$ denote the Lie algebras of G and K, and let $\mathfrak{S}$ be the radical of $\mathfrak{G}$. Let $\mathfrak{A}$ denote the last nonzero term in the derived series of $\mathfrak{S}$. Then $\mathfrak{A}$ is an ideal in $\mathfrak{G}$. We cannot have $\mathfrak{A} \subset \mathfrak{K}$ because G acts effectively on M, so $\mathfrak{K} \subsetneqq \mathfrak{K} + \mathfrak{A} \subset \mathfrak{G}$. Now $\mathbf{R}$-irreducibility of π says $\mathfrak{G} = \mathfrak{K} + \mathfrak{A}$. As $M = G/K$ is reductive we have an $\mathrm{ad}_G(K)$-stable decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ with $\mathfrak{M} \subset \mathfrak{A}$, and $\mathfrak{M}$ is an abelian Lie algebra because $\mathfrak{A}$ is abelian. Now $\mathfrak{G} = \mathfrak{K} +_\pi \mathfrak{R}^n$ semidirect sum where $\mathfrak{R}^n = \mathfrak{M}$ is the Lie algebra of the real vector group of dimension $n = \dim M$.

Let V be the analytic subgroup of G with Lie algebra $\mathfrak{M} = \mathfrak{R}^n$. Then $V = \mathbf{R}^n/D$, quotient of a vector group by a discrete additive subgroup. $\pi(K)$ acts on $\mathbf{R}^n$ *qua* $\mathfrak{R}^n$ and preserves D, and this linear action is irreducible. If $n > 1$ it follows that $D = \{1\}$, so V is the vector group $\mathbf{R}^n$; then $K \cap V = \{1\}$ and G is the semidirect product $K \times_\pi \mathbf{R}^n$ as asserted; in that case $(k, v) \to (k, v^{-1})$ is an involutive automorphism of G with fixed point set K so (G, K) is a symmetric pair. If $n = 1$ there is also the possibility that $D \neq \{1\}$. Then $G = V$ is a circle group and $K = \{1\}$, and it is immediate that (G, K) is a symmetric pair under the involutive automorphism $g \to g^{-1}$ of G, q.e.d.

The relevant special case of Lemma 1.2 is

(1.3) *Under conditions* (i) *through* (v), G *is semisimple.*

1.4. Lemma. *Let* $M = G/K$ *be a reductive coset space of connected real Lie groups such that* G *acts effectively on* M *and the linear isotropy representation of* K *is* $\mathbf{R}$*-irreducible. Suppose that* G *is semisimple but not simple. Then* K *is simple,* G *is locally isomorphic to* $K \times K$ *with* K *embedded diagonally, and* (G, K) *is a symmetric pair with symmetry* $(k_1, k_2) \to (k_2, k_1)$ $[k_i \in K]$.

Proof. We may divide out the center of G, assuming $G = G_1 \times \ldots \times G_r$ with G_i centerless and simple. Let β_i: $G \to G_i$ denote the projection.

If $\beta_i(K) \neq G_i$ for some index i then $K \subset \beta_i^{-1}\beta_i K \subsetneqq G$, so $K = \beta_i^{-1}\beta_i K$ by irreducibility of the linear isotropy representation. $r > 1$ because G is not simple, so there is an index $j \neq i$; then $G_j \subset K$ so G is not effective on M. That contradiction shows that $\beta_i(K) = G_i$ for every index i.

The Lie group K is reductive because the linear isotropy representation is faithful and fully reducible. Let K'' be the kernel of $\beta_1|_K$; now $K = K' \cdot K''$ local direct product, and β_1: $K' \cong G_1$. In particular K' is simple. If we have an index i with $\beta_i(K') \neq G_i$ then $\beta_i(K') = \{1\}$ so G_i is in the centralizer of K'. But K'' is the centralizer of K' and $G_i \not\subset K$.

Thus $\beta_i : K' \cong G_i$ for every index i. That shows that each $\beta_i(K'') = \{1\}$, so $K = K'$, K is simple and each $\beta_i : K \cong G_i$.

Identify G_i with K by β_i. Then $G = K \times \ldots \times K$ (r times) with K embedded diagonally. $K = \{(g_1, \ldots, g_r) \in G\colon g_1 = \ldots = g_r\}$. If $r > 2$ then $K \subsetneqq KG_r \subsetneqq G$, contradicting irreducibility of the linear isotropy representation. As $r > 1$ now $G = K \times K$ with K embedded diagonally. Then $(k_1, k_2) \to (k_2, k_1)$ is an involutive automorphism of G with fixed point set K and our assertions are proved, q.e.d.

In view of (1.3), the relevant special case of Lemma 1.4 is

(1.5) *Under conditions* (i) *through* (v), *G is simple.*

Proof of Theorem 1.1. (1.5) says that G is a simple Lie group. If G is noncompact we choose a maximal compactly embedded subalgebra $\mathfrak{L} \subsetneqq \mathfrak{G}$ such that $\mathfrak{K} \subset \mathfrak{L}$, and then $\mathfrak{K} = \mathfrak{L}$ by irreducibility of the linear isotropy representation; it follows that (G, K) is a symmetric pair. Thus G is compact, q.e.d.

The analysis of coset spaces $M = G/K$ satisfying conditions (i) through (v) is now reduced to a specific problem on compact simple Lie groups.

2. The case of equal ranks

If rank G = rank K the result is

2.1. THEOREM. *Let $M = G/K$ be a coset space of compact connected Lie groups with G acting effectively and* rank G = rank K. *Let χ be the linear isotropy representation of K on the tangent space of M. Then χ is $\mathbf{R}$-irreducible if and only if, either $M = G/K$ is an irreducible symmetric coset space, or the center of K is the cyclic group of order* 3. *In the latter case there are just six possibilities, as follows.*

	G	K	χ
1	$\mathbf{G}_2$	$\mathbf{SU}(3)$	$\overset{1}{\circ}-\circ\oplus\circ-\overset{1}{\circ}$
2	$\mathbf{F}_4$	$\mathbf{SU}(3)\cdot\mathbf{SU}(3)$	$(\overset{1}{\circ}-\circ\otimes\overset{2}{\circ}-\circ)\oplus(\circ-\overset{1}{\circ}\otimes\circ-\overset{2}{\circ})$
3	$\mathbf{E}_6/\mathbf{Z}_3$	$\mathbf{SU}(3)\cdot\mathbf{SU}(3)\cdot\mathbf{SU}(3)$	$(\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ)\oplus(\circ-\overset{1}{\circ}\otimes\circ-\overset{1}{\circ}\otimes\circ-\overset{1}{\circ})$
4	$\mathbf{E}_7/\mathbf{Z}_2$	$[\mathbf{SU}(3)\times\mathbf{SU}(6)]/\mathbf{Z}_6$	$(\circ-\overset{1}{\circ}\otimes\circ-\overset{1}{\circ}-\circ-\circ-\circ)\oplus(\overset{1}{\circ}-\circ\otimes\circ-\circ-\circ-\overset{1}{\circ}-\circ)$
5	$\mathbf{E}_8$	$\mathbf{SU}(9)/\mathbf{Z}_3$	$(\circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ-\circ-\circ)\oplus(\circ-\circ-\circ-\circ-\circ-\overset{1}{\circ}-\circ-\circ)$
6	$\mathbf{E}_8$	$[\mathbf{SU}(3)\times\mathbf{E}_6]/\mathbf{Z}_3$	$(\overset{1}{\circ}-\circ\otimes\overset{1}{\circ}-\circ-\overset{\overset{\circ}{\vert}}{\circ}-\circ-\circ)\oplus(\circ-\overset{1}{\circ}\otimes\circ-\circ-\overset{\overset{\circ}{\vert}}{\circ}-\circ-\overset{1}{\circ})$

Theorem 2.1 can be checked directly by computing the linear isotropy representation for each of the nonsymmetric pairs (G, K) listed by Borel and de Siebenthal [4]. That calculation is extremely unpleasant without *a priori* knowledge of irreducibility of χ, so we avoid the unpleasantness by using B. Kostant's result ([20], Theorem 8.13.3, p. 296):

2.2. THEOREM. *Let G be a connected reductive Lie group, let T be a Cartan subgroup, and let A be a subgroup of T. Let K be the identity component of the centralizer of A in G, let Z be the center of K, and suppose $Z_0 \subset A$. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$ be the orthogonal decomposition under the Killing form. Decompose $\mathfrak{M}^C = \Sigma\, \mathfrak{M}_i$ where $\mathrm{ad}_G(A)$ acts on $\mathfrak{M}_i$ by a multiple of an irreducible complex representation α_i and where the α_i are distinct characters on A. Then $\mathrm{ad}_G(K)$ preserves $\mathfrak{M}_i$, acting there by an irreducible complex representation π_i, and the π_i are mutually inequivalent.*

Proof. ad (K) preserves $\mathfrak{M}_i$ because K centralizes A. An equivalence $\pi_i \sim \pi_j$ would restrict to an equivalence $\alpha_i \sim \alpha_j$ and imply $\alpha_i = \alpha_j$; thus the π_i are mutually inequivalent. Now we need only prove that each π_i is irreducible.

Suppose π_i reducible and decompose $\pi_i = \Sigma\, \beta_a$ with β_a irreducible. Then $\mathfrak{M}_i = \Sigma\, \mathfrak{M}_{(a)}$ where $\mathfrak{M}_{(a)}$ is the representation space of β_a. Order the $\mathfrak{T}^C$-roots of $\mathfrak{K}^C$ and let λ_a denote the highest weight of β_a. Let $a \neq b$ be indices. Then λ_a and λ_b coincide on $\mathfrak{Z}^C$ because $Z_0 \subset A$, so they differ only on the intersection of $\mathfrak{T}^C$ with the semisimple part of the reductive Lie algebra $\mathfrak{K}^C$. There all highest weights are in the closure of the positive Weyl chamber, so $\langle \lambda_a, \lambda_b \rangle \geqslant 0$.

Suppose $\langle \lambda_a, \lambda_b \rangle > 0$. As λ_a and λ_b are $\mathfrak{T}^C$-roots of $\mathfrak{G}^C$ it follows that $\nu = \lambda_a - \lambda_b$ is a root. Choose nonzero root vectors $E_\nu \in \mathfrak{G}_\nu$, $E_a \in \mathfrak{G}_{\lambda_a}$ and $E_b \in \mathfrak{G}_{\lambda_b}$. Then $[E_\nu, E_b] = cE_a$ for some $c \neq 0$. If $g \in A$ then $c \cdot \alpha_i(g) E_a = c \cdot \mathrm{ad}(g) E_a = \mathrm{ad}(g)[E_\nu, E_b] = [\mathrm{ad}(g) E_\nu, \mathrm{ad}(g) E_b] = [\mathrm{ad}(g) E_\nu, \alpha_i(g) E_b]$, so $\mathrm{ad}(g) E_\nu = E_\nu$. Thus $E_\nu \in \mathfrak{K}^C$. Now $E_a \in \mathfrak{M}_{(a)}$ and $E_a = c^{-1}[E_\nu, E_b] \in [\mathfrak{K}^C, \mathfrak{M}_{(b)}] \subset \mathfrak{M}_{(b)}$ so $a = b$. In other words $a \neq b$ implies $\lambda_a \perp \lambda_b$.

Decompose $\mathfrak{K}^C = \Sigma\, \mathfrak{K}_r$, direct sum of its center $\mathfrak{Z}^C$ and its simple ideals. Then each $\beta_a = \otimes \beta_{a,r}$ with $\beta_{a,r}$ an irreducible complex representation of $\mathfrak{K}_r$, and each $\lambda_a = \sum \lambda_{a,r}$ where $\lambda_{a,r} \in \mathfrak{T}^C \cap \mathfrak{K}_r$ is the highest weight of $\beta_{a,r}$. If $a \neq b$ then $\lambda_a \perp \lambda_b$ says, for each index r, that at most one of the $\lambda_{a,r}$ can be nonzero. Now $\mathfrak{K}^C = \mathfrak{L}_1^C \oplus \mathfrak{L}_2^C$ and $\mathfrak{K} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ where $\mathfrak{L}_1^C$ is the sum of all $\mathfrak{K}_r$ for which $\lambda_{a,r} \neq 0$ and $\mathfrak{L}_2^C$ is the sum of the remaining $\mathfrak{K}_r$. This decomposes $\pi_i = (\tau_a \otimes 1) \oplus (1 \otimes \tau_a')$ where τ_a represents $\mathfrak{L}_1^C$ and τ_a' represents $\mathfrak{L}_2^C$. Thus $\tau_a \otimes 1 = \beta_a$ and $1 \otimes \tau_a' = \sum_{b \neq a} \beta_b$. Let L_1 and L_2 be the analytic subgroups of K with respective Lie algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Let Z_1 and Z_2 be their centers so $Z = Z_1 \cdot Z_2$; let A_1 and A_2 denote the projections of A on Z_1 and Z_2 so $A \subset A_1 \cdot A_2$. Then $1 \otimes \tau_a'$ annihilates A_1 because it annihilates L_1. This forces $\tau_a \otimes 1$ to annihilate A_1 because it and $1 \otimes \tau_a'$ both represent on $\mathfrak{M}_i$. Thus π_i annihilates

A_1. Similarly $\tau_a \otimes 1$, thus also $1 \otimes \tau_a'$, thus π_i, annihilates A_2. Now π_i annihilates $A \subset A_1 \cdot A_2$, so $\alpha_i = 1$, which is absurd. This contradiction shows that we cannot have two distinct summands β_a and β_b of π_i. In other words, π_i is irreducible, q.e.d.

Proof of Theorem 2.1. Let Z be the center of K and let T be a maximal torus of G such that $Z \subset T \subset K$. The rank condition says ([4], Theoreme 5; or [20], Theorem 8.10.2, p. 276) that K is the identity component of the centralizer of Z in G. Assume χ to be $\mathbf{R}$-irreducible. Then Schur's Lemma says that Z is a circle group or a cyclic group of some finite order m. If Z has an element of order 2 then $M = G/K$ is an irreducible symmetric coset space. If not, Z is cyclic of odd finite order $m > 1$. Then we apply Theorem 2.2 with $A = Z$ to obtain the decomposition $\mathfrak{M}^C = \sum \mathfrak{M}_\varepsilon$ where ε runs through a set of m-th roots of 1 and where we have chosen a generator z of Z such that $\alpha_\varepsilon(z) = \varepsilon$. As z has order m we have a primitive m-th root η of 1 such that $\mathfrak{M}_\eta \neq 0$. Then $\mathfrak{M}^C = \mathfrak{M}_\eta + \mathfrak{M}_{\bar\eta}$ by $\mathbf{R}$-irreducibility of χ. If $m > 3$ then $[\mathfrak{M}_\eta, \mathfrak{M}_{\bar\eta}] \subset \mathfrak{K}^C$, $[\mathfrak{M}_\eta, \mathfrak{M}_\eta] \subset \mathfrak{M}_{\eta^2} = 0$ and $[\mathfrak{M}_{\bar\eta}, \mathfrak{M}_{\bar\eta}] \subset \mathfrak{M}_{\bar\eta^2} = 0$; that implies $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{K}$ so that $M = G/K$ is an irreducible symmetric coset space. In other words, if $M = G/K$ is not an irreducible symmetric coset space then Z has order 3. Conversely if Z has order 3 then Theorem 2.2 shows that χ is $\mathbf{R}$-irreducible.

Consider the case where the center Z of K has order 3. The classification of all such pairs $(\mathfrak{G}, \mathfrak{K})$ is given by Borel and de Siebenthal [4] (or see [20], Theorem 8.10.9, p. 280). G is centerless, thus of the listed global form. $\chi = \beta \oplus \bar\beta$ for some irreducible complex representation β of K such that $\beta(K)$ has center of order 3 and β has degree $\deg \beta = \frac{1}{2} \dim M = \frac{1}{2} [\dim G - \dim K]$. In these low degrees there is no choice; β and χ are as listed because there are no other possibilities. Now K has the listed global form because β is faithful, q.e.d.

3. The case where G is exceptional and rank $G >$ rank K

Here the classifiction is given by

3.1 Theorem (E. B. Dynkin[1]). *The following is a complete list of the coset spaces G/K of compact connected Lie groups where* (a) *G acts effectively,* (b) rank $G >$ rank K, (c) *G is an exceptional group and* (d) *K acts irreducibly on the tangent space.*

$\mathbf{E}_6/\mathbf{A}_2$ is the only one for which the isotropy representation is not absolutely irreducible. $\mathbf{E}_6/\mathbf{C}_4$ and $\mathbf{E}_6/\mathbf{F}_4$ are the only ones which are symmetric. In $\mathbf{G}_2/\mathbf{A}_1$, the $\mathbf{A}_1$ is the principal three dimensional subgroup.

The result follows from Theorem 14.1 of E. B. Dynkin's paper [7]. Dynkin writes $\tilde{G}$ for our $\mathfrak{K}^C$, G for our $\mathfrak{G}^C$, $\chi_{\tilde{G}}$ for the representation of $\mathfrak{K}^C$ on the complexification of the tangent

(1) As will be seen from the proof, the result is essentially due to Dynkin.

	G	K	Isotropy representation of K on tangent space
1	$\mathbf{G}_2$	$\mathbf{A}_1$	$\overset{10}{\circ}$
2	$\mathbf{F}_4$	$\mathbf{A}_1\cdot\mathbf{G}_2$	$\overset{4}{\circ}\otimes\overset{1}{\bullet}\equiv\circ$
3	$\mathbf{E}_6$	$\mathbf{A}_2$	$\overset{1}{\circ}-\overset{4}{\circ}\oplus\overset{4}{\circ}-\overset{1}{\circ}$
4	$\mathbf{E}_6$	$\mathbf{G}_2$	$\overset{1}{\bullet}\equiv\overset{1}{\circ}$
5	$\mathbf{E}_6$	$\mathbf{A}_2\cdot\mathbf{G}_2$	$\overset{1}{\circ}-\overset{1}{\circ}\otimes\overset{1}{\bullet}\equiv\circ$
6	$\mathbf{E}_6$	$\mathbf{C}_4$	$\bullet-\bullet-\bullet=\overset{1}{\circ}$
7	$\mathbf{E}_6$	$\mathbf{F}_4$	$\overset{1}{\bullet}-\bullet=\circ-\circ$
8	$\mathbf{E}_7$	$\mathbf{A}_2$	$\overset{4}{\circ}-\overset{4}{\circ}$
9	$\mathbf{E}_7$	$\mathbf{G}_2\cdot\mathbf{C}_3$	$\overset{1}{\bullet}\equiv\circ\otimes\bullet-\overset{1}{\bullet}=\circ$
10	$\mathbf{E}_7$	$\mathbf{A}_1\cdot\mathbf{F}_4$	$\overset{2}{\circ}\otimes\bullet-\overset{1}{\bullet}=\circ-\circ$
11	$\mathbf{E}_8$	$\mathbf{G}_2\cdot\mathbf{F}_4$	$\overset{1}{\bullet}\equiv\circ\otimes\overset{1}{\bullet}-\bullet=\circ-\circ$

space of G/K. Thus we are looking for Dynkin's classification of pairs $(G, \tilde{G})$ consisting of a complex exceptional simple Lie algebra and a complex subalgebra $\tilde{G}$ such that (a) $\chi_{\tilde{G}}$ is absolutely irreducible or (b) $\chi_{\tilde{G}} = \beta \oplus \bar{\beta}$ where β is absolutely irreducible and has no nonzero symmetric bilinear invariant. $\tilde{G} = \mathfrak{K}^C$ will be a semisimple S-subalgebra in Dynkin's terminology because it is a maximal subalgebra which has lower rank. Following ([7], Theorem 14.1) now, the pair (G, K) is listed in our theorem under the number

1	if rank $K = 1$;
3, 4, 6, 7, 8	if rank $K > 1$ and K is simple;
2, 5, 9, 10, 11	if K is not simple.

This completes the proof that G/K is one of the spaces that we have listed. On the other hand, all the listed pairs $(\mathfrak{G}^C, \mathfrak{K}^C)$ exist, and given such a pair one can find a Cartan involution of $\mathfrak{G}^C$ which preserves $\mathfrak{K}^C$; then the pair $(\mathfrak{G}, \mathfrak{K})$ consists of the respective fixed point sets, so G/K exists, q.e.d.

4. The case where G is classical and K is not simple

The result is:

4.1. THEOREM. *The only nonsymmetric coset spaces G/K of compact connected Lie groups, where* (a) *G acts effectively,* (b) rank $G >$ rank K, (c) *G is a classical group,* (d) *K is not simple, and* (e) *K acts $\mathbf{R}$-irreducibly on the tangent space, are the*

$$\mathbf{SU}(pq)/\mathbf{SU}(p)\times\mathbf{SU}(q), \quad p>1, \quad q>1, \quad pq>4,$$

with the action of $\mathbf{SU}(pq)$ rendered effective.

Here the inclusion is the tensor product of the usual linear representations of $\mathbf{SU}(p)$ and $\mathbf{SU}(q)$, and the isotropy representation is the tensor product of the adjoint representations of $\mathbf{SU}(p)$ and $\mathbf{SU}(q)$. Let m be the least common multiple of p and q. Then globally

$$G = \mathbf{SU}(pq)/\mathbf{Z}_m \quad \text{and} \quad K = \{\mathbf{SU}(p)/\mathbf{Z}_p\}\times\{\mathbf{SU}(q)/\mathbf{Z}_q\}.$$

For the proof we first need some remarks on linear groups. Here * denotes dual representation, ad_L denotes the adjoint representation of a Lie group L, and 1_L denotes the trivial representation of degree 1.

(4.2) *Let δ: $\mathbf{SL}(n, \mathbf{C})\to\mathbf{GL}(n, \mathbf{C})$ denote the usual matrix representation of the complex special (determinant 1) linear group. Then $\delta\otimes\delta^* = 1_{\mathbf{SL}(n,\mathbf{C})}\oplus \mathrm{ad}_{\mathbf{SL}(n,\mathbf{C})}$.*

For the Lie algebra $\mathfrak{GL}(n, \mathbf{C})$ consists of all $n\times n$ complex matrices, so $\mathbf{SL}(n, \mathbf{C})$ acts on it by conjugation via $\delta\otimes\delta^*$. This action decomposes into the trivial action $1_{\mathbf{SL}(n,\mathbf{C})}$ on scalar matrices and the adjoint representation on matrices of trace zero.

(4.3) *Let δ: $\mathbf{Sp}(n, \mathbf{C})\to\mathbf{GL}(2n, \mathbf{C})$ denote the usual matrix representation of the complex symplectic group. Then $\mathrm{ad}_{\mathbf{Sp}(n,\mathbf{C})} = S^2(\delta)$, second symmetrization, which is the action on polynomials of degree 2.*

For $\mathrm{ad}_{\mathbf{Sp}(n,\mathbf{C})}$ is an irreducible summand of degree dim $\mathbf{Sp}(n, \mathbf{C}) = 2n^2+n$ in the representation $\delta\otimes\delta^* = \delta\otimes\delta$ on $\mathfrak{GL}(2n, \mathbf{C})$, hence contained in the representation on symmetric matrices or the representation on skew matrices. The latter has degree $2n^2-n$, which excludes it. The former is $S^2(\delta)$ and has degree $2n^2+n$, which yields our assertion.

(4.4) *Let δ: $\mathbf{SO}(n, \mathbf{C})\to\mathbf{GL}(n, \mathbf{C})$ denote the usual matrix representation of the complex special orthogonal group. Then $\mathrm{ad}_{\mathbf{SO}(n,\mathbf{C})} = \Lambda^2(\delta)$, second alternation, which is the action on differential forms of degree 2.*

For δ maps $\mathfrak{SO}(n, \mathbf{C})$ onto the set of all antisymmetric $n\times n$ complex matrices, and $\otimes\delta\delta^* = \delta\otimes\delta$.

(4.5) *Let K_i be groups, let F be a field of characteristic $\neq 2$, and let α_i: $K_i \to \mathbf{GL}(n_i, F)$ be linear representations. Then*

$$\Lambda^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes S^2(\alpha_2)\} \quad on \quad K_1 \times K_2;$$
$$S^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes S^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \quad on \quad K_1 \times K_2.$$

For let f and g be bilinear forms. Then so is $f \otimes g$. If f and g are both symmetric or both antisymmetric, one checks that $f \otimes g$ is symmetric. If one of $\{f, g\}$ is symmetric and the other is antisymmetric, one checks that $f \otimes g$ is antisymmetric. Now the assertion follows from the decomposition $\Lambda^2(\alpha_1 \otimes \alpha_2) + S^2(\alpha_1 \otimes \alpha_2) = (\alpha_1 \otimes \alpha_2) \otimes (\alpha_1 \otimes \alpha_2) = (\alpha_1 \otimes \alpha_1) \otimes (\alpha_2 \otimes \alpha_2) = \{\Lambda^2(\alpha_1) \oplus S^2(\alpha_1)\} \otimes \{\Lambda^2(\alpha_2) \oplus S^2(\alpha_2)\}$.

Proof of Theorem 4.1. According to Dynkin ([6], Theorems 1.3 and 1.4), $K^C \subset G^C$ is one of the inclusions

(1) $\mathbf{SL}(p_1, \mathbf{C}) \otimes \mathbf{SL}(p_2, \mathbf{C}) \subset \mathbf{SL}(p_1 p_2, \mathbf{C})$,
(2) $\mathbf{Sp}(p_1, \mathbf{C}) \otimes \mathbf{SO}(p_2, \mathbf{C}) \subset \mathbf{Sp}(p_1 p_2, \mathbf{C})$,
(3) $\mathbf{Sp}(p_1, \mathbf{C}) \otimes \mathbf{Sp}(p_2, \mathbf{C}) \subset \mathbf{SO}(4 p_1 p_2, \mathbf{C})$,
(4) $\mathbf{SO}(p_1, \mathbf{C}) \otimes \mathbf{SO}(p_2, \mathbf{C}) \subset \mathbf{SO}(p_1 p_2, \mathbf{C})$.

Here $K = K_1 \cdot K_2$ local direct product, K_i^C is a complex simple classical group with usual linear representation α_i: $K_i^C \to \mathbf{GL}(n_i, \mathbf{C})$ and $K_1^C \otimes K_2^C$ just denotes $(\alpha_1 \otimes \alpha_2)(K_1^C \otimes K_2^C)$. The cases are (1) $n_i = p_i$, (2) $n_1 = 2p_1$ and $n_2 = p_2$, (3) $n_i = 2p_i$, (4) $n_i = p_i$.

Let π be the representation of K on the tangent space of G/K. Then the representation ψ of K on $\mathfrak{G}$ decomposes as $\psi = \mathrm{ad}_K \oplus \pi = \{\mathrm{ad}_{K_1} \otimes 1_{K_2}\} \oplus \{1_{K_1} \otimes \mathrm{ad}_{K_2}\} \otimes \pi$. Now we check the four cases.

Case (1). Using (4.2) we have $1 \oplus \psi = (\alpha_1 \otimes \alpha_2) \otimes (\alpha_1 \otimes \alpha_2)^* = (\alpha_1 \otimes \alpha_1^*) \otimes (\alpha_2 \otimes \alpha_2^*) = (1_{K_1} \oplus \mathrm{ad}_{K_1}) \otimes (1_{K_2} \oplus \mathrm{ad}_{K_2}) = 1_{K_1 \times K_2} \oplus \{\mathrm{ad}_{K_1} \otimes 1_{K_2}\} \oplus \{1_{K_1} \otimes \mathrm{ad}_{K_2}\} \oplus \{\mathrm{ad}_{K_1} \otimes \mathrm{ad}_{K_2}\}$. Thus $\pi = \mathrm{ad}_{K_1} \otimes \mathrm{ad}_{K_2}$, absolutely irreducible. This is the case of the theorem.

Case (2). Using (4.3), (4.4) and (4.5), we have $\psi = S^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes S^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} = \{\mathrm{ad}_{K_1} \otimes [1_{K_2} \oplus \eta_2]\} \oplus \{[1_{K_1} \oplus \eta_1] \otimes \mathrm{ad}_{K_2}\} = \{\mathrm{ad}_{K_1} \otimes 1_{K_2}\} \oplus \{1_{K_1} \otimes \mathrm{ad}_{K_2}\} \oplus \{\mathrm{ad}_{K_1} \otimes \eta_2\} \oplus \{\eta_1 \otimes \mathrm{ad}_{K_2}\}$ for some representations η_i of K_i. Thus $\pi = \sigma \oplus \tau$ where $\sigma = \mathrm{ad}_{K_1} \otimes \eta_2$ and $\tau = \eta_1 \otimes \mathrm{ad}_{K_2}$. As π is irreducible over $\mathbf{R}$ we must have that (a) $\tau = \sigma^*$ and (b) σ has no symmetric bilinear invariant. But (a) says $\eta_2 = \mathrm{ad}_{K_2}$, which violates ($b$). Thus our case (2) is excluded.

Case (3). Using (4.3), (4.4) and (4.5), we have $\psi = \Lambda^2(\alpha_1 \otimes \alpha_2) = \{S^2(\alpha_1) \otimes \Lambda^2(\alpha_2)\} \oplus \{\Lambda^2(\alpha_1) \otimes S^2(\alpha_2)\} = \{\mathrm{ad}_{K_1} \otimes [1_{K_2} \oplus \eta_2]\} \oplus \{[1_{K_1} \oplus \eta_1] \otimes \mathrm{ad}_{K_2}\}$. As in case (2), this violates irreducibility of π over $\mathbf{R}$; thus case (3) is excluded.

Case (4) is also excluded by the argument used for case (2).

Finally, back in the admissible case (1), we have $p_i > 1$ so that $K_i^C = \mathbf{SL}(p_i, \mathbf{C})$ is semisimple, and we have $p_1 p_2 > 4$ because $\mathbf{SU}(4)/\mathbf{SU}(2) \otimes \mathbf{SU}(2) = \mathbf{SU}(4)/\mathbf{SO}(4)$, which is symmetric, q.e.d.

5. A problem in representation theory

Our classification problem for coset spaces G/K satisfying conditions (i)–(v), is now reduced to the case where G is a compact simple classical group and K is simple. On the Lie algebra level, $\mathfrak{G}$ is $\mathfrak{SU}(N)$, $\mathfrak{Sp}(N)$ or $\mathfrak{SO}(N)$ for some integer N, and we view the inclusion $\mathfrak{K} \to \mathfrak{G}$ as a linear representation π. If π is not absolutely irreducible it has image in a direct sum $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_2$ of Lie algebras of classical groups, and our simplicity conditions give $\mathfrak{K} \subsetneqq \mathfrak{L} \subsetneqq \mathfrak{G}$, contradicting irreducibility of K on the tangent space. Now π is absolutely irreducible, so it has a highest weight λ; thus $\pi = \pi_\lambda$. Let χ denote the representation of $\mathfrak{K}$ on the tangent space of G/K, so $\mathrm{ad}_G \circ \pi_\lambda = \mathrm{ad}_K \oplus \chi$. We must express $\mathbf{R}$-irreducibility of χ in terms of λ.

Let l be the rank of K. The choice of highest weight λ implied a choice of maximal torus $T \subset K$ and the choice of a system $\{\alpha_1, ..., \alpha_l\}$ of simple $\mathfrak{T}^C$-roots of $\mathfrak{K}^C$. Let ξ_r denote the linear form on $\mathfrak{T}^C$ specified by the conditions

$$\frac{2\langle \xi_r, \alpha_r \rangle}{\langle \alpha_r, \alpha_r \rangle} = 1, \quad \frac{2\langle \xi_r, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 0 \quad \text{for} \quad i \neq r.$$

Then the highest weights of absolutely irreducible representations of $\mathfrak{K}$ are just the linear forms $\eta = \sum n_i \xi_i$, n_i integers, $n_i \geqslant 0$. The representation of highest weight η is denoted π_η. The weights and representations ξ_i and π_{ξ_i} are called *basic.* The representation dual to π_η, which we denote π_η^*, has highest weight which we denote η^*. Note that $(\sum n_i \xi_i)^* = \sum n_i \xi_i^*$.

5.1. PROPOSITION. *If* $G = \mathbf{SU}(N)$, *then*

(1) $\lambda = k\xi_r$ *for some integer* $k \geqslant 1$ *and some basic weight* $\xi_r \neq \xi_r^*$,

(2) $\chi = \pi_{\lambda+\lambda^*}$, *absolutely irreducible, and*

(3) $N = \deg \pi_\lambda$ *satisfies* $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim K + 1$

Proof. If $\lambda = \lambda^*$ then π_λ maps K into a subgroup $L = \mathbf{SO}(N)$ or $\mathbf{Sp}(\frac{1}{2}N)$ of G. K does not map onto the subgroup because G/K is not symmetric, so $K \subsetneqq L \subsetneqq G$. That violates irreducibility. Thus $\lambda \neq \lambda^*$.

(4.2) says $\pi_\lambda \otimes \pi_{\lambda^*} = 1_K \oplus \mathrm{ad}_K \oplus \chi$. $\pi_{\lambda+\lambda^*}$ is a summand of $\pi_\lambda \otimes \pi_{\lambda^*}$, hence of ad_K or of χ. In the former case $\pi_{\lambda+\lambda^*} = \mathrm{ad}_K$. Let μ be the highest root so that $\mathrm{ad}_K = \pi_\mu$. Now $\mu = \lambda + \lambda^*$. As $\lambda \neq \lambda^*$, this says $\mu = \sum n_i \xi_i$ with at least two of the n_i nonzero. The only case is where

$\lambda+\lambda^*=\mu$: $\overset{1}{\underset{\alpha_1}{\circ}}-\underset{\alpha_2}{\circ}-\cdots-\overset{1}{\underset{\alpha_l}{\circ}}$. Then $K=\mathbf{SU}(l+1)$, λ is ξ_1 or ξ_l, and $\pi_\lambda(K)=G$. But $K\neq G$. Thus $\pi_{\lambda+\lambda^*}$ is a summand of χ. As χ is $\mathbf{R}$-irreducible and $\pi_{\lambda+\lambda^*}$ is real, now $\chi=\pi_{\lambda+\lambda^*}$, absolutely irreducible. Thus (2) is proved. (3) follows by taking degrees.

Decompose $\lambda=\sum n_i\xi_i$ and let m be the number of indices i with $n_i>0$. Then m is the multiplicity of ad_K in $\pi_\lambda\otimes\pi_{\lambda^*}$. We have just seen that ad_K has multiplicity zero in χ. Thus $m=1$, so λ has form $k\xi_r$, q.e.d.

The orthogonal case is more delicate:

5.2 Proposition. *If $G=\mathbf{SO}(N)$, then there are three cases:*

(*a*) *$\lambda=k\xi_r$ for some basic weight $\xi_r=\xi_r^*$, and $\chi=\pi_{2\lambda-\alpha_r}$, absolutely irreducible;*

(*b*) *$\lambda=k(\xi_r+\xi_r^*)$ for some basic weight $\xi_r\neq\xi_r^*$, and $\chi=\pi_{2\lambda-\alpha_r}\oplus\pi_{2\lambda-\alpha_r^*}$, not absolutely irreducible;*

(*c*) *$K=\mathbf{G}_2$ and $G=\mathbf{SO}(7)$, $\pi_\lambda=\chi=\pi_{\xi_1}$ $\overset{1}{\underset{\alpha_1}{\bullet}}\equiv\underset{\alpha_2}{\circ}$, and $\pi_{2\lambda-\alpha_1}=ad_K=\pi_{\xi_2}$ $\underset{\alpha_1}{\bullet}\equiv\overset{1}{\underset{\alpha_2}{\circ}}$.*

In all cases, $\lambda=\lambda^*$ with π_λ real, and $N=\deg\pi_\lambda$ satisfies $\frac{1}{2}(\deg\pi_\lambda)^2=\deg\chi+\frac{1}{2}\deg\pi_\lambda+$ dim K.

Proof. By (4.4), $\Lambda^2(\pi_\lambda)=\mathrm{ad}_K\oplus\chi$. As π_λ is orthogonal, this proves the last statement. Now $\chi=\beta_1\oplus\ldots\oplus\beta_p$ with β_i absolutely irreducible. As χ is $\mathbf{R}$-irreducible, either $p=1$, or $p=2$ with $\beta_1\neq\beta_2=\beta_1^*$, or $p=2$ with $\beta_1=\beta_1^*=\beta_2$ symplectic. Let α_i be a simple root not orthogonal to λ. Let V be the representation space of π_λ and choose weight vectors v_λ, $v_{\lambda-\alpha_i}$. Then $v_{\lambda-\alpha_i}\wedge v_\lambda\in\Lambda^2(V)$ is a weight vector of weight $2\lambda-\alpha_i$ for $\Lambda^2(\pi_\lambda)$ which is annihilated by every positive root space of $\mathfrak{K}^C$, so $\pi_{2\lambda-\alpha_i}$ is a summand of $\Lambda^2(\pi_\lambda)$.

If $\pi_{2\lambda-\alpha_i}=\mathrm{ad}_K$ then α_i is a terminal vertex on the Dynkin diagram of $\mathfrak{K}^C$. For otherwise we have two different simple roots α', α'' not orthogonal to α_i, so the highest root $\mu=2\lambda-\alpha_i$ is not orthogonal to α' nor to α''. That implies $\mathfrak{K}^C$ of type A_l, and then α' and α'' are terminal so $l=3$. Thus λ: $\circ-\overset{1}{\circ}-\circ$ so π_λ maps $\mathfrak{K}=\mathfrak{SU}(4)$ isomorphically onto $\mathfrak{G}=\mathfrak{SO}(6)$. As $\mathfrak{K}\neq\mathfrak{G}$ this is impossible. Thus α_i is terminal.

Let $\pi_{2\lambda-\alpha_i}=\mathrm{ad}_K$. Now α_i is terminal; let α' be the unique simple root not orthogonal to it. Then we have

$$\lambda:\ \overset{n_i}{\underset{\alpha_i}{\circ}}\sim\sim\overset{n'}{\underset{\alpha'}{\circ}}\!<\qquad\text{and}\qquad \mu=2\lambda-\alpha_i:\ \overset{2n_i-2}{\underset{\alpha_i}{\circ}}\sim\sim\overset{n'+e}{\underset{\alpha'}{\circ}}\!<$$

where e is 1, 2 or 3. If α' is not terminal, then μ: $\circ - \overset{1}{\circ} - \ldots - \circ - \left(- \circ = \bullet \text{ or } - \circ \lessdot \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right)$, so K is an orthogonal group and λ: $\overset{1}{\circ} - \circ - \ldots$. But then $\pi_\lambda(K) = G$, which is excluded. Thus α' is terminal. In other words, K has rank 2 with simple roots α_i and α'. The possibilites are

$$\text{(a)} \quad \underset{\alpha_i}{\circ} - \underset{\alpha'}{\circ} \qquad \text{(b)} \quad \begin{matrix} \circ = \bullet \\ \alpha_i \quad \alpha' \\ \bullet = \circ \end{matrix} \qquad \text{(c)} \quad \begin{matrix} \circ \equiv \bullet \\ \alpha_i \quad \alpha' \\ \bullet \equiv \circ \end{matrix}$$

In case (a), μ: $\overset{1}{\circ} - \overset{1}{\circ}$, so $2n_i - 2$ is odd. In case (b), μ: $\circ = \overset{2}{\bullet}$, so λ: $\overset{1}{\circ} = \bullet$ and $K = \mathbf{SO}(5) = G$. In case (c), μ: $\overset{1}{\circ} \equiv \bullet$, so λ: $\circ \equiv \overset{1}{\bullet}$; then we are in alternative (c) of the proposition.

Now we may assume $\pi_{2\lambda - \alpha_i} \neq \mathrm{ad}_K$ for every simple root not orthogonal to λ. Let $\lambda = k\xi_r$ for some r. Then $\lambda = \lambda^*$ says $\xi_r = \xi_r^*$. Now $\pi_{2\lambda-\alpha_r}$ is a summand of χ. If they are not equal then $\chi = \pi_{2\lambda-\alpha_r} \oplus (\pi_{2\lambda-\alpha_r})^* = \pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_r}$, so $2\lambda - \alpha_r$ has multiplicity $\geqslant 2$ in $\Lambda^2(\pi_\lambda)$. That being impossible, now $\chi = \pi_{2\lambda-\alpha_r}$. Now suppose λ not of the form $k\xi_r$. Then we have $\lambda = k\xi_r + t\xi_s$ and $\chi = \pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_s}$. The summands of χ must be dual, so $\xi_s = \xi_r^*$. They must be distinct because $2\lambda - \alpha_r$ has multiplicity 1 in $\Lambda^2(\pi_\lambda)$; so $\xi_r \neq \xi_r^*$. Now $\lambda = \lambda^*$ says $\lambda = k(\xi_r + \xi_r^*)$, and we have $\chi = \pi_{2\lambda-\alpha_r} \oplus \pi_{2\lambda-\alpha_r^*}$. q.e.d.

The symplectic case is more delicate:

5.3 Proposition. *If* $G = \mathbf{Sp}(N)$, *then*

(1) $\lambda = k\xi_r$ *for some basic weight* $\xi_r = \xi_r^*$, *and* π_λ *is not real on* K;

(2) $\chi = \pi_{2\lambda}$, *absolutely irreducible; and*

(3) $2N = \deg \pi_\lambda$ *satisfies* $\frac{1}{2}\{(\deg \pi_\lambda)^2 + \deg \pi_\lambda\} = \deg \pi_{2\lambda} + \dim K$.

Proof. (4.3) says $S^2(\pi_\lambda) = \mathrm{ad}_K \oplus \chi$, so $\pi_{2\lambda}$ is a summand of ad_K or of χ. If $\pi_{2\lambda} = \mathrm{ad}_K$ then the highest root $\mu = 2\lambda$, so there is a simple root α_i with μ: $\sim \overset{n_i}{\underset{\alpha_i}{\circ}} \sim$ and $n_i \geqslant 2$. That occurs only for $K = \mathbf{Sp}(l)$, and then μ: $\overset{2}{\bullet} - \ldots - \bullet = \circ$, so λ: $\overset{1}{\bullet} - \ldots - \bullet = \circ$ and $\pi_\lambda(K) = G$. That is excluded. Now $\pi_{2\lambda}$ is a summand of χ. As $\pi_{2\lambda}$ is real and χ is $\mathbf{R}$-irreducible, this shows $\chi = \pi_{2\lambda}$ absolutely irreducible.

Suppose that λ is not a multiple of a basic weight. Then we have distinct simple roots α' and α'' not orthogonal to λ. Let V be the representation space of π_λ; choose nonzero weight vectors $u \in V_\lambda$, $v \in V_{\lambda-\alpha'}$ and $w \in V_{\lambda-\alpha''}$; let $\{x_1, \ldots, x_t\}$ be a basis of $V_{\lambda-\alpha'-\alpha''}$. Let Y denote the weight space of weight $2\lambda - \alpha' - \alpha''$ for $\pi_\lambda \otimes \pi_\lambda$ on $V \otimes V$; now $Y \cap \Lambda^2(V)$ has basis $\{v \wedge w;\ u \wedge x_1, \ldots, u \wedge x_t\}$, so it has dimension $t+1$. B. Kostant's method for decomposing a tensor product shows that $\pi_{2\lambda-\alpha'-\alpha''}$ is a summand of multiplicity t in $\pi_\lambda \otimes \pi_\lambda$.

Its multiplicty in $\Lambda^2(\pi_\lambda)$ is at most $t-1$ because each of the subrepresentations $\pi_{2\lambda-\alpha'}$ and $\pi_{2\lambda-\alpha''}$ of $\Lambda^2(\pi_\lambda)$ has $2\lambda-\alpha'-\alpha''$ for a weight. Now $\pi_{2\lambda-\alpha'-\alpha''}$ is a subrepresentation of $S^2(\pi_\lambda)=\pi_{2\lambda}+\mathrm{ad}_K$. This shows that $\mathrm{ad}_K=\pi_{2\lambda-\alpha'-\alpha''}$.

Now the highest root $\mu=2\lambda-\alpha'-\alpha''$. If α is a simple root adjacent to α' or α'' in the Dynkin diagram of $\mathfrak{K}^C$, it follows that μ is not orthogonal to α, so $-\mu$ is joined to α in the extended Dynkin diagram. If K is not of type A_l, then there is a unique simple root α_0 joined to $-\mu$ in the extended diagram. α_0 is the only simple root adjacent to α' or α''; now it is adjacent to both, so it is interior to the diagram and satisfies $2\langle\mu,\alpha_0\rangle/\langle\alpha_0,\alpha_0\rangle\geqslant 2$. Those two properties contradict each other; thus K is of type A_l and μ: $\overset{1}{\circ}$—...—$\overset{1}{\circ}$. As $\mu=2\lambda-\alpha'-\alpha''$, now $l=2$ and μ: $\underset{\alpha'}{\overset{1}{\circ}}$——$\underset{\alpha''}{\overset{1}{\circ}}$. Thus $2\lambda=\mu+\alpha'+\alpha''$: $\overset{2}{\circ}$——$\overset{2}{\circ}$. Now λ: $\overset{1}{\circ}$——$\overset{1}{\circ}$ so π_λ is orthogonal. That is absurd. We have proved that λ is a multiple $k\xi_r$ of a basic weight.

$\xi_r=\xi_r^*$ because $\lambda=\lambda^*$, and (3) comes from $S^2(\pi_\lambda)=\pi_{2\lambda}\oplus\mathrm{ad}_K$ by taking degrees, q.e.d.

Propositions 5.1, 5.2 and 5.3 do several things. They identify χ in terms of λ, giving a formula for $\deg\pi_\lambda$. And they limit the possibilities for λ.

