Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 447-458

THE GEOMETRY OF GL(2,q) IN TRANSLATION PLANES OF EVEN ORDER q^2

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(Received August 5, 1977 and in revised form April 6, 1978)

ABSTRACT. In this article we show the following: Let π be a translation plane of even order q^2 that admits $GL(2,q)$ as a collineation group. Then π is either Desarguesian, Hall or Ott-Schaeffer.

KEY WORDS AND PHRASES. Translation planes, Desarguesian, Hall, Ott-Schaeffer planes.

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. 50D05, 05B25.

1. INTRODUCTION.

The author and Ostrom recently studied translation planes π of even order that admit $SL