Recall the H. Weyl degree formula:

$$\deg \pi_\nu=\prod_{\alpha>0}\frac{\langle\nu+g,\alpha\rangle}{\langle g,\alpha\rangle},\quad\text{where } g=\frac{1}{2}\sum_{\alpha>0}\alpha. \tag{5.4}$$

We need a modification involving some new notation. We have the system of simple roots $\{\alpha_1, \ldots, \alpha_l\}$. Given any positive root α, there is a unique expression $\alpha=\sum a_i\alpha_i$ where $a_i\geqslant 0$ are integers. Recall the *level* $l(\alpha)=\sum a_i$. Now define

$$\hat{a}_i=a_i\,\|\alpha_i\|^2;\quad \hat{l}(\alpha)=\sum\hat{a}_i,\ \textit{modified level.} \tag{5.5}$$

We calculate for $\nu=\sum n_i\,\xi_i$;

$$2\langle\nu,\alpha\rangle=\sum\left(2\langle\nu,\alpha_i\rangle\right)a_i=\sum\frac{2\langle\nu,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\cdot a_i\,\|\alpha_i\|^2=\sum n_i\,\hat{a}_i,$$

$$2\langle g,\alpha\rangle=\sum\left(2\langle g,\alpha_i\rangle\right)a_i=\sum\frac{2\langle g,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}\cdot a_i\,\|\alpha_i\|^2=\sum\hat{a}_i=\hat{l}(\alpha),$$

$$\frac{\langle\nu+g,\alpha\rangle}{\langle g,\alpha\rangle}=\frac{\langle\nu,\alpha\rangle}{\langle g,\alpha\rangle}+1=\frac{\sum n_i\,\hat{a}_i}{\hat{l}(\alpha)}+1=\frac{\hat{l}(\alpha)+\sum n_i\,\hat{a}_i}{\hat{l}(\alpha)}.$$

Substituting back into (5.4) we now have

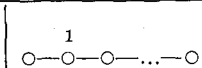
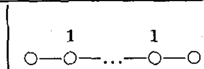
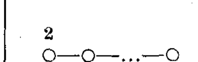
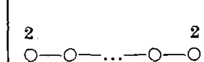
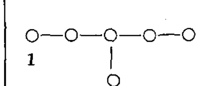
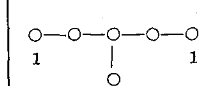
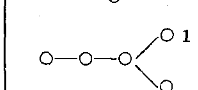
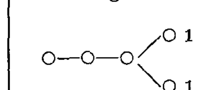
$$\deg \pi_\nu = \prod_{\alpha>0} \frac{\hat{l}(\alpha) + \sum n_i \hat{a}_i}{\hat{l}(\alpha)} \quad \text{where } \nu = \sum n_i \xi_i. \tag{5.6}$$

The next five sections consist of applying (5.6) to Propositions 5.1, 5.2 and 5.3, obtaining the classification of pairs (G, K) with G classical and K simple.

6. The case where G is unitary and K is simple

The result is:

6.1. THEOREM. *Let G be a special unitary group and let K be a compact connected non-symmetric subgroup. Let χ denote the representation of K on the tangent space of G/K. Then χ is irreducible over the real number field, if and only if (G, K) is one of the following*

G	K	π_λ	χ
$\mathbf{SU}\left(\frac{n(n-1)}{2}\right)$	$\mathbf{SU}(n)$ $(n \geq 5)$		
$\mathbf{SU}\left(\frac{n(n+1)}{2}\right)$	$\mathbf{SU}(n)$ $(n \geq 3)$		
$\mathbf{SU}(27)$	$\mathbf{E}_6$		
$\mathbf{SU}(16)$	$\mathbf{Spin}\ (10)$		

where the inclusion $K \to G$ is the absolutely irreducible representation π_λ of highest weight λ. In each case $\chi = \pi_{\lambda+\lambda^*}$, absolutely irreducible.

In each of the cases listed, χ is irreducible.

Now assume χ irreducible. Proposition 5.1 says that the inclusion $K \to G$ is an absolutely irreducible representation $\pi_{k\xi_r}$ for some basic weight $\xi_r \neq \xi_r^*$, and that

$$(\deg \pi_{k\xi_r})^2 = \deg \pi_{k(\xi_r+\xi_r^*)} + \dim K + 1. \tag{6.2}$$

To compute these degree we denote sets of positive roots by

$$P_r = \{\alpha = \sum a_i \alpha_i > 0 : a_r \neq 0 = a_{r^*}\} \quad \text{and} \quad S_r = \{\alpha = \sum a_i \alpha_i > 0 : a_r \neq 0 \neq a_{r^*}\},$$

where r^* is the integer t such that $\alpha_r^* = \alpha_t$. Now define

$$p_{r,k} = \prod_{P_r} \frac{\hat{l}(\alpha) + k\hat{a}_r}{\hat{l}(\alpha)}, \qquad s_{r,k} = \prod_{S_r} \frac{\hat{l}(\alpha) + k\hat{a}_r}{\hat{l}(\alpha)}, \qquad t_{r,k} = \prod_{S_r} \frac{\hat{l}(\alpha) + k\hat{a}_r + k\hat{a}_{r^*}}{\hat{l}(\alpha)}. \tag{6.3}$$

We compute from (5.6)

$$\deg \pi_{k\xi_r} = \prod_{\alpha>0} \frac{\hat{l}(\alpha)+k\hat{a}_r}{\hat{l}(\alpha)} = p_{r,k}\cdot s_{r,k},$$

$$\deg \pi_{k(\xi_r+\xi_r^*)} = \prod_{\alpha>0} \frac{\hat{l}(\alpha)+k\hat{a}_r+k\hat{a}_{r^*}}{\hat{l}(\alpha)} = p_{r,k}^{\,2}\cdot t_{r,k}.$$

Now (6.2) becomes
$$p_{r,k}^{\,2}\{s_{r,k}^{\,2}-t_{r,k}\} = 1+\dim K. \tag{6.4}$$

To limit k, we need a growth estimate:

6.5. Lemma. *If $1\leqslant h<k$, then $p_{r,h}^{\,2}\{s_{r,h}^{\,2}-t_{r,h}\} < p_{r,k}^{\,2}\{s_{r,k}^{\,2}-t_{r,k}\}$.*

Proof. As $p_{r,h}^{\,2}<p_{r,k}^{\,2}$ visibly, it suffices to show that $0\leqslant s_{r,h}^{\,2}-t_{r,h}\leqslant s_{r,k}^{\,2}-t_{r,k}$. From a glance at (6.3) we see that $s_{r,x}$ and $t_{r,x}$ are smooth functions of x for $x>0$. Thus we need only prove $d/dx\,(s_{r,x}^{\,2}-t_{r,x})\geqslant 0$ for $x\geqslant 1$. For every root $\beta=\sum b_j\alpha_j\in S_r$ we define

$$s_\beta(x) = 2\,\hat{b}_r\left\{\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)^2}\right\}\cdot\prod_{S_r-\beta}\frac{x^2\hat{a}_r^2+2\,x\hat{a}_r\hat{l}(\alpha)+\hat{l}(\alpha)^2}{\hat{l}(\alpha)^2}$$

$$t_\beta(x) = \frac{\hat{b}_r+\hat{b}_{r^*}}{\hat{l}(\beta)}\cdot\prod_{S_r-\beta}\frac{\hat{l}(\alpha)+x\hat{a}_r+x\hat{a}_{r^*}}{\hat{l}(\alpha)}$$

so that $d/dx\,(s_{r,x}^{\,2})=\Sigma_{S_r}s_\beta(x)$ and $d/dx\,(t_{r,x})=\Sigma_{S_r}t_\beta(x)$. Now we must prove $\Sigma_{S_r}s_\beta(x)\geqslant\Sigma_{S_r}t_\beta(x)$. For this it suffices to prove:

(A) if $\beta=\beta^*$, then $s_\beta(x)\geqslant t_\beta(x)$ for $x\geqslant 1$;
(B) if $\beta\neq\beta^*$, then $s_\beta(x)+s_{\beta^*}(x)\geqslant t_\beta(x)+t_{\beta^*}(x)$ for $x\geqslant 1$.

To prove (A) and (B) we first observe that

$$\frac{x^2\hat{a}_r^2+2\,x\hat{a}_r\hat{l}(\alpha)+\hat{l}(\alpha)^2}{\hat{l}(\alpha)^2}\geqslant\frac{x\hat{a}_r+x\hat{a}_{r^*}+\hat{l}(\alpha)}{\hat{l}(\alpha)} \quad \text{if } \alpha=\alpha^*, \tag{6.6}$$

$$\frac{x^2\hat{a}_r^2+2\,x\hat{a}_r\hat{l}(\alpha)+\hat{l}(\alpha)^2}{\hat{l}(\alpha)^2}\cdot\frac{x^2\hat{a}_{r^*}^2+2\,x\hat{a}_{r^*}\hat{l}(\alpha^*)+\hat{l}(\alpha^*)^2}{\hat{l}(\alpha^*)^2}$$
$$\geqslant\frac{x\hat{a}_r+x\hat{a}_{r^*}+\hat{l}(\alpha)}{\hat{l}(\alpha)}\cdot\frac{x\hat{a}_{r^*}+x\hat{a}_r+\hat{l}(\alpha^*)}{\hat{l}(\alpha^*)} \quad \text{if } \alpha\neq\alpha^*. \tag{6.7}$$

Inequality (6.6) is clear. For (6.7), observe that $\hat{l}(\alpha)=\hat{l}(\alpha^*)$ and expand. If $\beta=\beta^*$ we also have

$$2\,\hat{b}_r\left\{\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)^2}\right\}\geqslant\frac{2\,\hat{b}_r\,\hat{l}(\beta)}{\hat{l}(\beta)^2}=\frac{\hat{b}_r+\hat{b}_{r*}}{\hat{l}(\beta)};$$

with (6.6), this proves (A). Using (6.6) and (6.7), the proof of (B) reduces to checking that

$$2\,\hat{b}_r\left\{\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)^2}\right\}\left\{\frac{x^2\,\hat{b}_{r*}^2+2\,x\hat{b}_{r*}\hat{l}(\beta^*)+\hat{l}(\beta^*)^2}{\hat{l}(\beta^*)^2}\right\}+2\,\hat{b}_{r*}\left\{\frac{x\hat{b}_{r*}+\hat{l}(\beta^*)}{\hat{l}(\beta^*)^2}\right\}\left\{\frac{x^2\,\hat{b}_r^2+2\,x\hat{b}_r\hat{l}(\beta)+\hat{l}(\beta)^2}{\hat{l}(\beta)^2}\right\}$$

$$\geqslant\left\{\frac{\hat{b}_r+\hat{b}_{r*}}{\hat{l}(\beta)}\right\}\left\{\frac{\hat{l}(\beta^*)+x\hat{b}_r+x\hat{b}_{r*}}{\hat{l}(\beta^*)}\right\}+\left\{\frac{\hat{b}_{r*}+\hat{b}_r}{\hat{l}(\beta^*)}\right\}\left\{\frac{\hat{l}(\beta)+x\hat{b}_{r*}+x\hat{b}_r}{\hat{l}(\beta)}\right\}.$$

That inequality is checked by expanding out and using $\hat{l}(\beta)=\hat{l}(\beta^*)$, q.e.d.

We reformulate Lemma 6.5 as follows.

6.8. Lemma. *π_λ is among the following representations;*

$$K=\mathbf{SU}(n+1),\ n\geqslant 2,\quad and\quad \lambda:\ \ \underset{\alpha_1}{\overset{2}{\circ}}-\underset{\alpha_2}{\circ}-\cdots-\underset{\alpha_n}{\circ}$$

$$K=\mathbf{SU}(n+1),\ 1<r<\frac{n+1}{2},\quad and\quad \lambda:\ \ \underset{\alpha_1}{\circ}-\cdots-\underset{\alpha_r}{\overset{1}{\circ}}-\cdots-\underset{\alpha_n}{\circ}$$

$$K=\mathbf{Spin}\,(2n),\ n=2\,m+1\geqslant 5,\quad and\quad \lambda:\ \ \circ-\circ-\cdots-\circ\begin{matrix}\diagup\circ\ 1\\ \diagdown\circ\end{matrix}$$

$$K=\mathbf{E}_6,\quad and\quad \lambda:\ \ \underset{1}{\circ}-\circ-\underset{\substack{|\\ \circ}}{\circ}-\circ-\circ\qquad or\quad \lambda:\ \ \circ-\underset{1}{\circ}-\underset{\substack{|\\ \circ}}{\circ}-\circ-\circ\ .$$

Proof. As $\lambda\neq\lambda^*$, K must be of type A_n $(n\geqslant 2)$, D_n $(n=2m+1\geqslant 5)$ or E_6. If $\pi_{\xi_r}\otimes\pi_{\xi_r^*}=1_K\oplus\mathrm{ad}_K$, then K is of type A_n and ξ_r: $\overset{1}{\circ}-\circ-\cdots-\circ$. Then Lemma 6.5 says $p_{r,2}^{\,2}\{s_{r,2}^{\,2}+t_{r,2}\}\geqslant 1+\dim K$, so $k=2$ by (6.4) and Lemma 6.5. This is the first possibility listed in Lemma 6.8. Now suppose $\pi_{\xi_r}\otimes\pi_{\xi_r^*}\neq 1_K\oplus\mathrm{ad}_K$. Then the latter is a proper summand of $\pi_{\xi_r}\otimes\pi_{\xi_r^*}$ and we have $p_{r,1}\{s_{r,1}^{\,2}-t_{r,1}\}\geqslant 1+\dim K$. Then (6.4) and Lemma 6.5 say that $\lambda=\xi_r$, basic weight which is not self dual. These are the remaining possibilities listed in Lemma 6.8, q.e.d.

We now run through the cases of Lemma 6.8.

6.9. Lemma. *The representation π_λ given by λ: $\underset{\alpha_1}{\overset{2}{\circ}}-\underset{\alpha_2}{\circ}-\cdots-\underset{\alpha_n}{\circ}$, $n\geqslant 2$ maps $\mathbf{SU}(n+1)$ into $\mathbf{SU}\left(\dfrac{(n+1)(n+2)}{2}\right)$ and satisfies* $(\deg\pi_\lambda)^2=\deg\pi_{\lambda+\lambda^*}+\dim\,\mathbf{SU}(n+1)+1$.

Proof. The roots of $\mathbf{SU}(n+1)$ are the roots $\pm(\alpha_i+\alpha_{i+1}+\ldots+\alpha_j)$, $1 \leqslant i \leqslant j \leqslant n$, where $\{\alpha_1, \ldots, \alpha_n\}$ are the simple roots. We have $r=1$ and observe that

$$P_1=\{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\alpha_2+\alpha_{n-1}\} \text{ and } S_1=\{\alpha_1+\ldots+\alpha_n\}.$$

All roots have the same length, which we normalize to be 1, so $\hat{l}(\alpha_1+\ldots+\alpha_q)=q$ and $\hat{a}_i=a_i$. Now

$$p_{1,2}=\frac{1+2}{1}\cdot\frac{2+2}{2}\cdot\frac{3+2}{3}\cdots\frac{n-1+2}{n-1}=\frac{n(n+1)}{2},$$

$$s_{1,2}=\frac{n+2}{2} \quad \textit{and} \quad t_{1,2}=\frac{n+4}{2}.$$

Thus
$$p_{1,2}^2(s_{1,2}^2-t_{1,2})=\tfrac{1}{4}n^2(n+1)^2\left\{\frac{(n+2)^2}{n^2}-\frac{(n+4)}{n}\right\}$$
$$=\tfrac{1}{4}n^2(n+1)^2\cdot\frac{4}{n^2}=(n+1)^2=\dim \mathbf{SU}(n+1)+1, \text{ q.e.d.}$$

6.10. Lemma. *The representation* π_λ *of* $\mathbf{SU}(n+1)$ *given by* λ: $\underset{\alpha_1}{\circ}-\cdots-\overset{1}{\underset{\alpha_r}{\circ}}-\cdots-\underset{\alpha_n}{\circ}$, $1<r<n/2$, *satisfies* $(\deg \pi_\lambda)^2=\deg \pi_{\lambda+\lambda^*}+\dim \mathbf{SU}(n+1)+1$ *if and only if* $r=2$.

Proof. We go by induction on r. First let $r=2$. Then P_r consists of

root	α_2	$\alpha_1+\alpha_2$ $\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3$ $\alpha_2+\alpha_3+\alpha_4$	$\ldots$	$\alpha_1+\ldots+\alpha_{n-3}$ $\alpha_2+\ldots+\alpha_{n-2}$	$\alpha_1+\ldots+\alpha_{n-2}$
level	1	2	3	$\ldots$	$n-3$	$n-2$

so
$$p_{2,1}=\frac{2}{1}\cdot\left(\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n-2}{n-3}\right)^2\frac{n-1}{n-2}=2\left(\frac{n-2}{2}\right)^2\frac{n-1}{n-2}=\frac{(n-2)(n-1)}{2}.$$

And $S_r=\{\alpha_2+\ldots+\alpha_{n-1}, \alpha_1+\ldots+\alpha_{n-1}, \alpha_2+\ldots+\alpha_n, \alpha_1+\ldots+\alpha_n\}$ so that

$$s_{2,1}=\frac{n-1}{n-2}\left(\frac{n}{n-1}\right)^2\frac{n+1}{n}=\frac{n(n+1)}{(n-2)(n-1)} \text{ and } t_{2,1}=\frac{n}{n-2}\left(\frac{n+1}{n-1}\right)^2\frac{n+2}{n}=\frac{n+2}{n-2}\left(\frac{n+1}{n-1}\right)^2.$$

Thus
$$p_{2,1}^2(s_{2,1}^2-t_{2,1})$$
$$=\tfrac{1}{4}(n-2)^2(n-1)^2\left\{\frac{n^2(n+1)^2}{(n-2)^2(n-1)^2}-\frac{(n+2)(n+1)^2}{(n-2)(n-1)^2}\right\}$$
$$=\tfrac{1}{4}(n-2)^2(n-1)^2\left\{\frac{n^2(n+1)^2-(n-2)(n+2)(n+1)^2}{(n-2)^2(n-1)^2}\right\}$$
$$=\tfrac{1}{4}\{4n^2+8n+4\}=(n+1)^2=\dim \mathbf{SU}(n+1).$$

This proves the assertion for $r=2$.

Suppose $r \geqslant 3$ and suppose the lemma known for $r-1$ and all $n > 2r-2$. We decompose P_r into the subset P'_r given by $a_1 = 0$ and the complementary subset $P''_r = \{\alpha_1 + \ldots + \alpha_r, \alpha_1 + \ldots + \alpha_{r+1}, \ldots, \alpha_1 + \ldots + \alpha_{n-r}\}$. Then we have the factorization

$$p_{r,1} = p'_{r,1} \cdot p''_{r,1} = p'_{r,1} \cdot \frac{r+1}{r} \cdot \frac{r+2}{r+1} \cdots \frac{n-r+1}{n-r} = p'_{r,1} \cdot \frac{n-r+1}{r}.$$

Similarly S_r consists of the set S'_r given by $a_1 = a_n = 0$ and the complementary set

$$S''_r = \{\alpha_1 + \ldots + \alpha_{n-r+1},\ \alpha_r + \ldots + \alpha_n;\ \ldots;\ \alpha_1 + \ldots + \alpha_{n-1},\ \alpha_2 + \ldots + \alpha_n;\ \alpha_1 + \ldots + \alpha_n\}.$$

Thus
$$s_{r,1} = s'_{r,1} s''_{r,1} = s'_{r,1} \cdot \left\{\frac{n-r+2}{n-r+1} \cdot \frac{n-r+3}{n-r+2} \cdots \frac{n}{n-1}\right\}^2 \frac{n+1}{n}$$
$$= s'_{r,1} \cdot \frac{n(n+1)}{(n-r+1)^2}$$

and
$$t_{r,1} = t'_{r,1} t''_{r,1} = t'_{r,1} \cdot \left\{\frac{n-r+3}{n-r+1} \cdots \frac{n+1}{n-1}\right\}^2 \frac{n+2}{n} = t'_{r,1} \cdot \frac{n(n+1)^2(n+2)}{(n-r+1)^2(n-r+2)^2}.$$

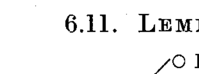
Now $q = n-r+1 < n$ satisfies $n(q+1)^2 > (n+2)q^2$, so

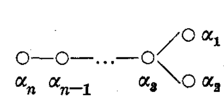
$$\frac{n^2(n+1)^2}{(n-r+1)^4} > \frac{n(n+1)^2(n+2)}{(n-r+1)^2(n-r+2)^2}.$$

This shows
$$(s_{r,1}^2 - t_{r,1}) > (s'^2_{r,1} - t'_{r,1}) \frac{n^2(n+1)^2}{(n-r+1)^4}.$$

The induction hypothesis, applied to the $\mathbf{SU}(n-1)$ with simple roots $\{\alpha_2, \ldots, \alpha_{n-1}\}$, says that $p'^2_{r,1}(s'^2_{r,1} - t'_{r,1}) \geqslant (n-1)^2$. Now we have

$$p_{r,1}(s_{r,1}^2 - t_{r,1}) > (n-1)^2 \left(\frac{n-r+1}{r}\right)^2 \frac{n^2(n+1)^2}{(n-r+1)^4} = \frac{n^2(n-1)^2}{r^2(n-r+1)^2}(n+1)^2$$
$$> (n+1)^2 = \dim \mathbf{SU}(n+1) + 1, \text{ q.e.d.}$$

6.11. Lemma. *The representation* π_λ *of* $\mathbf{Spin}(2n)$, $n = 2m+1 \geqslant 5$, *given by* λ*:* , *satisfies* $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathbf{Spin}(2n) + 1$ *if and only if* $n = 5$.

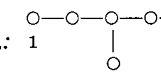
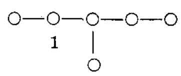
Proof. We label the simple roots , then $\lambda = \xi_1$ and π_λ is the half spin representation, $\deg \pi_\lambda = 2^{n-1}$. The usual representation $\pi_{\xi_n}\colon \mathbf{Spin}(2n) \to \mathbf{SO}(2n)$ satisfies

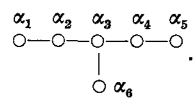
$$\Lambda^{n-1}(\pi_{\xi_n}) = \pi_{\xi_1+\xi_2} = \pi_{\lambda+\lambda^*}, \quad \text{degree} \binom{2n}{n-1}.$$

Thus our degree equation is $2^{2n-2} = \binom{2n}{n-1} + (2n^2 - n) + 1$, i.e.,

$$2^{2n-2} - \frac{(n+2)(n+3)\dots(2n)}{(n-1)!} = 2n^2 - n + 1,$$

where $n \geqslant 5$ is an odd integer. 5 is a solution, and one checks that the left side grows more than the right side when $n+2$ replaces n. Thus 5 is the only solution, q.e.d.

6.12. Lemma. *The representation* $\pi_\lambda : \mathbf{E}_6 \to \mathbf{SU}(27)$ *given by* λ:  *satisfies* $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathbf{E}_6 + 1$. *The representation* $\pi_\lambda : \mathbf{E}_6 \to \mathbf{SU}\,(351)$ *given by* λ:  *does not satisfy* $(\deg \pi_\lambda)^2 = \deg \pi_{\lambda+\lambda^*} + \dim \mathbf{E}_6 + 1$.

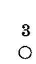
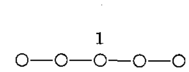
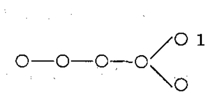
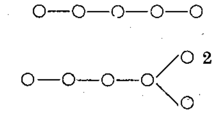
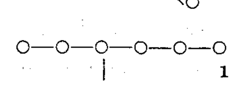
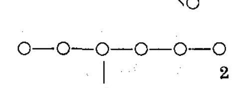
Proof. We label the simple roots . Then $\xi_5 = \xi_1^*$ and a calculation shows that $\deg \pi_{\xi_1} = 27$ and $\deg \pi_{\xi_1+\xi_1^*} = 650$, so that $(\deg \pi_{\xi_1})^2 = 27^2 = 729 = 650 + 78 + 1 = \deg \pi_{\xi_1+\xi_1^*} + \dim \mathbf{E}_6 + 1$. Also, $\xi_4 = \xi_2^*$ and a calculation shows that $\deg \pi_{\xi_2} = 351$ and $\deg \pi_{\xi_2+\xi_2^*} = 70\,070$, so that $(\deg \pi_{\xi_2})^2 = 123\,201 > 70\,070 + 78 + 1 = \deg \pi_{\xi_2+\xi_2^*} + \dim \mathbf{E}_6 + 1$, q.e.d.

Theorem 6.1 now follows from Lemmas 6.8, 6.9, 6.10, and 6.11.

7. The case where G is symplectic and K is simple

Here we have the classification:

7.1. Theorem. *Let G be a unitary symplectic group and let K be a compact connected simple subgroup. Then the representation χ of K on the tangent space of G/K is irreducible over the real number field, if and only if (G, K) is one of the following.*

G	K	π_λ	χ
$\mathbf{Sp}\,(2)$	$\mathbf{SU}\,(2)$		
$\mathbf{Sp}\,(7)$	$\mathbf{Sp}\,(3)$		
$\mathbf{Sp}\,(10)$	$\mathbf{SU}\,(6)$		
$\mathbf{Sp}\,(16)$	$\mathbf{Spin}\,(12)$		
$\mathbf{Sp}\,(28)$	$\mathbf{E}_7$		

where the inclusion $K \to G = \mathbf{Sp}(n) \subset \mathbf{GL}(2n, \mathbf{C})$ *is the absolutely irreducible representation* π_λ *of* K, *of highest weight* λ *and degree* $2n$. *In each case* $\chi = \pi_{2\lambda}$, *absolutely irreducible.*

In each of the cases listed, χ is irreducible.

Now assume χ irreducible. Proposition 5.3 shows that the inclusion $K \to G = \mathbf{Sp}(N)$ is an absolutely irreducible representation $\pi_{k\xi_r}$ for some basic weight $\xi_r = \xi_r^*$, that $2N = \deg \pi_{k_r \xi}$, that $\chi = \pi_{2k\xi_r}$, and that

$$\tfrac{1}{2}(\deg \pi_{k\xi_r})^2 + \tfrac{1}{2}\deg \pi_{k\xi_r} = \deg \pi_{2k\xi_r} + \dim K. \tag{7.2}$$

The fact that $\pi_{k\xi_r}$ is symplectic can be reformulated as follows using results of A. I. Mal'cev ([11], § 6); the details are carried out by E. B. Dynkin in Table 12 of [6].

7.3 Lemma. *The positive integer* k *is odd and the basic weight* ξ_r *is one of the following.*

Type of K	ξ_r	Conditions
A_n	○—○—…—○—…—○ (α_1 α_2 … α_r … α_n), 1 over α_r	$n = 4s+1$, $r = 2s+1$
B_n	○—○—…—○═● , 1 over the last node	$n = 4s+1$ or $4s+2$
C_n	●—●—…—●—…—●═○ (α_1 α_2 … α_r … α_{n-1} α_n), 1 over α_r	$r \geqslant 1$, r odd
D_n	○—○—…—○< (branch nodes ○ 1 and ○)	$n = 4s+2$, $s \geqslant 1$
E_7	○—○—○—○—○—○ with ○ below the fourth node, 1 at the first node *or* ○—○—○—○—○—○ with ○ below the fourth node, 1 at the third node *or* ○—○—○—○—○—○ with ○ below the fourth node, 1 at the node below	

This lemma is used with a precise growth estimate:

7.4. Lemma. *Either* $k = 1$ *or* $k = 3$. *If* $k = 3$ *and* rank $K > 1$, *then* $\dim K \geqslant \frac{22}{9}(\deg \pi_{\xi_r})^2 + \frac{5}{3}\deg \pi_{\xi_r}$.

Proof. Define $f(x) = \prod_{\alpha > 0} \dfrac{x\hat{a}_r + \hat{l}(\alpha)}{\hat{l}(\alpha)}$ where the product runs over the positive roots $\alpha = \sum a_i \alpha_i$ of K. Then (5.6) and (7.2) say $\frac{1}{2} f(k)^2 + \frac{1}{2} f(k) = f(2k) + \dim K$.

Let $x \geqslant 2$; we will prove that $F(x) = \frac{1}{2} f(x)^2 + \frac{1}{2} f(x) - f(2x)$ is a strictly increasing function of x. As the second term is increasing, it suffices to show that $d/dx\ \{\frac{1}{2} f(x)^2 - f(2x)\} \geqslant 0$, i.e., that

$$\sum_{\beta>0}\left\{\prod_{\substack{\alpha>0\\ \alpha\neq\beta}}\left(\frac{x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2\right\}\left\{\frac{\hat{b}_r}{\hat{l}(\beta)}\cdot\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)}\right\}\geqslant\sum_{\beta>0}\left\{\prod_{\substack{\alpha>0\\ \alpha\neq\beta}}\frac{2\,x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right\}\left\{\frac{2\,\hat{b}_r}{\hat{l}(\beta)}\right\}.$$

We will prove this inequality term by term. Dividing the β-term by $\hat{b}_r/\hat{l}(\beta)$, that amounts to showing

$$\left\{\prod_{\substack{\alpha>0\\ \alpha\neq\beta}}\left(\frac{x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2\right\}\left\{\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)}\right\}\geqslant 2\prod_{\substack{\alpha>0\\ \alpha\neq\beta}}\left(\frac{2\,x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right). \tag{7.5}$$

for every root $\beta>0$.

Let β be a fixed positive root and let S be any set of positive roots which does not contain β. We define

$$A_S=\prod_S\left(\frac{x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2,\qquad B_S=\frac{x\hat{b}_r+\hat{l}(\beta)}{\hat{l}(\beta)},\qquad \text{and } C_S=2\prod_S\frac{2\,x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}.$$

If $\alpha\notin S\cup\{\beta\}$ is a positive root, then

$$\left(\frac{x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2\geqslant\frac{2\,x\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}.$$

Thus, in order to prove (7.5) it suffices to find a set S such that $A_S B_S\geqslant C_S$.

If $\beta=\alpha_r$ we take S empty. Then $A_S=1$, $B_S=x+1\geqslant 2$ and $C_S=2$, so $A_S B_S\geqslant C_S$. Now suppose $\beta\neq\alpha_r$. Then rank $K>1$. Let α_s be a simple root adjacent to α_r in the Dynkin diagram of K, and consider $\gamma=\sum c_i\,\alpha_i$ defined as follows:

(1) $\|\alpha_r\|=\|\alpha_s\|$, $\quad\gamma=\alpha_r+\alpha_s$, $\quad\dfrac{x\hat{c}_r+\hat{l}(\gamma)}{\hat{l}(\gamma)}=\dfrac{x+2}{2}$;

(2) $\|\alpha_r\|^2=2\,\|\alpha_s\|^2$, $\quad\gamma=\alpha_r+2\,\alpha_s$, $\quad\dfrac{x\hat{c}_r+\hat{l}(\gamma)}{\hat{l}(\gamma)}=\dfrac{x+2}{2}$;

(3) $2\,\|\alpha_r\|^2=\|\alpha_s\|^2$, $\quad\gamma=2\,\alpha_r+\alpha_s$ $\quad\dfrac{x\hat{c}_r+\hat{l}(\gamma)}{\hat{l}(\gamma)}=\dfrac{x+2}{2}$.

If $\beta=\gamma$ we take $S=\{\alpha_r\}$; then $A_S=(x+1)^2$, $B_S=\frac{1}{2}(x+2)$ and $C_S=4x+2$; $x\geqslant 2$ says $A_S\geqslant 9$ so $A_S B_S\geqslant\frac{9}{2}x+9>4x+2=C_S$. Now we may assume $\alpha_r\neq\beta\neq\gamma$ and take $S=\{\alpha_r,\gamma\}$; then $A_S=\frac{1}{4}(x+1)^2(x+2)^2$, $B_S\geqslant 1$ and $C_S=(2x+1)(2x+2)$, so $A_S B_S\geqslant A_S\geqslant\frac{1}{4}(x+1)^2 4^2=4x^2+8x+4>4x^2+6x+2=C_S$. This completes the proof that $dF(x)/dx>0$ for $x\geqslant 2$.

$\pi_{3\xi_r}$ is symplectic because $\pi_{k\xi_r}$ is symplectic, and $\pi_{3\xi_r}(K)\subsetneqq \mathbf{Sp}(\frac{1}{2}\deg \pi_{3\xi_r})$. Thus $F(3)=\deg S^2(\pi_{3\xi_r})-\deg \pi_{6\xi_r}\geqslant \dim K$. As $F(x)$ is increasing for $x\geqslant 2$, as $F(k)=\dim K$, and as k is an odd positive integer, it follows that k is 1 or 3. This proves the first statement.

Suppose rank $K>1$. Then we have α_s and γ as defined above. Let V be the set of all positive roots except for α_s and γ. Now

$$(\deg \pi_{k\xi_r})^2=(k+1)^2\left(\frac{k+2}{2}\right)^2\prod_V\left(\frac{k\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2$$

$$\deg \pi_{2k\xi_r}=(2k+1)(k+1)\prod_V\frac{2k\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}$$

and we have termwise inequalities for the factors corresponding to roots $\alpha\in V$. Thus

$$4(2k+1)(k+1)(\deg \pi_{k\xi_r})^2\geqslant(k+1)^2(k+2)^2\deg \pi_{2k\xi_r} \tag{7.6}$$

Suppose $k\neq 1$, i.e., $k=3$. Then (7.6) says $\frac{7}{25}(\deg \pi_{3\xi_r})^2\geqslant \deg \pi_{6\xi_r}$, so the identity $F(3)=\dim K$ gives us

$$\dim K=\tfrac{1}{2}(\deg \pi_{3\xi_r})^2+\tfrac{1}{2}\deg \pi_{3\xi_r}-\deg \pi_{6\xi_r}\geqslant \tfrac{11}{50}(\deg \pi_{3\xi_r})^2+\tfrac{1}{2}\deg \pi_{3\xi_r}.$$

Suppose further that rank $K>1$. Looking at the α_r-term and the γ-term in the degree formula, we notice

$$\deg \pi_{3\xi_r}\geqslant\frac{3+1}{1+1}\cdot\frac{3+2}{1+2}\deg \pi_{\xi_r}=\tfrac{10}{3}\deg \pi_{\xi_r}.$$

Thus $$\tfrac{11}{50}(\deg \pi_{3\xi_r})^2+\tfrac{1}{2}\deg \pi_{3\xi}\geqslant\tfrac{22}{9}(\deg \pi_{\xi_r})^2+\tfrac{5}{3}\deg \pi_{\xi_r}.$$

This proves the second statement, q.e.d.

Now we can run through cases.

7.7. Lemma. *The representation π_λ of* $\mathbf{SU}(n+1)$ *given by* λ: $\underset{\alpha_1}{\circ}-\underset{\alpha_2}{\circ}-\dots-\underset{\alpha_r}{\overset{k}{\circ}}-\dots-\underset{\alpha_n}{\circ}$, $n=4s+1$, $r=2s+1$, $k=1$ *or* 3, *satisfies* $\frac{1}{2}(\deg \pi_\lambda)^2+\frac{1}{2}\deg \pi_\lambda=\deg \pi_{2\lambda}+\dim \mathbf{SU}(n+1)$, *if and only if* λ: $\overset{3}{\circ}$ *or* λ; $\circ-\circ-\overset{1}{\circ}-\circ-\circ$.

Proof. First suppose $k=3$. If $s=0$, then $n=1$ and $\deg \pi_{m\xi_1}=m+1$. Thus $\frac{1}{2}(\deg \pi_\lambda)^2+\frac{1}{2}\deg \pi_\lambda=\frac{1}{2}4^2+\frac{1}{2}4=10=7+3=\deg \pi_{2\lambda}+\dim \mathbf{SU}(2)$, which is our case λ: $\overset{3}{\circ}$. Now we prove by induction on s that $\frac{1}{2}(\deg \pi_{3\xi_r})^2+\frac{1}{2}\deg \pi_{3\xi_r}>\deg \pi_{6\xi_r}+\dim \mathbf{SU}(n+1)$ for $s\geqslant 1$.

By Lemma 7.4 it suffices to prove $\dim \mathbf{SU}(n+1) \leqslant \frac{22}{9} (\deg \pi_{\xi_r})^2$ for $s \geqslant 1$. For $s = 1$ this says $35 \leqslant \frac{22}{9} \cdot 20^2$, which is clear. The induction hypothesis is

$$(4s+2)^2 \leqslant \tfrac{22}{9} \binom{4s+2}{2s+1}^2, \quad \text{i.e.,}\ 36r^2 \leqslant 22 \binom{2r}{r}^2.$$

We write this in the form $(1 \cdot 2 \ldots \cdot r)^2 \cdot 36 r^2 \leqslant 22 \{(r+1)(r+2) \ldots (2r)\}^2$. If we raise s to $s+1$, r goes to $r+2$, and we multiply the left side by $\dfrac{(r+2)^4 (r+1)^2}{r^2}$, and the right side by $\dfrac{(2r+1)^2 (2r+2)^2 (2r+3)^2 (2r+4)^2}{(r+1)^2 (r+2)^2}$. The factor for the right side is larger, so the inequality persists.

We may now assume $k=1$, so $\lambda = \xi_r$ and $s \geqslant 1$. If $s=1$ then $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2}\deg \pi_\lambda = \frac{1}{2} \cdot 20^2 + \frac{1}{2} \cdot 20 = 210 = 175 + 35 = \deg \pi_{2\lambda} + \dim \mathbf{SU}(n+1)$. That is our case λ: $\circ\!-\!\circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ$. Now we prove by induction on s that $\frac{1}{2}(\deg \pi_{\xi_r})^2 + \frac{1}{2}\deg \pi_{\xi_r} > \deg \pi_{2\xi_r} + \dim \mathbf{SU}(2r)$ for $s \geqslant 2$. It suffices to show that $\frac{1}{2}(\deg \pi_{\xi_r})^2 \geqslant \deg \pi_{2\xi_r} + \dim \mathbf{SU}(2r)$. For $s=2$, $\deg \pi_{\xi_r} = 252$, $\deg \pi_{2\xi_r} = 19404$, and $\dim \mathbf{SU}(2r) = 99$, so the inequality is clear. Let $s > 2$, let S be the set of all roots $\alpha > 0$ where $a_r \neq 0$, and divide S into the set $T = \{\alpha \in S\colon a_1 = 0 = a_n\}$ and its complement $U = \{\alpha_1 + \ldots + \alpha_r, \alpha_r + \ldots + \alpha_n; \ldots; \alpha_1 + \ldots + \alpha_{n-1}, \alpha_2 + \ldots + \alpha_n; \alpha_1 + \ldots + \alpha_n\}$. Let L be the subgroup $\mathbf{SU}(n-1)$ of $\mathbf{SU}(n+1)$ with simple root system $\{\alpha_2, \ldots, \alpha_{n-1}\}$. By induction on r, $\frac{1}{2}(\deg \tau_{\xi_r})^2 \geqslant \deg \tau_{2\xi_r} + (2r-2)^2 - 1$ where τ_ν is the representation of L with highest weight ν. Now $\deg \pi_{\xi_r} = u \cdot \deg \tau_{\xi_r}$ and $\deg \pi_{2\xi_r} = v \cdot \deg \pi_{\xi_r}$ where

$$u = \prod_U \frac{1 + l(\alpha)}{l(\alpha)} = \left\{\frac{r+1}{r} \cdots \frac{2r-1}{2r-2}\right\}^2 \cdot \frac{2r}{2r-1} = \frac{4r-2}{r}$$

and
$$v = \prod_U \frac{2 + l(\alpha)}{l(\alpha)} = \left\{\frac{r+2}{r} \cdots \frac{2r}{2r-2}\right\}^2 \cdot \frac{2r+1}{2r-1} = 4\,\frac{(2r-1)(2r+1)}{(r+1)^2}.$$

Now $u^2 > v$ shows that $\frac{1}{2}(\deg \pi_{\xi_r})^2$ grows more than $\deg \pi_{2\xi_r}$ when r, hence when s, is raised. Also $u^2 > \dfrac{(2r)^2 - 1}{(2r-2)^2 - 1}$ so $\frac{1}{2}(\deg \pi_{\xi_r})^2$ grows faster than $\dim \mathbf{SU}(2r)$ when r, hence when s, is raised. Thus our inequalities persist when s is raised, q.e.d.

7.8 Lemma. *The group K is of type B_n or D_n, if and only if λ is given by $\overset{3}{\bullet}$ or by*

$\circ\!-\!\circ\!-\!\circ\!-\!\circ\!<^{\circ\ 1}_{\circ}$.

Proof. Let K be of type B_n. Then Lemma 7.3 says λ: $\underset{\alpha_n}{\circ}-\underset{\alpha_{n-1}}{\circ}-\cdots-\underset{\alpha_2}{\circ}=\overset{k}{\underset{\alpha_1}{\bullet}}$ and $n=4s+1$ or $4s+2$, and Lemma 7.4 says that $k=1$ or 3. Suppose $k=3$ and $n>1$. Then Lemma 7.4 says that $2n^2+n=\dim K>\frac{22}{9}(\deg \pi_{\xi_1})^2+\frac{5}{3}\deg \pi_{\xi_1}=\frac{22}{9}\cdot 2^{2n}+\frac{5}{3}\cdot 2^n$. That inequality has no integral solution $n>1$. Thus $k=3$ implies $n=1$; that is the case λ: $\overset{3}{\bullet}$ which occurs as λ: $\overset{3}{\circ}$ in Lemma 7.7. We may now assume $k=1$, so $\lambda=\xi_1$ and $n>1$. $\pi_{2\xi_1}$ can be obtained by composition of the inclusion $\mathbf{B}_n=\mathbf{SO}(2n+1)\subset\mathbf{SU}(2n+1)=\mathbf{A}_{2n}$ with the representation π_{ξ_n} of $\mathbf{A}_{2n}$; thus $\deg \pi_{2\xi_1}=\binom{2n+1}{n}$. As $\deg \pi_{\xi_1}=2^n$, our equation is $2^{2n-1}+2^{n-1}=\binom{2n+1}{n}+2n^2+n$. There are no integral solutions $n>1$.

Let K be of type D_n. Then Lemma 7.3 says that λ: $\underset{\alpha_n}{\circ}-\underset{\alpha_{n-1}}{\circ}-\cdots-\underset{\alpha_3}{\circ}\!<\!\begin{smallmatrix}\overset{k}{\circ}\,\alpha_1\\ \circ\,\alpha_2\end{smallmatrix}$ $n=4s+2$, $s\geqslant 1$; and Lemma 7.4 says that k is 1 or 3. If $k=3$, then Lemma 7.4 and $\deg \pi_{\xi_1}=2^{n-1}$ say that $2n^2-n=\dim K\geqslant\frac{22}{9}\cdot 2^{2n-2}+\frac{5}{3}\cdot 2^{n-1}$. There are no integral solutions $n\geqslant 6$. Thus $k=1$ and our equation is $2^{2n-3}+2^{n-2}=\deg \pi_{2\xi_1}+2n^2-n$. If $n=6$ then $\deg \pi_{2\xi_1}=462=2^9+2^4-72+6$, so we have the solution λ: $\circ-\circ-\circ-\circ\!<\!\begin{smallmatrix}\circ\,1\\ \circ\end{smallmatrix}$. Now we will prove by induction on n that, for $n>6$, the representation $\pi_{2\xi_1}$ of $\mathbf{D}_n$ satisfies

$$\deg \pi_{2\xi_1}<2^{2n-3}+2^{n-2}-2n^2+n,$$

i.e., that

$$2^{2n-3}>\deg \pi_{2\xi_1}-2^{n-2}+2n^2-n.$$

For when n is raised to $n+1$, the new roots $\alpha>0$ with $a_1\neq 0$ are $\{\alpha_1+\alpha_3+\alpha_4+\ldots+\alpha_{n+1}; \alpha_1+\alpha_2+\ldots+\alpha_{n+1}; \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\ldots+\alpha_{n+1};\ldots; \alpha_1+\alpha_2+2\alpha_3+\ldots+2\alpha_n+\alpha_{n+1}\}$, so $\deg \pi_{2\xi_1}$ is multiplied by $\frac{n+2}{n}\cdot\frac{n+3}{n+1}\cdot\ldots\cdot\frac{2n+1}{2n-1}=2\,\frac{2n+1}{n+1}<4$. Similarly 2^{n-2} is doubled and $2n^2-n$ is multiplied by a factor less than 4. But 2^{2n-3} is multiplied by 4. Thus our equality for $n=6$ becomes strict inequality for $n>6$, q.e.d.

7.9. Lemma. *The representation* π_λ *of* $\mathbf{Sp}(n)$ *given by* $\underset{\alpha_n}{\bullet}-\underset{\alpha_{n-1}}{\bullet}-\cdots-\overset{k}{\underset{\alpha_r}{\bullet}}-\cdots-\underset{\alpha_2}{\bullet}=\underset{\alpha_1}{\circ}$, $n-r+1$ *and* k *odd,* $1\leqslant r\leqslant n$, *satisfies* $\frac{1}{2}(\deg \pi_\lambda)^2+\frac{1}{2}\deg \pi_\lambda=\deg \pi_{2\lambda}+\dim \mathbf{Sp}(n)$ *if and only if* λ: $\bullet-\bullet=\overset{1}{\circ}$

Proof. As before, k is 1 or 3. Let $s=n-r+1$; then $\deg \pi_{\xi_r}=\binom{2n}{s}-\binom{2n}{s-2}$ with the usual convention that $\binom{m}{0}=1$ and $\binom{m}{-1}=0$. Thus $\deg \pi_{\xi_r} \geqslant 2n$. If $k=3$ then Lemma 7.4 says $2n^2+n=\dim \mathbf{Sp}(n) > \frac{22}{9}(\deg \pi_{\xi_r})^2 \geqslant \frac{22}{9}\cdot 4n^2 > 8n^2 > 2n^2+n$. Thus $k=1$.

Given an integer b with $1 \leqslant b \leqslant n$ we define

$$P(n,b)=\tfrac{1}{2}(\deg \pi_b)^2-\deg \pi_{2b} \quad \text{and} \quad Q(n,b)=\tfrac{1}{2}\deg \pi_b-\dim \mathbf{Sp}(n).$$

Then our degree equation is $P(n,r)+Q(n,r)=0$. Suppose $b<n$. Let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{Sp}(n-1)$ with simple root system $\{\alpha_1, \ldots, \alpha_{n-1}\}$. Then $\deg \pi_{\xi_b}=u\cdot\deg \tau_{\xi_b}$ and $\deg \pi_{2\xi_b}=v\cdot \deg \tau_{2\xi_b}$ where

$$u^2=\prod_{\substack{\alpha>0\\ a_n\neq 0}}\left(\frac{\hat{a}_b+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2 > \prod_{\substack{\alpha>0\\ a_n\neq 0}}\frac{2\hat{a}_b+\hat{l}(\alpha)}{\hat{l}(\alpha)}=v.$$

Thus: $\qquad$ *if* $b<n$ *then* $P(n,b)>P(n-1,b)$. $\qquad$ (7.10)

We observe $\dim \mathbf{Sp}(n)=w\cdot \dim \mathbf{Sp}(n-1)$ where $w=\dfrac{2n^2+n}{2n^2-3n+1}$. We compute $u=\dfrac{2n(2n+1)}{(n-b+1)(n+b+5)}$. Thus the condition for $u>w$ is $3n^2-12n-3+4b+b^2>0$. If $b=1$ this says $n\geqslant 4$; if $b=2$ it says $n\geqslant 4$; if $b\geqslant 3$ it is automatic.

Thus: $\qquad$ *if* $b<n$, *and if* $n\geqslant 4$ *or* $b\geqslant 3$, *then* $Q(n,b)>Q(n-1,b)$. $\qquad$ (7.11)

π_{ξ_n} maps $\mathbf{Sp}(n)$ onto $\mathbf{Sp}(\frac{1}{2}\deg \pi_{\xi_n})$; thus $r<n$. Let L denote the subgroup $\mathbf{Sp}(r+2)$ of $\mathbf{Sp}(n)$ with simple root system $\{\alpha_1, \ldots, \alpha_{r+2}\}$, and let τ denote its representation of highest weight ξ_r. Then τ maps L onto a proper subgroup of $\mathbf{Sp}(\frac{1}{2}\deg\tau)$; thus $P(r+2,r)+Q(r+2,r)\geqslant 0$. If $r+2<n$ then (7.10) and (7.11) say that $P(n,r)+Q(n,r)>0$. Thus $r=n-2$.

Suppose $n\geqslant 5$. Then $Q(n,n-2)\geqslant 0$. Define $U=\{\alpha_{n-2};\ \alpha_{n-1}+\alpha_{n-2},\ \alpha_{n-2}+\alpha_{n-3};\ \alpha_n+\alpha_{n-1}+\alpha_{n-2},\ \alpha_{n-1}+\alpha_{n-2}+\alpha_{n-3};\ \alpha_n+\alpha_{n-1}+\alpha_{n-2}+\alpha_{n-3}\}$ and let V be the complementary set of positive roots. As V contains the highest root, we define (recall $r=n-2$)

$$v_1=\prod_V\left(\frac{\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2 \quad \text{and} \quad v_2=\prod_V\frac{2\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}$$

and have $v_1>v_2$. We also define

$$u_1=\tfrac{1}{2}\prod_U\left(\frac{\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}\right)^2 \quad \text{and} \quad u_2=\prod_U\frac{2\hat{a}_r+\hat{l}(\alpha)}{\hat{l}(\alpha)}$$

so that $P(n,n-2)=u_1v_1-u_2v_2$. But

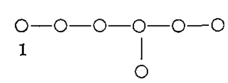
$$u_1 = \tfrac{1}{2}\cdot\tfrac{2}{1}\cdot\tfrac{2}{1}\cdot\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{5}{4}\cdot\tfrac{5}{4} = 50 \quad \text{and}$$

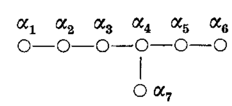
$$u_2 = \tfrac{3}{1}\cdot\tfrac{4}{2}\cdot\tfrac{4}{2}\cdot\tfrac{5}{3}\cdot\tfrac{5}{3}\cdot\tfrac{6}{4} = 50 = u_1.$$

Thus $P(n, n-2) > 0$. As $Q(n, n-2) \geqslant 0$, that contradicts $P(n, n-2) + Q(n, n-2) = 0$. Thus $n < 5$. As $n \geqslant 3$ now n must be 3 or 4.

Let $n=4$. Then λ: $\bullet\!-\!\bullet\!-\!\overset{1}{\bullet}\!=\!\circ$, so $\deg \pi_\lambda = 48$ and $\deg \pi_{2\lambda} = 825$, so $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2}(\deg \pi_\lambda) = 1176 > 825 + 36 = \deg \pi_{2\lambda} + \dim \mathbf{Sp}(4)$. Thus $n \neq 4$.

Now $n=3$ so λ: $\bullet\!-\!\bullet\!=\!\overset{1}{\circ}$. Here $\deg \pi_\lambda = 14$ and $\deg \pi_{2\lambda} = 84$, so $\frac{1}{2}(\deg \pi_\lambda)^2 + \frac{1}{2}\deg \pi_\lambda = 105 = 84 + 21 = \deg \pi_{2\lambda} + \dim \mathbf{Sp}(3)$, q.e.d.

7.12 Lemma. *If $K = \mathbf{E}_7$ then λ is given by* , *degree* 56.

Proof. We number the simple roots of $\mathbf{E}_7$ by . Then Lemmas 7.3 and 7.4 say that $\lambda = k\xi_r$; r is 1, 3 or 7; and k is 1 or 3. We compute $\deg \pi_{\xi_1} = 56$; $\deg \pi_{\xi_3} = 27\,664$; $\deg \pi_{\xi_7} = 912$; $\deg \pi_{2\xi_1} = 1\,463$; $\deg \pi_{2\xi_3} = 109\,120\,648$; $\deg \pi_{2\xi_7} = 84\,645$. Now $\dim \mathbf{E}_7 = 133$ and we have

$$\tfrac{22}{9}(27664)^2 > \tfrac{22}{9}(912)^2 > \tfrac{22}{9}(56)^2 > 133.$$

Thus Lemma 7.4 says $k=1$. Finally we compute

$$\tfrac{1}{2}(\deg \pi_{\xi_1})^2 + \tfrac{1}{2}(\deg \pi_{\xi_1}) = 1\,596 = \deg \pi_{2\xi_1} + \dim \mathbf{E}_7;$$

$$\tfrac{1}{2}(\deg \pi_{\xi_3})^2 + \tfrac{1}{2}(\deg \pi_{\xi_3}) = 382\,662\,280 > \deg.\ \pi_{2\xi_3} + \dim \mathbf{E}_7;$$

$$\tfrac{1}{2}(\deg \pi_{\xi_7})^2 + \tfrac{1}{2}(\deg \pi_{\xi_7}) = 416\,328 > \deg \pi_{2\xi_7} + \dim \mathbf{E}_7.$$

Thus $\lambda = \xi_1$, q.e.d.

Theorem 7.1 now follows from Lemmas 7.3, 7.7, 7.8, 7.9 and 7.12.

8. The case where G is orthogonal and χ reduces

As the first step in the classification for G orthogonal and K simple, we prove:

8.1. Theorem. *Let G be a simple*(1) *special orthogonal group and let K be a proper compact connected simple subgroup. Let χ be the representation of K on the tangent space of G/K. Then χ is irreducible over the real number field but not absolutely irreducible, if and only*

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(1) This means $G = \mathbf{SO}(n)$, $n > 2$, $n \neq 4$.

if the inclusion $K \to G$ *is given by* $\overset{1}{\underset{\alpha_1}{\circ}}\!-\!\ldots\!-\!\overset{1}{\underset{\alpha_n}{\circ}}$, *adjoint representation of* $\mathbf{SU}(n+1)$, $n \geq 2$. *In that case* χ *is given by* $\overset{3}{\circ}\!-\!\circ \oplus \circ\!-\!\overset{3}{\circ}$ *if* $n=2$, *by* $\circ\!-\!\overset{1}{\circ}\!-\!\ldots\!-\!\overset{2}{\circ} \oplus \overset{2}{\circ}\!-\!\ldots\!-\!\overset{1}{\circ}\!-\!\circ$ *if* $n>2$.

Note that the adjoint representation of $\mathbf{SU}(2)$ simply maps $\mathbf{SU}(2)$ onto $\mathbf{SO}(3)$, and thus is not interesting in the context of the theorem.

Proposition 5.2 says that the inclusion $K \to G = \mathbf{SO}(N)$ is an absolutely irreducible representation $\pi_{k(\xi_r+\xi_r^*)}$ for some basic weight $\xi_r \neq \xi_r^*$, that $N = \deg \pi_{k(\xi_r+\xi_r^*)}$, that $\chi = \pi_{2k(\xi_r+\xi_r^*)-\alpha_r} \oplus \pi_{2k(\xi_r+\xi_r^*)-\alpha_r^*}$, and that $\frac{1}{2}(\deg \pi_{k(\xi_r+\xi_r^*)})^2 - \frac{1}{2}\deg \pi_{k(\xi_r+\xi_r^*)} = \deg \chi + \dim K$. We write the latter in the form

$$\tfrac{1}{2}(\deg \pi_{k(\xi_r+\xi_r^*)})^2 = \tfrac{1}{2}\deg \pi_{k(\xi_r+\xi_r^*)} + 2\deg \pi_{2k(\xi_r+\xi_r^*)-\alpha_r} + \dim K. \tag{8.2}$$

We now make a growth estimate on k, proving:

8.3 LEMMA. *The integer k is equal to* 1.

Proof. For each integer $m>0$ we define

$$U_m = \tfrac{1}{2}(\deg \pi_{m(\xi_r+\xi_r^*)})^2, \quad V_m = \tfrac{1}{2}\deg \pi_{m(\xi_r+\xi_r^*)} \quad \text{and} \quad W_m = 2\deg \pi_{2m(\xi_r+\xi_r^*)-\alpha_r}.$$

We also define multipliers by

$$U_{m+1} = u_m U_m, \quad V_{m+1} = v_m V_m \quad \text{and} \quad W_{m+1} = w_m W_m.$$

$\xi_r \neq \xi_r^*$ shows that K is of type A_n, D_{2n+1} or E_6. Thus all simple roots have the same norm. Now a glance at (5.6) shows that

$$u_m = \prod_{\alpha>0} u_m(\alpha), \quad v_m = \prod_{\alpha>0} v_m(\alpha) \quad \text{and} \quad w_m = \prod_{\alpha>0} w_m(\alpha),$$

where

$$u_m(\alpha) = v_m(\alpha)^2, \quad v_m(\alpha) = \frac{(m+1)(a_r + a_{r^*}) + l(\alpha)}{m(a_r + a_{r^*}) + l(\alpha)} \quad \text{and}$$

$$w_m(\alpha) = \frac{2(m+1)a_{r^*} + 2ma_r + \sum a_i + l(\alpha)}{2ma_{r^*} + 2(m-1)a_r + \sum a_i + l(\alpha)}$$

and the summation $\sum a_i$ is extended over all simple roots adjacent to α_r in the Dynkin diagram of K.

Observe $v_m(\alpha_r) = \dfrac{m+2}{m+1} = v_m(\alpha_{r^*})$. Also $w_m(\alpha_r) = \dfrac{2m+1}{2m-1}$, $w_m(\alpha_{r^*}) = \dfrac{2m+3}{2m+1}$ if $a_{r^*} \perp \alpha_r$, and $w_m(\alpha_{r^*}) = \dfrac{m+2}{m+1}$ if α_{r^*} and α_r are adjacent. Now $(m+2)^4(2m-1)(2m+1) = 4m^6 +$

$32\,m^5+95\,m^4+120\,m^3+40\,m^2-32\,m-16>4\,m^6+24\,m^5+59\,m^4+76\,m^3+54\,m^2+20\,m+3=(m+1)^4\,(2m+1)\,(2m+3)$. Thus $\left(\dfrac{m+2}{m+1}\right)^4>\dfrac{2\,m+1}{2\,m-1}\cdot\dfrac{2\,m+3}{2\,m+1}>\dfrac{2\,m+1}{2\,m-1}\cdot\dfrac{m+2}{m+1}$. This proves

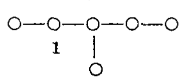
$$u_m(\alpha_r)\cdot u_m(\alpha_{r^*})>w_m(\alpha_r)\cdot w_m(\alpha_{r^*}). \tag{8.4}$$

Let α be a positive root, $\alpha_r \neq \alpha \neq \alpha_{r^*}$. Denote $a=a_r+a_{r^*}$, $s=\sum a_i$ and $l=(\alpha)$. Then

$$u_m(\alpha)=1+\frac{(2\,m+1)\,a^2+2\,a\,l}{(ma+l)^2} \quad \text{and} \quad w_m(\alpha)=1+\frac{2\,a}{2\,(m-1)\,a+2\,a_{r^*}+s+l}.$$

We will prove that $u_m(\alpha)\geqslant w_m(\alpha)$, i.e., that

$$(*) \qquad \{(2\,m+1)\,a^2+2\,al\}\,\{2\,(m-1)\,a+2\,a_{r^*}+s+l\}\geqslant 2\,a\,(ma+l)^2.$$

For $m=1$ this inequality is $(3a^2+2al)(2a_{r^*}+s+l)\geqslant 2a^3+4a^2l+2al^2$, i.e., $6a^2a_{r^*}+3a^2s+4laa_{r^*}+2las\geqslant la^2+2a^3$. If $a_r=0$, then $a_{r^*}=a$ and the inequality follows; if $a_r>1$ then ξ_r:  so $s\geqslant a_r$ and the inequality follows; now suppose $a_r=1$. If $a_{r^*}>0$ then $2a_{r^*}\geqslant a$ and the inequality follows, so suppose $a_{r^*}=0$. Then the inequality says $3s+2ls\geqslant l+2$; as $\alpha\neq\alpha_r$ we have $s\geqslant 1$ and $l\geqslant 2$ so this is clear. Now (*) is proved for $m=1$.

To prove (*) for $m>1$ we let m range as a real variable and we differentiate. Thus we must prove $2a^2\{2(m-1)a+2a_{r^*}+s+l\}+\{(2m+1)a^2+2al\}2a\geqslant 4ma^3+2a^2l$ which is clear by inspection. This completes the proof of

$$u_m(\alpha)\geqslant w_m(\alpha) \quad \text{for} \quad \alpha>0, \quad \alpha_r\neq\alpha\neq\alpha_{r^*}. \tag{8.5}$$

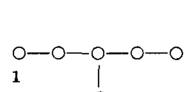
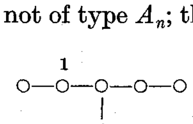
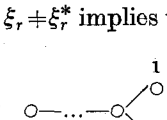
Combining (8.4) and (8.5) we have $u_m>w_m$. And $v_m>1$ shows $u_m>v_m$. This says

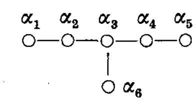
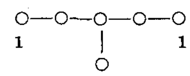
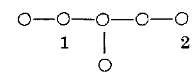
$$U_{m+1}-\{V_{m+1}+W_{m+1}+\dim K\}>U_m-\{V_m+W_m+\dim K\}. \tag{8.6}$$

Let $\nu=\xi_r+\xi_r^*$. Then π_ν is orthogonal and $\pi_\nu(K)\subsetneqq \mathbf{SO}\,(\deg\pi_\nu)$. As $\pi_{2\nu-\alpha_r}$ and $\pi_{2\nu-\alpha_{r^*}}$ are summands of $\Lambda^2(\pi_\lambda)$, this shows that $U_1\geqslant V_1+W_1+\dim K$. If $k>1$ then repetition of (8.6) says $U_k>V_k+W_k+\dim K$. But (8.2) says $U_k=V_k+W_k+\dim K$. This proves $k=1$, q.e.d.

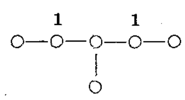
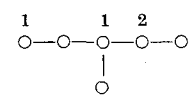
8.7. Lemma. *K is of type A_n, $1\leqslant r\leqslant n/2$.*

Proof. Suppose that K is not of type A_n; then $\xi_r\neq\xi_r^*$ implies that ξ_r is given by

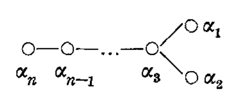
 or  or  (n odd, $n\geqslant 5$).

Suppose $K=\mathbf{E}_6$. We number the simple roots  as before. If $r=1$, then $\lambda=\xi_1+\xi_5$:  and $2\lambda-\alpha_1=\xi_2+2\xi_5$: .

Then we calculate $\deg\pi_\lambda=650$ and $\deg\pi_{2\lambda-\alpha_1}=78\,975$. Thus $\frac{1}{2}(\deg\pi_\lambda)^2-\frac{1}{2}\deg\pi_\lambda=210\,925>158\,028=2\deg\pi_{2\lambda-\alpha_1}+\dim K$. Now $r\neq 1$. If $r=2$, then

$\lambda=\xi_2+\xi_4$:  and $2\lambda-\alpha_2=\xi_1+\xi_3+2\xi_4$: 

We then calculate $\deg\pi_\lambda=70\,070$ and $\deg\pi_{2\lambda-\alpha_2}=252\,808\,452$. Thus $\frac{1}{2}(\deg\pi_\lambda)^2-\frac{1}{2}\deg\pi_\lambda=2\,454\,867\,415>505\,616\,982=2\deg\pi_{2\lambda-\alpha_2}+\dim K$. We conclude $K\neq\mathbf{E}_6$.

Now suppose $K=\mathbf{D}_n$, $n\geqslant 5$. We number the simple roots , and will prove by induction on n that

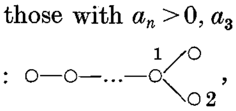
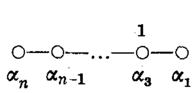
$$(*)\qquad \tfrac{1}{2}(\deg\pi_{\xi_1+\xi_2})^2>\tfrac{1}{2}\deg\pi_{\xi_1+\xi_2}+2\deg\pi_{2\xi_1+2\xi_2-\alpha_1}+\dim\mathbf{D}_n.$$

For $n=5$ we have $\deg\pi_{\xi_1+\xi_2}=210$ and $\deg\pi_{2\xi_1+2\xi_2-\alpha_1}=6\,930$; thus $\frac{1}{2}(\deg\pi_{\xi_1+\xi_2})^2=22\,050>14\,010=\frac{1}{2}\deg\pi_{\xi_1+\xi_2}+2\deg\pi_{2\xi_1+2\xi_2-\alpha_1}+\dim\mathbf{D}_n$. Now suppose $n>5$. Let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{D}_{n-1}$ with simple root system $\{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}\}$. Then we have multipliers defined by

$$\tfrac{1}{2}(\deg\pi_{\xi_1+\xi_2})^2=t\cdot\tfrac{1}{2}(\deg\tau_{\xi_1+\xi_2})^2,\quad \deg\pi_{\xi_1+\xi_2}=u\cdot\deg\tau_{\xi_1+\xi_2},$$
$$2\deg\pi_{2\xi_1+2\xi_2-\alpha_1}=v\cdot 2\deg\tau_{2\xi_1+2\xi_2-\alpha_1}$$

and $\dim\mathbf{D}_n=w\cdot\dim\mathbf{D}_{n-1}$. From $\pi_{\xi_1+\xi_2}=\Lambda^{n-1}(\pi_{\xi_n})$, $\tau_{\xi_1+\xi_2}=\Lambda^{n-2}(\tau_{\xi_{n-1}})$ and $\dim\mathbf{D}_q=2q^2-q$, we have

$$t=u^2,\quad u=\frac{2n(2n-1)}{(n+1)(n-1)}\quad\text{and}\quad w=\frac{2n^2-n}{2n^2-5n+3}.$$

The positive roots of $\mathbf{D}_n$ which contribute to v are (a) those with $a_n>0$, $a_3>0$ and $a_2=0$, and (b) those with $a_n>0$ and $a_2>0$. As $2\xi_1+2\xi_2-\alpha_1$: , those satisfying (a) form a system  and contribute a factor of $\binom{n}{2}\Big/\binom{n-1}{2}=\frac{n}{n-2}$ to v. The roots $\alpha>0$ which satisfy (b) are $\{\alpha_2+\alpha_3+\ldots+\alpha_n,\ \alpha_1+\alpha_2+\ldots+\alpha_n,\ \alpha_1+\alpha_2+$

$2\alpha_3+\alpha_4+\ldots+\alpha_n,\ldots,\ a_1+a_2+2\alpha_3+\ldots+2\alpha_{n-1}+\alpha_n\}$. They contribute to v a factor of

$$\frac{n+2}{n-1}\cdot\frac{n+3}{n}\left\{\frac{n+5}{n+1}\cdots\frac{2n+1}{2n-3}\right\}=\frac{4(2n-1)(2n+1)}{(n+1)(n+4)}.$$

Thus
$$v=\frac{4n(4n^2-1)}{(n-2)\ (n+1)(n+4)}.$$

Notice $n(2n-1)=2n^2-n>2n^2-n-1=(2n+1)(n-1)$, and $(n-2)(n+4)=n^2+2n-8>n^2-1$ as $n>5$; now this shows $t>v$. And $t>u$ and $t>w$ are clear. Thus, using the induction hypothesis on $\mathbf{D}_{n-1}$, we have $\frac{1}{2}(\deg\pi_{\xi_1+\xi_2})^2=t\cdot\frac{1}{2}(\deg\tau_{\xi_1+\xi_2})^2>t\cdot\frac{1}{2}\deg\tau_{\xi_1+\xi_2}+t\cdot 2\deg\tau_{2\xi_1+2\xi_2-\alpha_1}+t\cdot\dim\mathbf{D}_{n-1}>\frac{1}{2}\deg\pi_{\xi_1+\xi_2}+2\deg\pi_{2\xi_1+2\xi_2-\alpha_1}+\dim\mathbf{D}_n$. Now (*) is proved. This shows $K\neq\mathbf{D}_n$, completing the proof of the lemma, q.e.d.

8.8. Lemma. *A representation π_λ of* $\mathbf{SU}(n+1)$, $\lambda=\xi_r+\xi_r^*$ *with* $\xi_r\neq\xi_r^*$, *satisfies* $\frac{1}{2}(\deg\pi_\lambda)^2=\frac{1}{2}\deg\pi_\lambda+2\deg\pi_{2\lambda-\alpha_r}+\dim\mathbf{SU}(n+1)$ *if and only if* λ: $\overset{1}{\circ}\!-\!\ldots\!-\!\overset{1}{\circ}$.

Proof. We label the simple roots $\underset{\alpha_1}{\circ}\!-\!\underset{\alpha_2}{\circ}\!-\!\ldots\!-\!\underset{\alpha_n}{\circ}$. Now $\xi_r^*=\xi_{n+1-r}$, so we may assume $1\leqslant r\leqslant\frac{1}{2}n$.

First suppose $r=1$. If $n=2$ then $\deg\pi_\lambda=8$ and $\deg\pi_{2\lambda-\alpha_1}=10$; thus $\frac{1}{2}(\deg\pi_\lambda)^2=32=4+2\cdot 10+8=\frac{1}{2}\deg\pi_\lambda+2\deg\pi_{2\lambda-\alpha_1}+\dim\mathbf{A}_2$. Now let $n>2$. π_λ is the adjoint representation,

$$\deg\pi_\lambda=n^2+2n=\dim\mathbf{SU}(n+1).$$

$2\lambda-\alpha_1=\xi_2+2\xi_n$; we compute $\deg\pi_{2\lambda-\alpha_1}=\frac{1}{4}(n-1)n(n+2)(n+3)$. Thus $\frac{1}{2}(\deg\pi_\lambda)^2=\frac{1}{2}n^2(n+2)^2=\frac{1}{2}n(n+2)\{1+(n-1)(n+3)+2\}=\frac{1}{2}\deg\pi_\lambda+2\deg\pi_{2\lambda-\alpha_1}+\dim\mathbf{SU}(n+1)$. This proves the equality for $r=1$.

Suppose $r>1$ and let $\mathbf{SU}(n-1)$ denote the subgroup of that type with simple root system $\{\alpha_2,\ldots,\alpha_{n-1}\}$. Let τ_ν denote the representation of $\mathbf{SU}(n-1)$ with highest weight ν. Finally define multipliers by

$$\deg\pi_\lambda=x\cdot\deg\tau_\lambda,\ \deg\pi_{2\lambda-\alpha_r}=y\cdot\deg\tau_{2\lambda-\alpha_r}\ \text{and}\ \dim\mathbf{SU}(n+1)=z\cdot\dim\mathbf{SU}(n-1).\quad(8.9)$$

Let $r=\frac{1}{2}n$. If $r=2$ then λ: $\circ\!-\!\overset{1}{\circ}\!-\!\overset{1}{\circ}\!-\!\circ$ and $2\lambda-\alpha_r$: $\circ\!-\!\overset{1}{\circ}\!-\!\overset{3}{\circ}\!-\!\circ$. For that case we calculate $\frac{1}{2}(\deg\pi_\lambda)^2=\frac{1}{2}(75)^2>\frac{1}{2}(75)+2(700)+24=\frac{1}{2}\deg\pi_\lambda+2\deg\pi_{2\lambda-\alpha_r}+\dim\mathbf{SU}(n+1)$. Now suppose $r>2$. The roots of $\mathbf{SU}(n+1)$ which are not roots of $\mathbf{SU}(n-1)$ are the $\alpha_1+\ldots+\alpha_m$ and the $\alpha_l+\ldots+\alpha_n$. Thus

$$x=\left\{\frac{r+1}{r}\cdot\frac{r+3}{r+1}\cdot\frac{r+4}{r+2}\cdots\frac{2r+1}{2r-1}\right\}^2\cdot\frac{2r+2}{2r}=4\,\frac{(r+1)(2r+1)^2}{r(r+2)^2}\text{ and}$$

$$y=\left\{\frac{r}{r-1}\cdot\frac{r+1}{r}\cdot\frac{r+5}{r+1}\right\}\left\{\frac{r+3}{r}\cdot\frac{r+4}{r+1}\right\}\left\{\frac{r+6}{r+2}\cdot\frac{r+7}{r+3}\cdot\ldots\cdot\frac{2r+3}{2r-1}\right\}^2\frac{2r+4}{2r}$$

$$=16\,\frac{r+1}{r-1}\cdot\frac{(2r+1)^2(2r+3)^2}{(r+2)(r+3)(r+4)(r+5)}.$$

By expanding we check $(r-1)(r+3)(r+4)\geqslant r(r+2)^2$ and $(2r+1)^2(r+5)(r+1)>r(r+2)(2r+3)^2$; it follows that $x^2>y$. That $x^2>x$ and $x^2>z$ are clear.

Now suppose $1<r<n/2$. Then we compute

$$x=\left\{\frac{r+1}{r}\cdot\frac{r+2}{r+1}\cdot\ldots\cdot\frac{n-r+1}{n-r}\right\}^2\left\{\frac{n-r+3}{n-r+1}\cdot\frac{n-r+4}{n-r+2}\cdot\ldots\cdot\frac{n+1}{n-1}\right\}^2\cdot\frac{n+2}{n}=\frac{n(n+1)^2(n+2)}{r^2(n-r+2)^2}$$

and

$$y=\left\{\frac{r}{r-1}\cdot\frac{r+1}{r}\cdot\frac{r+3}{r+1}\cdot\frac{r+4}{r+2}\cdot\ldots\cdot\frac{n-r+2}{n-r}\cdot\frac{n-r+5}{n-r+1}\right\}\left\{\frac{r+2}{r}\cdot\frac{r+3}{r+1}\cdot\ldots\cdot\frac{n-r+1}{n-r-1}\cdot\frac{n-r+3}{n-r}\right.$$

$$\left.\cdot\frac{n-r+4}{n-r+1}\right\}\cdot\left\{\frac{n-r+6}{n-r+2}\cdot\frac{n-r+7}{n-r+3}\cdot\ldots\cdot\frac{n+3}{n-1}\right\}^2\cdot\frac{n+4}{n}$$

$$=\frac{n(n+1)^2(n+2)^2(n+3)^2(n+4)}{(r-1)r(r+1)(r+2)(n-r+2)(n-r+3)(n-r+4)(n-r+5)}.$$

An extremely unpleasant expansion shows $x^2>y$. Again $x^2>x$ and $x^2>z$ are clear. We have proved:

$$x^2>y,\quad x^2>x\quad\text{and}\quad x^2>z\quad\text{for}\quad 2\leqslant r\leqslant\tfrac{1}{2}n. \tag{8.10}$$

We have proved $\frac{1}{2}(\deg\tau_\lambda)^2=\frac{1}{2}\deg\tau_\lambda+2\deg\tau_{2\lambda-\alpha_r}+\dim \mathbf{SU}(n-1)$ for $r=2$. By induction, we have $\frac{1}{2}(\deg\tau_\lambda)^2>\frac{1}{2}\deg\tau_\lambda+2\deg\tau_{2\lambda-\alpha_r}+\dim\mathbf{SU}(n-1)$ for $r>2$. Now (8.9) and (8.10) give us $\frac{1}{2}(\deg\pi_\lambda)^2>\frac{1}{2}\deg\pi_\lambda+2\deg\pi_{2\lambda-\alpha_r}+\dim\mathbf{SU}(n+1)$ for $r>1$, q.e.d.

Theorem 8.1 is immediate from Lemmas 8.3, 8.7 and 8.8.

9. The estimate for the case where G is orthogonal with χ absolutely irreducible

The estimate is:

9.1 PROPOSITION. *Let ξ_r be a basic weight of a compact connected simple Lie group K. For every integer $m\geqslant 1$, define*

$$U_m=\tfrac{1}{2}(\deg\pi_{m\xi_r})^2,\quad V_m=\tfrac{1}{2}\deg\pi_{m\xi_r}\quad\textit{and}\quad W_m=\deg\pi_{2m\xi_r-\alpha_r}.$$

If $m \geqslant 2$ *then* $U_{m+1} - V_{m+1} - W_{m+1} > U_m - V_m - W_m$. *If* $\pi_{\xi_r}(K) \subsetneqq \mathbf{SO}\,(\deg \pi_{\xi_r})$ *in case* α_r *is a terminal vertex*(¹) *on the Dynkin diagram of* K, *then* $U_2 - V_2 - W_2 > U_1 - V_1 - W_1$.

Proof. We define multipliers by

$$U_{m+1} = u_m U_m, \quad V_{m+1} = v_m V_m \quad \text{and} \quad W_{m+1} = w_m W_m.$$

Given a root $\alpha > 0$ we define $a = \hat{a}_r$ and $l = \hat{l}(\alpha)$. Let S be the set of all simple roots adjacent to α_r in the Dynkin diagram of K, and let $\alpha_i \in S$. Then $n_i = \dfrac{2\langle \alpha_i, 2m\xi_r - \alpha_r\rangle}{\langle \alpha_i, \alpha_i\rangle} = -\dfrac{2\langle \alpha_i, \alpha_r\rangle}{\langle \alpha_i, \alpha_i\rangle}$ is 1 if $\|\alpha_i\|^2 \geqslant \|\alpha_r\|^2$, and is $\dfrac{\|\alpha_r\|^2}{\|\alpha_i\|^2}$ otherwise. Observe $k\xi_r - \alpha_r = (k-2)\,\xi_r + \sum_S n_i\,\xi_i$. For our given α we define $s = \sum_S n_i\,\hat{a}_i$. Now a glance at (5.6) shows

$$w_m = \prod_{\alpha>0} w_m(\alpha), \quad w_m(\alpha) = \frac{2ma + s + l}{2(m-1)\,a + s + l} = 1 + \frac{2a}{2(m-1)\,a + s + l},$$

$$v_m = \prod_{\alpha>0} v_m(\alpha) \qquad v_m(\alpha) = \frac{(m+1)\,a + l}{ma + l}, \quad \text{and}$$

$$u_m = \prod_{\alpha>0} u_m(\alpha), \quad u_m(\alpha) = v_m(\alpha)^2 = 1 + \frac{(2m+1)\,a^2 + 2\,la}{(ma+l)^2}.$$

If $m \geqslant 2$ then $\{(2m+1)\,a^2 + 2\,la\}\{2\,(m-1)\,a + s + l\} \geqslant 2\,a\,(ma+l)^2$, with strict inequality when $a > 0$. Thus $u_m(\alpha) \geqslant w_m(\alpha)$, and $u_m(\alpha) > w_m(\alpha)$ in case $a > 0$. Now

$$\textit{if } m \geqslant 2 \textit{ then } u_m > w_m. \tag{9.2}$$

Now let $m = 1$. We compute

$$u_1(\alpha) = 1 + \frac{3\,a^2 + 2\,la}{(a+l)^2} \quad \text{and} \quad w_1(\alpha) = 1 + \frac{2\,a}{s+l}.$$

Suppose $a > 0$. If $s \geqslant \frac{2}{3}a$, then $3as + 2ls > al + 2a^2$, so $(3a^2 + 2la)(s+l) > 2a(a+l)^2$. Thus

$$\textit{if } s \geqslant \tfrac{2}{3}a > 0 \quad \textit{then} \quad u_1(\alpha) > w_1(\alpha). \tag{9.3}$$

Suppose $s < \frac{2}{3}a$. If $a_r \geqslant 3$ then α must be one of a few roots of exceptional groups, and one easily checks that

$$\alpha = 3\alpha_1 + \alpha_2 \quad \text{for} \quad K = \mathbf{G}_2 \quad \overset{\alpha_1}{\bullet}\!\equiv\!\overset{\alpha_2}{\circ} \quad \text{and} \quad r = 1 \tag{9.4}$$

is the only possibility. Now suppose $a_r = 2$. If α_r is a terminal vertex of the Dynkin diagram

(¹) In other words, there is no condition if α_r is interior to the Dynkin diagram. But if α_r is not interior, then π_{ξ_r} must be orthogonal with image $\neq \mathbf{SO}\,(\deg \pi_{\xi_r})$.

and α_0 denotes the unique adjacent root, then $s<\frac{2}{3}a$ says $n_0 a_0\|\alpha_0\|^2<\frac{4}{3}\|\alpha_r\|^2$. If $\|\alpha_r\|^2\geqslant\|\alpha_0\|^2$ then n_0 is the quotient so $a_0<\frac{4}{3}$. As $a_0\neq 0$ because $2\alpha_r$ is not a root, now $a_0=1$ and $\alpha_0+2\alpha_r$ is a root, which contradicts $\|\alpha_r\|^2\geqslant\|\alpha_0\|^2$. Now $\|\alpha_r\|^2<\|\alpha_0\|^2$, so $n_0=1$ and $a_0<\frac{4}{3}\|\alpha_r\|^2/\|\alpha_0\|^2\leqslant\frac{2}{3}$. That says $a_0=0$, contradicting the fact that $2\alpha_r$ is not a root. Thus α_r must be interior to the Dynkin diagram of K, and by the argument for terminal vertices we have α_1 and α_2 in S with $a_1>0$ and $a_2>0$. If $\|\alpha_r\|^2\geqslant\|\alpha_1\|^2$ then $a_1\|\alpha_r\|^2=n_1\hat{a}_1<s<\frac{4}{3}\|\alpha_r\|^2$ so $a_1=1$. If also $\|\alpha_r\|^2\geqslant\|\alpha_2\|^2$ then $a_2=1$ and $s\geqslant 2\|\alpha_r\|^2>\frac{2}{3}a$; thus $\|\alpha_r\|^2<\|\alpha_2\|^2$, and now $\|\alpha_2\|^2=2\|\alpha_r\|^2$ because rank $K\geqslant 3$. We calculate $\|\alpha_r\|^2+2a_2\|\alpha_r\|^2=n_1\hat{a}_1+n_2\hat{a}_2\leqslant s<\frac{4}{3}\|\alpha_r\|^2$; that is impossible. Thus $\|\alpha_r\|^2<\|\alpha_1\|^2$. Similarly $\|\alpha_r\|^2<\|\alpha_2\|^2$. But $\overset{\alpha_1}{\circ}\!=\!\overset{\alpha_r}{\bullet}\!=\!\overset{\alpha_2}{\circ}$ cannot be contained in a Dynkin diagram. We have proved $a_r\neq 2$. Finally suppose $a_r=1$. Then $s<\frac{2}{3}a$ says $s<\frac{2}{3}\|\alpha_r\|^2$. Let $\alpha_0\in S$. If $\|\alpha_r\|^2\geqslant\|\alpha_0\|^2$ then $n_0\hat{a}_0=a_0\|\alpha_r\|^2\leqslant s<\frac{2}{3}\|\alpha_r\|^2$; thus $a_0=0$. If $\|\alpha_r\|^2<\|\alpha_0\|^2$ then $n_0\hat{a}_0>a_0\|\alpha_r\|^2$ so again $a_0=0$. We have proved:

$$\textit{if } s<\tfrac{2}{3}a, \textit{ then either } \alpha=\alpha_r \textit{ or } \alpha \textit{ is given by } (9.4). \tag{9.5}$$

We eliminate the odd case (9.4). There $\deg\pi_{\xi_1}=7$ and $\deg\pi_{2\xi_1}=27$, so $u_1=729/49$. Also $2\xi_1-\alpha_1=\xi_2$ and $\deg\pi_{\xi_2}=14$, and $4\xi_1-\alpha_1=2\xi_1+\xi_2$ and $\pi_{2\xi_1+\xi_2}$ has degree 189, so $w_1=189/14$. As $729\cdot 14=10\,206>9\,261=49\cdot 189$, this shows:

$$\textit{if } K=\mathbf{G}_2\ \ \overset{\alpha_1}{\bullet}\!\equiv\!\overset{\alpha_2}{\circ}\ \textit{ and }\ r=1, \textit{ then } u_1>w_1. \tag{9.6}$$

Now we need roots to overcome α_r. We look for a set Γ of positive roots such that

$$\alpha_r\in\Gamma \quad \textit{and} \quad \prod_{\Gamma} u_1(\alpha)>\prod_{\Gamma} w_1(\alpha). \tag{9.7}$$

If rank $K\geqslant 4$ and α_r is not a terminal vertex on the Dynkin diagram of K, then we choose a subdiagram Δ of rank 4 with α_r interior to Δ. We run through the possibilities for Δ.

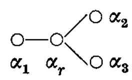
(1) Δ: $\overset{\alpha_1}{\circ}\!-\!\overset{\alpha_2}{\circ}\!-\!\overset{\alpha_r}{\circ}\!-\!\overset{\alpha_4}{\circ}$. Here we define $\Gamma=\{\alpha_r,\ \alpha_2+\alpha_r,\ \alpha_r+\alpha_4,\ \alpha_2+\alpha_r+\alpha_4,\ \alpha_1+\alpha_2+\alpha_r+\alpha_4\}$ and calculate $\prod_\Gamma u_1(\alpha)=16>140/9=\prod_\Gamma w_1(\alpha)$.

(2) Δ: contains $\overset{\alpha_1}{\circ}\!-\!\overset{\alpha_r}{\circ}\!=\!\overset{\alpha_2}{\bullet}$. Define $\Gamma=\{\alpha_r,\ \alpha_r+\alpha_2,\ \alpha_r+2\alpha_2,\ \alpha_1+\alpha_r,\ \alpha_1+\alpha_r+\alpha_2,\ \alpha_1+\alpha_r+2\alpha_2,\ \alpha_1+2\alpha_r+2\alpha_2\}$. Then $\prod_\Gamma u_1(\alpha)=49>286/7=\prod_\Gamma w_1(\alpha)$.

(3) Δ: contains $\overset{\alpha_1}{\bullet}\!-\!\overset{\alpha_r}{\bullet}\!=\!\overset{\alpha_2}{\circ}$. Define $\Gamma=\{\alpha_r,\ \alpha_1+\alpha_r,\ \alpha_r+\alpha_2,\ 2\alpha_r+\alpha_2,\ \alpha_1+\alpha_r+\alpha_2,\ \alpha_1+2\alpha_r+\alpha_2,\ 2\alpha_1+2\alpha_r+\alpha_2\}$. Then $\prod_\Gamma u_1(\alpha)=2025/49>286/7=\prod_\Gamma w_1(\alpha)$.

(4) Δ: $\overset{\alpha_1}{\bullet}\!=\!\overset{\alpha_2}{\circ}\!-\!\overset{\alpha_r}{\circ}\!-\!\overset{\alpha_4}{\circ}$. Here we imitate case (1), defining $\Gamma=\{\alpha_r,\ \alpha_2+\alpha_r,\ \alpha_r+\alpha_4,\ \alpha_2+\alpha_r+\alpha_4,\ 2\alpha_1+\alpha_2+\alpha_r+\alpha_4\}$ and calculating $\prod_\Gamma u_1(\alpha)=16>140/9=\prod_\Gamma w_1(\alpha)$

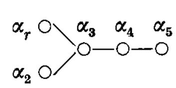
(5) Δ: $\overset{\alpha_1}{\circ}\!=\!\overset{\alpha_2}{\bullet}\!-\!\overset{\alpha_r}{\bullet}\!-\!\overset{\alpha_4}{\bullet}$. Again we imitate case (1), defining $\Gamma=\{\alpha_r,\ \alpha_2+\alpha_r,\ \alpha_r+\alpha_4,\ \alpha_2+\alpha_r+\alpha_4,\ \alpha_1+2\alpha_2+2\alpha_r+2\alpha_4\}$ and calculating $\prod_\Gamma u_1(\alpha)=16>140/9=\prod_\Gamma w_1(\alpha)$.

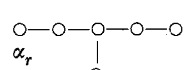
(6) Δ: . Define $\Gamma=\{\alpha_r;\ \alpha_1+\alpha_r,\ \alpha_2+\alpha_r,\ \alpha_3+\alpha_r;\ \alpha_1+\alpha_2+\alpha_r,\ \alpha_2+\alpha_3+\alpha_r,\ \alpha_3+\alpha_1+\alpha_r;\ \alpha_1+\alpha_2+\alpha_3+\alpha_r\}$. Then $\prod_\Gamma u_1(\alpha)=625/9>49=\prod_\Gamma w_1(\alpha)$.

This covers all possibilities for Δ. We have proved: if rank $K\geqslant 4$ and α_r is interior to the Dynkin diagram of K, then there exists a set Γ satisfying (9.7). In fact, in considering cases (2) and (3), we also proved this for K of type B_3 or C_3. As A_3 is the only other type of rank 3, and as the rank must be at least 3 if there is to be a root interior to the diagram we summarize as follows.

If $K\neq A_3$ and α_r is interior to the diagram, then a set Γ exists satisfying (9.7). (9.8)

Now let α_r be a terminal vertex on the Dynkin diagram of K. We examine some possibilities Ψ for a subdiagram containing α_r; the numbering is continued from the cases for Δ listed above.

(7) Ψ: . Here we define $\Gamma=\{\alpha_r,\ \alpha_r+\alpha_3,\ \alpha_r+\alpha_2+\alpha_3,\ \alpha_r+\alpha_2+\alpha_3+\alpha_4,\ \alpha_r+\alpha_2+2\alpha_3+\alpha_4,\ \alpha_r+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_r+\alpha_2+2\alpha_3+\alpha_4+\alpha_5,\ \alpha_r+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5\}$. Then $\prod_\Gamma u_1(\alpha)=441/16>55/2=\prod_\Gamma w_1(\alpha)$.

(8) Ψ: . In this case we define Γ to be all positive roots on the system Ψ which involve α_r. Then $\prod_\Gamma u_1(\alpha)=169>154=\prod_\Gamma w_1(\alpha)$.

(9) Ψ: $\overset{\alpha_r}{\bullet}\!=\!\overset{\alpha_2}{\circ}\!-\!\overset{\alpha_1}{\circ}$. We define $\Gamma=\{\alpha_r,\ \alpha_r+\alpha_2,\ 2\alpha_r+\alpha_2,\ \alpha_r+\alpha_2+\alpha_1,\ 2\alpha_r+\alpha_2+\alpha_1,\ 2\alpha_r+2\alpha_2+\alpha_1\}$ and calculate $\prod_\Gamma u_1(\alpha)=1225/64>18=\prod_\Gamma w_1(\alpha)$.

(10) Ψ: $\overset{\alpha_r}{\circ}\!=\!\overset{\alpha_2}{\bullet}\!-\!\overset{\alpha_1}{\bullet}$. We define $\Gamma=\{\alpha_r,\ \alpha_r+\alpha_2,\ \alpha_r+\alpha_2+\alpha_1,\ \alpha_r+2\alpha_2,\ \alpha_r+2\alpha_2+\alpha_1,\ \alpha_r+2\alpha_2+2\alpha_1\}$ and calculate $\prod_\Gamma u_1(\alpha)=36>273/10=\prod_\Gamma w_1(\alpha)$.

(11) Ψ: $\overset{\alpha_r}{\bullet}\!-\!\bullet\!=\!\circ\!-\!\circ$. Then $u_1=\dfrac{9^3\cdot 11^4}{2^4\cdot 5^2\cdot 13^2}>\dfrac{1287}{10}=w_1$.

(12) Ψ: $\bullet\!-\!\bullet\!=\!\circ\!-\!\overset{\alpha_r}{\circ}$. Then $u_1=\dfrac{6561}{16}>\dfrac{26163}{98}=w_1$.

(13) Ψ: $\bullet\!\equiv\!\circ\,\alpha_r$. Then $u_1=\dfrac{121}{4}>\dfrac{221}{11}=w_1$.

(14) Ψ': α_r o—o—o—o—o—o—o (with a vertex o attached below the sixth vertex). Then $u_1 = 15^6 > \dfrac{3^5 \cdot 5^2 \cdot 13 \cdot 23}{2^2 \cdot 7^2} = w_1$.

If the terminal vetex α_r is not contained as shown in one of the configurations Ψ' just considered, then the position of α_r in the Dynkin diagram of K must (by classification) be one of the following

(u) $\overset{\alpha_r}{\circ}$—...—o , rank $K > 1$

(f_1) $\overset{\alpha_r}{\circ}$—...—o=●, rank $K > 1$

(f_2) $\overset{\alpha_r}{\circ}$—...—o<(o, o), rank $K > 3$

(s_1) $\overset{\alpha_r}{\bullet}$—...—●=o, rank $K > 1$

(s_2) $\overset{\alpha_r}{\circ}$, rank $K = 1$

(s_3) o—o—o—o—o—o (with a vertex o attached below the third vertex; α_r the last vertex)

except for the case which is settled by (9.6). In case (u), π_{ξ_r} is unitary, not self dual, hence not orthogonal. In cases (f_i), $\pi_{\xi_r}(K)$ is the full **SO** $(\deg \pi_{\xi_r})$. In cases (s_i), π_{ξ_r} is symplectic, hence not orthogonal. Thus those cases are excluded by hypothesis in considering the inequality of Proposition 9.1 for $m=1$. We summarize as follows.

(9.9) *If α_r is a terminal vertex with $\pi_{\xi_r}(K) \subsetneqq$* **SO** $(\deg \pi_{\xi_r})$, *then a set Γ exists satisfying* (9.7).

Combine (9.3), (9.5), (9.8) and (9.9), using the fact that $u_1(\alpha) = 1 = w_1(\alpha)$ whenever $a = 0$. This gives:

(9.10) *If $\xi_r \neq$ o—$\overset{1}{\circ}$—o, and if $\pi_{\xi_r}(K) \subsetneqq$* **SO** $(\deg \pi_{\xi_r})$ *in case α_r is terminal, then $u_1 > w_1$.*

We also notice

(9.11) *If $\xi_r =$ o—$\overset{1}{\circ}$—o, then $U_2 - V_2 - W_2 = 15 > 0 = U_1 - V_1 - W_1$.*

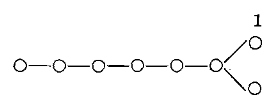
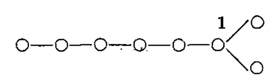
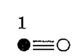
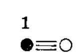
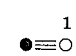
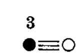
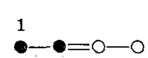
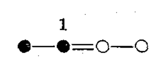
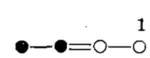
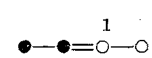
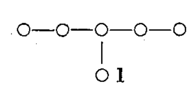
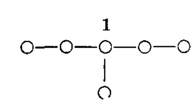
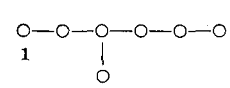
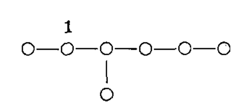
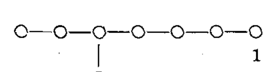
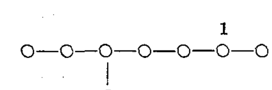
We complete the proof of Proposition 9.1. Let $m \geqslant 1$ be an integer. If $m = 1$, suppose $\xi_r \neq$ o—$\overset{1}{\circ}$—o, and further assume $\pi_{\xi_r}(K) \subsetneqq$ **SO** $(\deg \pi_{\xi_r})$ if α_r if a terminal vertex. Then $u_m > w_m$ by (9.2) and (9.10). Notice also $u_m > v_m > 1$. Now $U_{m+1} - V_{m+1} - W_{m+1} = u_m U_m - v_m V_m - w_m W_m > u_m (U_m - V_m - W_m) > U_m - V_m - W_m$. With (9.11), this proves our assertions, q.e.d.

10. The classification for the case where G is orthogonal with χ absolutely irreducible

The classification is:

10.1 Theorem. *Let G be a simple special orthogonal group and let K be a proper compact connected simple subgroup. Then the representation χ of K on the tangent space of G/K is absolutely irreducible, if and only if (G, K) is one of the following*

G	K	π_λ	χ
$\mathbf{SO}(5)$	$\mathbf{SU}(2)/\mathbf{Z}_2$	$\overset{4}{\circ}$	$\overset{6}{\circ}$
$\mathbf{SO}(20)$	$\mathbf{SU}(4)/\mathbf{Z}_4$	$\circ-\overset{2}{\circ}-\circ$	$\overset{1}{\circ}-\overset{2}{\circ}-\overset{1}{\circ}$
$\mathbf{SO}(70)$	$\mathbf{SU}(8)/\mathbf{Z}_4$	$\circ-\circ-\circ-\overset{1}{\circ}-\circ-\circ-\circ$	$\circ-\circ-\overset{1}{\circ}-\circ-\overset{1}{\circ}-\circ-\circ$
$\mathbf{SO}(2n^2+n)$	$\mathbf{SO}(2n+1)$	$\circ=\overset{2}{\bullet}$ if $n=2$ $\circ-\overset{1}{\circ}-\cdots-\circ=\bullet$ if $n>2$ (adjoint)	$\overset{1}{\circ}=\overset{2}{\bullet}$ if $n=2$ $\overset{1}{\circ}-\circ=\overset{2}{\bullet}$ if $n=3$ $\overset{2}{\circ}-\circ-\overset{1}{\circ}-\cdots-\circ=\bullet$ if $n>3$
$\mathbf{SO}(2n^2+3n)$	$\mathbf{SO}(2n+1)$	$\overset{2}{\circ}-\cdots-\circ=\bullet$ $n\geqslant 2$	$\overset{2}{\circ}=\overset{2}{\bullet}$ if $n=2$ $\overset{2}{\circ}-\overset{1}{\circ}-\cdots-\circ=\bullet$ if $n>2$
$\mathbf{SO}(16)$	$\mathbf{Spin}(9)$	$\circ-\circ-\circ=\overset{1}{\bullet}$	$\circ-\circ-\overset{1}{\circ}=\bullet$
$\mathbf{SO}(2n^2-n-1)$	$\mathbf{Sp}(n)/\mathbf{Z}_2$	$\bullet-\overset{1}{\bullet}-\cdots-\bullet=\circ$ $n\geqslant 3$	$\overset{1}{\bullet}-\bullet=\overset{1}{\circ}$ if $n=3$ $\overset{1}{\bullet}-\bullet-\overset{1}{\bullet}-\cdots-\bullet=\circ$ if $n>3$
$\mathbf{SO}(2n^2+n)$	$\mathbf{Sp}(n)/\mathbf{Z}_2$	$\overset{2}{\bullet}-\bullet-\cdots-\bullet=\circ$ $n\geqslant 3$ (adjoint)	$\overset{2}{\bullet}-\overset{1}{\bullet}-\cdots-\bullet=\circ$ $n\geqslant 3$
$\mathbf{SO}(42)$	$\mathbf{Sp}(4)/\mathbf{Z}_2$	$\bullet-\bullet-\bullet=\overset{1}{\circ}$	$\bullet-\bullet-\overset{2}{\bullet}=\circ$
$\mathbf{SO}(2n^2-n)$	$\mathbf{SO}(2n)/\mathbf{Z}_2$	$\circ-\overset{1}{\circ}-\cdots-\circ\genfrac{}{}{0pt}{}{\diagup\circ}{\diagdown\circ}$ $n\geqslant 4$ (adjoint)	$1\,\circ-\circ\genfrac{}{}{0pt}{}{\diagup\circ\,1}{\diagdown\circ\,1}$ if $n=4$ $\overset{1}{\circ}-\circ-\overset{1}{\circ}-\cdots-\circ\genfrac{}{}{0pt}{}{\diagup\circ}{\diagdown\circ}$ if $n>4$
$\mathbf{SO}(2n^2+n-1)$	$\mathbf{SO}(2n)/\mathbf{Z}_2$	$\overset{2}{\circ}-\circ-\cdots-\circ\genfrac{}{}{0pt}{}{\diagup\circ}{\diagdown\circ}$ $n\geqslant 4$	$\overset{2}{\circ}-\overset{1}{\circ}-\cdots-\circ\genfrac{}{}{0pt}{}{\diagup\circ}{\diagdown\circ}$ $n\geqslant 4$

G	K	π_λ	χ
$\mathbf{SO}(128)$	$\mathbf{Spin}(16)/\mathbf{Z}_2$		
$\mathbf{SO}(7)$	$\mathbf{G}_2$		
$\mathbf{SO}(14)$	$\mathbf{G}_2$	 (adjoint)	
$\mathbf{SO}(26)$	$\mathbf{F}_4$		
$\mathbf{SO}(52)$	$\mathbf{F}_4$	 (adjoint)	
$\mathbf{SO}(78)$	$\mathbf{E}_6/\mathbf{Z}_3$	 (adjoint)	
$\mathbf{SO}(133)$	$\mathbf{E}_7/\mathbf{Z}_2$	 (adjoint)	
$\mathbf{SO}(248)$	$\mathbf{E}_8$	 (adjoint)	

where the inclusion $K \to G = \mathbf{SO}(N) \subset \mathbf{GL}(N, \mathbf{C})$ *is the absolutely irreducible representation* π_λ *of* K *with highest weight* λ *indicated in the chart.*

Remark. $\mathbf{SO}(5)/\{\mathbf{SU}(2)/\mathbf{Z}_2\} = \{\mathbf{Sp}(2)/\mathbf{Z}_2\}/\{\mathbf{SU}(2)/\mathbf{Z}_2\}$.

If K is not of type A_n then we notice that the adjoint representation of K is one of the possibilities for π_λ. Combining this with Theorem 8.1 and with the fact that $\dim K \neq 4$ for K simple, we have the following.

10.2 Corollary. *Let K be a compact connected simple Lie group, $n = \dim K$. Let $\pi\colon K \to \mathbf{SO}(n)$ denote the adjoint representation and let χ denote the representation of K on the tangent space of $\mathbf{SO}(n)/\pi(K)$. Then χ is irreducible over the real number field, and χ is absolutely irreducible if and only if K is not of type A_l for $l \geq 2$.*

We go on to prove Theorem 10.1.

Let χ be absolutely irreducible. We will eliminate all possibilities for π_λ except those listed in the theorem, and in the process we will check that χ is absolutely irreducible for the listed π_λ.

For the moment we set aside the case where $K = \mathbf{G}_2$ and $G = \mathbf{SO}(7)$. Then Proposition 5.2 says that the inclusion $K \to G$ is an absolutely irreducible representation $\pi_{k\xi_r}$ for some basic weight $\xi_r = \xi_r^*$ of K, and that

$$\tfrac{1}{2}\,(\deg \pi_{k\xi_r})^2 = \tfrac{1}{2}\deg \pi_{k\xi_r} + \deg \pi_{2k\xi_r - \alpha_r} + \dim K. \tag{10.3}$$

If $\pi_{\xi_r}(K) \subsetneqq \mathbf{SO}\,(\deg \pi_{\xi_r})$, then $\tfrac{1}{2}\,(\deg \pi_{\xi_r})^2 + \tfrac{1}{2}\deg \pi_{\xi_r} \geqslant \deg \pi_{2\xi_r - \alpha_r} + \dim K$, so Proposition 9.1 says $k = 1$. If either $\pi_{\xi_r}(K) = \mathbf{SO}(\deg \pi_{\xi_r})$ or $\pi_{\xi_r}(K) \not\subset \mathbf{SO}\,(\deg \pi_{\xi_r})$, then we still have $\pi_{2\xi_r}(K) \subsetneqq \mathbf{SO}\,(\deg \pi_{2\xi_r})$ because $\xi_r = \xi_r^*$, so Proposition 9.1 says $k = 2$. We now run through cases. λ denotes $k\xi_r$.

10.4 Lemma. *If K is of type A_n, then* (1) $n = 1$ *with* λ: $\overset{4}{\circ}$, *or* (2) $n = 3$ *with* λ: $\circ\!-\!\overset{2}{\circ}\!-\!\circ$ *or* (3) $n = 7$ *with* λ: $\circ\!-\!\circ\!-\!\circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ$.

Remark. In case (1), $G/K = \mathbf{SO}(5)/(\mathbf{SU}(2)/\{\pm I\}) = \mathbf{Sp}(2)/\mathbf{SU}(2)$; in the latter, $\mathbf{A}_1 \to \mathbf{C}_2$ is given by $\overset{3}{\circ}$.

Proof. $\xi_r = \xi_r^*$ says that $n = 2r - 1$ and K has diagram $\underset{\alpha_1}{\circ}\!-\!\underset{\alpha_2}{\circ}\!-\!\cdots\!-\!\underset{\alpha_r}{\circ}\!-\!\cdots\!-\!\underset{\alpha_n}{\circ}$. Now orthogonality of $\pi_{k\xi_r}$ is equivalent to $kr \equiv 0$ (modulo 2).

Let $r = 1$ so $n = 1$. The representation with highest weight $\overset{m}{\circ}$ has degree $m + 1$. Now (10.3) becomes $\tfrac{1}{2}(k+1)^2 = \tfrac{1}{2}(k+1) + (2k-1) + 3$, which has solutions 4 and -1. Thus $k = 4$, λ: $\overset{4}{\circ}$.

Let $r = 2$ so $n = 3$. Then ξ_r: $\circ\!-\!\overset{1}{\circ}\!-\!\circ = \overset{1}{\circ}\!<\!\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}$ so $\pi_{\xi_r}(K) = \mathbf{SO}(6)$. Thus $k = 2$, and we check for λ: $\circ\!-\!\overset{2}{\circ}\!-\!\circ$ that $\tfrac{1}{2}\,(\deg\,\pi_\lambda)^2 = 200 = 10 + 175 + 15 = \tfrac{1}{2}\,(\deg\,\pi_\lambda) + \deg\,\pi_{2\lambda - \alpha_2} + \dim \mathbf{A}_3$.

If $r = 3$ then $n = 5$ and $kr \equiv 0$ (mod 2) says $k = 2$. But for λ: $\circ\!-\!\circ\!-\!\overset{2}{\circ}\!-\!\circ\!-\!\circ$ we compute $\tfrac{1}{2}\,(\deg \pi_\lambda)^2 = \tfrac{1}{2}(175)^2 > \tfrac{1}{2}(175) + 11\,340 + 35 = \tfrac{1}{2}\deg \pi_\lambda + \deg \pi_{2\lambda - \alpha_3} + \dim \mathbf{A}_5$.

If $r = 4$ then $n = 7$ and $k = 1$. We check for λ: $\circ\!-\!\circ\!-\!\circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ$ that $\tfrac{1}{2}\,(\deg \pi_\lambda)^2 - \tfrac{1}{2}\deg \pi_\lambda = 2415 = 2352 + 63 = \deg \pi_{2\lambda - \alpha_4} + \dim \mathbf{A}_7$.

Let $r > 4$; we will prove $\tfrac{1}{2}\,(\deg \pi_{\xi_r})^2 > \tfrac{1}{2}\,(\deg \pi_{\xi_r}) + \deg \pi_{2\xi_r - \alpha_r} + \dim \mathbf{A}_{2r-1}$. Let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{A}_{2r-3}$ with simple root system

$\{\alpha_2, a_3, \ldots, \alpha_{2r-2}\}$. By induction if $r>5$, and as just seen if $r=5$, we have $\frac{1}{2}(\deg \tau_{\xi_r})^2 \geqslant \frac{1}{2}\deg \tau_{\xi_r} + \deg \tau_{2\xi_r-\alpha_r} + \dim \mathbf{A}_{2r-3}$. Now define multipliers by

$$\deg \pi_{\xi_r} = x \cdot \deg \tau_{\xi_r}, \quad \deg \pi_{2\xi_r-\alpha_r} = y \cdot \deg \tau_{2\xi_r-\alpha_r} \quad \text{and} \quad \dim \mathbf{A}_{2r-1} = z \cdot \dim \mathbf{A}_{2r-3}.$$

We compute $$x = \left\{\prod_{a_1>0=a_n} \frac{a_r + l(\alpha)}{l(\alpha)}\right\}\left\{\prod_{a_n>0=a_1} \frac{a_r + l(\alpha)}{l(\alpha)}\right\} \frac{n+1}{n} = \frac{4r-2}{r}.$$

To calculate y we note $2\xi_r - \alpha_r = \xi_{r-1} + \xi_{r+1}$. The roots involving α_1 and just one of $\{\alpha_{r-1}, \alpha_{r+1}\}$ are $\alpha_1 + \ldots + \alpha_{r-1}$ and $\alpha_1 + \ldots + \alpha_r$; those involving α_n and just one of $\{\alpha_{r-1}, \alpha_{r+1}\}$ are $\alpha_r + \ldots + \alpha_n$ and $\alpha_{r+1} + \ldots + \alpha_n$.

$$y = \left\{\frac{r}{r-1} \cdot \frac{r+1}{r}\right\}^2 \cdot \left\{\prod_{l=r+1}^{2r-2} \frac{l+2}{l}\right\}^2 \cdot \frac{2r+1}{2r-1} = \frac{4r^2(2r-1)(2r+1)}{(r-1)^2(r+2)^2}.$$

Observe (check for $r=5$ and differentiate; iterate three times) that $(2r-1)(r-1)^2(r+2)^2 - r^4(2r+1) = 2r^4 - 8r^3 - 5r^2 + 12r - 4$ is positive for $r \geqslant 5$. It follows that $x^2 > y$. And $x>1$ gives $x^2 > x > 1$. Finally notice that $z = (4r^2-1)/(4r^2-8r+3) < x^2$. Now $\frac{1}{2}(\deg \pi_{\xi_r})^2 - \frac{1}{2}\deg \pi_{\xi_r} - \deg \pi_{2\xi_r-\alpha_r} - \dim \mathbf{A}_{2r-1} > x^2\{\frac{1}{2}(\deg \tau_{\xi_r})^2 - \frac{1}{2}\deg \tau_{\xi_r} - \deg \tau_{2\xi_r-\alpha_r} - \dim \mathbf{A}_{2r-3}\} \geqslant 0$, which proves our assertion. Proposition 9.1 shows that we may replace ξ_r by any multiple and retain the inequality. Thus we cannot have $r>4$, q.e.d.

10.5. Lemma. *If K is of type B_n, $n \geqslant 2$, then*

(1) *π_λ is the adjoint representation, given by λ: $\circ\!=\!\overset{2}{\bullet}$ for $n=2$ and by λ: $\circ\!-\!\overset{1}{\circ}\!-\!\ldots\!-\!\circ\!=\!\bullet$ for $n>2$, or*

(2) *π_λ has degree $2n^2+3n$, given by λ: $\overset{2}{\circ}\!-\!\circ\!-\!\ldots\!-\!\circ\!=\!\bullet$, or*

(3) *$n=4$ and π_λ is the spin representation, λ: $\circ\!-\!\circ\!-\!\circ\!=\!\overset{1}{\bullet}$.*

Proof. We first examine $\mathbf{B}_2$: $\overset{\alpha_1}{\bullet}\!=\!\overset{\alpha_2}{\circ}$. If $\lambda = k\xi_1$ then $k=2$ because π_{ξ_1} is symplectic; we check $\frac{1}{2}(\deg \pi_{2\xi_1})^2 = 50 = 5 + 35 + 10 = \frac{1}{2}(\deg \pi_{2\xi_1}) + \deg \pi_{4\xi_1-\alpha_1} + \dim \mathbf{B}_2$. If $\lambda = k\xi_2$ then $k=2$ because $\pi_{\xi_2}(\mathbf{B}_2) = \mathbf{SO}(5)$; we check $\frac{1}{2}(\deg \pi_{2\xi_2})^2 = 98 = 7 + 81 + 10 = \frac{1}{2}\deg \pi_{2\xi_2} + \deg \pi_{4\xi_2-\alpha_2} + \dim \mathbf{B}_2$. Our assertions are proved for $n=2$.

Now assume $n>2$. We number the simple roots by $\overset{\alpha_1}{\bullet}\!=\!\overset{\alpha_2}{\circ}\!-\!\ldots\!-\!\overset{\alpha_n}{\circ}$. Suppose $\lambda = \xi_1$. Then $\deg \pi_\lambda = 2^n$ and $\deg \pi_{2\lambda-\alpha_1} = \deg \pi_{\xi_2} = \binom{2n+1}{n-1}$, so (10.3) says $\frac{1}{2}\{2^{2n} - 2^n\} - \left\{\binom{2n+1}{n-1} + 2n^2 + n\right\} = 0$. The only solution is $n=4$, giving case (3) of the lemma. Now suppose $\lambda = 2\xi_1$. Let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{B}_{n-1}$ with simple root system $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. Define multipliers by

$$\deg \pi_{2\xi_1} = x \cdot \deg \tau_{2\xi_1}, \quad \deg \pi_{4\xi_1-\alpha_1} = y \cdot \deg \tau_{4\xi_1-\alpha_1} \quad \text{and} \quad \dim \mathbf{B}_n = z \cdot \dim \mathbf{B}_{n-1}.$$

The roots involving α_1 and α_n are $\{\alpha_1 + \ldots + \alpha_n,\ 2\alpha_1 + \alpha_2 + \ldots + \alpha_n,\ \ldots,\ 2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n\}$. And $\alpha_1 + \ldots + \alpha_{n-1}$ is the only root involving α_1 and α_{n-1} but not α_n. Thus

$$x = \frac{2n+1}{2n-1}\left\{\frac{2n+4}{2n}\,\frac{2n+6}{2n+2}\cdots\frac{4n}{4n-4}\right\} = 2\,\frac{2n+1}{n+1};$$

$$y = \frac{2n}{2n-2}\cdot\frac{2n+3}{2n-1}\cdot\frac{2n+6}{2n}\left\{\frac{2n+10}{2n+2}\cdot\frac{2n+12}{2n+4}\cdot\ldots\cdot\frac{4n+4}{4n-4}\right\} = 4\,\frac{n(2n+1)(2n+3)}{(n-1)(n+2)(n+4)} \quad \text{if } n>5;$$

$$y = \frac{715}{63} \text{ if } n=5, \quad y = 11 \text{ if } n=4, \quad y = \frac{54}{5} \text{ if } n=3.$$

If $n>5$ then $(n-1)(n+4)>(n+1)^2$ and $(n+2)(2n+1)>n(2n+3)$, implying $x^2>y$. If $n=5$ then $x^2=121/9>715/63=y$; if $n=4$ then $x^2=324/25>11=y$; if $n=3$ then $x^2=49/4>54/5=y$. Thus we have $x^2>y$. Note that $x>1$ so $x^2>x>1$ and that $z=(2n^2+n)/(2n^2-3n+1)<3<x<x^2$. By induction if $n>3$, and as we proved if $n=2$, $\frac{1}{2}(\deg \tau_{2\xi_1})^2 \geqslant \frac{1}{2}\deg \tau_{2\xi_1} + \deg \tau_{4\xi_1-\alpha_1} + \dim \mathbf{B}_{n-1}$. Now $\frac{1}{2}(\deg \pi_{2\xi_1})^2 - \{\frac{1}{2}\deg \pi_{2\xi_1} + \deg \pi_{4\xi_1-\alpha_1} + \dim \mathbf{B}_n\} > x^2\{\frac{1}{2}(\deg \tau_{2\xi_1})^2 - \frac{1}{2}\deg \tau_{2\xi_1} - \deg \tau_{4\xi_1-\alpha_1} - \dim \mathbf{B}_{n-1}\} \geqslant 0$, violating (10.3). Thus $\lambda \neq 2\xi_1$ for $n>2$.

Suppose $\lambda = k\xi_n$. Then $k=2$ because $\pi_{\xi_n}(\mathbf{B}_n) = \mathbf{SO}(2n+1)$. Now we compute $\deg \pi_{2\xi_n} = 2n^2+3n$ and $\deg \pi_{4\xi_n-\alpha_n} = \frac{1}{2}n(2n-1)(n+1)(2n+5)$. Thus $\frac{1}{2}(\deg \pi_{2\xi_n})^2 = \frac{1}{2}(4n^4+12n^3+9n^2) = \frac{1}{2}(2n^2+3n) + \frac{1}{2}(4n^4+12n^3+3n^2-5n) + \frac{1}{2}(4n^2+2n) = \frac{1}{2}\deg \pi_{2\xi_n} + \deg \pi_{4\xi_n-\alpha_n} + \dim \mathbf{B}_n$.

Suppose $\lambda = k\xi_{n-1}$. Then $k=1$ because $\pi_{\xi_{n-1}}(\mathbf{B}_n) \subsetneqq \mathbf{SO}(2n^2+n)$, so π_λ is the adjoint representation $\pi_{\xi_{n-1}}$. Calculating separately for $n=3$ $(2\xi_{n-1}-\alpha_{n-1}$: $\overset{1}{\circ}-\circ=\overset{2}{\bullet})$ and $n>3$ $(2\xi_{n-1}-\alpha_{n-1}$: $\circ-\overset{1}{\circ}-\overset{1}{\circ}-\ldots-\circ=\bullet)$, $\deg \pi_{2\xi_{n-1}-\alpha_{n-1}} = \frac{1}{2}(n-1)(2n+3)n(2n+1)$. Thus $\frac{1}{2}(\deg \pi_{\xi_{n-1}})^2 = \frac{1}{2}(4n^4+4n^3+n^2) = \frac{1}{2}(2n^2+n) + \frac{1}{2}(4n^4+4n^3-5n^2-3n) + \frac{1}{2}(4n^2+2n) = \frac{1}{2}\deg \pi_{\xi_{n-1}} + \deg \pi_{2\xi_{n-1}-\alpha_{n-1}} + \dim \mathbf{B}_n$.

We summarize the last four paragraphs as follows: Lemma 10.5 gives precisely those cases for which $\lambda = k\xi_r$ with $r=1$, $n-1$, or n. Thus we need only show that $2 \leqslant r \leqslant n-2$ violates (10.3).

Suppose $r=2 \leqslant n-2$. Then $\lambda = \xi_2$. We retain the notation that τ_ν denotes the representation of highest weight ν for the $\mathbf{B}_{n-1}$ with simple root system $\{\alpha_1, \ldots, \alpha_{n-1}\}$. If $n=4$ we check $\frac{1}{2}(\deg \pi_{\xi_2})^2 = 3528 > 42 + 2772 + 36 = \frac{1}{2}(\deg \pi_{\xi_2}) + \deg \pi_{2\xi_2-\alpha_2} + \dim \mathbf{B}_4$. Now assume $n>4$; by induction we have $\frac{1}{2}(\deg \tau_{\xi_2})^2 - \{\frac{1}{2}\deg \tau_{\xi_2} + \deg \tau_{2\xi_2-\alpha_2} + \dim \mathbf{B}_{n-1}\} > 0$. Define multipliers by

$$\deg \pi_{\xi_2} = u \cdot \deg \tau_{\xi_2} \quad \text{and} \quad \deg \pi_{2\xi_2-\alpha_2} = v \cdot \deg \tau_{2\xi_2-\alpha_2}.$$

Then we calculate

$$u=\frac{2n(2n+1)}{(n+2)(n-1)} \quad \text{and} \quad v=\frac{4n(2n+1)(2n+3)}{(n-2)(n+2)(n+5)}<u^2.$$

As $u^2>u>(\dim \mathbf{B}_n)/(\dim \mathbf{B}_{n-1})>0$, we now have

$$\tfrac{1}{2}(\deg \pi_{\xi_2})^2-\{\tfrac{1}{2}\deg \pi_{\xi_2}+\deg \pi_{2\xi_2-\alpha_2}+\dim \mathbf{B}_n\}$$
$$>u^2[\tfrac{1}{2}(\deg \tau_{\xi_2})^2-\{\tfrac{1}{2}\deg \tau_{\xi_2}+\deg \tau_{2\xi_2-\alpha_2}+\dim \mathbf{B}_{n-1}\}]>0,$$

violating (10.3). Thus $r\neq 2$ for $n\geqslant 4$. Now we need only show that $3\leqslant r\leqslant n-2$ violates (10.3).

Suppose $3\leqslant r\leqslant n-2$. Then $n\geqslant 5$ and $\lambda=\xi_r$. As before we define multipliers by

$$\deg \pi_{\xi_r}=s\cdot\deg \tau_{\xi_r} \quad \text{and} \quad \deg \pi_{2\xi_r-\alpha_r}=t\cdot\deg \tau_{2\xi_r-\alpha_r}.$$

Now we compute

$$s=\frac{\binom{2n+1}{n-r+1}}{\binom{2n-1}{n-r}}=\frac{(n+r+1)\dots(2n+1)}{(n-r+1)!}\cdot\frac{(n-r)!}{(n+r)\dots(2n-1)}=\frac{2n(2n+1)}{(n+r)(n-r+1)} \quad \text{and}$$

$$t=\frac{2n(2n+1)(2n+2)(2n+3)}{(n-r)(n+r)(n-r+3)(n+r+3)}<s^2.$$

If $r<n-2$ we use induction, and if $r=n-2$ we use our verification for the adjoint representation of $\mathbf{B}_{n-1}$, concluding that $\tfrac{1}{2}(\deg \pi_{\xi_r})^2-\{\tfrac{1}{2}\deg \pi_{\xi_r}+\deg \pi_{2\xi_r-\alpha_r}+\dim \mathbf{B}_n\}>s^2[\tfrac{1}{2}(\deg \tau_{\xi_r})^2-\{\tfrac{1}{2}\deg \tau_{\xi_r}+\deg \tau_{2\xi_r-\alpha_r}+\dim \mathbf{B}_{n-1}\}]>0$. This violates (10.3). Thus we cannot have $3\leqslant r\leqslant n-2$, q.e.d.

10.6 Lemma. *If K is of type C_n, $n\geqslant 3$, then*

(1) *π_λ is the adjoint representation, given by λ:* $\overset{2}{\bullet}-\bullet-\dots-\bullet\!=\!\circ$, *or*

(2) *π_λ has degree $2n^2-n-1$, given by λ:* $\bullet-\overset{1}{\bullet}-\dots-\bullet\!=\!\circ$, *or*

(3) *$n=4$ and λ:* $\bullet-\bullet-\bullet\!=\!\overset{1}{\circ}$.

Remark. Here the similarity between type B_n and C_n is striking. It suggests the possibility of a cohomological treatment of our results on representations.

Proof. We number the simple roots $\overset{\alpha_1}{\circ}\!=\!\overset{\alpha_2}{\bullet}-\dots-\overset{\alpha_{n-1}}{\bullet}-\overset{\alpha_n}{\bullet}$. Suppose $\lambda=k\xi_n$. Then $k=2$ because π_{ξ_n} is symplectic, so $\lambda=2\xi_n$ and π_λ is the adjoint representation. We check $\tfrac{1}{2}(\deg \pi_{2\xi_n})^2=\tfrac{1}{2}(4n^4+4n^3+n^2)=\tfrac{1}{2}(2n^2+n)+\tfrac{1}{2}(4n^4+4n^3-5n^2-3n)+\tfrac{1}{2}(4n^2+2n)=\tfrac{1}{2}\deg \pi_{2\xi_n}+\deg \pi_{4\xi_n-\alpha_n}+\dim \mathbf{C}_n$.

Suppose $\lambda = k\xi_{n-1}$. Then $k=1$ because $\pi_{\xi_{n-1}}(\mathbf{C}_n) \subsetneqq \mathbf{SO}(2n^2-n-1) = \mathbf{SO}(\deg \pi_{\xi_{n-1}})$. Now we check $\frac{1}{2}(\deg \pi_{\xi_{n-1}})^2 = \frac{1}{2}(4n^4-4n^3-3n^2+2n+1) = \frac{1}{2}(2n^2-n-1)+\frac{1}{2}(4n^4-4n^3-9n^2+n+2) + \frac{1}{2}(4n^2+2n) = \frac{1}{2}\deg \pi_{\xi_{n-1}} + \deg \pi_{2\xi_{n-1}-\alpha_{n-1}} + \dim \mathbf{C}_n$.

Let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{C}_{n-1}$ with simple root system $\{\alpha_1, \ldots, \alpha_{n-1}\}$. If $n=4$ we compute $\frac{1}{2}(\deg \pi_{\xi_1})^2 = 882 = 21+825+36 = \frac{1}{2}\deg \pi_{\xi_1} + \deg \pi_{2\xi_1-\alpha_1} + \dim \mathbf{C}_4$; this gives case (3) of the lemma. Now let $n>4$. As just seen for $n=5$, and by induction for $n>5$, we have $\frac{1}{2}(\deg \tau_{\xi_1})^2 \geqslant \frac{1}{2}\deg \tau_{\xi_1} + \deg \tau_{2\xi_1-\alpha_1} + \dim \mathbf{C}_{n-1}$. Define multipliers by

$$\deg \pi_{\xi_1} = a\cdot \deg \tau_{\xi_1} \quad \text{and} \quad \deg \pi_{2\xi_1-\alpha_1} = b\cdot \deg \tau_{2\xi_1-\alpha_1}.$$

We compute $a = 2\,\dfrac{2n+1}{n+2}$ and $b = 4\,\dfrac{(n+1)(2n+1)(2n+3)}{(n-1)(n+4)(n+5)}$. One checks $a^2>b$ for $n\geqslant 5$. As before, it follows that $\frac{1}{2}(\deg \pi_{\xi_1})^2 > \frac{1}{2}\deg \pi_{\xi_1} + \deg \pi_{2\xi_1-\alpha_1} + \dim \mathbf{C}_n$. If $\lambda = k\xi_1$ then this shows $k=1$ by case (10) for Ψ in the proof of Proposition 9.1, and the latter violates (10.3). Thus $\lambda \neq k\xi_1$ for $n\geqslant 5$.

Let $\lambda = k\xi_1$. As just seen, this implies $n\leqslant 4$, so n is 3 or 4. If $n=3$ then $k=2$ because $\overset{1}{\circ}{=}\bullet{-}\bullet$ is symplectic, and we calculate $\frac{1}{2}(\deg \pi_{2\xi_1})^2 = 3528 > 42+1638+21 = \frac{1}{2}(\deg \pi_{2\xi_1}) + \deg \pi_{4\xi_1-\alpha_1} + \dim \mathbf{C}_3$; thus $n=4$. If $n=4$ then $k=1$ because $\overset{1}{\circ}{=}\bullet{-}\bullet{-}\bullet$ is orthogonal, and we have already checked (10.3) for $\lambda=\xi_1$ with $n=4$.

The proof of Lemma 10.6 is now reduced to showing that we cannot have $2\leqslant r\leqslant n-2$.

Let $r=2\leqslant n-2$ and define multipliers by

$$\deg \pi_{\xi_2} = c\cdot \deg \tau_{\xi_2} \quad \text{and} \quad \deg \pi_{2\xi_2-\alpha_2} = d\cdot \deg \tau_{2\xi_2-\alpha_2}.$$

We compute $c = \dfrac{2n(2n+1)}{(n-1)(n+3)}$. If $n\geqslant 6$ we calculate $d = 4\,\dfrac{n(2n+1)(2n+3)}{(n-2)(n+3)(n+6)} < c^2$; if $n=5$ then $d = 195/18 < 3025/256 = c^2$; if $n=4$ then $d = 396/35 < 576/49 = c^2$; now $c^2>d$ for $n\geqslant 4$. And trivially $c^2 > c > (\dim \mathbf{C}_n)/(\dim \mathbf{C}_{n-1}) > 0$. As has been checked when $n=4$, and by induction if $n>4$, $\frac{1}{2}(\deg \tau_{\xi_2})^2 \geqslant \frac{1}{2}\deg \tau_{\xi_2} + \deg \tau_{2\xi_2-\alpha_2} + \dim \mathbf{C}_{n-1}$; it follows that $\frac{1}{2}(\deg \pi_{\xi_2})^2 > \frac{1}{2}\deg \pi_{\xi_2} + \deg \pi_{2\xi_2-\alpha_2} + \dim \mathbf{C}_n$. Now Proposition 9.1 shows that $\lambda = k\xi_2$, $n\geqslant 4$, would contradict (10.3). The proof of Lemma 10.6 is thus reduced to showing that we cannot have $3\leqslant r\leqslant n-2$. As before, we define multipliers by

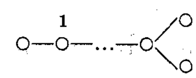
$$\deg \pi_{\xi_r} = u\cdot \deg \tau_{\xi_r} \quad \text{and} \quad \deg \pi_{2\xi_r-\alpha_r} = v\cdot \deg \tau_{2\xi_r-\alpha_r};$$

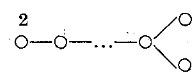
we compute

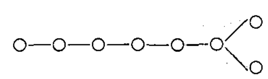
$$u = \frac{2n(2n+1)}{(n-r+1)(n+r+2)} \quad \text{and} \quad v = \frac{2n(2n+1)(2n+2)(2n+3)}{(n-r)(n-r+3)(n+r+2)(n+r+5)}$$

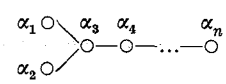
and we check $u^2 > v$ and $u^2 > u > (\dim \mathbf{C}_n)/(\dim \mathbf{C}_{n-1}) > 1$. As we saw for $r = n-2$, and by induction on $n-r$ if $r < n-2$, we have $\frac{1}{2}(\deg \tau_{\xi_r})^2 \geqslant \frac{1}{2}\deg \tau_{\xi_r} + \deg \tau_{2\xi_r - \alpha_r} + \dim \mathbf{C}_{n-1}$. It follows that $\lambda = \xi_r$ violates (10.3). Now Proposition 9.1 says $\lambda = \xi_r$ if $\lambda = k\xi_r$. Thus we cannot have $3 \leqslant r \leqslant n-2$, q.e.d.

10.7 LEMMA. *If K is of type D_n, $n \geqslant 4$, then*

(1) *π_λ is the adjoint representation, given by λ:*  , *or*

(2) *π_λ has degree $2n^2+n+1$, given by λ:*  , *or*

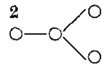
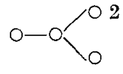
(3) *$n = 8$ and π_λ is the half spin representation given by λ:* 

Proof. We label the simple roots by  and let τ_ν denote the representation of highest weight ν for the subgroup $\mathbf{D}_{n-1}$ with simple root system $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$.

If $\lambda = k\xi_n$ then $k = 2$ because $\pi_{\xi_n}(\mathbf{D}_n) = \mathbf{SO}(2n)$. And we check $\frac{1}{2}(\deg \pi_{2\xi_n})^2 = \frac{1}{2}(4n^4 + 4n^3 - 3n^2 - 2n + 1) = \frac{1}{2}(2n^2+n-1) + \frac{1}{2}(4n^4+4n^3-9n^2-n+2) + \frac{1}{2}(4n^2-2n) = \frac{1}{2}\deg \pi_{2\xi_n} + \deg \pi_{4\xi_n - \alpha_n} + \dim \mathbf{D}_n$.

If $\lambda = k\xi_{n-1}$ then $k = 1$ because $\pi_{\xi_{n-1}}(\mathbf{D}_n) \subsetneqq \mathbf{SO}(2n^2-n)$. Then $\pi_\lambda = \pi_{\xi_{n-1}}$, adjoint representation, and we check $\frac{1}{2}(\deg \pi_\lambda)^2 = \frac{1}{2}(4n^4 - 4n^3 + n^2) = \frac{1}{2}(2n^2-n) + \frac{1}{2}(4n^4 - 4n^3 - 5n^2 + 3n) + \frac{1}{2}(4n^2-2n) = \frac{1}{2}\deg \pi_\lambda + \deg \pi_{2\lambda - \alpha_{n-1}} + \dim \mathbf{D}_n$.

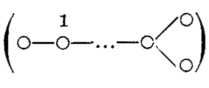
If $\lambda = \xi_1$ then (10.3) says $2^{2n-3} = 2^{n-2} + \binom{2n}{n-2} + (2n^2 - n)$; $n = 8$ is the only solution, and π_{ξ_1} *is* orthogonal for $n = 8$; this gives case (3) of the lemma.

Define multipliers by $\deg \pi_{2\xi_1} = p \cdot \deg \tau_{2\xi_1}$ and $\deg \pi_{4\xi_1 - \alpha_1} = q \cdot \deg \tau_{4\xi_1 - \alpha_1}$. Then $p = 2\,\dfrac{2n-1}{n}$, and $q = 4\,\dfrac{n(2n-1)(2n+1)}{(n-2)(n+1)(n+4)}$ for $n \geqslant 5$. So $p^2 > q$ for $n \geqslant 5$. As $p^2 > p > (\dim \mathbf{D}_n)/(\dim \mathbf{D}_{n-1}) > 1$, we have $\frac{1}{2}(\deg \pi_{2\xi_1})^2 - \{\frac{1}{2}\deg \pi_{2\xi_1} + \deg \pi_{4\xi_1 - \alpha_1} + \dim \mathbf{D}_n\} > p^2[\frac{1}{2}(\deg \tau_{2\xi_1})^2 - \{\frac{1}{2}\deg \tau_{2\xi_1} + \deg \tau_{4\xi_1 - \alpha_1} + \dim \mathbf{D}_{n-1}\}] \geqslant 0$, the last inequality having checked on  $\cong$  for $n = 5$, and being the induction hypothesis of $n > 5$. Now (10.3) shows $\lambda \neq 2\xi_1$ if $n > 4$.

We have just seen that $\lambda = k\xi_1$ implies $n = 8$ and $k = 1$; or $n = 4$ and $k = 2$; the latter is included in case (2) of the lemma. If $\lambda = k\xi_2$ we change notation, coming back to the case $\lambda = k\xi_1$. If $\lambda = k\xi_r$, $3 \leqslant r \leqslant n-2$, then $k = 1$ because $\pi_{\xi_r} = \Lambda^{n-r+1}(\pi_{\xi_n})$ maps $\mathbf{D}_n$ onto a proper subgroup of $\mathbf{SO}(\deg \pi_{\xi_r})$. Now we need only check that $\lambda \neq \xi_r$ whenever $3 \leqslant r \leqslant n-2$.

Let $\lambda=\xi_r$, $3\leqslant r\leqslant n-2$. Define multipliers by $\deg\pi_{\xi_r}=u\cdot\deg\tau_{\xi_r}$ and $\deg\pi_{2\xi_r-\alpha_r}=v\cdot\deg\tau_{2\xi_r-\alpha_r}$. Then, calculating separately for $r=3$ and $r>3$,

$$u=\frac{2n(2n-1)}{(n+r-1)(n-r+1)}\quad\text{and}\quad v=\frac{4n(n+1)(2n-1)(2n+1)}{(n-r)(n-r+3)(n+r-1)(n+r+2)}.$$

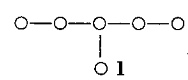
It follows that $u^2>v$. Now $\frac{1}{2}(\deg\pi_{\xi_r})^2-\{\frac{1}{2}\deg\pi_{\xi_r}+\deg\pi_{2\xi_r-\alpha_r}+\dim\mathbf{D}_n\}>u^2\,[\frac{1}{2}(\deg\tau_{\xi_r})^2-\{\frac{1}{2}\deg\tau_{\xi_r}+\deg\tau_{2\xi_r-\alpha_r}+\dim\mathbf{D}_{n-1}\}]\geqslant 0$, where the last inequality was checked  for $n-r=2$ and is the induction hypothesis if $n-r>2$. This contradicts (10.3). Thus $\lambda\neq\xi_r$ for $3\leqslant r\leqslant n-2$, q.e.d.

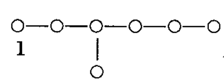
We finally come to the easy case.

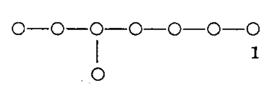
10.8. Lemma. *If K is an exceptional group, then*

(1) $K=\mathbf{G}_2$ *and* λ: $\bullet\!\equiv\!\overset{1}{\circ}$ *or* λ: $\overset{1}{\bullet}\!\equiv\!\circ$; *or*

(2) $K=\mathbf{F}_4$ *and* λ: $\bullet\!-\!\bullet\!=\!\circ\!-\!\overset{1}{\circ}$ *or* λ: $\overset{1}{\bullet}\!-\!\bullet\!=\!\circ\!-\!\circ$; *or*

(3) $K=\mathbf{E}_6$ *and* λ:  ; *or*

(4) $K=\mathbf{E}_7$ *and* λ:  ; *or*

(5) $K=\mathbf{E}_8$ *and* λ: .

In each case, the first-mentioned possibility for λ is the case where π_λ is the adjoint representation.

Proof. Let $K=\mathbf{G}_2$: $\overset{\alpha_1}{\bullet}\!\equiv\!\overset{\alpha_2}{\circ}$. We compute $\frac{1}{2}(\deg\pi_{\xi_2})^2=98=7+77+14=\frac{1}{2}\deg\pi_{\xi_2}+\deg\pi_{2\xi_2-\alpha_2}+\dim\mathbf{G}_2$. Thus $\lambda=\xi_2$ is admissible. $\lambda=\xi_1$ is case (*c*) of Proposition 5.2, and

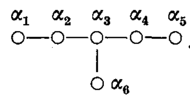
$$\tfrac{1}{2}(\deg\pi_{2\xi_1})^2=\tfrac{1}{2}(729)>\tfrac{1}{2}(27)+189+14=\tfrac{1}{2}\deg\pi_{2\xi_1}+\deg\pi_{4\xi_1-\alpha_1}+\dim\mathbf{G}_2.$$

Now λ is ξ_1 or ξ_2 by Proposition 9.1.

Let $K=\mathbf{F}_4$: $\overset{\alpha_4}{\bullet}\!-\!\overset{\alpha_3}{\bullet}\!=\!\overset{\alpha_2}{\circ}\!-\!\overset{\alpha_1}{\circ}$. We compute

$$\tfrac{1}{2}(\deg\pi_{\xi_1})^2=1352=26+1274+52=\tfrac{1}{2}\deg\pi_{\xi_1}+\deg\pi_{2\xi_1-\alpha_1}+\dim\mathbf{F}_4,$$
$$\tfrac{1}{2}(\deg\pi_{\xi_2})^2=831\,538>637+420\,147+52=\tfrac{1}{2}\deg\pi_{\xi_2}+\deg\pi_{2\xi_2-\alpha_2}+\dim\mathbf{F}_4,$$
$$\tfrac{1}{2}(\deg\pi_{\xi_3})^2=\tfrac{1}{2}(74\,529)>\tfrac{1}{2}(273)+19\,278+52=\tfrac{1}{2}\deg\pi_{\xi_3}+\deg\pi_{2\xi_3-\alpha_3}+\dim\mathbf{F}_4,$$
$$\tfrac{1}{2}(\deg\pi_{\xi_4})^2=338=13+273+52=\tfrac{1}{2}\deg\pi_{\xi_4}+\deg\pi_{2\xi_4-\alpha_4}+\dim\mathbf{F}_4.$$

Now Proposition 9.1 says that λ is ξ_1 or ξ_4.

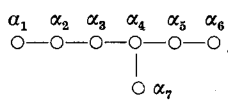
Let $K = \mathbf{E}_6$: 

. Then $\lambda = k\xi_r$, $\xi_r = \xi_r^*$, says $r=3$ or $r=6$. We calculate

$$\tfrac{1}{2}(\deg \pi_{\xi_3})^2 - \tfrac{1}{2}\deg \pi_{\xi_3} = 4\,276\,350 > 2\,453\,814 + 78 = \deg \pi_{2\xi_3-\alpha_3} + \dim \mathbf{E}_6$$

$$\tfrac{1}{2}(\deg \pi_{\xi_6})^2 - \tfrac{1}{2}\deg \pi_{\xi_6} = 3003 = 2925 + 78 = \deg \pi_{2\xi_6-\alpha_6} + \dim \mathbf{E}_6.$$

Now Proposition 9.1 says $\lambda = \xi_6$.

Let $K = \mathbf{E}_7$: 

. Then π_{ξ_1}, π_{ξ_3} and π_{ξ_7} are symplectic while π_{ξ_2}, π_{ξ_4}, π_{ξ_5} and π_{ξ_6} are orthogonal. We compute

$$\tfrac{1}{2}(\deg \pi_{2\xi_1})^2 - \tfrac{1}{2}\deg \pi_{2\xi_1} = 1\,069\,453 > 915\,705 + 133 = \deg \pi_{4\xi_1-\alpha_1} + \dim \mathbf{E}_7,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_2})^2 - \tfrac{1}{2}\deg \pi_{\xi_2} = 1\,183\,491 > 980\,343 + 133 = \deg \pi_{2\xi_2-\alpha_2} + \dim \mathbf{E}_7,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_3})^2 - \tfrac{1}{2}\deg \pi_{\xi_3} = 382\,634\,616 > 209\,868\,813 + 133 = \deg \pi_{2\xi_3-\alpha_3} + \dim \mathbf{E}_7,$$

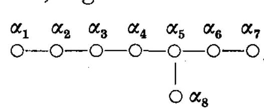
$$\tfrac{1}{2}(\deg \pi_{\xi_4})^2 - \tfrac{1}{2}\deg \pi_{\xi_4} = 66\,886\,348\,375 > 19\,903\,763\,880 + 133 = \deg \pi_{2\xi_4-\alpha_4} + \dim \mathbf{E}_7,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_5})^2 - \tfrac{1}{2}\deg \pi_{\xi_5} = 37\,363\,690 > 24\,386\,670 + 133 = \deg \pi_{2\xi_5-\alpha_5} + \dim \mathbf{E}_7,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_6})^2 - \tfrac{1}{2}\deg \pi_{\xi_6} = 8778 = 8645 + 133 = \deg \pi_{2\xi_6-\alpha_6} + \dim \mathbf{E}_7,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_7})^2 - \tfrac{1}{2}\deg \pi_{\xi_7} = 415\,416 > 365\,750 + 133 = \deg \pi_{2\xi_7-\alpha_7} + \dim \mathbf{E}_7.$$

Now Proposition 9.1, together with case (7) of Ψ' in its proof, shows that $\lambda = \xi_6$.

Let $K = \mathbf{E}_8$: 

. Then we calculate [1]

$$\deg \pi_{\xi_1} = 248; \quad \deg \pi_{\xi_2} = 30\,380; \quad \deg \pi_{\xi_3} = 2\,450\,240; \quad \deg \pi_{\xi_4} = 146\,325\,270;$$

$$\deg \pi_{\xi_5} = 6\,899\,079\,264; \quad \deg \pi_{\xi_6} = 6\,696\,000; \quad \deg \pi_{\xi_7} = 3875; \quad \deg \pi_{\xi_8} = 147\,250.$$

Now we compute

$$\tfrac{1}{2}(\deg \pi_{\xi_1})^2 - \tfrac{1}{2}\deg \pi_{\xi_1} = 30\,628 = 30\,380 + 248 = \deg \pi_{2\xi_1-\alpha_1} + \dim \mathbf{E}_8,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_2})^2 - \tfrac{1}{2}\deg \pi_{\xi_2} = 461\,457\,010 > 344\,452\,500 + 248 = \deg \pi_{2\xi_2-\alpha_2} + \dim \mathbf{E}_8$$

$$\tfrac{1}{2}(\deg \pi_{\xi_3})^2 - \tfrac{1}{2}\deg \pi_{\xi_3} = 3\,001\,836\,803\,680 > 1\,283\,242\,632\,840 + 248 = \deg \pi_{2\xi_3-\alpha_3} + \dim \mathbf{E}_8,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_4})^2 - \tfrac{1}{2}\deg \pi_{\xi_4} = 10\,705\,542\,247\,123\,815 > 2\,118\,568\,836\,696\,000 + 248 = \deg \pi_{2\xi_4-\alpha_4} + \dim \mathbf{E}_8,$$

(1) Note that this does not agree with Dynkin's table 30 in [6], which is incorrect for $\mathbf{E}_8$.

$$\tfrac{1}{2}(\deg \pi_{\xi_5})^2 - \tfrac{1}{2}\deg \pi_{\xi_5} = 23\,798\,647\,342\,027\,851\,216$$
$$> 1\,704\,723\,757\,359\,480\,000 + 248 = \deg \pi_{2\xi_5-\alpha_5} + \dim \mathbf{E}_8,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_6})^2 - \tfrac{1}{2}\deg \pi_{\xi_6} = 22\,418\,204\,652\,000$$
$$> 7\,723\,951\,192\,125 + 248 = \deg \pi_{2\xi_6-\alpha_6} + \dim \mathbf{E}_8,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_7})^2 - \tfrac{1}{2}\deg \pi_{\xi_7} = 7\,505\,875 > 6\,696\,000 + 248 = \deg \pi_{2\xi_7-\alpha_7} + \dim \mathbf{E}_8,$$

$$\tfrac{1}{2}(\deg \pi_{\xi_8})^2 - \tfrac{1}{2}\deg \pi_{\xi_8} = 10\,841\,207\,625 > 6\,899\,079\,264 + 248 = \deg \pi_{2\xi_8-\alpha_8} + \dim \mathbf{E}_8.$$

Now Proposition 9.1 says $\lambda = \xi_1$, q.e.d.

Theorem 10.1 now follows from Lemmas 10.4, 10.5, 10.6, 10.7 and 10.8.

11. Summary and global formulation

We summarize and reformulate the results of Chapter I as follows.

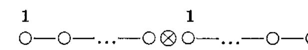
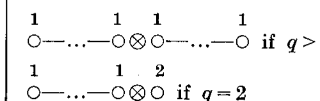
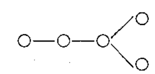
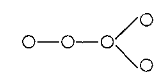
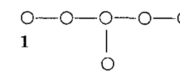
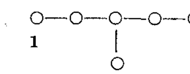
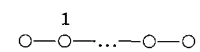
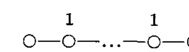
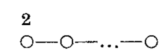
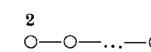
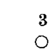
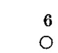
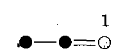
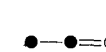
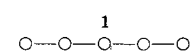
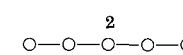
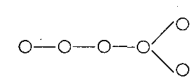
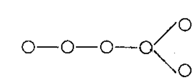
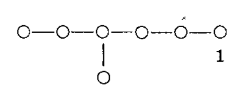
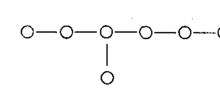
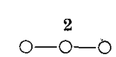
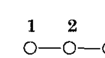
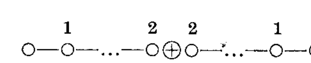
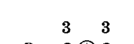
11.1 THEOREM. *The table on pages 107–110 gives a complete list of the nonsymmetric simply connected coset spaces $M = G/K$, where* (1) *G is a connected Lie group acting effectively on M and* (2) *K is a compact subgroup whose linear isotropy action χ on the tangent space of M is irreducible over the real number field.* (Explanation of table: *π denotes the inclusion $K \to G$; if G is locally isomorphic to a classical linear group, then π is listed as a linear representation of $\mathfrak{K}$; if G is exceptional, then π is given as a linear representation α of $\mathfrak{G}$ and a linear representation β of $\mathfrak{K}$ such that $\beta(\mathfrak{K}) \subset \alpha(\mathfrak{G})$. The center of G is denoted Z, and $N_G(K)$ is the normalizer of K in G.*)

Let $M' = G'/K'$ be an effective coset space of a connected Lie group by a compact subgroup, where the identity component K_0' acts irreducibly on the tangent space of M'. Then there is an entry $M = G/K$ in the table, a central subgroup $Q \subset Z$ of G, and a subgroup $K'' \subset N_G(K)$ with $K = K_0'' = N_G(K)_0$, such that $G' = G/Q$ and $K' = (Q \cdot K'')/Q$.

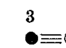
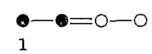
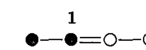
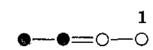
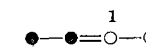
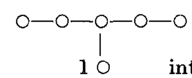
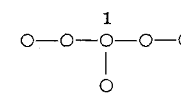
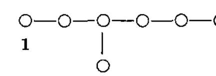
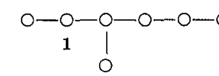
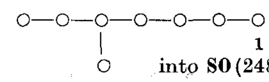
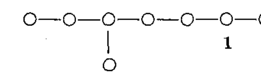
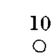
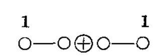
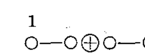
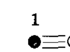
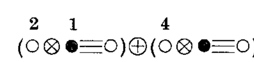
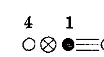
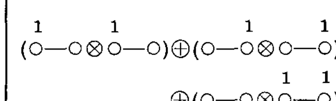
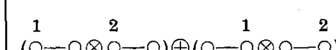
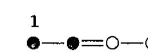
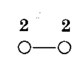
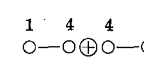
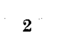
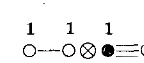
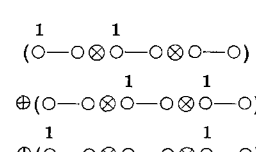
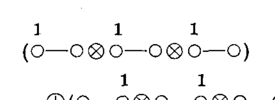
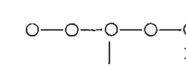
Proof. Theorems 1.1, 2.1, 3.1, 4.1, 6.1, 7.1 and 8.1 give us all the information in the chart except for (*a*) the global isomorphism classes of G and K within their respective local isomorphism classes (*b*) Z and (*c*) $N_G(K)/Z$. As Z is specified (and so listed) by G, we need only check items (*a*) and (*c*).

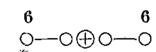
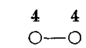
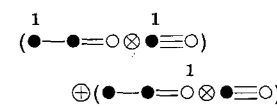
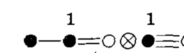
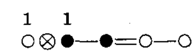
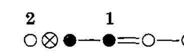
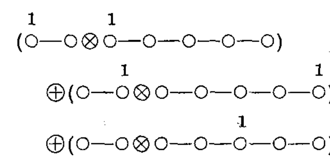
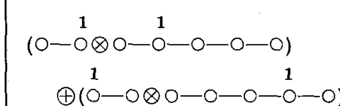
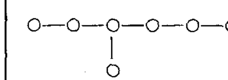
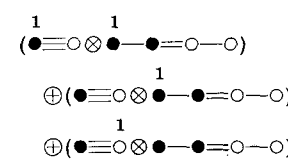
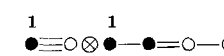
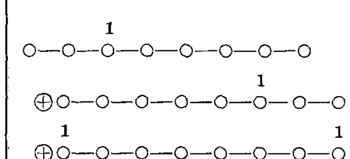
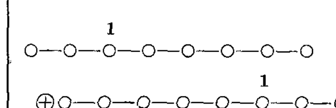
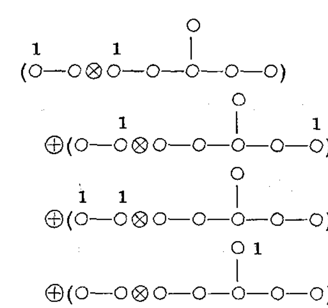
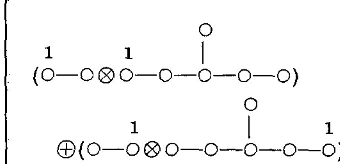
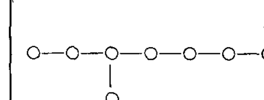
Let rank G = rank K. Then G is centerless and K has center of order 3. This specifies the listed forms of G and K.

Let rank $G >$ rank K. Then K is centerless. Let $\tilde{G}$ denote the simply connected group with Lie algebra $\mathfrak{G}$, let $\tilde{K}$ be the subgroup generated by $\mathfrak{K}$, and let Q denote the center of $\tilde{K}$. Then $G = \tilde{G}/Q$, $\tilde{K} = K/Q$ and $Z = \tilde{Z}/Q$ where $\tilde{Z}$ is the center of $\tilde{G}$.

G	K	Z	$N_G(K)/ZK$	π	χ	Conditions
$\mathbf{SU}(pq)/\mathbf{Z}_m$	$\{\mathbf{SU}(p)/\mathbf{Z}_p\} \times \{\mathbf{SU}(q)/\mathbf{Z}_q\}$	$\mathbf{Z}_{pq}/m$	$\{1\}$		 if $q>2$; if $q=2$	$p \geqslant q \geqslant 2$, $pq>4$ $m = l.c.m.\ \{p, q\}$
$\mathbf{SU}(16)/\mathbf{Z}_4$	$\mathbf{SO}(10)/\mathbf{Z}_2$	$\mathbf{Z}_4$	$\{1\}$			—
$\mathbf{SU}(27)/\mathbf{Z}_3$	$\mathbf{E}_6/\mathbf{Z}_3$	$\mathbf{Z}_9$	$\{1\}$			—
$\mathbf{SU}\left(\frac{n(n-1)}{2}\right)/\mathbf{Z}_m$	$\mathbf{SU}(n)/\mathbf{Z}_n$	$\mathbf{Z}_{n(n-1)/2m}$	$\{1\}$			$n \geqslant 5$ n even: $m=n/2$ n odd: $m=n$
$\mathbf{SU}\left(\frac{n(n+1)}{2}\right)/\mathbf{Z}_m$	$\mathbf{SU}(n)/\mathbf{Z}_n$	$\mathbf{Z}_{n(n+1)/2m}$	$\{1\}$			$n \geqslant 3$, m as above
$\mathbf{Sp}(2)/\mathbf{Z}_2$	$\mathbf{SO}(3)$	$\{1\}$	$\{1\}$			—
$\mathbf{Sp}(7)/\mathbf{Z}_2$	$\mathbf{Sp}(3)/\mathbf{Z}_2$	$\{1\}$	$\{1\}$			—
$\mathbf{Sp}(10)/\mathbf{Z}_2$	$\mathbf{SU}(6)/\mathbf{Z}_6$	$\{1\}$	$\mathbf{Z}_2$			—
$\mathbf{Sp}(16)/\mathbf{Z}_2$	$\mathbf{SO}(12)/\mathbf{Z}_2$	$\{1\}$	$\{1\}$			—
$\mathbf{Sp}(28)/\mathbf{Z}_2$	$\mathbf{E}_7/\mathbf{Z}_2$	$\{1\}$	$\{1\}$			—
$\mathbf{SO}(20)$	$\mathbf{SU}(4)/\mathbf{Z}_4$	$\mathbf{Z}_2$	$\mathbf{Z}_2$			—
$\mathbf{SO}(70)/\mathbf{Z}_2$	$\mathbf{SU}(8)/\mathbf{Z}_8$	$\{1\}$	$\mathbf{Z}_2$			—
$\mathbf{Spin}(n^2-1)$, n odd	$\mathbf{SU}(n)/\mathbf{Z}_n$	$\mathbf{Z}_2 \times \mathbf{Z}_2$, n odd	$\{1\}$, $n \equiv 1$ or $n \equiv 2$		 if $n>3$	$n \geqslant 3$
$\mathbf{SO}(n^2-1)$, n even		$\{1\}$, n even	$\mathbf{Z}_2$, $n \equiv 3$ or $n \equiv 0$ (mod 4)	into $\mathbf{SO}(n^2-1)$	 if $n=3$	

G	K	Z	$N_G(K)/ZK$	π	χ	Conditions
$\mathbf{Spin}(16)/\mathbf{Z}_2\ (\neq \mathbf{SO}(16))$	$\mathbf{SO}(9)$	$\mathbf{Z}_2$	$\{1\}$	○—○—○═●[1] into $\mathbf{SO}(16)$	○—○—○[1]═●	—
$\mathbf{SO}(2n^2+n)$	$\mathbf{SO}(2n+1)$	$\{1\}$, n odd $\mathbf{Z}_2$, n even	$\{1\}$	○═●[2] if $n=2$ ○—○[1]—…—○═● if $n>2$	○[1]═●[2] if $n=2$ ○[1]—○═●[2] if $n=3$ ○[1]—○—○[1]—…—○═● if $n>3$	$n \geqslant 2$
$\mathbf{SO}(2n^2+3n)$	$\mathbf{SO}(2n+1)$	$\{1\}$, n odd $\mathbf{Z}_2$, n even	$\{1\}$	○[2]—…—○═●	○[2]═●[2] if $n=2$ ○[2]—○[1]—…—○═● if $n>2$	$n \geqslant 2$
$\mathbf{Spin}(42)$	$\mathbf{Sp}(4)/\mathbf{Z}_2$	$\mathbf{Z}_4$	$\{1\}$	●—●—●═○ 1 into $\mathbf{SO}(42)$	●—●—●[2]═○	—
$\mathbf{Spin}(2n^2-n-1)$, $n\equiv 0$ or $n\equiv 1 \pmod 4$ $\mathbf{SO}(2n^2-n-1)$, $n\equiv 2$ or $n\equiv 3 \pmod 4$	$\mathbf{Sp}(n)/\mathbf{Z}_2$	$\mathbf{Z}_2\ n\equiv 0$ $\mathbf{Z}_2\times\mathbf{Z}_2$, $n\equiv 1$ $\{1\}$, $n\equiv 2$ $\mathbf{Z}_2$, $n\equiv 3$	$\{1\}$	●—●[1]—…—●═○ into $\mathbf{SO}(2n^2-n-1)$	●[1]—●[1]═○ if $n=3$ ●[1]—●—●[1]—…—●═○ if $n>3$	$n \geqslant 3$
$\mathbf{Spin}(2n^2+n)$, $n\equiv 0$ or $n\equiv 3 \pmod 4$ $\mathbf{SO}(2n^2+n)$, $n\equiv 1$ or $n\equiv 2 \pmod 4$	$\mathbf{Sp}(n)/\mathbf{Z}_2$	$\mathbf{Z}_2\times\mathbf{Z}_2$, $n\equiv 0$ or $n\equiv 3$; $\{1\}$, $n\equiv 1$; $\mathbf{Z}_2$, $n\equiv 2$	$\{1\}$	●[2]—●—…—●═○ into $\mathbf{SO}(2n^2+n)$	●[2]—●[1]—…—●═○	$n \geqslant 3$
$\mathbf{Spin}(128)/\mathbf{Z}_2\ (\neq \mathbf{SO}(128))$	$\mathbf{SO}(16)/\mathbf{Z}_2$	$\mathbf{Z}_2$	$\{1\}$	○—○—○—○—○—○<(○ 1, ○) into $\mathbf{SO}(128)$	○—○—○—○—○—○[1]<(○, ○)	—
$\mathbf{Spin}(2n^2-n)$, n even $\mathbf{SO}(2n^2-n)$, n odd	$\mathbf{SO}(2n)/\mathbf{Z}_2$	$\mathbf{Z}_2\times\mathbf{Z}_2$, $n\equiv 0$ $\mathbf{Z}_4$, $n\equiv 2$ $\{1\}$, n odd	$\{1\}$, n even $\mathbf{Z}_2$, n odd	○—○[1]—…—○<(○, ○) into $\mathbf{SO}(2n^2-n)$	1 ○—○<(○ 1, ○ 1) if $n=4$ ○[1]—○—○[1]—…—○<(○, ○) if $n>4$	$n \geqslant 4$
$\mathbf{Spin}(2n^2+n-1)$, n even $\mathbf{SO}(2n^2+n-1)$, n odd	$\mathbf{SO}(2n)/\mathbf{Z}_2$	$\mathbf{Z}_2$	$\{1\}$	○[2]—○—…—○<(○, ○) into $\mathbf{SO}(2n^2+n-1)$	○[2]—○[1]—…—○<(○, ○)	$n \geqslant 4$
$\mathbf{Spin}(7)$	$\mathbf{G}_2$	$\mathbf{Z}_2$	$\{1\}$	●[1]≡○ into $\mathbf{SO}(7)$	●[1]≡○	—

G	K	Z	$N_G(K)/ZK$	π	χ	Conditions
Spin (14)	$\mathbf{G_2}$	$\mathbf{Z_4}$	$\{1\}$	 into **SO (14)**		—
Spin (26)	$\mathbf{F_4}$	$\mathbf{Z_4}$	$\{1\}$	 into **SO (26)**		—
Spin (52)	$\mathbf{F_4}$	$\mathbf{Z_2 \times Z_2}$	$\{1\}$	 into **SO (52)**		—
Spin (78)	$\mathbf{E_6/Z_3}$	$\mathbf{Z_4}$	$\{1\}$	 into **SO(78)**		—
SO (133)	$\mathbf{E_7/Z_2}$	$\{1\}$	$\{1\}$			—
Spin (248)	$\mathbf{E_8}$	$\mathbf{Z_2 \times Z_2}$	$\{1\}$	 into **SO (248)**		—
$\mathbf{G_2}$	**SO (3)**	$\{1\}$	$\{1\}$			$\mathbf{G_2} \subset$ **SO (7) by**
$\mathbf{G_2}$	**SU (3)**	$\{1\}$	$\mathbf{Z_2}$			
$\mathbf{F_4}$	**SO (3)** $\times\ \mathbf{G_2}$	$\{1\}$	$\{1\}$			$\mathbf{F_4} \subset$ **SO (26) by**
$\mathbf{F_4}$	$\{\mathbf{SU(3)} \times \mathbf{SU(3)}\}/\mathbf{Z_3}$	$\{1\}$	$\mathbf{Z_2}$			
$\mathbf{E_6}$	**SU (3)**$/\mathbf{Z_3}$	$\mathbf{Z_3}$	$\{1\}$			$\mathbf{E_6} \subset$ **SU (27)**
$\mathbf{E_6}$	$\mathbf{G_2}$	$\mathbf{Z_3}$	$\{1\}$			by
$\mathbf{E_6/Z_3}$	$\{\mathbf{SU(3)/Z_3}\} \times \mathbf{G_2}$	$\{1\}$	$\{1\}$			
$\mathbf{E_6/Z_3}$	$\dfrac{\mathbf{SU(3) \times SU(3) \times SU(3)}}{\mathbf{Z_3 \times Z_3}}$	$\{1\}$	$\{1\}$			

G	K	Z	$N_G(K)/ZK$	π	χ	Conditions
$\mathbf{E}_7$	$\mathbf{SU}(3)/\mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{Z}_2$			$\mathbf{E}_7 \subset \mathbf{Sp}(28) \subset \mathbf{SU}(56)$
$\mathbf{E}_7/\mathbf{Z}_2$	$\{\mathbf{Sp}(3)/\mathbf{Z}_2\} \times \mathbf{G}_2$	$\{1\}$	$\{1\}$			by
$\mathbf{E}_7/\mathbf{Z}_2$	$\mathbf{SO}(3) \times \mathbf{F}_4$	$\{1\}$	$\{1\}$			
$\mathbf{E}_7/\mathbf{Z}_2$	$\{\mathbf{SU}(3) \times \mathbf{SU}(6)\}/\mathbf{Z}_6$	$\{1\}$	$\mathbf{Z}_2$			
$\mathbf{E}_8$	$\mathbf{G}_2 \times \mathbf{F}_4$	$\{1\}$	$\{1\}$			$\mathbf{E}_8 \subset \mathbf{SO}(248)$
$\mathbf{E}_8$	$\mathbf{SU}(9)/\mathbf{Z}_3$	$\{1\}$	$\mathbf{Z}_2$			by
$\mathbf{E}_8$	$\{\mathbf{SU}(3) \times \mathbf{E}_6\}/\mathbf{Z}_3$	$\{1\}$	$\mathbf{Z}_2$			

Suppose that G is of type A_l, C_l, G_2, F_4, E_6, E_7 or E_8. Then we have $\tilde{G}$ realized as a simply connected linear group and the inclusion $\pi\colon \tilde{K}\to\tilde{G}$ is given as a linear representation π_λ. Now Q is cyclic of some order q. Let L_{rt} be the root lattice of $\mathfrak{K}^C$ and let L_{wt} be the weight lattice. Then the class $[\lambda]$ of λ in $\Lambda = L_{wt}/L_{rt}$ has order q in the finite abelian group Λ which is isomorphic to the center of the simply connected version of K. This specifies the listed forms of G and K.

Suppose that G is of type D_l or B_l, locally isomorphic to $\mathbf{SO}(m)$ with m being $2l$ or $2l+1$. In order to repeat the trick used above, we must replace $\mathbf{SO}(m)$ by its two-sheeted covering $\tilde{G} = \mathbf{Spin}(m)$. As a Lie algebra inclusion, π is given as an absolutely irreducible representation $\pi_\lambda\colon \mathfrak{K}\to\mathfrak{SO}(m)$. Now we compose π_λ with the spin representation σ of $\mathfrak{D}_l$ or $\mathfrak{B}_l$, looking at the highest summand π_ν of $\sigma\cdot\pi_\lambda$. If $\delta_1 > \dots > \delta_m$ are the weights of the usual representation of $\mathfrak{SO}(m)$, then $\frac{1}{2}\sum_{i=1}^{l}\delta_i$ is the highest weight of σ. Now let $\lambda = \lambda_1 > \lambda_2 \geqslant \dots \geqslant \lambda_m$ be the weights of π_λ; it follows that

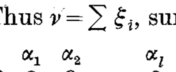
$$\nu = \tfrac{1}{2}(\lambda_1 + \lambda_2 + \dots + \lambda_l).$$

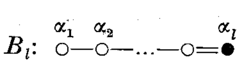
If γ is any weight of $\sigma\cdot\pi_\lambda$, then $\gamma = \frac{1}{2}(\pm\lambda_1 \pm \dots \pm \lambda_l)$ for some choice of signs, so $\gamma \equiv \nu$ modulo the lattice generated by the λ_i. If π_λ has a zero weight, i.e., if $\pi_\lambda(K)$ is centerless, then it follows that $[\gamma] = [\nu]$ in Λ, so the projection $\sigma\cdot\pi_\lambda(\mathfrak{K})\to\pi_\nu(\mathfrak{K})$ is an isomorphism on the group level. In that case we will usually use $\pi_\nu(\mathfrak{K})$ rather than $\sigma\cdot\pi_\lambda(\mathfrak{K})$ to find the order q of the center Q of K. Note there that Q is cyclic.

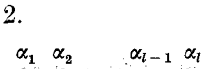
Let K' denote the subgroup of $\mathbf{SO}(m)$ generated by $\pi_\lambda(\mathfrak{K})$ and let Q' be its center. The orders q and q' of Q and Q' are related by $q = q'$ if $[\nu] = [\lambda]$ or by $q = 2q'$ if $[2\nu] = [\lambda]$, and we can read off q' from the diagram of λ. Thus we need only check which of the two situations apply.

If K is of type A_l ($l+1$ odd), G_2, F_4, E_6 or E_8, then the simply connected group with Lie algebra $\mathfrak{K}$ has center of odd order. Thus q and q' are odd, so $q = q'$ because $q = 2q'$ is impossible. Now we need only check the cases where K is of type A_{2r-1}, B_l, C_l, D_l or E_7.

Suppose that π_λ is the adjoint representation. Then $q' = 1$, λ is the highest root and ν is half the sum of the positive roots. As noted in § 5, this implies $\dfrac{2\langle\nu, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle} = 1$ for every simple root α_i. Thus $\nu = \sum \xi_i$, sum of the basic weights.

Type A_{2r-1}: . Then $\Lambda = \{z\} \cong \mathbf{Z}_{2r}$ and $[\xi_k] = z^k$. Thus $[\nu] = [\xi_r] = z^r$ so $q = 2$.

Type B_l: . Then $\Lambda = \{z\} \cong \mathbf{Z}_2$, $[\xi_k] = 1$ for $k < l$ and $[\xi_l] = z$. Thus $[\nu] = [\xi_l] = z$ so $q = 2$.

Type C_l: . Then $\Lambda = \{z\} \cong \mathbf{Z}_2$ and $[\xi_k] = z^k$. If $l = 4r$ or $l = 4r+3$ then $[\nu] = 1$ so $q = 1$; if $l = 4r+1$ or $l = 4r+2$ then $[\nu] = z$ so $q = 2$.

Type D_{2r}: $\overset{\alpha_1}{\circ}-\overset{\alpha_2}{\circ}-\cdots-\circ\langle{}^{\circ\,\alpha_{2r-1}}_{\circ\,\alpha_{2r}}$. Then $\Lambda=\{z_1\}\times\{z_2\}\cong\mathbf{Z}_2\times\mathbf{Z}_2$, $[\xi_k]=z_1^k$ for $k<2r-1$, $[\xi_{2r-1}]=z_1z_2$ and $[\xi_r]=z_2$. If $r=2t$ then $[\nu]=1$ so $q=1$; if $r=2t+1$ then $[\nu]=z_1$ so $q=2$.

Type E_7: $\overset{\alpha_1}{\circ}-\circ-\circ-\underset{\underset{\circ}{|}}{\circ}-\circ-\circ$. We compute $2\nu=27\alpha_1+\sum_2^7 n_i\alpha_i$; thus $[\nu]\neq 1$ in $\Lambda\cong\mathbf{Z}_2$ so $q=2$.

We extend the method:

11.2 Lemma. *If α_i is a simple root of K let h_i be the non-negative integer defined by: $\{0, \alpha_i, \ldots, h_i\alpha_i\}$ are weights of π_λ but $(h_i+1)\alpha_i$ is not. Then $\nu=\sum\{\frac{1}{2}h_i(h_i+1)\}\xi_i$.*

Proof of lemma. Let $\{\psi-p\alpha_i, \ldots, \psi+q\alpha_i\}$ be a maximal α_i-string of weights of π_λ. Let $\mathfrak{G}_i$ be the simple three dimensional subalgebra of $\mathfrak{K}$ with positive root α_i. $\pi_\lambda(\mathfrak{G}_i)$ has trace 0 on the sum of the weight spaces of the string; thus $\langle\sum_{j=-p}^{q}\psi+j\alpha_i, \alpha_i\rangle=0$. If $\psi>0$, ψ not a multiple of α_i, then each $\psi+j\alpha_i>0$. Now the positive weight system decomposes into $S_0\cup\bigcup_{j=1}^{r}S_j$, where $S_0=\{(0), \alpha_i, \ldots, h_i\alpha_i\}$ and the S_j are strings. Thus

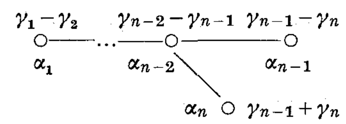
$$\frac{2\langle\nu, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}=\sum_{j=0}^{h_i}\frac{\langle j\alpha_i, \alpha_i\rangle}{\langle\alpha_i, \alpha_i\rangle}=\frac{1}{2}h_i(h_i+1). \quad \text{Q.e.d.}$$

If π_λ is given by $\circ-\overset{2}{\circ}-\circ$ or by $\bullet-\bullet-\bullet\!=\!\overset{1}{\circ}$ then we calculate that $h_i=1$ for each i, so $\nu=\sum\xi_i$. Thus

if π_λ: $\circ-\overset{2}{\circ}-\circ$ *then* $q=2=2q'$; *if* π_λ: $\bullet-\bullet-\bullet\!=\!\overset{1}{\circ}$ *then* $q=1=q'$.

Let π_λ: $\overset{2}{\circ}-\cdots-\circ\!=\!\bullet$. The usual matrix representation π_γ: $\overset{1}{\circ}-\cdots-\circ\!=\!\bullet$ of $\mathfrak{B}_n$ as $\mathfrak{SO}(2n+1)$ has weights $\gamma_1>\gamma_2>\ldots>\gamma_{2n+1}$, $\gamma_{n+1}=0$, $\gamma_j+\gamma_{2n+2-j}=0$, $\{\gamma_1, \ldots, \gamma_n\}$ linearly independent. $\Lambda^2(\pi_\gamma)$ is the adjoint representation, so $\{\pm\gamma_i, \pm\gamma_i\pm\gamma_j\}_{1\leq i, j\leq n; i\neq j}$ are the roots, and the simple roots are given by $\underset{\alpha_1}{\overset{\gamma_1-\gamma_2}{\circ}}-\underset{\alpha_2}{\overset{\gamma_2-\gamma_3}{\circ}}-\cdots-\underset{\alpha_{n-1}}{\overset{\gamma_{n-1}-\gamma_n}{\circ}}\!=\!\underset{\alpha_n}{\overset{\gamma_n}{\bullet}}$. $S^2(\pi_\gamma)=\pi_\lambda\oplus 1$, so $\{\pm\gamma_1, \pm\gamma_i\pm\gamma_j\}_{1\leq i, j\leq n}$ are the weights of π_λ. Thus $\{\alpha_1, \ldots, \alpha_n, 2\alpha_n\}$ are weights of π_λ while $\{2\alpha_1, \ldots, 2\alpha_{n-1}\}$ are not. Now $\nu=\xi_1+\ldots+\xi_{n-1}+3\xi_n$. Thus $[\nu]\neq 1\in\Lambda$, so $q=2q'=2$.

Let π_λ: $\overset{2}{\circ}-\cdots-\circ\langle{}^{\circ}_{\circ}$. The usual matrix representation π_γ: $\overset{1}{\circ}-\cdots-\circ\langle{}^{\circ}_{\circ}$ has weights $\gamma_1>\ldots>\gamma_{2n}$, $\gamma_j+\gamma_{2n+1-j}=0$, $\{\gamma_1, \ldots, \gamma_n\}$ linearly independent. $\Lambda^2(\pi_\gamma)$ is the adjoint representation, so $\{\pm\gamma_i\pm\gamma_j\}_{1\leq i, j\leq n; i\neq j}$ are the roots and the simple roots are given by



$S^2(\pi_\gamma)=\pi_\gamma\oplus 1$, so $\{\pm\gamma_i\pm\gamma_j\}_{1\leqslant i,j\leqslant n}$ are the weights of π_λ. Thus $\{\alpha_1, ..., \alpha_n\}$ are weights of π_λ while $\{2\alpha_1, ..., 2\alpha_n\}$ are not. Now $\nu=\sum\xi_i$. Thus $[\nu]=1$ and $q=1=q'$ if n is even, $[\nu]\neq 1$ and $q=2=2q'$ if n is odd.

Let π_λ: $\bullet\!-\!\overset{1}{\bullet}\!-\!\cdots\!-\!\bullet\!=\!\circ$. The usual matrix representation π_γ: $\overset{1}{\bullet}\!-\!\bullet\!-\!\cdots\!-\!\bullet\!=\!\circ$ of $\mathfrak{C}_n$ has weights $\gamma_1>...>\gamma_{2n}$, $\gamma_j+\gamma_{2n+1-j}=0$, $\{\gamma_1, ..., \gamma_n\}$ linearly independent. $S^2(\pi_\gamma)$ is the adjoint representation, so $\{\pm\gamma_i\pm\gamma_j\}_{1\leqslant i,j\leqslant n}$ are the roots together with $\pm\gamma_i\mp\gamma_i=0$, and the simple roots are given by $\underset{\alpha_1}{\overset{\gamma_1-\gamma_2}{\bullet}}\!-\!\underset{\alpha_2}{\overset{\gamma_2-\gamma_3}{\bullet}}\!-\!\cdots\!-\!\underset{\alpha_{n-1}}{\overset{\gamma_{n-1}-\gamma_n}{\bullet}}\!=\!\underset{\alpha_n}{\overset{2\gamma_n}{\circ}}$. $\Lambda^2(\pi_\gamma)=\pi_\lambda\oplus 1$, so $\{\pm\gamma_i\pm\gamma_j\}_{1\leqslant i,j\leqslant n, i\neq j}$ are the nonzero weights of π_λ. Thus $\{\alpha_1, ..., \alpha_{n-1}\}$ are weights of π_λ while $\{2\alpha_1, ..., 2\alpha_{n-1}, \alpha_n\}$ are not. Now $\nu=\sum_1^{n-1}\xi_i$. If n has form $4r$ or $4r+1$ then $[\nu]=1$ and $q=1=q'$; if n has form $4r+2$ or $4r+3$, then $[\nu]\neq 1$ so $q=2=2q'$.

There remain only the three cases in which π_λ has no zero weight, i.e., in which K' has nontrivial center.

Let π_λ: $\circ\!-\!\circ\!-\!\circ\!=\!\overset{1}{\bullet}$. Then K' is simply connected, so $q>q'$ is impossible. Thus $q=q'$.

Let π_λ: $\circ\!-\!\circ\!-\!\circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ$. Then $K'=\mathbf{SU}(8)/\mathbf{Z}_4$ in $G'=\mathbf{SO}(70)$. Let g generate the center of $\tilde{G}=\mathbf{Spin}(70)$. Then z has order 4 and $z^2\in\tilde{K}$. If $z\notin K$, then $Q=\{1, z^2\}$, so $G=\tilde{G}/Q=\mathbf{SO}(70)\supset K=\tilde{K}/Q=\mathbf{SU}(8)/\mathbf{Z}_8$. That inclusion contradicts the original setup. Thus $z\in\tilde{K}$ and $q=2q'=4$.

Finally let π_λ: $\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!<^{\circ\,1}_{\circ}$, half spin representation of $\mathfrak{D}_8$. Let π_γ: $\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ\!<^{\circ}_{\circ}$ denote the usual representation as $\mathfrak{SO}(16)$ and let $\gamma_1>...>\gamma_{16}$ denote its weights. Then $\lambda=\frac{1}{2}(\gamma_1+...+\gamma_8)$. Now let $\lambda=\lambda_1\geqslant...\geqslant\lambda_{64}$ denote the *positive* weights of π_λ; they are just the $\frac{1}{2}(\gamma_1\pm\gamma_2\pm...\pm\gamma_8)$, where the number of minus signs is even. Let $S_{\varepsilon,j}$ consist of those of the form $\frac{1}{2}(\gamma_1+\varepsilon\gamma_2+\varepsilon_3\gamma_3+...+\varepsilon_8\gamma_8)$, $\varepsilon=\pm 1$, $\varepsilon_k=\pm 1$, in which just j of the signs ε_k are -1; let $\sum_{\varepsilon,j}$ be the sum of the elements of $S_{\varepsilon,j}$; note that $S_{+,j}$ is empty for j odd and $S_{-,j}$ is empty for j even. Now $\nu=\sum\lambda_i$ is given by

$$\nu=(\textstyle\sum_{+,0}+\sum_{+,6})+(\sum_{+,2}+\sum_{+,4})+\sum_{-,1}+\sum_{-,3}+\sum_{-,5}.$$

Given $\lambda_i=\frac{1}{2}(\gamma_1+\gamma_2+\sum_3^8\varepsilon_j\gamma_j)\in S_{+,k}$ we have $'\lambda_i=\frac{1}{2}(\gamma_1+\gamma_2-\sum_3^8\varepsilon_j\gamma_j)\in S_{+,6-k}$, and $'('\lambda_i)=\lambda_i$ and $\lambda_i+{}'\lambda_i=\gamma_1+\gamma_2$. As $S_{+,0}$ has just one element λ and as $S_{+,2}$ has $\binom{6}{2}=15$ elements, this shows

$$(\sum_{+,0}+\sum_{+,6})+(\sum_{+,2}+\sum_{+,4})=16\gamma_1+16\gamma_2.$$

$S_{-,j}$ has $\binom{6}{j}$ elements. The coefficient of γ_i is -1 for $\binom{5}{j-1}$ of them, $+1$ for the others. Thus

$$\sum_{-,1}=3\gamma_1-3\gamma_2+2(\gamma_3+\ldots+\gamma_8),\ \sum_{-,3}=10\gamma_1-10\gamma_2 \text{ and }$$
$$\sum_{-,5}=3\gamma_1-3\gamma_2-2(\gamma_3+\ldots+\gamma_8).$$

This shows $\nu=32\gamma_1$. Let L be the lattice spanned by the λ_i. σ is the spin representation of $\mathfrak{SO}(128=\deg\pi_\lambda)$; the weights of $\sigma\cdot\pi_\lambda$ are the $\frac{1}{2}(\pm\lambda_1\pm\ldots\pm\lambda_{64})$, so any two of them differ by an element of L. Let L_{rt} be the root lattice. $2\gamma_1$ is an integral linear combination of roots; thus $\nu=32\gamma_1\in L_{rt}\subset L$, so L contains every weight of $\sigma\cdot\pi_\lambda$. This shows that K and K' are isomorphic, so $q=q'$.

This completes the verification of the material in the first three columns of the chart.

The description of normalizers is based on a simple remark:

11.3 LEMMA. *Let A be the group of all automorphisms of K which extend to inner automorphisms of G. Then A_0 consists of the inner automorphisms of K, and $g\to\mathrm{ad}(g)|_K$ maps $N_G(K)/ZK$ isomorphically onto A/A_0. If* rank $K<$rank G, *then Z is the centralizer of K in G, and $g\to\mathrm{ad}(g)|_K$ maps $N_G(K)/Z$ isomorphically onto A.*

Proof. Let β denote the map $g\to\mathrm{ad}(g)|_K$. Then β is a homomorphism of $N_G(K)$ onto A, and the kernel of β is the centralizer of K in G. If g is in that centralizer but $g\notin Z$, then K is the connected centralizer of g in G, so $g\in K$. Thus $\ker\beta\subset ZK$. As $\beta(ZK)=A_0$, this shows β: $N_G(K)/ZK\cong A/A_0$. Now if rank $K<$rank G then K is not a connected centralizer, so $Z=\ker\beta$ and β: $N_G(K)/Z\cong A$, q.e.d.

Now suppose rank $K<$rank G until we state the contrary.

If K has no outer automorphism, then Lemma 11.3 says $N_G(K)/ZK=\{1\}$.

Let $\tilde{K}$ have a central element z of order $m>2$. Let α be an automorphism of G which is outer on K. Then α lifts to $\tilde{G}$ and $\alpha(z)=z^{-1}\neq z$. As z is central in $\tilde{G}$, now α is outer on $\tilde{G}$. Thus α is outer on G and $N_G(K)/ZK=\{1\}$.

Let K be of type D_n, $n>4$, where π is the half spin representation. Let α be an automorphism of G which is outer on K. Then α interchanges the two half spin representations, so it is not defined on $\pi(K)$. If α were inner on G it would be defined on $\pi(K)$. Thus $N_G(K)/ZK=\{1\}$.

Let π: $K\to G$ come from the adjoint representation of K, ad: $K\to\mathbf{SO}(p)$, where $p=\dim K$. Let $\alpha\in\mathbf{O}(p)$ be any outer automorphism of K. Then $\alpha\in A$ if and only if $\det\alpha=1$, Let q be the number of positive roots of K and let $l=\operatorname{rank} K$, so $p=l+2q$. If α is not the triality automorphism of $\mathbf{D}_4$, we may multiply it by an inner automorphism and assume that α is $-I$ on the Cartan subalgebra and simple interchange $E_\varphi \leftrightarrow E_{-\varphi}$ of root vectors. Then $\det \alpha=(-1)^{l+q}$, so $N_G(K)/ZK$ is $\{1\}$ if $l+q$ is even, $\mathbf{Z}_2$ if $l+q$ is odd. Now we notice

$$A_{n-1}:\ l=n-1,\quad p=n^2-1,\quad q=\frac{n(n-1)}{2},\quad l+q=\frac{(n+2)(n-1)}{2};$$

$$D_n:\ l=n,\ p=2n^2-n,\ q=n^2-n,\ l+q=n^2;$$

$$E_6:\ l=6,\ p=78,\ q=36,\ l+q=42.$$

Finally, for $\mathbf{D}_4$, triality now has determinant 1 because its cube has determinant 1.

Let π: $\underset{\alpha_1}{\circ}$—…—$\overset{1}{\underset{\alpha_q}{\circ}}$—…—$\underset{\alpha_{2q-1}}{\circ}$. Then $\pi(\mathbf{SU}(2q))$ preserves the bilinear form on $\Lambda^q(\mathbf{C}^{2q})$ given by $(v_1\wedge\ldots\wedge v_q,\, w_1\wedge\ldots\wedge w_q)=v_1\wedge\ldots\wedge v_q\wedge w_1\wedge\ldots\wedge w_q$ where $\Lambda^{2q}(\mathbf{C}^{2q})$ is identified with $\mathbf{C}$, and the linear version of G is in the symplectic or special orthogonal group of that form. Let $\underset{\sim}{a}$ denote complex conjugation of $\mathbf{C}^{2q}$ and $\mathbf{C}$; it extends to the outer automorphism α of K. Note that $\underset{\sim}{a}$ preserves the form. In the symplectic case (q odd), this puts $\underset{\sim}{a}$ in the linear version of G, so $N_G(K)/ZK\cong\mathbf{Z}_2$. In the orthogonal case (q even), $\underset{\sim}{a}$ acts on the real form of $\mathbf{C}^q$ with determinant $(-1)^{q/2}$. In the case $\circ$—$\circ$—$\circ$—$\overset{1}{\circ}$—$\circ$—$\circ$—$\circ$ we have $q=4$, so $N_G(K)/ZK=\mathbf{Z}_2$. Now look at $\circ$—$\overset{2}{\circ}$—$\circ$. There $q=2$ so we have $\underset{\sim}{a}\in\mathbf{SO}(6)$, and $\underset{\sim}{a}$ persists from $\circ$—$\overset{1}{\circ}$—$\circ$ to $\circ$—$\overset{2}{\circ}$—$\circ$, so again $N_G(K)/ZK\cong\mathbf{Z}_2$.

In the usual representation $\overset{1}{\circ}$—$\circ$—…—$\circ\!<\!{}^{\circ}_{\circ}$ of $\mathbf{SO}(2n)$, the outer automorphism is conjugation α by $\underset{\sim}{a}=\operatorname{diag}\{-1;1,\ldots,1\}\in\mathbf{O}(2n)$, using a basis $\{v_1,\ldots,v_{2n}\}$ of $\mathbf{R}^{2n}$. In $S^2(\mathbf{R}^{2n})$ the (-1)-eigenvectors of $S^2(\underset{\sim}{a})$ are $\{v_1v_2, v_1v_3,\ldots,v_1v_{2n}\}$; there are $2n-1$ of them, so $S^2(\underset{\sim}{a})\notin\mathbf{SO}(2n^2-n)$. Thus $N_G(K)/ZK=\{1\}$ for π: $\overset{2}{\circ}$—$\circ$—…—$\circ\!<\!{}^{\circ}_{\circ}$.

Let $G=\mathbf{E}_6$ and $K=\mathbf{SU}(3)/\mathbf{Z}_3$, and let $\alpha=\operatorname{ad}(g)$ be an inner automorphism of G which is outer on K. An outer automorphism β of G is induced by a complex-antilinear map of $\mathbf{C}^{27}$, sending $\begin{smallmatrix}\circ&—&\circ&—&\circ&—&\circ&—&\overset{1}{\circ}\\&&&&|\\&&&&\circ\end{smallmatrix}$ to $\begin{smallmatrix}\overset{1}{\circ}&—&\circ&—&\circ&—&\circ&—&\circ\\&&&&|\\&&&&\circ\end{smallmatrix}$; it preserves $\underset{2}{\circ}$—$\underset{2}{\circ}$ so we may assume $\beta(K)=K$. If β is inner on K, then we may assume $\beta|_K=1$, so K is in the fixed point set F_β of β. Then $K=F_\beta$. But the fixed point set of an outer automorphism of $\mathbf{E}_6$ has rank 4. Thus β is outer on K. Similarly $\beta^{-1}\alpha$ is outer on K. Now $\alpha=\beta\cdot\beta^{-1}\alpha$ is inner on K. That is absurd, so α cannot exist.

Let $G=\mathbf{E}_7$ and $K=\mathbf{SU}(3)/\mathbf{Z}_3$. Let ν denote the complex conjugation automorphism of $\mathbf{SU}(56)$ with fixed point set $\mathbf{SO}(56)$. We may assume embeddings chosen so that ν preserves each of $\mathbf{Sp}(28) \supset G \supset K$. ν is necessarily inner on G, but it interchanges the summands $\overset{6}{\circ}\!-\!\circ$ and $\circ\!-\!\overset{6}{\circ}$ of $K \to \mathbf{SU}(56)$, so it is outer on K. Thus $N_G(K)/ZK \cong \mathbf{Z}_2$.

This completes the determination of $N_G(K)/ZK$ for the cases rank $K <$ rank G.

Let $G = \mathbf{E}_6/\mathbf{Z}_3$ and $K = (\mathbf{SU}(3))^3/(\mathbf{Z}_3)^2$. Then $Z = \{1, z, z^{-1}\}$ is the center of K. Let α be an automorphism, inner on G and outer on K. Then $\alpha(z) = z^{-1}$. Choose a maximal torus T containing z and let β be the outer automorphism of G which is given on T by $x \to x^{-1}$. Then $\beta(K) = K$ and β is outer on K. Now $(\alpha\beta)(z) = z$, so $\alpha\beta$ is inner on K, whence $\alpha\beta$ is trivial on a maximal torus $S \subset K$. Thus $\alpha\beta$ is inner on G. It follows that β is inner on G. Now $N_G(K)/ZK = \{1\}$.

Finally let rank $K =$ rank G with $G \neq \mathbf{E}_6/\mathbf{Z}_3$. Then $-I$ is in the Weyl group of G, so every element is conjugate to its inverse. Let $\{1, z, z^{-1}\} = Z$; there exists $g \in G$ with $gzg^{-1} = z^{-1}$, and so $\mathrm{ad}(g)$ is outer on Z. Thus $N_G(K)/ZK \cong \mathbf{Z}_2$.

This completes the proof of Theorem 11.1, q.e.d.

12. Extension to noncompact isotropy subgroup

The extension *per se* is

12.1 THEOREM. *Let $M_u = G_u/H_u$ be an effective reductive coset space where H_u is a compact connected group with $\mathbf{R}$-irreducible linear isotropy representation χ_u. Let σ be an involutive automorphism of $\mathfrak{G}_u$ which preserves $\mathfrak{H}_u$. Decompose $\mathfrak{G}_u = \mathfrak{J}_u + \mathfrak{P}_u$ into (± 1)-eigenspaces of σ, and define $\mathfrak{G} = \mathfrak{J}_u + \sqrt{-1}\,\mathfrak{P}_u$ and $\mathfrak{H} = (\mathfrak{H}_u \cap \mathfrak{J}_u) + \sqrt{-1}(\mathfrak{H}_u \cap \mathfrak{P}_u)$. Let $H \subset G$ and $H_u^C \subset G_u^C$ denote the connected Lie groups with Lie algebras $\mathfrak{H} \subset \mathfrak{G}$ and $\mathfrak{H}_u^C \subset \mathfrak{G}_u^C$, respectively, such that $M = G/H$ and $M_u^C = G_u^C/H_u^C$ are simply connected effective coset spaces.*(1) *Let χ and χ_u^C denote their respective linear isotropy representations. Then there are only three possibilities, as follows.*

1. *χ_u and χ are absolutely irreducible while χ_u^C is $\mathbf{R}$-irreducible but not absolutely irreducible.*
2. *$\chi_u = \beta \oplus \bar{\beta}$ with $\beta \nsim \bar{\beta}$, χ_u^C is not $\mathbf{R}$-irreducible, and χ is $\mathbf{R}$-irreducible if and only if $\beta \cdot \sigma \sim \beta$.*
3. *$\chi_u = \beta \oplus \beta$, G_u is not semisimple, χ is $\mathbf{R}$-irreducible, and χ_u^C is not $\mathbf{R}$-irreducible.*

Proof. First suppose χ_u absolutely irreducible. As χ has the same extension to a complex representation of $\mathfrak{H}_u^C$, it too is absolutely irreducible. If χ_u^C reduces over $\mathbf{R}$ then the complex extension of χ_u reduces over $\mathbf{C}$; thus χ_u^C is $\mathbf{R}$-irreducible.

(1) For example $G = G'/Z$ and $H = H'/Z$, where G' is the connected simply connected Lie group with Lie algebra $\mathfrak{G}$, H' is the analytic subgroup with Lie algebra $\mathfrak{H}$, and Z is the kernel of the action of G' on $M = G'/H' = G/H$.

Now suppose χ_u not absolutely irreducible. Then it has form $\beta \oplus \bar{\beta}$. Decompose $\mathfrak{G}_u = \mathfrak{H}_u + \mathfrak{M}_u$ and $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$; $\mathfrak{M}_u^C = \mathfrak{M}^C = \mathfrak{M}' + \mathfrak{M}''$, where β has representation space $\mathfrak{M}'$ and $\bar{\beta}$ has representation space $\mathfrak{M}''$. Let τ and τ_u be the respective conjugations of $\mathfrak{G}^C = \mathfrak{G}_u^C$ over $\mathfrak{G}$ and $\mathfrak{G}_u$. Now τ_u interchanges $\mathfrak{M}'$ and $\mathfrak{M}''$ and $\tau = \tau_u \sigma$. If $\beta \nsim \bar{\beta}$ then $\sigma\mathfrak{M}'$ is $\mathfrak{M}'$ or $\mathfrak{M}''$. In the first case τ interchanges $\mathfrak{M}'$ and $\mathfrak{M}''$ so χ is $\mathbf{R}$-irreducible. In the second case τ preserves both $\mathfrak{M}'$ and $\mathfrak{M}''$ so χ reduces over $\mathbf{R}$. If $\beta \sim \bar{\beta}$ then we can choose $\mathfrak{M}'$ and $\mathfrak{M}''$ to be σ-stable, so τ interchanges them and χ is $\mathbf{R}$-irreducible. This proves the statements on χ. On the other hand reduction of χ_u over $\mathbf{C}$ amounts to reduction of χ_u^C over $\mathbf{R}$, so the statements on χ_u^C are immediate, q.e.d.

In order to use the extension to reduce enumeration problems to the compact case we need its converse.

12.2 Theorem. *Let $M = G/H$ be an effective reductive coset space of connected Lie groups, where H has $\mathbf{R}$-irreducible linear isotropy representation χ.*

If G is not semisimple then it is the semidirect product $H \times_\omega V$, where ω is a faithful $\mathbf{R}$-irreducible linear representation of H on a vector group V or $H = \{1\}$ and G is a circle group.

If G is semisimple then it has an involutive automorphism σ, unique up to $\mathrm{ad}_G(H)$-conjugacy in the automorphism group of G, such that

(i) *the fixed point set F of σ is a maximal compactly embedded subgroup*[1] *of G,*

(ii) *$\sigma(H) = H$, and*

(iii) *the fixed point set $F \cap H$ of $\sigma|_H$ is a maximal compact subgroup of H.*

Decompose $\mathfrak{G} = \mathfrak{F} + \mathfrak{P}$ into (± 1)-eigenspaces of σ. Define $\mathfrak{G}_u = \mathfrak{F} + \sqrt{-1}\,\mathfrak{P}$ and $\mathfrak{H}_u = (\mathfrak{H} \cap \mathfrak{F}) + \sqrt{-1}\,(\mathfrak{H} \cap \mathfrak{P})$. Let $H_u \subset G_u$ denote the connected Lie groups with Lie algebras $\mathfrak{H}_u \subset \mathfrak{G}_u$ such that the "compact version" $M_u = G_u/H_u$ of M is a simply connected effective coset space. Then G_u, H_u and M_u are compact and there are only two possibilities, as follows.

1. *The linear isotropy representation χ_u of H_u is $\mathbf{R}$-irreducible.*

2. *There is a simply connected effective coset space A/B of compact connected Lie groups such that B has absolutely irreducible linear isotropy representation, $G_u = A \times A$, $H_u = B \times B$, $M_u = (A/B) \times (A/B)$, $\mathfrak{G} = \mathfrak{A}^C$ and $\mathfrak{H} = \mathfrak{B}^C$.*

Proof. If G is not semisimple then the assertion is contained in Lemma 1.2.

We now assume G semisimple. H is a reductive subgroup of G because its linear isotropy representation is faithful and fully reducible. Now a result of Mostow on Cartan involutions [12] gives the existence of σ satisfying (i), (ii) and (iii). Compactness of G_u, and thus of H_u and M_u, is immediate.

[1] This means $F = \mathrm{ad}^{-1}(F')$ for some maximal compact subgroup F' of $\mathrm{ad}\,(G)$.

Assume $\mathbf{R}$-reducibility of χ_u; we must check the statements of (2). Lemma 1.4 shows that G is simple. If G_u is not simple then G must be a complex Lie group *qua* real Lie group. In that case the inclusion $\mathfrak{H}\to\mathfrak{G}$ defines a homomorphism over $\mathbf{C}$, ϕ: $\mathfrak{H}^C\to\mathfrak{G}$. Let $\mathfrak{L}=\phi(\mathfrak{H}^C)$; $\mathbf{R}$-irreducibility of χ says that either $\mathfrak{H}=\mathfrak{L}$ or $\mathfrak{L}=\mathfrak{G}$. If $\mathfrak{L}=\mathfrak{G}$ it follows that $\mathfrak{H}$ is a real form of $\mathfrak{G}$ and that χ_u is absolutely irreducible; thus $\mathfrak{H}=\mathfrak{L}$. Now $\mathfrak{H}\subset\mathfrak{G}$ is an inclusion of complex Lie algebras, $\mathfrak{G}_u=\mathfrak{A}\oplus\mathfrak{A}$, where $\mathfrak{A}$ is the compact real form of $\mathfrak{G}$, and $\mathfrak{H}_u=\mathfrak{B}\oplus\mathfrak{B}$, where $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. The assertions of (2) follow from simple connectivity of M_u. Now the proof is reduced to the demonstration that G_u cannot be simple when χ_u is $\mathbf{R}$-reducible.

If G_u was simple with χ_u reducible over $\mathbf{R}$, and if we decomposed $\chi_u=\beta\oplus\gamma$, the real representations β and γ would be equivalent. On the other hand, H_u would be a maximal subgroup of G_u because of $\mathbf{R}$-irreducibility of χ, so we could not have $G_u/H_u=\mathbf{Spin}(8)/\mathbf{G}_2$. Thus the following lemma would give a contradiction, q.e.d. modulo lemma.

12.3 Lemma. *Let $M_u=G_u/H_u$ be a simply connected effective coset space of connected compact Lie groups. Suppose that H_u has linear isotropy representation $\chi_u=\beta_1\oplus\beta_2$ with β_i absolutely irreducible and $\beta_1\sim\beta_2$. Then $G_u=\mathbf{Spin}(8)$, $H_u=\mathbf{G}_2$, M_u is the product $\mathbf{S}^7\times\mathbf{S}^7$ of spheres, and χ_u: $\overset{1}{\bullet}\equiv\circ\oplus\overset{1}{\bullet}\equiv\circ$; or $G_u=M_u=\mathbf{T}^2$ and $H_u=\{1\}$.*

Proof. If rank $G_u=$rank H_u then H_u is the connected centralizer of its center, and Theorem 2.2 says $\beta_1\not\sim\beta_2$. Now rank $H_u<$rank G_u. If H_u has a central element $z\neq1$ then $\beta_1(z)=\beta_2(z)\neq I$ so H_u is the connected centralizer of z; that violates the rank condition. Now H_u is centerless.

Suppose that $\mathfrak{H}_u$ is a maximal subalgebra of $\mathfrak{G}_u$. The rank condition says that $\mathfrak{G}_u$ is semisimple. Now Lemma 1.4 says that $\mathfrak{G}_u$ is simple. As $\mathfrak{H}_u\subset\mathfrak{G}_u$ does not appear on Dynkin's list ([7], page 231), now $\mathfrak{G}_u$ is classical simple, so (*a*) $\mathfrak{G}_u=\mathfrak{SU}(N)$, (*b*) $\mathfrak{G}_u=\mathfrak{SO}(N)$, or (*c*) $\mathfrak{G}_u=\mathfrak{Sp}(N)$. We view the inclusion $\mathfrak{H}_u\to\mathfrak{G}_u$ as a linear representation π. If it is not absolutely irreducible then maximality of $\mathfrak{H}_u$ shows M_u irreducible symmetric with χ_u absolutely irreducible. Now $\pi=\pi_\lambda$ for some highest weight λ of $\mathfrak{H}_u$. Let ψ denote $\mathrm{ad}_{G_u}\cdot\pi$, so $\psi=\mathrm{ad}_{H_u}\oplus\beta_1\oplus\beta_2$. We go by cases.

(*a*) $\mathfrak{G}_u=\mathfrak{SU}(N)$. Then $\psi\oplus 1_{H_u}=\pi_\lambda\otimes\pi_{\lambda^*}$, so $\pi_{\lambda+\lambda^*}$ is a summand of ψ. $\lambda\neq\lambda^*$ because $\mathfrak{SO}(N)\neq\mathfrak{G}_u\neq\mathfrak{Sp}(N/2)$. If $\pi_{\lambda+\lambda^*}$ is one of the β_i, say β_1, then $\beta_1\sim\beta_2$ says that $\lambda+\lambda^*$ is a weight of multiplicity $\geqslant 2$ in $\pi_\lambda\otimes\pi_{\lambda^*}$. Now $\pi_{\lambda+\lambda^*}$ is a summand of ad_{H_u}, so $\mathfrak{H}_u=\mathfrak{G}_u$. That is impossible.

(*b*) $\mathfrak{G}_u=\mathfrak{SO}(N)$. Then $\psi=\Lambda^2(\pi_\lambda)$, so $\pi_{2\lambda-\alpha_i}$ is a summand of ψ for each simple root α_i not orthogonal to λ. Every weight of the form $2\lambda-\alpha_i$ has multiplicity 1 in $\Lambda^2(\pi_\lambda)$, so

$\pi_{2\lambda-\alpha_i}+\beta_j$; thus each $\pi_{2\lambda-\alpha_i}$ is a summand of ad_{H_u}. If $\mathfrak{H}_u$ is not simple it has form $\mathfrak{Sp}(n_1)\oplus\mathfrak{Sp}(n_2)$, $4n_1n_2=N$, or $\mathfrak{SO}(n_1)\oplus\mathfrak{SO}(n_2)$, $n_1n_2=N$, as in the proof of Theorem 4.1. Then π_λ is the tensor product of usual vector representations of the summands and each $2\lambda-\alpha_i$ has nonzero values on both summands. That prevents $\pi_{2\lambda-\alpha_i}$ from being a summand of ad_{H_u}. Now $\mathfrak{H}_u$ is simple, λ is a multiple $k\xi_r$ of a basic weight, and $\mathrm{ad}_{H_u}=\pi_{2k\xi_r-\alpha_r}$. Thus $\mathfrak{H}_u$ has highest root $2k\xi_r-\alpha_r$ and the second and third paragraphs of the proof of Proposition 5.2 show $G_u/H_u=\mathbf{Spin}(7)/\mathbf{G}_2$. That implies absolute irreducibility of χ_u, which is impossible.

(*c*) $\mathfrak{G}_u=\mathfrak{Sp}(N)$. Then $\psi=S^2(\pi_\lambda)$, which has $\pi_{2\lambda}$ as a summand. As 2λ is a weight of multiplicity 1 in $S^2(\pi_\lambda)$, $\pi_{2\lambda}$ cannot be one of the β_i, so $\pi_{2\lambda}$ is a summand of ad_{H_u}. As π_λ is faithful this shows that H_u is simple and $\mathrm{ad}_{H_u}=\pi_{2\lambda}$. Now 2λ is the highest root of H_u and it follows from classification that either 2λ: $\overset{2}{\circ}$ or 2λ: $\overset{2}{\bullet}-\bullet-\ldots-\bullet\!=\!\circ$. That says that either λ: $\overset{1}{\circ}$ or λ: $\overset{1}{\bullet}-\bullet-\ldots-\bullet\!=\!\circ$. Now $\mathfrak{H}_u=\mathfrak{G}_u$. That is impossible.

For purposes of Theorem 12.2 we could stop at this point. But we continue because the lemma is relevant to the results of § 14.

We have proved that $\mathfrak{H}_u$ is not a maximal subalgebra of $\mathfrak{G}_u$. Thus we have $\mathfrak{G}_u=\mathfrak{H}_u+\mathfrak{M}_1+\mathfrak{M}_2$, where $\mathfrak{M}_i$ is the representation space of β_i and $\mathfrak{L}=\mathfrak{H}_u+\mathfrak{M}_1$ is an algebra. Let γ denote the representation of $\mathfrak{L}$ on $\mathfrak{M}_2$, so $\beta_2=\gamma|_{\mathfrak{H}u}$; thus γ is absolutely irreducible and Theorem 2.2 shows rank $\mathfrak{L}<$rank $\mathfrak{G}$. Similarly Theorem 2.2 and absolute irreducibility of β_1 shows rank $\mathfrak{H}_u<$rank $\mathfrak{L}$. If $\mathfrak{H}_u$ contains a nonzero ideal of $\mathfrak{L}$ then that ideal is killed by β_1, hence also by β_2, contradicting effectiveness of G_u on M_u; now L/H_u is effective and isotropy irreducible.

If L is not simple then Lemma 1.4 says that $\mathfrak{L}\cong\mathfrak{H}_u\oplus\mathfrak{H}_u$ with $\mathfrak{H}_u$ simple and embedded diagonally, so β_1 is the adjoint representation of H_u. Then $\beta_2\sim\mathrm{ad}_{H_u}$ and Corollary 10.2 says that $\gamma(L)$ is the full $\mathbf{SO}(\mathfrak{M}_2)$. As γ is faithful on the diagonal H_u of L , it is faithful on L; thus $L\cong\mathbf{SO}(m)$, $m=\dim\mathfrak{M}_i$, and G_u/L is effective. That says $G_u=\mathbf{SO}(m+1)$. Note that $m=4$ because L is semisimple but not simple. Now $H_u\subset L\subset G_u$ is given by $\mathbf{SU}(2)\subset\mathbf{SO}(4)\subset\mathbf{SO}(5)$, so $m=4$ is equal to $\dim\mathbf{SU}(2)=3$. That is absurd. Thus L is simple. If G_u is not simple then $\mathfrak{G}_u=\mathfrak{L}\oplus\mathfrak{L}$ and $\dim\mathfrak{M}_2=\dim\mathfrak{L}>\dim\mathfrak{M}_1$; thus G_u is also simple.

Suppose that H_u is not simple. Then Theorem 11.1 and the classification of symmetric spaces A/B (A simple, rank $B<$rank A) show that L/H_u is one of

(*a*) $\mathbf{SU}(pq)/\mathbf{SU}(p)\cdot\mathbf{SU}(q)$, $p\geqslant 2$, $q\geqslant 2$; $N=(p^2-1)(q^2-1)$.

(*b*) $\mathbf{F}_4/\mathbf{SO}(3)\times\mathbf{G}_2$; $N=35$.

(*c*) $\mathbf{E}_6/\mathbf{SU}(3)\times\mathbf{G}_2$; $N=46$.

(*d*) $\mathbf{E}_7/\mathbf{Sp}(3)\times\mathbf{G}_2$; $N=98$.

(*e*) $\mathbf{E}_7/\mathbf{SU}(2)\times\mathbf{F}_4$; $N=78$.

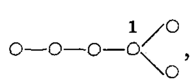
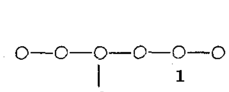
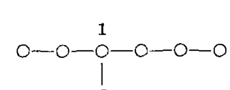
(*f*) $\mathbf{E}_8/\mathbf{G}_2\times\mathbf{F}_4$; $N=182$.

Here N is the dimension. Note that $N=\dim\mathfrak{M}_i=\dim G_u/L$. $\mathbf{SU}(m)$ $(m\geqslant 4)$, $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ cannot be symmetric subgroups of lower rank in a simple group. If G_u/L is symmetric, now it must be $\mathbf{E}_6/\mathbf{F}_4$, which has dimension 26 $(\neq 35)$. Thus G_u/L is not symmetric, so it is listed in Theorem 11.1. Now $L\neq\mathbf{E}_8$ because $\dim\mathbf{SO}(248)/\mathbf{E}_8=30\,132>182$. And $L\neq\mathbf{E}_7$ because $\dim\mathbf{SO}(133)/\mathbf{E}_7=8645>\dim\mathbf{Sp}(28)/\mathbf{E}_7=1463>98>78$. Similarly $L\neq\mathbf{E}_6$ because $\dim\mathbf{SO}(78)/\mathbf{E}_6=2925>\dim\mathbf{SU}(27)/\mathbf{E}_6=705>46$, and $L\neq\mathbf{F}_4$ because $\dim\mathbf{SO}(52)/\mathbf{F}_4=1274>\dim\mathbf{SO}(26)/\mathbf{F}_4=283>35$. Thus $L=\mathbf{SU}(m)$, $m=pq$ not prime, with $N=(p^2-1)(q^2-1)<m^2$. As $m\neq 3$ and rank $L<$ rank G_u now G_u/L is (a_1) $\mathbf{SO}(20)/\mathbf{SU}(4)$, (a_2) $\mathbf{SO}(70)/\mathbf{SU}(8)$, (a_3) $\mathbf{SO}(m^2-1)/\mathbf{SU}(m)$, (a_4) $\mathbf{Sp}(10)/\mathbf{SU}(6)$ or (a_5) $\mathbf{SU}(\frac{1}{2}m(m\pm 1))/\mathbf{SU}(m)$. Each case is eliminated because it has dimension $>m^2$. Thus H_u is simple.

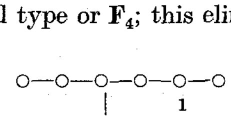
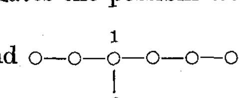
Now H_u, L and G_u are all simple, and we have $\beta_2(H_u)\subsetneqq\gamma(L)\subset\mathbf{SO}(\mathfrak{M}_2)$. Suppose $\gamma(L)\neq\mathbf{SO}(\mathfrak{M}_2)$. Then (Dynkin [6], pages 253 and 364) γ is given

by $\overset{k}{\bullet}\equiv\circ$ with $k\geqslant 1$, by $\overset{k}{\bullet}=\circ-\cdots-\circ$ with $n\geqslant 1$ and $k\geqslant 1$ and $k(n+1)$ odd,

by $\overset{6}{\circ}$, by $\circ-\overset{1}{\circ}-\circ-\overset{1}{\circ}-\circ$, by $\overset{1}{\bullet}=\circ-\circ-\overset{1}{\circ}$, by $\circ=\overset{2}{\bullet}-\bullet$,

by , by , or by .

If G_u/L is not symmetric then it is listed in Theorem 11.1 with $\chi=\gamma$; the only cases are $\mathbf{SO}(14)/\mathbf{G}_2$ with γ; $\overset{3}{\bullet}\equiv\circ$, $\mathbf{SO}(7)/\mathbf{G}_2$ with γ: $\overset{1}{\bullet}\equiv\circ$, $\mathbf{Sp}(2)/\mathbf{SU}(2)$ with γ: $\overset{6}{\circ}$, and $\mathbf{SU}(15)/\mathbf{SU}(6)$ with γ: $\circ-\overset{1}{\circ}-\circ-\overset{1}{\circ}-\circ$. If L is of type G_2 then L/H_u has dimension 11; this eliminates the first two cases. It L is of type A_1 then H_u is $\{1\}$; this eliminates the third case. If L is of type A_5 then $\dim L/H_u<\dim L=35<\deg\gamma$; this eliminates the last case. Now G_u/L is symmetric. A simple symmetric subgroup of lower rank is a simple group is of classical type or $\mathbf{F}_4$; this eliminates the possibilities

$\overset{k}{\bullet}\equiv\circ$,  and 

for γ. If L is of type A_n then $G_u/L=\mathbf{SU}(3)/\mathbf{SO}(3)$, so $H_u=\{1\}$; that is impossible. If L is of type B_n then G_u/L is $\mathbf{SU}(2n+1)/\mathbf{SO}(2n+1)$ or $\mathbf{SO}(2n+2)/\mathbf{SO}(2n+1)$; the first has isotropy representation $\overset{2}{\circ}-\circ-\cdots-\circ=\bullet$ and the second has $\overset{1}{\circ}-\circ-\cdots-\circ=\bullet$; those are not possibilities for γ. If L is of type $C_n (n > 2)$ then $G_u/L = \mathbf{SU}(2n)/\mathbf{Sp}(n)$ which has isotropy representation $\bullet\!-\!\overset{1}{\bullet}\!-\!\cdots\!-\!\bullet\!=\!\circ$ not among the choices of γ. If L is of type D_n then G_u/L and the isotropy representations are $\mathbf{SU}(2n)/\mathbf{SO}(2n)$ and $\overset{2}{\circ}\!-\!\circ\!-\!\cdots\!-\!\circ\!\genfrac{}{}{0pt}{}{\diagup\circ}{\diagdown\circ}$ which is not a possibility for γ. Now γ does not exist. This contradicts the assumption $\gamma(L) \subsetneqq \mathbf{SO}(\mathfrak{M}_2)$.

Now H_u, L and G_u are simple with $\gamma(L) = \mathbf{SO}(\mathfrak{M}_2)$. Thus G_u/L is of the form $\mathbf{SO}(2m)/\mathbf{SO}(2m-1)$ with $\deg \beta_i = 2m-1$. As $\mathbf{B}_{m-1}$ has no outer automorphism, L/H_u is listed in Theorem 11.1 with $L = \mathbf{SO}(2m-1)$ or $L = \mathbf{Spin}(2m-1)$, and $\dim H_u = (2m-1)(m-2)$. The latter says $\dim H_u = [(m-2)/(m-1)] \dim L$. That shows $L/H_u = \mathbf{Spin}(7)/\mathbf{G}_2$ and $G_u/L = \mathbf{Spin}(8)/\mathbf{Spin}(7)$, so $M_u = G_u/H_u = \mathbf{Spin}(8)/\mathbf{G}_2 = \mathbf{S}^7 \times \mathbf{S}^7$, q.e.d.

12.4 *Remark.* The proof of Lemma 12.3 actually shows: *If $M_u = G_u/H_u$ is an effective coset space of compact connected Lie groups, and if H_u is a maximal proper connected subgroup of G_u and has linear isotropy representation which is a sum of copies of the same irreducible complex representation, then the linear isotropy representation of H_u is absolutely irreducible.*

The classification of simply connected isotropy irreducible reductive coset spaces $M = G/H$, G connected and effective on M, now splits into three parts.

1. The case where G is not semisimple. Here all spaces M are constructed as follows. Let β be a faithful irreducible complex representation of a reductive Lie algebra $\mathfrak{H}$. If β is equivalent to a real representation, define $\pi = \beta$, $n = \deg \beta$, and let H be the analytic subgroup of $\mathbf{GL}(n, \mathbf{R})$ with Lie algebra $\beta(\mathfrak{H})$. Otherwise, define $\pi = \beta \oplus \bar{\beta}$, $n = 2\deg \beta$, and let H be the analytic subgroup of $\mathbf{GL}(n, \mathbf{R})$ with Lie algebra $(\beta \oplus \bar{\beta})(\mathfrak{H})$. Then H is a closed subgroup of $\mathbf{GL}(n, \mathbf{R})$ and $G = H \times_\pi \mathbf{R}^n$. The various possibilities for the pair $(\beta, \mathfrak{H})$ are known from É. Cartan's theory of representations of real semisimple Lie algebras.

2. The case where G is semisimple and χ is absolutely irreducible. These are the spaces $M = G/H$ constructed as follows. $M_u = G_u/H_u$ either is an arbitrary nonhermitian compact simply connected irreducible symmetric space, or is any space listed in Theorem 11.1 with linear isotropy representation χ_u absolutely irreducible. For each such M_u one must find all $\mathrm{ad}(H_u)$-conjugacy classes of involutive automorphisms σ of $\mathfrak{G}_u$ which preserve $\mathfrak{H}_u$. For each such triple $(\mathfrak{G}_u, \mathfrak{H}_u, \sigma)$ one has the space $M = G/H$ constructed in Theorem 12.1. All possible $M = G/H$ are constructed this way.

3. The case where G is semisimple and χ is not absolutely irreducible. All such spaces $M = G/H$ are constructed, as in Theorems 12.1 and 12.2, from the compact version $M_u = G_u/H_u$ and an involutive automorphism σ of $\mathfrak{G}_u$ which preserves $\mathfrak{H}_u$, as follows.

(3a) G is a complex simple Lie group, A is a compact real form (hence a maximal compact subgroup), $M_u = G_u/H_u = (A \times A)/(B \times B) = (A/B) \times (A/B)$, and σ is interchange of the two compact simple factors of $G_u = A \times A$. Here A/B either is an arbitrary non-hermitian compact simply connected irreducible symmetric space or is any of the spaces listed in Theorem 11.1 with absolutely irreducible linear isotropy representation.

(3b) $M_u = G_u/H_u$ is an irreducible hermitian symmetric space and σ is any involutive automorphism of $\mathfrak{G}_u$ which preserves $\mathfrak{H}_u$ and does not interchange the two inequivalent irreducible summands of χ_u.

(3c) $M_u = G_u/H_u$ is any space listed in Theorem 11.1 with linear isotropy representation χ_u which is not absolutely irreducible, and σ is any involutive automorphism of G_u which preserves H_u and does not interchange the two summands of χ_u.

Problem 1 was settled by É. Cartan, as mentioned above. Problem 2 is straightforward and quite tedious. The techniques relevant to problem 2 are all needed for problem 3, which we will settle in § 13 in the context of invariant almost complex structures.

Chapter II. Invariant structures on isotropy irreducible coset spaces

In this chapter we study complex and quaternionic structures on isotropy irreducible coset spaces. Complex structures are considered in § 13. There we see that an isotropy irreducible coset space G/H carries an invariant complex structure if and only if either it is hermitian symmetric or G and H are complex Lie groups. We see that G/H carries an invariant almost complex structure if and only if the linear isotropy representation is not absolutely irreducible, and it then turns out that H must be connected, except when G and H are complex groups. These characterizations lead to an easy classification. Quaternionic structures are considered in § 14, partly because they are needed later in our description of linear holonomy groups, and partly to illustrate the general notions of invariant structure and commuting structure. Invariant quaternionic structures turn out to exist only on those isotropy irreducible coset spaces which are the quaternionic symmetric spaces of [18] and their noncompact versions.

13. Complex structures

We first settle the case of compact isotropy subgroup:

13.1 THEOREM. *Let $M = G/K$, where G is a connected Lie group acting effectively, K is compact, and the linear isotropy representation of the identity component K_0 is* $\mathbf{R}$*-irreducible. Then M has a G-invariant complex structure if and only if it is a hermitian symmetric space. If M is not euclidean then the following conditions are equivalent.*

1. *M has a G-invariant almost complex structure.*

2. *M has precisely two G-invariant almost complex structures.*

3. *K is connected and its linear isotropy representation is not absolutely irreducible (so necessarily $\chi=\beta\oplus\bar{\beta}$, β irreducible complex, $\beta \not\simeq \bar{\beta}$).*

13.2 Corollary. *Let $M=G/K$ where G is a connected Lie group acting effectively, K is compact, and the identity component K_0 has $\mathbf{R}$-irreducible linear isotropy representation χ. Suppose that M has a G-invariant almost complex structure but that M is not hermitian symmetric. Then $G=\tilde{G}/E$ and $\tilde{K}=\tilde{K}E/E$ where E is an arbitrary central subgroup of $\tilde{G}$ and all possibilities are given as follows.*

$\tilde{G}$	$\tilde{K}$	Center of G	χ
$\mathbf{Spin}\,(n^2-1)$, n odd, $n>2$	$\mathbf{SU}(n)/\mathbf{Z}_n$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$\circ - \overset{3}{\circ} \oplus \overset{3}{\circ} - \circ$ if $n=3$
$\mathbf{SO}\,(n^2-1)$, n even, >3	$\mathbf{SU}(n)/\mathbf{Z}_n$	$\{1\}$	$\circ - \overset{1}{\circ} - \cdots - \overset{2}{\circ} \oplus \overset{2}{\circ} - \cdots - \overset{1}{\circ} - \circ$ if $n>3$
$\mathbf{G}_2$	$\mathbf{SU}(3)$	$\{1\}$	$\overset{1}{\circ} - \circ \oplus \circ - \overset{1}{\circ}$
$\mathbf{F}_4$	$\{\mathbf{SU}(3)\times\mathbf{SU}(3)\}/\mathbf{Z}_3$	$\{1\}$	$(\overset{1}{\circ} - \circ \otimes \overset{2}{\circ} - \circ) \oplus (\circ - \overset{1}{\circ} \otimes \circ - \overset{2}{\circ})$
$\mathbf{E}_6$	$\mathbf{SU}(3)/\mathbf{Z}_3$	$\mathbf{Z}_3$	$\overset{1}{\circ} - \overset{4}{\circ} \oplus \overset{4}{\circ} - \overset{1}{\circ}$
$\mathbf{E}_6/\mathbf{Z}_3$	$\{\mathbf{SU}(3)\times\mathbf{SU}(3)\times\mathbf{SU}(3)\}/\{\mathbf{Z}_3\times\mathbf{Z}_3\}$	$\{1\}$	$(\overset{1}{\circ} - \circ \otimes \overset{1}{\circ} - \circ \otimes \overset{1}{\circ} - \circ)$ $\oplus (\circ - \overset{1}{\circ} \otimes \circ - \overset{1}{\circ} \otimes \circ - \overset{1}{\circ})$
$\mathbf{E}_7/\mathbf{Z}_2$	$\{\mathbf{SU}(3)\times[\mathbf{SU}(6)/\mathbf{Z}_2]\}/\mathbf{Z}_3$	$\{1\}$	$(\circ - \overset{1}{\circ} \otimes \circ - \overset{1}{\circ} - \circ - \circ - \circ)$ $\oplus (\overset{1}{\circ} - \circ \otimes \circ - \circ - \circ - \overset{1}{\circ} - \circ)$
$\mathbf{E}_8$	$\mathbf{SU}(9)/\mathbf{Z}_3$	$\{1\}$	$\circ - \circ - \overset{1}{\circ} - \circ - \circ - \circ - \circ - \circ$ $\oplus\, \circ - \circ - \circ - \circ - \circ - \overset{1}{\circ} - \circ - \circ$
$\mathbf{E}_8$	$\{\mathbf{SU}(3)\times\mathbf{E}_6\}/\mathbf{Z}_3$	$\{1\}$	$\left(\overset{1}{\circ} - \circ \otimes \overset{1}{\circ} - \circ - \overset{\overset{\circ}{\vert}}{\circ} - \circ - \circ\right)$ $\oplus \left(\circ - \overset{1}{\circ} \otimes \circ - \circ - \overset{\overset{\circ}{\vert}}{\circ} - \circ - \overset{1}{\circ}\right)$

Proof. We first check equivalence of the three conditions listed in the theorem. First assume (3). Decomposing $\chi = \beta \oplus \bar{\beta}$ and glancing through Theorem 11.1 we see that $\beta \nsim \bar{\beta}$. Thus the commuting algebra of χ (the algebra of linear transformations of M_x which commute with every element of $\chi(K)$) is $\mathbf{C}$, which has precisely two elements of square -1. Thus (3) implies (2).

(2) implies (1) at a glance.

Assume (1). If χ were absolutely irreducible its commuting algebra would be $\mathbf{R}$, which has no element of square -1, so (1) would fail; thus χ is not absolutely irreducible. Suppose that K is not connected. Then M is not simply connected, so G/K is not hermitian symmetric. If rank $G =$ rank K it follows that the center of K_0 is generated by an element z of order 3 and, replacing z by z^{-1} if necessary, the almost complex structure J satisfies $\chi(z) = \cos(2\pi/3) I + \sin(2\pi/3) J$. If $k \in K$ then stability of J under k implies $kz = zk$. Pre-images of k and z in the universal covering group of G still commute; as the pre-image of K_0 there is the full centralizer of any pre-image of z, and is connected, it follows that $k \in K_0$. Thus disconnectedness of K implies rank $G >$ rank K. If $k \in K$, $k \notin K_0$, now $\mathrm{ad}(k)$ gives an outer automorphism of K_0 because K_0 is a maximal connected subgroup of lower rank. K_0 is simple by Theorem 11.1; it follows that $\beta \cdot \mathrm{ad}(k) \sim \bar{\beta} \nsim \beta$, so K has commuting algebra $\mathbf{R}$, which is ridiculous. This contradiction proves K connected, completing the proof that (1) implies (3).

We have proved equivalence of the three conditions of the theorem. If M had a G-invariant complex structure it would be a C-space in the sencse of H.-C. Wang [14] and K would be contained in the centralizer of a toral subgroup of G. Thus K could not be semisimple, so M would be hermitian symmetric. The theorem is proved.

The corollary follows from the theorem and a glance at Theorem 11.1, q.e.d.

In the case of equal ranks, passage to noncompact isotropy is based on

13.3 THEOREM. *Let $M = G/K$ where G is a compact connected Lie group acting effectively and K is a closed connected subgroup of maximal rank. Let α be an automorphism of G which preserves K, thus acts on M, and which preserves some G-invariant almost complex structure J on M.*

1. *The following conditions are equivalent, and each implies that α preserves every G-invariant almost complex structure on M.*

(1*a*) *α is an inner automorphism of G.*

(1*b*) *$\alpha|_K$ is an inner automorphism of K.*

(1*c*) *α is conjugation $\mathrm{ad}_G(k)$ by some element $k \in K$,*

2. *If G is simple and α is an outer automorphism of G, then*

(2*a*) $G = \mathbf{SU}(2n)/\mathbf{Z}_{2n}$, $K = \mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$, $\alpha|_K$ *interchanging the two* $\mathbf{U}(n)$*; or*

(2*b*) $G = \mathbf{SO}(2n)/\mathbf{Z}_2$, $K = \{\mathbf{U}(n_1) \times \ldots \times \mathbf{U}(n_s) \times \mathbf{SO}(2m)\}/\mathbf{Z}_2$, $n_1 + \ldots + n_s + m = n$, $m \geqslant 2$, α *conjugation by* $\operatorname{diag}\{P_1, \ldots, P_s; Q\}$, $P_i \in \mathbf{U}(n_i)$, $Q \in \mathbf{O}(2m)$, $\det Q = -1$*; or*

(2*c*) $G = \mathbf{E}_6/\mathbf{Z}_3$, $K = \{\mathbf{SU}(3) \times \mathbf{SU}(3) \times L_i\}/\{\mathbf{Z}_3 \times \mathbf{Z}_3\}$, $1 \leqslant i \leqslant 3$, α *exchanging the two* $\mathbf{SU}(3)$, $\alpha(L_i) = L_i$, *where* $L_1 \subset L_2 \subset L_3$ *is* $\mathbf{T}^2 \subset \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(2)) \subset \mathbf{SU}(3)$.

Proof. We first prove (1). Let $\alpha|_K$ be an inner automorphism of K. Then $\alpha|_T$ is the identity for some maximal torus T of K. As T is a maximal torus of G, now α is an inner automorphism of G. So $\alpha = \operatorname{ad}(g)$ for some $g \in G$ which centralizes T. Thus $g \in T \subset K$. We have just seen that (1*b*) implies (1*a*) and (1*c*). As (1*c*) visibly implies (1*b*), we now need only check that (1*a*) implies (1*b*).

Let Z be the center of K. If $z \in Z$ has odd order then $\alpha(J) = J$ and Theorem 2.2 show that $\alpha(z) = z$. Let Z_0 be the identity component of Z, central toral subgroup of K; now $\alpha|_{Z_0} = 1$. Let L be the centralizer of Z_0 in G. As α is inner now $\alpha = \operatorname{ad}(g)$ for some $g \in L$. Decompose $\mathfrak{L} = \mathfrak{L}' \oplus \mathfrak{Z}$ and $\mathfrak{K} = \mathfrak{K}' \oplus \mathfrak{Z}$ into the derived algebras and the centers; the semisimple parts $\mathfrak{K}' \subset \mathfrak{L}'$, this reduces the proof that (1*a*) implies (1*b*) to the case where K is semisimple.

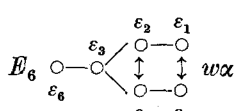
Now K is semisimple and Z is finite. If K were not maximal among the connected subgroups of maximal rank in G, say $K \subset L \subset G$, induction on codimension would prove $\alpha|_L$ inner on L and then $\alpha|_K$ inner on K. Now we may assume K maximal among the connected subgroups of G. If Z had even order, K would be a nonhermitian symmetric subgroup of G, contradicting the existence of the invariant almost complex structure. Thus Z has odd order and $\alpha|_Z = 1$. This proves $g \in K$, so $\alpha|_K$ is inner.

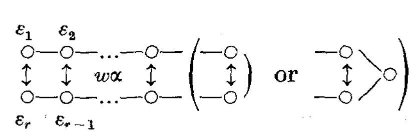
Part (1) of the theorem is proved.

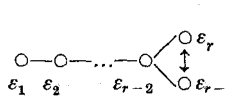
We now assume G simple and α outer. Again, Z is the center of K, Z_0 the identity component of Z, and L the centralizer of Z_0 in G. If $L = G$ then Z is finite and K is semisimple; [9] shows that G is an exceptional group and then G must be $\mathbf{E}_6/\mathbf{Z}_3$ because it is centerless simple and admits an outer automorphism. Then it is immediate [9] that $K = \{\mathbf{SU}(3) \times \mathbf{SU}(3) \times \mathbf{SU}(3)\}/\{\mathbf{Z}_3 \times \mathbf{Z}_3\}$ with α interchanging the first two factors and preserving the third. If $K \neq L$, so G has a connected subgroup of maximal rank which is not the centralizer of a torus, then [4] G is exceptional, hence again of type E_6, and it follows [9] that $L = G$. In the proof of part (2) of the theorem, now, we may assume $K = L$, so that K is the centralizer of the torus Z_0.

If $z \in Z$ has odd order then $\alpha(J) = J$ and Theorem 2.2 show $\alpha(z) = z$; thus $\alpha|_{\mathfrak{Z}} = 1$. Decompose $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$, $\mathfrak{K} = \mathfrak{Z} + \sum \mathfrak{K}_s$, $\mathfrak{M}^C = \sum \mathfrak{M}_i$, where the $\mathfrak{K}_s$ are the simple ideals of $\mathfrak{K}$ and where $\mathfrak{K}$ acts on $\mathfrak{M}_i$ by an irreducible representation π_i. Theorem 2.2 and the existence of the almost complex structure show that the π_i are mutually inequivalent and that each $\pi_i \neq \bar{\pi}_i$. Triviality of α on $\mathfrak{Z}$ now shows that $\pi_i \cdot \alpha \sim \pi_i$ and $\alpha(\mathfrak{M}_i) = \mathfrak{M}_i$. Decompose $\pi_i = \theta_i \otimes (\otimes \eta_{is})$ where θ_i represents $\mathfrak{Z}$ and η_{is} represents $\mathfrak{K}_s$. If $\alpha(\mathfrak{K}_s) = \mathfrak{K}_s$ then $\eta_{is} \cdot \alpha \sim \eta_{is}$.

Choose a maximal torus T of G such that $Z_0 \subset T \subset K$; choose a system $B = \{\beta_1, \ldots, \beta_r\}$ of simple roots of G such that $\mathfrak{Z}$ has equation $\beta_1 = \ldots = \beta_t = 0$ in $\mathfrak{T}$. Then $B' = \{\beta_1, \ldots, \beta_t\}$ is a simple root system for the derived algebra $\mathfrak{K}' = \sum \mathfrak{K}_s$. These choices of simple root systems $B' \subset B$ amount to choices of positive Weyl chambers $\mathfrak{D}' = \mathfrak{D} \cap \sqrt{-1}\mathfrak{T}' \subset \mathfrak{D}$, $\mathfrak{T}' = \mathfrak{T} \cap \mathfrak{K}'$. Let w be an element of the Weyl group of $\mathfrak{K}'$ which carries $\alpha(\mathfrak{D}')$ back to $\mathfrak{D}'$. Then $w\alpha$ is the identity on $\mathfrak{Z}$, permutes B' and preserves $\mathfrak{T}'$. Let $x \in \mathfrak{D}$, say $x = z + x'$ with $z \in \sqrt{-1}\,\mathfrak{Z}$ and $x' \in \sqrt{-1}\,\mathfrak{T}'$, x regular, $\beta_i(z) >> \beta_i(x')$ for $i > t$. Then it is immediate that every $\beta_i(w\alpha x) > 0$. Thus $w\alpha(\mathfrak{D}) = \mathfrak{D}$. As α is outer, now $w\alpha$ is a nontrivial automorphism of the Dynkin diagram of G; it preserves the diagram of K, which is obtained from that of G by deleting the vertices β_i with $i > t$, and induces a nontrivial automorphism there. If $i > t$, so β_i is not a root of K, then $w\alpha$ leaves β_i fixed; for if $\{x_j\}$ is a dual basis of $\sqrt{-1}\,\mathfrak{T}$ relative to the Killing form and the basis $\{\beta_j\}$, so that $\mathfrak{Z}$ has basis consisting of the $\sqrt{-1}\,x_j$ for $t < j \leq r$, then triviality of $w\alpha$ on $\mathfrak{Z}$ shows $w\alpha\ (\sqrt{-1}x_i) = \sqrt{-1}x_i$. In other words, B' contains every root of B which is moved by $w\alpha$.

We run through the list of simple groups G which admit outer automorphisms. If G is of type E_6  then B' contains ε_1, ε_2, ε_4 and ε_5, for those are the roots moved by $w\alpha$. If B' also contains ε_3, then $B' \neq B$ shows that $K' = \mathbf{SU}(6)/\mathbf{Z}_2$ globally, so $\eta \cdot \alpha \sim \bar{\eta}_i \neq \eta_i$. Thus $\varepsilon_3 \notin B'$. Now B' is $\{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5\}$ or $\{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$, and both possibilities occur.

Let G be of type A_r, $r > 1$ . Then $B' \neq B$ says that $r = 2v + 1$ and $B' = \{\varepsilon_1, \ldots, \varepsilon_v;\ \varepsilon_{v+2}, \ldots, \varepsilon_r\}$ for the latter are the roots moved by $w\alpha$. This case occurs geometrically as orthocomplementation on the grassmannian of $(v+1)$-planes in $\mathbf{C}^{2v+2}$.

Let G be of type D_r, $r > 3$  $w\alpha$. Then B' contains ε_{r-1} and ε_r. Thus $G = \mathbf{SO}(2r)/\mathbf{Z}_2$ and $K = \{\mathbf{U}(r_1) \times \mathbf{U}(r_2) \times \ldots \times \mathbf{U}(r_v) \times \mathbf{SO}(2u)\}/\mathbf{Z}_2$, where $r_1 + \ldots + r_v + u = r$ and $u \geq 2$. α preserves each of the $\mathbf{U}(r_s)$, inducing inner automorphisms on them because $\eta_{is} \cdot \alpha \sim \eta_{is}$. Thus α is conjugation by an orthogonal matrix diag $\{P_1, \ldots, P_v, Q\}$, where $P_s \in \mathbf{U}(r_s)$ is a $2r_s \times 2r_s$ block and $Q \in \mathbf{O}(2u)$ has determinant -1, q.e.d.

Now we can settle the case of noncompact isotropy subgroup.

13.4 THEOREM. *Let $M = G/H$ be an effective reductive coset space in which G is a connected semisimple Lie group and the linear isotropy representation χ of the identity component H_0 is $\mathbf{R}$-irreducible.*

1. *M has a G-invariant complex structure if and only if*

(1a) *$M = G/K$ is a hermitian symmetric coset space, or*

(1b) *G is a complex Lie group and H is a complex subgroup.*

2. *If G is a complex Lie group, then H is a complex subgroup, $\mathfrak{G} = \mathfrak{A}^C$ and $\mathfrak{H} = \mathfrak{B}^C$, where A/B is a coset space of compact connected Lie groups which either is an irreducible nonhermitian symmetric coset space, or is listed in Theorem 11.1 with absolutely irreducible linear isotropy representation β. $\mathfrak{H}$ has real linear isotropy representation $\chi = \beta \oplus \bar{\beta}$ (conjugation over $\mathfrak{G}$) with commuting algebra $\mathbf{C}$, and M carries just two G-invariant almost complex structures, both of which are integrable.*

3. *If G is not a complex Lie group then the following conditions are equivalent.*

(3a) *M has a G-invariant almost complex structure.*

(3b) *M has precisely two G-invariant almost complex structures.*

(3c) *H is connected and χ is not absolutely irreducible (so necessarily $\chi = \beta \oplus \bar{\beta}$ with β complex irreducible, $\beta \not\simeq \bar{\beta}$).*

(3d) *H is connected, the compact version $M_u = G_u/H_u$ either is hermitian symmetric or is listed in Corollary 13.2, and $(\mathfrak{G}, \mathfrak{H})$ is defined from $(\mathfrak{G}_u, \mathfrak{H}_u)$ [as in Theorem 12.1] by an involutive automorphism σ of $\mathfrak{G}_u$ which preserves both $\mathfrak{H}_u$ and the two G_u-invariant almost complex structures on M_u.*

(3e) *G/H is listed in Table 13.5, 13.6 or 13.7 with the following convention. A second subscript (e.g. the D_8 in E_{8,D_8}) denotes Cartan classification type of the maximal compact subgroup, and then (in contrast to the compact case, where it means the simply connected group) boldface means the centerless group if it stands alone as in $\mathbf{E}_{6,A_5A_1}$, or the group with cyclic center of order m if it occurs in an expression of the type $[\mathbf{E}_{6,A_5A_1} \times \mathbf{T}^1]/\mathbf{Z}_m$; in that type of expression the $\mathbf{Z}_m$ is diagonal between the circle group and the center of the simple group.*

13.5. *Table.* M_u hermitian symmetric.

M_u	G	K	Conditions
$\mathbf{SU}(p+q)/\mathbf{S}[\mathbf{U}(p) \times \mathbf{U}(q)]$	$\mathbf{SU}^{u+v}(p+q)/\mathbf{Z}_{p+q}$	$\mathbf{S}[\mathbf{U}^u(p) \times \mathbf{U}^v(q)]$	$0 \leqslant 2u \leqslant p \leqslant q,\ 0 \leqslant 2v \leqslant q$
$\mathbf{SU}(2n)/\mathbf{S}[\mathbf{U}(n) \times \mathbf{U}(n)]$	$\mathbf{SL}(n, \mathbf{Q})/\mathbf{Z}_2$ or $\mathbf{SL}(2n, \mathbf{R})/\mathbf{Z}_2$	$[\mathbf{SL}(n, \mathbf{C}) \times \mathbf{T}^1]/\mathbf{Z}_n$	$n > 1$
$\mathbf{SO}(2n)/\mathbf{U}(n)$	$\mathbf{SO}^{2r}(2n)/\mathbf{Z}_2$ $\mathbf{SO}^*(2n)/\mathbf{Z}_2$	$\mathbf{U}^r(n)/\mathbf{Z}_2$	$0 \leqslant 2r \leqslant n,\ n \geqslant 3$

M_u	G	K	Conditions
$\mathbf{Sp}(n)/\mathbf{U}(n)$	$\mathbf{Sp}^r(n)/\mathbf{Z}_2$	$\mathbf{U}^r(n)/\mathbf{Z}_2$	$0 \leqslant 2r \leqslant n$
$\mathbf{SO}(2m+2)/\mathbf{SO}(2m) \times \mathbf{SO}(2)$	$\mathbf{SO}^r(2m+2)/\mathbf{Z}_2$	$\{\mathbf{SO}^r(2m) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	$0 \leqslant r \leqslant m$
	$\mathbf{SO}^{r+2}(2m+2)/\mathbf{Z}_2$	$\{\mathbf{SO}^r(2m) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	$0 \leqslant r \leqslant m$
	$\mathbf{SO}^*(2m+2)/\mathbf{Z}_2$	$\{\mathbf{SO}^*(2m) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
$\mathbf{SO}(2m+3)/\mathbf{SO}(2m+1) \times \mathbf{SO}(2)$	$\mathbf{SO}^r(2m+3)$ $\mathbf{SO}^{r+2}(2m+3)$	$\mathbf{SO}^r(2m+1) \times \mathbf{SO}(2)$	$0 \leqslant r \leqslant m$
$[\mathbf{E}_6/\mathbf{Z}_3]/[\{\mathbf{SO}(10) \times \mathbf{SO}(2\}/\mathbf{Z}_2]$	$\mathbf{E}_6/\mathbf{Z}_3$	$\{\mathbf{SO}(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
	$\mathbf{E}_{6, A_5 A_1}$	$\{\mathbf{SO}^*(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
		$\{\mathbf{SO}^4(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
	$\mathbf{E}_{6, D_5 T^1}$	$\{\mathbf{SO}(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
		$\{\mathbf{SO}^*(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
		$\{\mathbf{SO}^2(10) \times \mathbf{SO}(2)\}/\mathbf{Z}_2$	—
$[\mathbf{E}_7/\mathbf{Z}_2]/[\{\mathbf{E}_6 \times \mathbf{T}^1\}/\mathbf{Z}_3]$	$\mathbf{E}_7/\mathbf{Z}_2$	$\{\mathbf{E}_6 \times \mathbf{T}^1\}/\mathbf{Z}_3$	—
	$\mathbf{E}_{7, A_7}$	$\{\mathbf{E}_{6, A_1 A_5} \times \mathbf{T}^1\}/\mathbf{Z}_2$	—
	$\mathbf{E}_{7, A_1 D_6}$	$\{\mathbf{E}_{6, A_1 A_5} \times \mathbf{T}^1\}/\mathbf{Z}_2$	—
		$\{\mathbf{E}_{6, D_5 T^1} \times \mathbf{T}^1\}/\mathbf{Z}_2$	—
	$\mathbf{E}_{7, E_6 T^1}$	$\{\mathbf{E}_{6, D_5 T^1} \times \mathbf{T}^1\}/\mathbf{Z}_2$	—
		$\{\mathbf{E}_6 \times \mathbf{T}^1\}/\mathbf{Z}_3$	—

13.6. *Table*. rank H = rank G, but M_u not hermitian symmetric.

M_u	G	H
$\mathbf{G}_2/\mathbf{SU}(3)$	$\mathbf{G}_2$	$\mathbf{SU}(3)$
	$\mathbf{G}_{2, A_1 A_1}$	$\mathbf{SU}^1(3)$
$\mathbf{F}_4/\mathbf{SU}(3) \cdot \mathbf{SU}(3)$	$\mathbf{F}_4$	$[\mathbf{SU}(3) \times \mathbf{SU}(3)]/\mathbf{Z}_3$
	$\mathbf{F}_{4, B_4}$	$[\mathbf{SU}^1(3) \times \mathbf{SU}(3)]/\mathbf{Z}_3$
	$\mathbf{F}_{4, C_3 C_1}$	$[\mathbf{SU}(3) \times \mathbf{SU}^1(3)]/\mathbf{Z}_3$ and $[\mathbf{SU}^1(3) \times \mathbf{SU}^1(3)]/\mathbf{Z}_3$
$\mathbf{E}_6/\mathbf{SU}(3) \cdot \mathbf{SU}(3) \cdot \mathbf{SU}(3)$	$\mathbf{E}_6/\mathbf{Z}_3$	$[\mathbf{SU}(3) \times \mathbf{SU}(3) \times \mathbf{SU}(3)]/[\mathbf{Z}_3 \times \mathbf{Z}_3]$
	$\mathbf{E}_{6, A_1 A_5}$	$[\mathbf{SU}^1(3) \times \mathbf{SU}(3) \times \mathbf{SU}(3)]/[\mathbf{Z}_3 \times \mathbf{Z}_3]$ and $[\mathbf{SU}^1(3) \times \mathbf{SU}^1(3) \times \mathbf{SU}^1(3)]/[\mathbf{Z}_3 \times \mathbf{Z}_3]$
	$\mathbf{E}_{6, D_5 T^1}$	$[\mathbf{SU}^1(3) \times \mathbf{SU}^1(3) \times \mathbf{SU}(3)]/[\mathbf{Z}_3 \times \mathbf{Z}_3]$
	$\mathbf{E}_{6, F_4}$	$[\mathbf{SL}(3, \mathbf{C}) \times \mathbf{SU}(3)]/\mathbf{Z}_3$
	$\mathbf{E}_{6, C_4}$	$[\mathbf{SL}(3, \mathbf{C}) \times \mathbf{SU}^1(3)]/\mathbf{Z}_3$
$\mathbf{E}_7/\mathbf{SU}(3) \cdot \mathbf{SU}(6)$	$\mathbf{E}_7/\mathbf{Z}_2$	$[\mathbf{SU}(3) \times \mathbf{SU}(6)]/\mathbf{Z}_6$
	$\mathbf{E}_{7, A_7}$	$[\mathbf{SU}(3) \times \mathbf{SU}^1(6)]/\mathbf{Z}_6$ and $[\mathbf{SU}^1(3) \times \mathbf{SU}^3(6)]/\mathbf{Z}_6$
	$\mathbf{E}_{7, A_1 D_6}$	$[\mathbf{SU}^1(3) \times \mathbf{SU}(6)]/\mathbf{Z}_6$, $[\mathbf{SU}(3) \times \mathbf{SU}^2(6)]/\mathbf{Z}_6$, and $[\mathbf{SU}^1(3) \times \mathbf{SU}^2(6)]/\mathbf{Z}_6$
	$\mathbf{E}_{7, E_6 T^1}$	$[\mathbf{SU}^1(3) \times \mathbf{SU}^1(6)]/\mathbf{Z}_6$ and $[\mathbf{SU}(3) \times \mathbf{SU}^3(6)]/\mathbf{Z}_6$
$\mathbf{E}_8/\mathbf{SU}(3) \cdot \mathbf{E}_6$	$\mathbf{E}_8$	$[\mathbf{SU}(3) \times \mathbf{E}_6]/\mathbf{Z}_3$
	$\mathbf{E}_{8, D_8}$	$[\mathbf{SU}(3) \times \mathbf{E}_{6, D_5 T^1}]/\mathbf{Z}_3$ and $[\mathbf{SU}^1(3) \times \mathbf{E}_{6. A_1 A_5}]/\mathbf{Z}_3$
	$\mathbf{E}_{8, A_1 E_7}$	$[\mathbf{SU}^1(3) \times \mathbf{E}_6]/\mathbf{Z}_3$, $[\mathbf{SU}^1(3) \times \mathbf{E}_{6, D_5 T^1}]/\mathbf{Z}_3$, $[\mathbf{SU}(3) \times \mathbf{E}_{6, A_1 A_5}]/\mathbf{Z}_3$
$\mathbf{E}_8/[\mathbf{SU}(9)/\mathbf{Z}_3]$	$\mathbf{E}_8$	$\mathbf{SU}(9)/\mathbf{Z}_3$
	$\mathbf{E}_{8, D_8}$	$\mathbf{SU}^1(9)/\mathbf{Z}_3$ and $\mathbf{SU}^4(9)/\mathbf{Z}_3$
	$\mathbf{E}_{8, A_1 E_7}$	$\mathbf{SU}^2(9)/\mathbf{Z}_3$ and $\mathbf{SU}^3(9)/\mathbf{Z}_3$

13.7. *Table.* rank $G >$ rank H. *Then* $G = \bar{G}/Z$ *and* $H = \bar{H}Z/Z$, *where* Z *is an arbitrary central subgroup of* $\bar{G}$ *and where* $\bar{G}$ *and* $\bar{H}$ *are given as follows.*

M_u	$\bar{G}$	Center of $\bar{G}$	$\bar{H}$	Conditions
$\mathbf{Spin}(n^2-1)/\mathrm{ad}\ \mathbf{SU}(n)$	$\mathbf{Spin}(n^2-1)$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{SU}(n)/\mathbf{Z}_n$	n odd, $n > 2$
	$\mathbf{SO}^{2r(n-r)}(n^2-1)$	$\mathbf{Z}_2$	$\mathbf{SU}^r(n)/\mathbf{Z}_n$	n odd, $n > 2$, $0 < 2r \leqslant n$
$\mathbf{SO}(n^2-1)/\mathrm{ad}\ \mathbf{SU}(n)$	$\mathbf{SO}^{2r(n-r)}(n^2-1)$	1	$\mathbf{SU}(n)/\mathbf{Z}_n$	n even, $n > 3$, $0 \leqslant 2r \leqslant n$
$\mathbf{E}_6/[\mathbf{SU}(3)/\mathbf{Z}_3]$	simply connected (six-sheeted) covering group of $\mathbf{E}_{6,A_5A_1}$	$\mathbf{Z}_6$	$\mathbf{SU}^1(3)/\mathbf{Z}_3$	—
	$\mathbf{E}_6$	$\mathbf{Z}_3$	$\mathbf{SU}(3)/\mathbf{Z}_3$	—

Proof. We first dispose of the case where G is a complex Lie group. There Theorem 12.2 tells us that H_0 is a complex analytic subgroup and that $\mathfrak{G} = \mathfrak{A}^C$ and $\mathfrak{H} = \mathfrak{B}^C$ where (i) $\mathfrak{B} \subset \mathfrak{A}$ are the compact real forms of $\mathfrak{H} \subset \mathfrak{G}$, (ii) $\mathfrak{G}_u = \mathfrak{A} \oplus \mathfrak{A}$ and $\mathfrak{H}_u = \mathfrak{B} \oplus \mathfrak{B}$, and (iii) $\mathfrak{B}$ has absolutely irreducible linear isotropy representation (say β) in A/B. Let T denote the real tangent space of M, so $T^C = T' \oplus T''$, where T' is the holomorphic tangent space and $T'' = \overline{T'}$ is the antiholomorphic tangent space. Now H_0 acts on T' by β, on T'' by $\bar{\beta}$. But H is a complex subgroup of G, so its linear isotropy action preserves T' and T'', and we may view β and $\bar{\beta}$ as inequivalent (one is holomorphic, the other antiholomorphic) representations of H. Let $\mathbf{A}$ denote the commuting algebra of H on T, real division algebra by Schur's Lemma, $\neq \mathbf{R}$ because the complex structure gives it an element of square -1, and $\neq \mathbf{Q}$ because β and $\bar{\beta}$ are inequivalent on H. Then $\mathbf{A} = \mathbf{C}$ by elimination of all other possibilities. Thus M carries precisely two G-invariant almost complex structures, and these must be the ones defined by the natural complex structure and its conjugate. We have proved all our assertions for the case of a complex group G.

From now on, G is not a complex Lie group.

Suppose that χ is not absolutely irreducible. Then $\chi = \beta \oplus \bar{\beta}$ and a glance at Theorem 11.1 shows $\beta \not\simeq \bar{\beta}$; in particular χ has commuting algebra $\mathbf{C}$ so there are precisely two G-invariant almost complex structures on G/H_0. On the other hand, if M carries a G-invariant almost complex structure than χ has commuting algebra $\neq \mathbf{R}$, so χ cannot be absolutely irreducible. This shows equivalence of $(3a)$, $(3b)$ and $(3c)$, except that it remains to show that $(3a)$ implies connectivity of H.

Let M carry a G-invariant almost complex structure. Then $\mathfrak{G}=\mathfrak{H}+\mathfrak{M}$ and $\mathfrak{M}^C=\mathfrak{M}'+\mathfrak{M}''$ where, $\mathfrak{M}''=\overline{\mathfrak{M}'}$, H acts irreducibly (say by β) on $\mathfrak{M}'$ and by $\bar{\beta}$ on $\mathfrak{M}''$, and $\chi=\beta\oplus\bar{\beta}$ with $\beta \not\sim \bar{\beta}$. Let σ be the involutive automorphism of $\mathfrak{G}_u$ preserving $\mathfrak{H}_u$ which defines $\mathfrak{G}$ and $\mathfrak{H}$ as in Theorem 12.1. Theorem 12.1(2) shows that σ preserves $\mathfrak{M}'$ and $\mathfrak{M}''$. Let $h\in H$. Then $\mathrm{ad}(h)$ induces an automorphism of $\mathfrak{G}_u^C$ preserving $\mathfrak{H}^C$, $\mathfrak{M}'$ and $\mathfrak{M}''$, the latter two because the almost complex structure of M is preserved. Glancing through the cases of Corollary 13.2 we see that $\mathrm{ad}(h)|_{\mathfrak{H}}$ is an inner automorphism because $\mathrm{ad}(h)$ preserves $\mathfrak{M}'$ and $\mathfrak{M}''$. Thus we may replace h by an element of hH_0 and assume $\mathrm{ad}(h)|_{\mathfrak{H}}=1$. Now $\mathrm{ad}(h)$ has eigenvalues 1 on $\mathfrak{H}$, and $\mathrm{ad}(h)|_{\mathfrak{M}'}$ and $\mathrm{ad}(h)|_{\mathfrak{M}''}$ are in the commuting algebras of β and $\bar{\beta}$ respectively. Thus there is a complex number $c\neq 0$ such that $\mathrm{ad}(h)$ is multiplication by c on $\mathfrak{M}'$ and by $\bar{c}$ on $\mathfrak{M}''$. If rank $K=$ rank G then Theorem 2.2 implies $[\mathfrak{M}', \mathfrak{M}'']=\mathfrak{H}^C$ so $[\mathfrak{M}', \mathfrak{M}'']\neq 0$ and it follows that $c\bar{c}=1$. If rank $K<$ rank G we check from Corollary 13.2 that $[\mathfrak{M}', \mathfrak{M}'']\neq 0$ and it follows that $|c\bar{c}|$ is $|c|$ or 1. In either case, now, $|c|=1$, so h is contained in a maximal compact subgroup of H. Extending σ to $\mathfrak{G}$ and G we now have $\sigma(h)=h$. On the group level now we have $h\in G_u$, viewing both G and G_u as $\mathbf{R}$-analytic subgroups of a complex group with Lie algebra $\mathfrak{G}^C$. $G_u/(H_u\cup h\cdot H_u)$ has an invariant almost complex structure; now $h\in H_u$ by Theorem 13.1, and so $\mathrm{ad}(h)|_{\mathfrak{H}}=1$ implies that h is central in H_u. If rank $K<$ rank G it follows that $h=1$. If rank $K=$ rank G and σ is inner, $\sigma=\mathrm{ad}(k)$ for some $k\in H_u$ by Theorem 13.3, h is contained in any maximal torus T of H_u containing k, so $h\in T\subset H_0$. If rank $K=$ rank G and σ is outer then in each of the three cases of Theorem 13.3(2) h is contained in a toral subgroup T of H_u which is fixed by σ, so $h\in T\subset H_0$. In any case we have shown that H is connected. This completes the proof of equivalence of (3a), (3b) and (3c), which are clearly equivalent to (3d) by means of Theorem 12.1.

If M has a G-invariant complex structure then [8] $\mathfrak{G}^C=\mathfrak{L}+\bar{\mathfrak{L}}$ where $\mathfrak{L}$ is a complex subalgebra with $\mathfrak{H}^C=\mathfrak{L}\cap\bar{\mathfrak{L}}$. Then we may take $\mathfrak{L}=\mathfrak{H}^C+\mathfrak{M}''$ and $\bar{\mathfrak{L}}=\mathfrak{H}^C+\mathfrak{M}'$ in the notation above, and we have a map $f\colon M_u\to G^C/L$ given by $f(gH_u)=gL$. f is G_u-equivariant and maps $\mathfrak{M}$ isomorphically onto $\mathfrak{G}^C/\mathfrak{L}$; thus f is a nonsingular differentiable map with open image. As M_u, and thus $f(M_u)$, is compact, f must be surjective. Now f is a complex analytic covering. Theorem 13.1 shows M_u hermitian symmetric. Thus M is (indefinite) hermitian symmetric.

Now we need only check the listing in Tables 13.5, 13.6 and 13.7. If G and H have the same rank, the same is true (see Theorem 13.3) for their maximal compact subgroups, so G is centerless and G/H is simply connected. Now we need only check the equal rank case (Tables 13.5 and 13.6) on the Lie algebra level. We use the notation $\sigma=\mathrm{ad}\ (s)$, $s\in H_u$, $s^2=1$, in the case where σ is inner.

$M_u = \mathbf{SU}(p+q)/\mathbf{S}[\mathbf{U}(p)\times\mathbf{U}(q)]$. If σ is inner then the matrix s' representing s has square cI where $c^{p+q}=1$. Thus s' is a scalar multiple of $\operatorname{diag}[-I_u, I_{p-u}, -I_v, I_{p-v}]$ with $2u\leqslant p$ and $2v\leqslant q$. Then $\mathfrak{G}=\mathfrak{SU}^{u+v}(p+q)$ and $\mathfrak{H}=\mathfrak{S}[\mathfrak{U}^u(p)\times\mathfrak{U}^v(q)]$. If σ is outer then Theorem 13.3 says that p and q have a common value n and that σ interchanges the two local $\mathbf{U}(n)$ factors of H_u. Thus $\mathfrak{H}=\mathfrak{SL}(n,\mathbf{C})\cdot\mathfrak{Z}_0$, where its connected center Z_0 is a circle group because it is common to H_u, and $\mathfrak{G}$ is either $\mathfrak{SL}(n,\mathbf{Q})$ or $\mathfrak{SL}(2n,\mathbf{R})$ according to whether ${}^tg=-g$ or ${}^tg=g$, where $\sigma=\alpha\cdot\operatorname{ad}(g)$ and α is complex conjugation; see [21].

$M_u=\mathbf{SO}(2n)/\mathbf{U}(n)$. Theorem 13.3 says that σ is inner. The matrix s' representing s has square I or $-I$ and is in $\mathbf{U}(n)$, so it is $\mathbf{U}(n)$-conjugate to

$$\begin{pmatrix}-I_{2r} & 0\\ 0 & I_{2n-2r}\end{pmatrix} \text{ or } \begin{pmatrix}-J_r & 0\\ 0 & J_{n-r}\end{pmatrix}, \qquad 2r\leqslant n,$$

where J_t is the $2t\times 2t$ matrix with 2×2 blocks $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ down the diagonal and zeros elsewhere. Thus $\mathfrak{G}=\mathfrak{SO}^{2r}(2n)$ [resp. $\mathfrak{SO}^*(2n)$] and $\mathfrak{H}=\mathfrak{U}^r(n)$.

$M_u=\mathbf{Sp}(n)/\mathbf{U}(n)$. Theorem 13.3 says that σ is inner. The matrix s' is diagonable over $\mathbf{Q}$, so we may take it to be $\begin{pmatrix}-I_r & 0\\ 0 & I_{n-r}\end{pmatrix}$ and then $\mathfrak{G}=\mathfrak{Sp}^r(n)$ and $\mathfrak{H}=\mathfrak{U}^r(n)$.

$M_u=\mathbf{SO}(n+2)/\mathbf{SO}(n)\times\mathbf{SO}(2)$, $n>2$, $n\neq 4$. If σ is inner then s'^2 is I or $-I$, so we may conjugate s in H_u and assume that s' is given by

$$\begin{pmatrix}-I_{2r} & 0 & 0\\ 0 & I_{n-2r} & 0\\ 0 & 0 & \pm I_2\end{pmatrix} \text{ or } \begin{pmatrix}0 & I_m & 0 & 0\\ -I_m & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.$$

Then $\mathfrak{G}=\mathfrak{SO}^{m \text{ or } m+2}(n+2)$ and $\mathfrak{H}=\mathfrak{SO}^m(n)\oplus\mathfrak{SO}(2)$ with $2m\leqslant n$ and $m=2r$ or $m=n-2r$; or $n=2m$ with $\mathfrak{G}=\mathfrak{SO}^*(n+2)$ and $\mathfrak{H}=\mathfrak{SO}^*(n)\oplus\mathfrak{SO}(2)$. If σ is outer we are in case (2b) of Theorem 13.3 with $s=1$ and $n_1=1$; that is the same as the first of the two cases directly above except that $2r$ is replaced by an odd number.

If G is exceptional and rank $K=$ rank G, the possibilities for σ are quickly listed up to $\operatorname{ad}(H_u)$-conjugacy by means of Theorem 13.3; given such a σ, one looks at its action on a Weyl basis of $\mathfrak{G}^C$ and calculates the dimension of the fixed point set; the local form of G is specified by that dimension. These calculations are carried out in a somewhat more general context in [21] and the results are as recorded in Tables 13.5 and 13.6. This completes our checking for the case of equal ranks.

Now suppose rank $H<$ rank G. Then Corollary 13.2 says that $M_u=G_u/H_u$ is $\mathbf{Spin}(n^2-1)/\operatorname{ad}\mathbf{SU}(n)$ with $n>2$ odd, $\mathbf{SO}(n^2-1)/\operatorname{ad}\mathbf{SU}(n)$ with $n>3$ even, or $\mathbf{E}_6/[\mathbf{SU}(3)/\mathbf{Z}_3]$. $\sigma|_{H_u}$ is inner because it does not interchange the two summands of the linear isotropy representation. Thus there exists $s \in H_u$ such that $\mathrm{ad}(s)\cdot\sigma$ acts trivially on H_u. If $G_u = \mathbf{E}_6$ and σ is outer, then the fixed point set $\mathfrak{F}_u$ of $\mathrm{ad}(s)\cdot\sigma$ on $\mathfrak{G}_u$ has rank 4, so it properly contains the maximal subalgebra $\mathfrak{H}_u$; thus σ is inner if $G = \mathbf{E}_6$. If $G_u = \mathbf{SO}(n^2-1)$ with n even, $n = 2m$, then G_u is of type B_{2m^2-1} and so necessarily σ is inner. If $G_u = \mathbf{Spin}(n^2-1)$ with $n > 2$ odd and if σ is outer, then $n = 2m+1$ and the fixed point set $\mathfrak{F}_u$ of ad $(s)\cdot\sigma$ on $\mathfrak{G}_u$ has rank equal to rank $G_u - 1 = \frac{1}{2}(n^2-1) - 1 = 2m^2+2m-1$ and contains the maximal subalgebra $\mathfrak{H}_u$ of rank $n-1 = 2m$. Then $2m^2+2m-1 = 2m$, so $2m^2 = 1$, which is ridiculous. Thus σ is inner. Now we have verified that σ and $\sigma|_{H_u}$ are inner. So $\sigma = \mathrm{ad}(g)$ for some $g \in G_u$, and $\sigma|_{H_u} = \mathrm{ad}(s)|_{H_u}$ for some $s \in H_u$. Thus gs^{-1} centralizes H_u and consequently is central in G_u. It follows that $\sigma = \mathrm{ad}(s)$, $s \in H_u$, $s^2 = 1$.

$M_u = [\mathbf{Spin} \text{ or } \mathbf{SO}](n^2-1)/\mathrm{ad}\mathbf{SU}(n)$. As we work first on the Lie algebra level we may first assume $M_u = \mathbf{SO}(n^2-1)/\mathrm{ad}\mathbf{SU}(n)$, $n > 2$ even or odd. s is $\mathrm{ad}(H_u)$-conjugate to an element of H_u represented by the matrix $s' = (-1)^r \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} \in \mathbf{SU}(n)$, $2r \leqslant n$. Thus $\mathfrak{H} = \mathfrak{S}\mathfrak{U}^r(n)$. H is centerless so it must be $\mathbf{SU}^r(n)/\mathbf{Z}_n$ globally. The inclusion $\pi\colon \mathfrak{H}_u \to \mathfrak{G}_u$ is the adjoint representation. The $(+1)$-eigenspace of $\pi(s')$ is $\mathfrak{S}[\mathfrak{U}(r)\times\mathfrak{U}(n-r)]$ which has dimension $r^2+(n-r)^2-1$, so the (-1)-eigenspace of $\pi(s')$ has dimension $n^2-1-[r^2+(n-r)^2-1] = 2r(n-r)$. Thus $\mathfrak{G} = \mathfrak{S}\mathfrak{D}^{2r(n-r)}(n^2-1)$.

Let $\theta\colon \bar{G} \to G$ be a covering group, $\bar{H} = \theta^{-1}(H)_0$ and $\bar{M} = \bar{G}/\bar{H}$, such that $\bar{G}$ acts effectively on $\bar{M}$ and $\bar{M}$ is simply connected. Then $G = \bar{G}/Z$ and $H = (\bar{H}Z)/Z$, where Z is an arbitrary central subgroup of $\bar{G}$ which is characterized up to isomorphism by $Z \cong \pi_1(M)$. To find $\bar{G}$ we start with the existence of a central Z' in $\bar{G}$ such that $G' = \bar{G}/Z'$ is $\mathbf{SO}^{2r(n-r)}(n^2-1)$ and $H' = (\bar{H}Z')/Z'$ is the linear group ad $\mathbf{SU}^r(n)$. Then $Z' = \pi_1(G'/H') = \pi_1(K/L)$, where $L \subset K$ are the maximal compact subgroups of $H' \subset G'$. Now $L = \mathbf{S}[\mathbf{U}(r)\times\mathbf{U}(n-r)]/\mathbf{Z}_n$ and $K = K_1 \times K_2$, where $K_1 = \mathbf{SO}(r^2+(n-r)^2-1)$ and $K_2 = \mathbf{SO}(2r(n-r))$. As a linear representation, the inclusion $L \subset K$ is $\omega_1 \oplus \omega_2$, where $\omega_1 = \mathrm{ad}_L$ maps into K_1, and $\omega_2 = [\alpha_r \otimes \bar{\alpha}_{n-r}] \oplus [\bar{\alpha}_r \otimes \alpha_{n-r}]$ (where α_m is the usual vector representation of degree m of $\mathbf{U}(m)$) maps into K_2. If $r = 0$ we know G and K; now assume $r > 0$ so ω_2 is faithful. $\pi_1(K_2) = \mathbf{Z}_2$ because $n > 2$, so $\mathbf{Spin}(2r(n-r)) \to K_2$ is the universal covering. If $K_2/\omega_2(L)$ is not simply connected, now ω_2 lifts to a $\mathbf{Spin}(2r(n-r))$-valued representation. Then if b_t is a 1-parameter subgroup of L, $b_t = 1$ if and only if t is an integer, the lift of $\omega_2(b_t)$ to the Clifford algebra is 1 if and only if t is an integer. We test this with the 1-parameter subgroup $b_t = \mathrm{diag}\,[\varepsilon_t, \ldots, \varepsilon_t]$, $\varepsilon_t = e^{2\pi\sqrt{-1}t}$, of the subgroup $\mathbf{U}(r) \subset L$. In an orthonormal basis $\{v_i\}$ of $\mathbf{R}^{2r(n-r)}$, $\omega_2(b_t)$ has matrix diag $[E_t, \ldots, E_t]$, $E_t = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$. Thus the lift to the Clifford algebra on $\mathbf{R}^{2r(n-r)}$ is given by $b_t \to \prod v_{2j-1} \cdot (\cos \pi t v_{2j} + \sin \pi t v_{2j-1})$ where j ranges from 1 to $r(n-r)$ in the product. Thus $b_1 \to v_1 \cdot v_2 \cdot \ldots \cdot v_{2r(n-r)-1} \cdot v_{2r(n-r)}$. In other words, the representation ω_2 does not lift. If G'/H' is not simply connected, then $\bar{G} = \mathbf{Spin}^{2r(n-r)}(n^2-1)$ with maximal compact subgroup $\bar{K} = \bar{K}_1 \cdot \bar{K}_2$, $\bar{K}_i = \mathbf{Spin}(m)$ covering $K_i = \mathbf{SO}(m)$, so ω_2 lifts. That proves G'/H' simply connected. Thus $\bar{G} = G' = \mathbf{SO}^{2r(n-r)}(n^2-1)$ for $r > 1$.

$M_u = \mathbf{E}_6/[\mathbf{SU}(3)/\mathbf{Z}_3]$. If $\sigma \neq 1$, so $\sigma = \mathrm{ad}(s)$ with $s \in H_u$ of order 2, we may conjugate and assume that s is represented by $s' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{SU}(3)$. Then $H = \mathbf{SU}^1(3)/\mathbf{Z}_3$. Let α denote complex conjugation on $\mathbf{SU}(27)$; we view $G_u = \mathbf{E}_6$ as an α-stable subgroup of $\mathbf{SU}(27)$ The inclusion $H_u \to G_u$ is given by the real representation $\overset{2}{\circ}\!-\!\overset{2}{\circ}$ so any element of the image may be conjugated and assumed fixed under α. Thus we may assume $\alpha(s) = s$. Then α and σ commute, so α preserves the fixed point set K of σ, and σ preserves the fixed point set F of α. For convenience let $L = F \cap K$. Now K is of type D_5T^1 or A_5A_1, F is of type F_4 or C_4, and L is a group of rank 4 which is a symmetric subgroup in both K and F. Recall the symmetric subgroups of rank 4 in those groups.

C_4: A_3T^1, C_1C_3 and C_2C_2. F_4: B_4 and C_3C_1

A_5A_1: D_3A_1, D_3T^1, C_3A_1 and C_3T^1. D_5T^1: B_1D_3, B_2D_2 and B_3D_1.

Here note $D_3 = A_3$, $B_2 = C_2$, $D_2 = A_1A_1$, $D_1 = T^1$ and $A_1 = B_1 = C_1$. Despite this, D_5T^1 is eliminated as a possibility for K. Thus G is of type E_{6,A_5A_1}.

As before $\bar{G}/\bar{H} = \bar{M}$ is the simply connected covering group of M and $G = \bar{G}/Z$, $H = \bar{H}Z/Z$ where Z is an arbitrary central subgroup of G. We start on the matrix level with $G' = G/Z'$ which has maximal compact subgroup $K = [\mathbf{SU}(2) \times \mathbf{SU}(6)]/\mathbf{Z}_2$ embedded in $G' \subset \mathbf{SL}(27, \mathbf{C})$ by the representation $(\overset{1}{\circ} \otimes \overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ\!-\!\circ) \oplus (\circ \otimes \circ\!-\!\overset{1}{\circ}\!-\!\circ\!-\!\circ\!-\!\circ)$. Then the fundamental group $\pi_1(G') = \pi_1(K) = Z_2$. Let $G'' \to G'$ be the universal 2-fold covering, H'' the identity component of the inverse image of $H' = HZ'/Z'$. H' is centerless so H'' has center of order 1 or 2. But $H' = \mathbf{SU}^1(3)/\mathbf{Z}_3$ does not have a covering group with center of order 2. Thus G''/H'' is simply connected and effective. We have now proved that $G'' = \bar{G}$ is the simply connected group of type E_{6,A_5A_1}, that it has center $\mathbf{Z}_6$, and that $H'' = \bar{H} = \mathbf{SU}^1(3)/\mathbf{Z}_3$, q.e.d.

14. Invariant division algebras

Let A be an associative algebra of linear transformations of a real vector space V. By an *A-structure* on a differentiable manifold M, we mean a family $\{A_x\}_{x \in M}$, where A_x is an algebra of linear transformations of the tangent space M_x, and there exist linear isomorphisms $V \to M_x$ carrying A to A_x. If F is a division algebra $\mathbf{R}$ (reals), $\mathbf{C}$ (complex numbers) or $\mathbf{Q}$ (quaternions), then we view F^n as a real vector space of dimension $n \cdot (\dim_R F)$; this realizes F as an associative algebra of linear transformations of a real vector space and allows us to speak of an F-structure. Note that a $\mathbf{C}$-structure is more general than an almost complex structure, but that locally it defines an almost complex structure up to sign.

Let G be a differentiable transformation group on M. Then an A-structure $\{A_x\}_{x \in M}$ is called *G-invariant* if $x \in M$ and $g \in G$ imply that g carries A_x to $A_{g(x)}$.

Suppose that G is transitive on M, so $M = G/H$ where $H = \{g \in G: g(x_0) = x_0\}$. If $\{A_x\}_{x \in M}$ is a G-invariant A-structure on M, then A_{x_0} satisfies

(i) if $h \in H$, then $h_* A_{x_0} h_*^{-1} = A_{x_0}$,

and A_{x_0} defines the structure by means of

(ii) $A_{g(x_0)}$ is the image of A_{x_0} under g.

Conversely, if A_{x_0} is an algebra of linear transformations of M_{x_0}, then it defines an A_{x_0}-structure if and only if it satisfies (i), and in that case a G-invariant structure is defined by (ii). In particular:

14.1 LEMMA. *Let M be a coset space G/H, where H is the isotropy subgroup at a point x. Then M has a G-invariant $\mathbf{C}$-structure if and only if M_x has a complex vector space structure for which every tangent map h_{*x}, $h \in H$, is either $\mathbf{C}$-linear or $\mathbf{C}$-antilinear. M has a G-invariant $\mathbf{Q}$-structure if and only if M_x has a quaternionic vector space structure for which every h_{*x}, $h \in H$, is the product of a $\mathbf{Q}$-linear map and a $\mathbf{Q}$-scalar map.*

Now let $M = G/H$ be an effective reductive coset space such that the linear isotropy action of H_0 is an $\mathbf{R}$-irreducible representation χ. If χ is not absolutely irreducible Theorem 13.4 shows that its commuting algebra A is a complex number field; A is normalized by the non-identity components of H and thus extends to a G-invariant $\mathbf{C}$-structure on M. That is the unique G-invariant $\mathbf{C}$-structure on M. If χ is absolutely irreducible, so $A = \mathbf{R}$, then Theorem 13.4 shows that there is no G-invariant $\mathbf{C}$-structure on M. In other words

14.2 THEOREM. *Let $M = G/H$ be an effective reductive coset space, where G is a connected Lie group, H is a closed subgroup, and H_0 has $\mathbf{R}$-irreducible linear isotropy representation χ. Suppose that M is not euclidean. Then M has a G-invariant $\mathbf{C}$-structure if and only if χ is not absolutely irreducible, and in that case the structure is unique.*

It is known [18] that certain compact riemannian symmetric spaces have invariant $\mathbf{Q}$-structures. Essentially they are the ones which are base spaces of 2-sphere fibrations of compact complex homogeneous contact manifolds. We say 'essentially' because there is a mild complication which involves the notion of scalar part, which we now define.

Let $M=G/H$ be an effective reductive coset space with a G-invariant $\mathbf{Q}$-structure $\{A_z\}_{z\in M}$. Then the linear isotropy group H at x is a local direct product, $H_0=H'\cdot H''$, where the linear isotropy representation sends H' into transformations which commute with every element of A_x and sends H'' into A_x. H' is the $\mathbf{Q}$-*linear part* of H_0 and H'' is the $\mathbf{Q}$-*scalar part* of H_0. These parts are well defined because G/H is effective and reductive. We use the notation that $\mathbf{R}^*$, $\mathbf{C}^*$ and $\mathbf{Q}^*$ are the multiplicative groups of nonzero reals, complex numbers and quaternions, respectively; that $\mathbf{R}'$, $\mathbf{C}'$ and $\mathbf{Q}'$ are the respective subgroups consisting of elements of absolute value 1; and that $\mathbf{R}^*_+$ and $\mathbf{R}'_+$ are the respective subgroups of $\mathbf{R}^*$ and $\mathbf{R}'$ consisting of positive numbers. Now there are three types of possibilities for the analytic subgroup H'' of H, which we name and list as follows.

(i) The linear isotropy representation maps H'' into $\mathbf{R}^*$. Then H'' is isomorphic to $\mathbf{R}'_+=\{1\}$ or to $\mathbf{R}^*_+$, and we say that H has *real scalar part.*

(ii) The linear isotropy representation maps H'' into $\mathbf{C}^*$ but not into a real subfield. Then H'' is isomorphic to $\mathbf{C}'=\mathbf{T}^1$ or to $\mathbf{C}^*=\mathbf{C}'\times\mathbf{R}^*_+$, and we say that H has *complex scalar part.*

(iii) The linear isotropy representation maps H'' into $\mathbf{Q}^*$ but not into a complex subalgebra. Then H'' is isomorphic to $\mathbf{Q}'=\mathbf{Sp}(1)$ or to $\mathbf{Q}^*=\mathbf{Q}'\times\mathbf{R}^*_+$, and we say that H has *quaternionic scalar part.*

14.3 THEOREM. *Let $M=G/H$ be an effective reductive coset space, where G is a connected Lie group, H is a closed subgroup, and H_0 has* $\mathbf{R}$-*irreducible linear isotropy representation χ. Suppose that M is not euclidean.*

1. *M has no G-invariant* $\mathbf{Q}$-*structure for which H has real scalar part.*

2. *M has a G-invariant* $\mathbf{Q}$-*structure for which H has complex scalar part, if and only if the compact version $M_u=G_u/H_u$ is the complex projective plane.*

3. *M has a G-invariant* $\mathbf{Q}$-*structure for which H has quaternionic scalar part, if and only if* (3*a*) *$M_u=G_u/H_u$ is one of the compact quaternionic symmetric spaces classified in* [18, Theorem 5.4] *and the involutive automorphism σ of G_u which gives the Cartan involution of G is trivial on the subgroup of H_u corresponding to the* $\mathbf{Q}$-*scalar part of H, or* (3*b*) $\mathfrak{G}=\mathfrak{A}^C$, $\mathfrak{H}=\mathfrak{B}^C$, *$G_u=A\times A$, $H_u=B\times B$ and $M_u=(A/B)\times(A/B)$, where A/B is a nonhermitian compact quaternionic symmetric space listed in* [18, Theorem 5.4].

Proof. We may assume H connected. (1) is immediate from Theorem 12.1, which shows that χ cannot have commuting algebra $\mathbf{Q}$.

Let M have a G-invariant $\mathbf{Q}$-structure with complex scalar part. Then Theorem 13.4

shows that H'' is a circle group $\mathbf{C}' = \mathbf{T}^1$, so M_u is hermitian symmetric, and [18, Theorem 3.7] says that M_u is the complex projective plane. On the other hand, if M_u is the complex projective plane, then $G_u = \mathbf{SU}(3)/Z_3$ and $H_u = \mathbf{S}[\mathbf{U}(1) \times \mathbf{U}(2)]/\mathbf{Z}_3 \cong \mathbf{U}(2)$, and the first line of Table 13.5 shows that either $G = G_u$ and $H = H_u$, or $G = \mathbf{SU}^1(3)/\mathbf{Z}_3$ and $H = \mathbf{S}[\mathbf{U}(1) \times \mathbf{U}^1(2)]/\mathbf{Z}_3 \cong \mathbf{U}^1(2)$; in both cases M has a G-invariant $\mathbf{Q}$-structure for which H has complex scalar part.

Let M have a G-invariant $\mathbf{Q}$-structure for which H has quaternionic scalar part. If H is not semisimple then Theorem 12.1(2) says that the center of H is a circle group; thus H'' is $\mathbf{Sp}(1) = \mathbf{Q}'$, represented by $\overset{1}{\circ}$ in χ. Now suppose that G/H is not symmetric. Then a glance through the list of Theorem 11.1 eliminates the possibility that G and H are complex groups, so G_u/H_u is listed in Theorem 11.1. Then either χ is absolutely irreducible with $\chi = \chi|_{H'} \otimes \chi|_{H''}$ and $\chi|_{H''} = \overset{1}{\circ}$, or $\chi = \beta \oplus \bar{\beta}$ with $\beta = \beta|_{H'} \otimes \beta|_{H''}$ and $\beta|_{H''} = \overset{1}{\circ}$; no such spaces are listed in Theorem 11.1. In other words, G/H is symmetric. Theorems 12.1 and 12.2, with Theorem 5.4 of [18], now show that either G_u/H_u is one of the symmetric spaces listed in [18, Theorem 5.4] which have quaternionic structure such that H_u has quaternionic scalar part, or there is a nonhermitian quaternionic symmetric space A/B listed in [18, Theorem 5.4] such that $\mathfrak{G} = \mathfrak{A}^C$, $\mathfrak{H} = \mathfrak{B}^C$, $G_u = A \times A$, $H_u = B \times B$ and $M_u = (A/B) \times (A/B)$, q.e.d.

The *commuting structure* on a coset space G/H is the G-invariant structure $\{A_x\}_{x \in G/H}$ where A_x is the commuting algebra of the linear isotropy group at x. Theorems 14.2 and 14.3 say, for a simply connected noneuclidean reductive isotropy irreducible coset space G/H, that the commuting structure is an $\mathbf{R}$-structure if the linear isotropy representation is absolutely irreducible, is a $\mathbf{C}$-structure otherwise, and cannot be a $\mathbf{Q}$-structure.

Note that the commuting structure is the structure of the algebra of $n \times n$ real matrices if and only if $\chi = \beta_1 \oplus \ldots \oplus \beta_n$ with β_i absolutely irreducible real and all the β_i equivalent. In this context see Lemma 12.3 and Remark 12.4.

Chapter III. Riemannian geometry on isotropy irreducible coset spaces

An isotropy irreducible coset space $M = G/K$, with K compact, has a riemannian metric which is unique up to a constant scalar factor. In § 15 we see that M is an Einstein manifold and that sectional curvature keeps its sign, and we determine when two such riemannian manifolds are isometric. In § 16 we determine the linear holonomy group of M. § 17 contains the determination of the full group of isometries; if M has invariant almost complex structure we also determine the full group of almost hermitian isometries and study the group of almost-analytic diffeomorphisms. Finally, in § 18, we study locally isotropy irreducible riemannian manifolds and their relations to isotropy irreducible coset spaces.

Most of the results extend immediately to indefinite metric.

15. Curvature and equivalence

The elementary properties of isotropy irreducible riemannian homogeneous spaces are given by the following theorem.

15.1 THEOREM. *Let M be an effective coset space G/K of a connected Lie group by a compact subgroup, where K is* **R***-irreducibile on the tangent space.*

1. *If ds^2 and $d\sigma^2$ are G-invariant riemannian metrics on M, then $ds^2 = c \cdot d\sigma^2$ for some constant $c > 0$. In particular ds^2 and $d\sigma^2$ have the same Levi–Civitá connection.*

2. *Choose a G-invariant riemannian metric ds^2 on M, let $\mathbf{r}$ denote the Ricci tensor, and let r denote the scalar curvature. Then (M, ds^2) is an Einstein space, $\mathbf{r} = \frac{r}{n}\, ds^2$ with r constant and $n = \dim M$.*

3. *If the identity component K_0 is* **R***-irreducible on the tangent space, then*

(3a) $r < 0$, (M, ds^2) is a riemannian symmetric space of noncompact type, and every sectional curvature satisfies $\varkappa \leqslant 0$; or

(3b) $r = 0$ and (M, ds^2) is a euclidean space; or

(3c) $r > 0$, M is compact, and (M, ds^2) has every sectional curvature $\varkappa \geqslant 0$.

Remark. Now Theorem 11.1 gives many new examples of Einstein spaces which are neither symmetric nor kaehlerian.

Remark. By uniqueness, the Levi–Cività connection on M must be the first canonical connection for G/K.

Proof. K is the isotropy subgroup at some point $x \in M$. Let χ be the representation of K on M_x. As χ is **R**-irreducible, any nonzero $\chi(K)$-invariant symmetric bilinear form on M_x is definite and any two are proportional. Thus $ds_x^2 = c \cdot d\sigma_x^2$ for some $c > 0$. If $z \in M$, $z = g^{-1}(x)$, then $ds_z^2 = g^* ds_x^2 = c \cdot g^* d\sigma_x^2 = c \cdot d\sigma_z^2$. This proves (1). Similarly $\mathbf{r} = f \cdot ds^2$ for some constant f, and $f = r/n$ by definition of r; this proves (2).

Let $\chi(K_0)$ be **R**-irreducible on M_x. If G is not semisimple then Lemma 1.2 shows (M, ds^2) isometric to euclidean space; in particular $r = 0$. If G is noncompact and semisimple then K is a maximal compact subgroup, so (M, ds^2) is a riemannian symmetric space of noncompact type; in particular $r < 0$ and every sectional curvature $\varkappa \leqslant 0$. If G is compact and semisimple then M is compact. By uniqueness, ds_x^2 is the restriction of a negative multiple of the Killing form of $\mathfrak{G}$ to the orthocomplement of $\mathfrak{K}$; in particular every sectional curvature $\varkappa \geqslant 0$ and some $\varkappa > 0$; it follows that $r > 0$, q.e.d.

Distinct coset spaces may give isometric manifolds. For example, we have euclidean $(2n)$-space given as $\mathbf{SO}(2n)\cdot\mathbf{R}^{2n}/\mathbf{SO}(2n)$, $\mathbf{SU}(n)\cdot\mathbf{R}^{2n}/\mathbf{SU}(n)$ and $\mathbf{U}(n)\cdot\mathbf{R}^{2n}/\mathbf{U}(n)$. This is not a phenomenon restricted to euclidean spaces, for $\mathbf{Spin}(7)/\mathbf{G}_2$ is isometric to the 7-sphere $\mathbf{S}^7$. Thus we need the uniqueness theorem:

15.2 THEOREM. *Let G/K and A/B be simply connected effective coset spaces of connected Lie groups by compact subgroups with* $\mathbf{R}$*-irreducible linear isotropy representations. Let each carry an invariant riemannian metric and suppose that they are isometric. Then*

(i) *there is an isomorphism of G onto A which carries K onto B; or*

(ii) *G/K and A/B are euclidean spaces of the same dimension; or*

(iii) *G/K and A/B are the two presentations* $\mathbf{Spin}(7)/\mathbf{G}_2$ *and* $\mathbf{SO}(8)/\mathbf{SO}(7)$ *of the sphere* $\mathbf{S}^7$*; or*

(iv) *G/K and A/B are the two presentations* $\mathbf{G}_2/\mathbf{SU}(3)$ *and* $\mathbf{SO}(7)/\mathbf{SO}(6)$ *of the sphere* $\mathbf{S}^6$.

Proof. If one (thus both) of G/K and A/B is a euclidean space then we are in case (ii). If G/K and A/B are both noneuclidean symmetric coset spaces then we are in case (i). Now we may assume that A/B is a nonsymmetric coset space, so it is listed in Theorem 11.1, and that G/K either is a compact irreducible symmetric coset space or is listed in Theorem 11.1.

Let M be the common riemannian manifold of G/K and A/B and write $M=U/V$, where U is the largest connected group of isometries. Then $G\subset U$ and $A\subset U$, and we may assume $K\subset V$ and $B\subset V$. It suffices to prove our assertion in the case where U is G or A; for then $G\neq U\neq A$ implies that either $M=\mathbf{S}^7$ with G and A as conjugates of $\mathbf{Spin}(7)$ in $\mathbf{SO}(8)$, or $M=\mathbf{S}^6$ with G and A as conjugates of $\mathbf{G}_2$ in $\mathbf{SO}(7)$, and we are in case (i). Thus we are reduced to considering the case $A\subsetneqq G$ with $B=A\cap K$. These situations are classified by A. L. Oniščik ([22], Table 7, p. 219 [p. 29 in the translation], except that Oniščik writes: G' for K and G'' for A, or G' for A and G'' for K), U for B, and $Sp(2n)$ for $\mathbf{Sp}(n)$, A and G are simple and B is semisimple. Thus $\pi_2(A/B)=\pi_2(G/K)$ is finite, so K is semisimple. Now we need only run through the entries on Oniščik's list which have G'' and U semisimple, checking for isotropy irreducibility. Doing that, we find that we are in case (iii) or case (iv) of the theorem, q.e.d.

16. Holonomy

Let M be a riemannian manifold. If $x\in M$, then $\mathbf{O}(M_x)$ denotes the orthogonal group of the tangent space and $\mathbf{SO}(M_x)$ is the subgroup consisting of proper rotations. If σ is a sectionally smooth curve in M with both endpoints at x, then the parallel translation about σ is an element $\tau_\sigma^M\in\mathbf{O}(M_x)$. All such transformations τ_σ^M compose the *linear holonomy* *group* $H(M, x)$ at x. $H(M, x)$ carries the subspace topology from its inclusion in $\mathbf{O}(M_x)$; the arc component of I is a closed Lie subgroup $H_0(M, x) \subset \mathbf{SO}(M_x)$ which is called the *restricted linear holonomy group* of M at x. $H_0(M, x)$ consists of all τ_σ^M for which σ is homotopic (with fixed endpoints) to the trivial curve at x. If φ: $M' \to M$ is a riemannian covering, $\varphi(x') = x$, then $\tau_\sigma^{M'} \to \tau_{\varphi\sigma}^M$ defines an injection φ_*: $H(M', x') \to H(M, x)$ which is equivariant with the tangent map φ_*: $M'_{x'} \to M_x$. Thus restricted linear holonomy is invariant under riemannian coverings.

The holonomy of symmetric spaces is well known, although I cannot find the global statement in the literature:

16.1 PROPOSITION. *Let $M = G/K$ be an effective symmetric coset space with a G-invariant riemannian metric, where G is a connected Lie group and K is a compact subgroup. Let χ be the linear isotropy action of K on M_x. Decompose $M \sim M^O \times M'$ locally as the product of a euclidean space M^O and a product M' of irreducible spaces, so $x = (x^O, x')$ and $M_x = M^O_{x^O} \oplus M'_{x'}$. Then $K = K^O \times K'$ and $\chi = \chi^O \oplus \chi'$ where $\chi^O(K^O)$ acts on $M^O_{x^O}$, $\chi'(K')$ acts on $M'_{x'}$, and $H(M, x) = \chi'(K')$.*

The proof is immediate, by means of the universal riemannian covering, from ([17], § 7) and ([16], § 3).

By way of contrast, the holonomy of isotropy irreducible nonsymmetric coset spaces s much less complicated:
i

16.2 THEOREM. *Let $M = G/K$ be a nonsymmetric effective coset space with a G-invariant riemannian metric, where G is a connected Lie group and K is a compact subgroup. Suppose that the linear isotropy action χ of K_0 on M_x is $\mathbf{R}$-irreducible. Then*

$$H(M, x) = \mathbf{SO}(M_x) \text{ if } M \text{ is orientable,}$$
$$H(M, x) = \mathbf{O}(M_x) \text{ if } M \text{ is not orientable.}$$

An immediate consequence is:

16.3 COROLLARY. *If Ψ is a nonzero parallel differential form on M, then either Ψ is a scalar constant, or M is orientable and Ψ is a constant multiple of the volume element.*

Proof of theorem. M is orientable if and only if $H(M, x) \subset \mathbf{SO}(M_x)$. Thus we need only prove $H_0(M, x) = \mathbf{SO}(M_x)$, and for this we may assume M simply connected. The de Rham decomposition of M as a product of a euclidean space and some irreducible riemannian manifolds, decomposes the largest connected group of isometries; thus $\mathbf{R}$-irreducibility of χ implies that M is an irreducible riemannian manifold. Now ([1] or [13]) either M is isometric to an irreducible riemannian symmetric space, or $H(M, x)$ is transitive on the unit sphere in M_x.

Let M be isometric to an irreducible riemannian symmetric space. Then Theorem 15.2 says that M is isometric to a sphere $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, and Proposition 16.1 now implies that $H(M, x) = \mathbf{SO}(M_x)$.

Let $H(M, x)$ be transitive on the unit sphere $\mathbf{S}^{n-1}$ in M_x. Then $H(M, x)$ must be (i) $\mathbf{SO}(n) = \mathbf{SO}(M_x)$, (ii) $\mathbf{SU}\left(\frac{n}{2}\right)$, (iii) $\mathbf{U}\left(\frac{n}{2}\right)$, (iv) $\mathbf{Sp}\left(\frac{n}{4}\right)$, (v) $\mathbf{Sp}\left(\frac{n}{4}\right)\cdot\mathbf{T}^1$, (vi) $\mathbf{Sp}\left(\frac{n}{4}\right)\cdot\mathbf{Sp}(1)$, (vii) $\mathbf{G}_2$ with $n=7$, (viii) $\mathbf{Spin}(7)$ with $n=8$; or (ix) $\mathbf{Spin}(9)$ with $n=16$.

If the commuting algebra of $H(M, x)$ on M_x has an element J of square $-I$, then J defines a kaehlerian structure on M. By compactness, the cohomology $\mathbf{H}^2(M; \mathbf{R}) \neq 0$, so $\pi_2(M)$ is infinite by the Hurewicz Theorem. The exact homotopy sequence of $G \to G/K = M$ then shows $\pi_1(K)$ infinite, contradicting semisimplicity of K. This excludes the possibilities (ii), (iii), (iv) and (v) for $H(M, x)$.

In the possibilities remaining, $H(M, x)$ is its own connected normalizer in $\mathbf{SO}(M_x)$. Thus $\chi(K) \subset H(M, x)$.

If M_x has an $H(M, x)$-stable structure as a quaternionic vector space, then we have a G-invariant $\mathbf{Q}$-structure on M, contradicting Theorem 14.3. This excludes the possibility (vi) for $H(M, x)$.

If M has dimension 7 and K is isomorphic to a subgroup of $\mathbf{G}_2$ then Theorem 11.1 says $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$, so $H(M, x) = \mathbf{SO}(M_x)$. This excludes possibility (vii) for $H(M, x)$.

Following Dynkin ([6], Table 5), a connected semisimple subgroup of $\mathbf{Spin}(2m+1)$ absolutely irreducible on $\mathbf{R}^{2m}$ must be all of $\mathbf{Spin}(2m+1)$. As $\circ\!-\!\circ\!=\!\overset{1}{\bullet}$ and $\circ\!-\!\circ\!-\!\circ\!=\!\overset{1}{\bullet}$ are not among the possibilities for χ listed in Theorem 11.1, this excludes possibilities (viii) and (ix) for $H(M, x)$ *in the case where χ is absolutely irreducible*. But if χ is not absolutely irreducible, Theorem 11.1 shows that M cannot have dimension 8 or 16. This excludes the possibilities (viii) and (ix) for $H(M, x)$, q.e.d.

17. Isometries

Let M be a riemannian manifold. Then $\mathbf{I}(M)$ denotes the group of all isometries of M onto itself. As is now standard, we let $\mathbf{I}(M)$ carry the compact-open topology, and then $\mathbf{I}(M)$ is a Lie transformation group on M. The identity component is denoted $\mathbf{I}_0(M)$.

Given an effective coset space $M = G/K$ with a G-invariant riemannian metric, one knows $G \subset \mathbf{I}(M)$. But in general one does not know how to determine $\mathbf{I}(M)$, or even $\mathbf{I}_0(M)$, and this can be troublesome.

The determination of $\mathbf{I}(M)$ is reduced to the simply connected case as follows. Let φ: $M' \to M$ be the universal riemannian covering. Then $M = M'/\Gamma$, where $\Gamma \subset \mathbf{I}(M')$ is the group of deck transformations of the covering. Every $g \in \mathbf{I}(M)$ lifts to a φ-fibre preserving map $g' \in \mathbf{I}(M')$, and the φ-fibre maps in $\mathbf{I}(M')$ induce isometries of M. An element $g' \in \mathbf{I}(M')$ maps a φ-fibre $\Gamma(x')$ to another (necessarily $\Gamma(g'x')$) if and only if $(g'\Gamma)(x') = (\Gamma g')(x')$. Let N_Γ be the normalizer of Γ in $\mathbf{I}(M')$, so its identity component N_Γ^0 is the centralizer of Γ. It follows that

$$\mathbf{I}(M) = N_\Gamma/\Gamma \quad \text{and} \quad \mathbf{I}_0(M) = (\Gamma \cdot N_\Gamma^0)/\Gamma.$$

If $M = G/K$ is a simply connected symmetric coset space, then the determination of $\mathbf{I}_0(M)$ and $\mathbf{I}(M)$ from (G, K) is due to Cartan (see [16, § 2]); in the irreducible case $G_0 = \mathbf{I}_0(M)$, so $K_0 = K \cap \mathbf{I}_0(M)$, and $\mathbf{I}(M)$ is constructed from the pair (G_0, K_0) by examining automorphisms of K_0 which extend to G_0. It turns out that Cartan's idea works for isotropy irreducible spaces:

17.1 Theorem. *Let $M = G/K$ be a noneuclidean simply connected effective coset space with a G-invariant riemannian metric, where G is a connected Lie group and K is a (necessarily connected) compact subgroup. Suppose that the linear isotropy action χ of K on M_x is $\mathbf{R}$-irreducible. Suppose* $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$. Then $G = \mathbf{I}_0(M)$.

Let $\mathrm{Aut}(K)^G$ *denote the group of all automorphisms of K which extend to G, and let* $\mathrm{Inn}(K)^G$ *denote the normal subgroup of finite index consisting of inner automorphisms of K; let* $\mathrm{Aut}(K)^G = \bigcup_{i=1}^r k_i \cdot \mathrm{Inn}(K)^G$ *be the coset decomposition. Define*

$\bar{G} = G \cup s \cdot G$ *and* $\bar{K} = K \cup s \cdot K$ *if* rank $G >$ rank K *and G/K is symmetric with symmetry s,*

$\bar{G} = G$ *and* $\bar{K} = K$ *otherwise.*

If $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$, *then*

$$\mathbf{I}(M) = \bigcup_{i=1}^r k_i \cdot \bar{G}, \text{ and } \bigcup_{i=1}^r k_i \cdot \bar{K} \text{ is the isotropy subgroup at } x.$$

Remark. If $G/K = \mathbf{G}_2/\mathbf{SU}(3)$, then $M = \mathbf{S}^6$ must be rewritten as $\mathbf{SO}(7)/\mathbf{SO}(6)$ to apply the theorem. If $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$ then $M = \mathbf{S}^7$ must be re-written as $\mathbf{SO}(8)/\mathbf{SO}(7)$.

Proof. Let $A = \mathbf{I}_0(M)$ and let B be the isotropy subgroup at x. Then $M = A/B$, and $K \subset B$ shows B to be $\mathbf{R}$-irreducible on M_x. If $A \neq G$, then Theorem 15.2 says that either $G/K = \mathbf{G}_2/\mathbf{SU}(3)$ with $A/B = \mathbf{SO}(7)/\mathbf{SO}(6)$, or $G/K = \mathbf{Spin}(7)/\mathbf{G}_2$ with $A/B = \mathbf{SO}(8)/\mathbf{SO}(7)$. Thus $\mathbf{G}_2/\mathbf{SU}(3) \neq G/K \neq \mathbf{Spin}(7)/\mathbf{G}_2$ implies $G = \mathbf{I}_0(M)$.

Now we must prove:

(17.2) *Suppose $G = \mathbf{I}_0(M)$. Let K' be the isotropy subgroup of $\mathbf{I}(M)$ at x. Let $k' \in K'$. Then $k' \in \bar{K}$ if and only if $\mathrm{ad}(k')|_K$ is an inner automorphism of K.*

(17.2) is meaningful because $\bar{K} \subset K'$ and K is the identity component of K'. If $k' \in \bar{K}$, then either $k' \in K$, or $k's \in K$, where s is the symmetry. In the latter case $\mathrm{ad}(k's)|_K = \mathrm{ad}(k')|_K$. Thus $\mathrm{ad}\,(k')|_K$ is inner.

Conversely let $\mathrm{ad}\,(k')|_K$ be inner. Then K has an element k such that $\mathrm{ad}(k'k)|_K$ is trivial. Let χ' be the isotropy representation of K' on M_x and let A be the commuting algebra of χ. Then $\chi'(k'k) \in A$. If $\chi'(k'k) = I$ then $k'k = 1$ and $k' \in \bar{K}$. If $\chi'(k'k) = -I$ then M is symmetric and $k'k = s \in \bar{K}$, so $k' \in \bar{K}$.

Now suppose $\chi'(k'k) \neq \pm I$. Then $A \not\cong \mathbf{R}$, so χ is not absolutely irreducible and $A \cong \mathbf{C}$; $\chi'(k'k) \in A$ corresponds to a non-real element $\varepsilon \in \mathbf{C}$ of norm 1. Let Z be the center of K. If M is symmetric, it is hermitian symmetric and $\chi(Z) \subset A$ corresponds to the set of all elements of norm 1 in $\mathbf{C}$; thus $k'k \in Z$ and so $k' \in \bar{K}$.

Now assume M nonsymmetric. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{M}$, $\mathfrak{K}^C = \mathfrak{M}_0$ and $\mathfrak{M}^C = \mathfrak{M}_\varepsilon + \mathfrak{M}_{\bar{\varepsilon}}$, where $\mathrm{ad}(k'k)$ is scalar multiplication by α on $\mathfrak{M}_\alpha$. Note that $\mathrm{ad}(k'k)$ is scalar multiplication by $\alpha\beta$ on $[\mathfrak{M}_\alpha, \mathfrak{M}_\beta]$. Thus $[\mathfrak{M}_\varepsilon, \mathfrak{M}_{\bar{\varepsilon}}] \subset \mathbf{K}^C$ and $[\mathfrak{M}_\varepsilon, \mathfrak{M}_\varepsilon] \subset \mathfrak{M}_{\varepsilon^2}$. As $\varepsilon^2 \neq \varepsilon$, and as $[\mathfrak{M}^C, \mathfrak{M}^C] \not\subset \mathfrak{K}^C$ by nonsymmetry, now $\varepsilon^2 = \bar{\varepsilon}$. Thus $\varepsilon = e^{\pm 2\pi i/3}$. If rank G = rank K, then Z has order 3 and so $\chi'(k'k) \in \chi(Z)$; thus $k'k \in Z$ and $k' \in \bar{K}$.

Finally suppose rank $G >$ rank K. G/K is $\mathbf{E}_6/\{\mathbf{SU}(3)/\mathbf{Z}_3\}$ or $\{(\mathbf{Spin}\text{ or }\mathbf{SO})\,(n^2-1)\}/\mathrm{ad}\mathbf{SU}(n)$. If τ is an outer automorphism of order 3 on G, it follows that $G/K = \mathbf{Spin}(8)/\mathrm{ad}\mathbf{SU}(3)$ and τ is triality; then τ has fixed point set $\mathbf{G}_2$ which does not contain the centerless version of $\mathbf{SU}(3)$. As $k'k$ has order 3 it follows that $\mathrm{ad}(k'k)|_G$ is an inner automorphism. Let $g \in G$ such that $\mathrm{ad}(k'k)|_G = \mathrm{ad}(g)$ and let L be the connected centralizer of g. Then $K \subset L$ and rank L = rank $G >$ rank K; thus $L = G$. Now $\mathrm{ad}(k'k)|_M = \chi'(k'k) = I$, so $k'k = 1$ and $k' \in \bar{K}$. This completes the proof of (17.2).

Now $\bar{K} \subset K' \subset \bigcup_{i=1}^r k_i \cdot \bar{K}$. Re-ordering the k_i, it follows that

$$K' = \bigcup_{i=1}^{v} k_i \cdot \bar{K}, \text{ and thus } \mathbf{I}(M) = \bigcup_{i=1}^{v} k_i \cdot \bar{G}.$$

Define groups $K'' = \bigcup_{i=1}^r k_i \cdot \bar{K}$ and $G'' = \bigcup_{i=1}^r k_i \cdot \bar{G}$. Then G is transitive on G''/K'' and $K = G \cap K''$. Thus we identify M with G''/K''. Let V be the kernel of the action of G'' on M. Then $V \subset K''$ is normalized by K, and V is finite because $V \cap K = V \cap K_0'' = \{1\}$. As K is connected it follows that V centralizes K. Now $V \subset \bar{K} \subset K'$, so $V = \{1\}$. Thus G'' is effective on M. As K'' is compact, M has a G''-invariant riemannian metric. That metric is G-invariant, hence proportional to the original one. Thus $G'' \subset \mathbf{I}(M)$. But we just saw $\mathbf{I}(M) \subset G''$, so now $\mathbf{I}(M) = G''$ and $K' = K''$, q.e.d.

A similar result holds for isometries which preserve an almost-complex structure. The symmetric case is due to Cartan.

17.3 THEOREM. *Let $M = G/K$ be a noneuclidean simply connected effective coset space, with a G-invariant riemannian metric ds^2 and a G-invariant almost complex structure J. Suppose that G is a connected Lie group and K is a compact subgroup whose linear isotropy action is* $\mathbf{R}$*-irreducible. Given $z \in M$ and tangent vectors $X, Y \in M_z$, define $\omega_z(X, Y) = ds_z^2(X, J_z Y)$ and define $h_z = ds_z^2 + i\omega_z$. Then h is an almost-hermitian metric on M.*

Let $\mathbf{H}(M)$ be the group of all almost-hermitian isometries of M and let $\mathbf{H}_0(M)$ be the identity component. Then $G = \mathbf{H}_0(M)$. If $G/K \neq \mathrm{ad}(\mathbf{E}_6)/\mathbf{A}_2\cdot\mathbf{A}_2\cdot\mathbf{A}_2$, then $G = \mathbf{H}(M)$. If $G/K = \mathrm{ad}\,(\mathbf{E}_6)/\mathbf{A}_2\cdot\mathbf{A}_2\cdot\mathbf{A}_2$, then $\mathbf{H}(M) = G \cup \varphi\cdot G$, where $\mathrm{ad}\,(\varphi)|_G$ is an involutive outer automorphism with fixed point set $\mathbf{F}_4$. M has an isometry λ which sends J to $-J$. If $G/K \neq \mathbf{G}_2/\mathbf{SU}(3)$, then $\mathbf{I}(M) = \mathbf{H}(M) \cup \lambda\cdot\mathbf{H}(M)$.

Proof. Let χ be the representation of K on M_x. Then the commuting algebra of χ is $\mathbf{C}$ and J_x is one of its two elements of square $-I$. The unimodular elements of $\mathbf{C}$ are in $\mathbf{SO}(M_x)$. Thus h_x is a hermitian inner product on M_x. Now h is an almost-hermitian metric on M.

Let $\mathbf{H}(M)$ be the group of all almost-hermitian isometries of M. Then $G \subset \mathbf{H}(M) \subset \mathbf{I}(M)$. If $G/K \neq \mathbf{G}_2/\mathbf{SU}(3)$, then $G = \mathbf{I}_0(M)$ and so $G = \mathbf{H}_0(M)$. If $G/K = \mathbf{G}_2/\mathbf{SU}(3)$, then $\mathbf{I}_0(M) = \mathbf{SO}(7)$ has $G = \mathbf{G}_2$ as a maximal connected subgroup, and $\mathbf{I}_0(M) \not\subset \mathbf{H}(M)$; it follows that $G = \mathbf{H}_0(M)$. Now $G = \mathbf{H}_0(M)$ in general.

Let L be the isotropy subgroup of $\mathbf{H}(M)$ at x. Then $K = L_0$. If $k' \in L$, then k'_* commutes with J_x, so $\mathrm{ad}(k')|_K$ does not interchange the two irreducible summands of χ. If $G/K = \mathrm{ad}(\mathbf{E}_6)/\mathbf{A}_2\cdot\mathbf{A}_2\cdot\mathbf{A}_2$, assume further that $\mathrm{ad}(k')|_G$ is inner. Then a case by case check shows that $\mathrm{ad}(k')|_K$ is inner. Let $k \in K$ so that $\mathrm{ad}(k'k)|_K$ is trivial. As in the proof of Theorem 17.1, it follows that $k'k$ is central in K, so $k' \in K$. On the other hand, as noted in the proof of Theorem 13.6, $\mathrm{ad}(\mathbf{E}_6)/\mathbf{A}_2\cdot\mathbf{A}_2\cdot\mathbf{A}_2$ has an almost complex involutive automorphism φ such that $\mathrm{ad}(\varphi)$ is outer on $\mathbf{E}_6$. Thus $G = \mathbf{H}(M)$ for $G/K \neq \mathrm{ad}(\mathbf{E}_6)/\mathbf{A}_2\cdot\mathbf{A}_2\cdot\mathbf{A}_2$ and $\mathbf{H}(M) = G \cup \varphi\cdot G$ in the exceptional case.

We find λ. First suppose G compact with rank $G >$ rank K. If $G/K = \mathbf{E}_6/\mathrm{ad}\;\mathbf{SU}(3)$ then λ is the outer automorphism of G which preserves K. If $G/K = \mathbf{SO}(n^2-1)/\mathrm{ad}\;\mathbf{SU}(n)$, then λ is the outer automorphism of $\mathfrak{SU}(n)$ viewed as an element of $\mathbf{GL}(n^2-1, \mathbf{R})$ which normalizes $\mathbf{SO}(n^2-1)$. Now suppose G compact with rank $G =$ rank K. Embed the center Z of K in a maximal torus $T \subset K$, and let λ be the automorphism of order 2 on G which is $-I$ on $\mathfrak{T}$. λ preserves K because K is the connected centralizer of Z. Finally suppose G noncompact. Then G/K is hermitian symmetric of noncompact type, and we have an automorphism λ_u of the compact form G_u which preserves K and is inversion on Z. Extend λ_u from $\mathfrak{G}_u$ to $\mathfrak{G}^C$ by linearity and let λ be its restriction to $\mathfrak{G}$.

Finally suppose $G/K \neq \mathbf{G}_2/\mathbf{SU}(3)$. If k is in the isotropy subgroup of $\mathbf{I}(M)$ at x, then k_* either commutes or anticommutes with J_x, for $k_* J k_*^{-1}$ is another almost complex structure on M. In the commuting case, $k \in \mathbf{H}(M)$. In the anticommuting case, $k\lambda \in \mathbf{H}(M)$. Thus $\mathbf{I}(M) = \mathbf{H}(M) \cup \lambda \cdot \mathbf{H}(M)$, q.e.d.

The analysis of $\mathbf{H}(M)$ allows us to study the group $\mathbf{A}(M)$ of all almost-complex diffeomorphisms of M:

17.4 THEOREM. *Let $M = G/K$ be a noneuclidean effective coset space with a G-invariant almost-complex structure, where G is a connected Lie group and K is a compact subgroup whose identity component is $\mathbf{R}$-irreducible on the tangent space. Choose a G-invariant riemannian metric on M. Then:*

1. *If G/K is noncompact then $\mathbf{A}(M) = \mathbf{H}(M) = \mathbf{I}_0(M) = G$.*
2. *If G/K is compact, then $\mathbf{A}(M)$ is a simple Lie group with maximal compact subgroup $\mathbf{H}(M)$, and $\mathbf{A}_0(M)$ is a simple Lie group with finite center and maximal compact subgroup $G = \mathbf{H}_0(M)$.*
3. *If G/K is compact with* rank $G =$ rank K, *then either G/K is hermitian symmetric with $\mathbf{A}(M) = \mathbf{H}(M)^C = G^C$, or G/K is nonsymmetric with $\mathbf{A}(M) = \mathbf{H}(M)$ and $\mathbf{A}_0(M) = G$.*

Proof. If G/K is noncompact it is a hermitian symmetric space of noncompact type. Then, in the Harish-Chandra realization as a bounded domain with Bergman metric, every analytic automorphism is an isometry; so $\mathbf{A}(M) = \mathbf{H}(M)$ and our assertions follow from Theorem 17.3

Now assume G/K compact. Let A denote $\mathbf{A}_0(M)$ and let B denote the isotropy subgroup; so $G \subset A$ and $K = G \cap B$. $\mathbf{A}(M)$ is a Lie group [2]; now $\mathbf{H}(M)$ must be a maximal compact subgroup. In particular G is a maximal compact subgroup of A. Whenever S is a closed connected subgroup of A normalized by G, we have $G/K = (G \cdot S)/(K \cdot S)$; if $G \not\subset S$ then simplicity of G and effectiveness of A show that $S = \{1\}$. Take S to be the connected radical of A; now A is semisimple. If A has two simple factors take S to be one which does not contain G; now A is simple. If A has infinite center take $\mathfrak{S}$ to be the one dimensional vector group orthogonal to $\mathfrak{G}$ in a maximal compactly embedded subalgebra of $\mathfrak{A}$; now A has finite center. Thus A is a simple linear group with G as maximal compact subgroup.

Suppose rank $K =$ rank G. As $\mathfrak{K}$ is its own normalizer in $\mathfrak{G}$, it is an algebraic subalgebra of $\mathfrak{G}$; thus A has an Iwasawa decomposition GSN for some maximal $\mathbf{R}$-split algebraic torus S, such that $B = KSN$. If $A = G^C$, then A has a complex Cartan subgroup H such that $H \cap K$ is a maximal torus $T \subset K$ and $H = T \cdot S$. Now B contains the Borel subgroup TSN

of A, so B is a parabolic subgroup of A and $M = A/B$ has a natural A-invariant complex structure. Of course this structure is G-invariant. Now Theorem 13.1 says that G/K is hermitian symmetric. Conversely $G^C \subset \mathbf{A}(M)$ if $M = G/K$ is hermitian symmetric. Thus *M is (hermitian) symmetric if and only if $A = G^C$*

Let G/K be $\mathbf{G}_2/\mathbf{A}_2$, $\mathbf{E}_6/\mathbf{A}_2\mathbf{A}_2\mathbf{A}_2$, $\mathbf{E}_7/\mathbf{A}_2\mathbf{A}_5$, $\mathbf{E}_8/\mathbf{A}_8$ or $\mathbf{E}_8/\mathbf{A}_2\mathbf{E}_6$. Then $A \neq G^C$ and there is no noncompact absolutely simple group with G as maximal compact subgroup. Thus A is compact. Now $A = G$.

Let $G/K = \mathbf{F}_4/\mathbf{A}_2\mathbf{A}_2$. Then $\mathbf{E}_{6,F_4}$ is the only noncompact absolutely simple group with G as maximal compact subgroup. Suppose $A = \mathbf{E}_{6,F_4}$ and let σ be the involutive automorphism with fixed point set G. Then $L = (B \cap \sigma B)_0$ is reductive in A with maximal compact subgroup K. If L is almost effective on L/K, it follows that the semisimple part $L' = K^C$, so rank $L \geqslant 8$ in contradiction to $L \subset A$. Now $L = L_1 \cdot \mathbf{A}_2$, where $\mathbf{A}_2$ is the second factor in $K = \mathbf{A}_2 \cdot \mathbf{A}_2$, and $L/K = L_1/\mathbf{A}_2$, where the $\mathbf{A}_2$ is the first factor. If $L' = K$ then $L = K \cdot S = K \times S$ in the notation of the Iwasawa decomposition $A = GSN$, and K is the semisimple part of the centralizer of S. That is false.(1) Thus $L = \mathbf{A}^C \cdot \mathbf{A}_2$. Now A/L is the noncompact almost complex isotropy irreducible space derived from $\mathbf{E}_6/\mathbf{A}_2\mathbf{A}_2\mathbf{A}_2$ by the involution σ. Thus L is irreducible on the orthocomplement of $\mathfrak{L}$ in $\mathfrak{A}$. As $L \subset B$ normalizes B, and as $B \neq A$, that says $L = B$; but then $\dim A/B = 54 > 36 = \dim G/K$ contradicts $A/B \cong M \cong G/K$. This proves $A \neq \mathbf{E}_{6,F_4}$. As $A \neq G^C$ we conclude $A = G$, q.e.d.

Remark. The method shows that $\mathbf{A}_0(\mathbf{E}_6/\mathrm{ad}\,\mathbf{SU}(3))$ is $\mathbf{E}_6$ or $\mathbf{E}_6^C$, and that $\mathbf{A}_0(\mathbf{SO}(n^2-1)/\mathrm{ad}\,\mathbf{SU}(n))$ is $\mathbf{SO}(n^2-1)$, $\mathbf{SO}(n^2-1, \mathbf{C})$, $\mathbf{SL}(n^2-1, \mathbf{R})$ or $\mathbf{SO}^1(n^2)$. It seems probable that $\mathbf{A}_0(G/K) = G$ in both cases, just as for the non-integrable almost complex spaces of equal rank.

18. Local structure

Let M be a riemannian manifold. If $x \in M$, then $K^{(x)}$ denotes the group of all isometries of neighborhoods of x which fix x, where we identify two isometries if they coincide on a neighborhood of x. $K^{(x)}$ is called the *group of local isometries at x*. $K^{(x)}$ is a compact Lie

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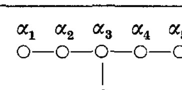
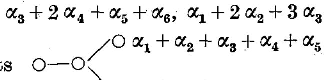
(1) $\mathbf{E}_6$ has simple roots 

, $\mathbf{F}_4$ has simple roots $\{\frac{1}{2}(\alpha_1+\alpha_5), \frac{1}{2}(\alpha_2+\alpha_4), \alpha_3, \alpha_6\}$, and S is spanned by $\{\alpha_1-\alpha_5, \alpha_2-\alpha_4\}$. Thus the centralizer of S in $\mathbf{E}_6$ has positive roots $\{\alpha_3, \alpha_6, \alpha_3+\alpha_6, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4+\alpha_6, \alpha_2+2\alpha_3+\alpha_4+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6\}$, so it has semisimple part of type D_4 with simple roots 

group, for its linear isotropy representation χ on M_x is faithful, and $\chi(K^{(x)})$ consists of all the orthogonal linear transformations of M_x which preserve all covariant differentials $(\nabla^m R)_x$, $m \geqslant 0$, of the curvature tensor at x. We say that M is *locally isotropy irreducible at* x if $\chi(K_0^{(x)})$ is $\mathbf{R}$-irreducible on M_x. M is said to be *locally isotropy irreducible* if it is locally isotropy irreducible at each of its points.

Let $\mathfrak{G}^{(x)}$ denote the Lie algebra of germs of Killing vector fields at x. Then $\mathfrak{K}^{(x)}$ is naturally identified with the subalgebra consisting of all elements of $\mathfrak{G}^{(x)}$ which vanish at x. If φ is a local isometry carrying x to z, then φ sends $\mathfrak{G}^{(x)}$ isomorphically onto $\mathfrak{G}^{(z)}$ and carries $K^{(x)}$ to $K^{(z)}$. If M is locally homogeneous, it follows that the isomorphism classes of the pair $(\mathfrak{G}^{(x)}, \mathfrak{K}^{(x)})$ and the group $K^{(x)}$ do not depend on the choice of x.

18.1 THEOREM. *Let M be a connected locally isotropy irreducible riemannian manifold. Then M is locally homogeneous.*

Choose $x \in M$. Let G/K be the simply connected effective coset space with $\mathfrak{G} = \mathfrak{G}^{(x)}$, $\mathfrak{K} = \mathfrak{K}^{(x)}$ and G connected. Then K is $\mathbf{R}$-irreducible on the tangent space of G/K. For $X \in \mathfrak{G}^{(x)}$ near 0, *define $f(\exp_G(X)K) = \exp_M(X)\cdot x$. Then there is a G-invariant riemannian metric on G/K such that f is an isometry of a neighborhood of $K \in G/K$ onto a neighborhood of $x \in M$. If M is complete, then f extends to a riemannian covering.*

Before proving this theorem we note some consequences.

18.2 COROLLARY. *Let M be a complete connected simply connected locally isotropy irreducible riemannian manifold. Then M is homogeneous, so $M = G/K$ with $G = \mathbf{I}_0(M)$, and*

(i) *M is a euclidean space; or*

(ii) *M is an irreducible riemannian symmetric space; or*

(iii) *G/K is listed in Theorem* 11.1.

For G/K coincides with the coset space of Theorem 18.1.

18.3 COROLLARY. *Let M_1 and M_2 be complete connected locally isotropy irreducible riemannian manifolds, with M_1 simply connected. Let $f\colon U_1 \to U_2$ be an isometry, where U_i is a connected simply connected open submanifold of M_i. Then f extends to a riemannian covering $\bar{f}\colon M_1 \to M_2$. If M_2 is simply connected then $\bar{f}$ is an isometry.*

For let $\pi\colon M_2' \to M_2$ be the universal riemannian covering and let $f'\colon U_1 \to U_2' = f'(U_1) \subset M_2'$ be a lift of f. Then f' is an isometry. Let $M_1 = G_1/K_1$ and $M_2' = G_2/K_2$ as in Corollary 18.2; now f' induces an isomorphism of G_1 onto G_2 which carries K_1 to K_2, and thus f' induces an isometry $\bar{f}'\colon M_1 \to M_2'$. Define $\bar{f} = \pi \cdot \bar{f}'$ and the assertions follow.

Proof of theorem. If $z \in M$, then $V_z = \{X_z \colon X \in \mathfrak{G}^{(z)}\}$ is a $K^{(z)}$-stable subspace of M_z. By local isotropy irreducibility, either $V_z = 0$ or $V_z = M_z$. If $V_z = 0$ we choose $w \in M$ such that $z \neq w$ but z is in a normal neighborhood of w. Let $\{w_t\}_{0 \leqslant t \leqslant 1}$ be the unique minimizing geodesic from w to z, and let $W \in M_w$ be its tangent vector at w. If $K_0^{(w)}(z) = z$ then $K_0^{(w)}$ fixes W, contradicting isotropy irreducibility at w. Now $K_0^{(w)}$ has a one parameter subgroup $\exp(tX)$, $X \in \mathfrak{K}^{(w)}$, which moves z, and so $0 \neq X_z \in V_z$. This proves $V_z = M_z$ for every $z \in M$.

Let $x_0, x_1 \in M$. Let $\{x_t\}$ be a smooth curve in M from x_0 to x_1. Given $0 \leqslant t \leqslant 1$, every element of M_{x_t} is the value of some Killing vector field on a neighborhood of x_t, so there is an open set $U_t \ni x_t$ consisting of images of x_t under local isometries. The Heine–Borel Theorem gives $0 = t_0 < \ldots < t_k = 1$ such that $\bigcup_{i=0}^{k} U_{t_i}$ contains the curve $\{x_t\}$. Now a composition of k local isometries carries x_0 to x_1. This proves that M is locally homogeneous.

$\mathfrak{G}^{(x)} = \mathfrak{K}^{(x)} + \mathfrak{M}$ where $\mathfrak{M}$ is an ad $(\mathfrak{K}^{(x)})$-stable subspace identified with M_x under $X \to X_x$. As $K_0^{(x)}$ is $\mathbf{R}$-irreducible on M_x, now $\mathfrak{K}^{(x)}$ is $\mathbf{R}$-irreducible on $\mathfrak{M}$, and thus K is $\mathbf{R}$-irreducible on the tangent space of G/K. Lift the metric from M_x to $\mathfrak{M}$; then it defines a G-invariant riemannian metric on G/K. Choose a convex open neighborhood $\mathfrak{S}$ of 0 in $\mathfrak{G}^{(x)}$ such that $\exp_M(-Y_2)$ is defined at $\exp_M(Y_1)\cdot x$ whenever $Y_1, Y_2 \in \mathfrak{S}$. Then we have linear isometries

$$(G/K)_{\exp_G(Y)\cdot K} \xrightarrow{\exp_G(-Y)} (G/K)_K \xrightarrow{f_*} M_x \xrightarrow{\exp_M(Y)} M_{\exp_M(Y)\cdot x}$$

and $f_*\colon (G/K)_{\exp_G(Y)\cdot K} \to M_{\exp_M(Y)\cdot x}$ is their composition. Thus f is an isometry on neighborhoods.

Let M be complete and let $\pi\colon M' \to M$ denote the universal riemannian covering. We cut f down to an isometry $g\colon U \to V$ of simply connected neighborhoods and then lift it to an isometry $g'\colon U \to g(U) = V' \subset M'$. As $G = \mathbf{I}_0(G/K)$ we can develope g' along smooth curves. As G/K is real analytic it follows that M is real analytic. Now ([10], p. 256) g' extends to an isometry $\bar{g}$, and $\pi \cdot \bar{g}$ is a riemannian covering. $\pi \cdot \bar{g}$ agrees with f on the domain of g, so they agree on the domain of f by analyticity. Thus $\pi \cdot \bar{g}$ extends f, q.e.d.

Added in Proof

On 8 August 1967, Professor C. T. C. Wall informed me of the following generalization of Corollary 10.2. Let $S = A/B$ be a compact simply connected irreducible symmetric space, $n = \dim S$ and $A = \mathbf{I}_0(S)$, S not a real or quaternionic Grassmann manifold. $\beta, B \to \mathbf{SO}(n)$ is the linear isotropy representation. Decompose $B = K \cdot L$, $\beta = \pi \otimes \omega$, $\pi\colon K \to G$ where (*i*) S is neither hermitian nor quaternionic [18], $B = K$ and $G = \mathbf{SO}(n)$; or (*ii*) S is hermitian, L is a circle, $K = [B, B]$ and $G = \mathbf{SU}(n/2)$; or (*iii*) S is quaternionic, $L = \mathbf{SU}(2)$ with $\omega\colon \overset{1}{\circ}$ and $G = \mathbf{Sp}(n/4)$. Then $G/\pi(K)$ is a nonsymmetric isotropy irreducible coset space, G classical, $\mathbf{SO}(7)/\mathbf{G}_2 \neq G/\pi(K) \neq \mathbf{SO}(20)/[\mathbf{SU}(4)/\mathbf{Z}_4]$; and conversely every nonsymmetric isotropy irreducible coset space $G/\pi(K)$, G classical, $\mathbf{SO}(7)/\mathbf{G}_2 \neq G/\pi(K) \neq \mathbf{SO}(20)/[\mathbf{SU}(4)/\mathbf{Z}_4]$, is constructed as above from a compact irreducible symmetric space S which is not a real or quaternionic grassmannian. This observation is checked by classification. An *a priori* proof will be valuable, but it will also be difficult because of the exceptions.

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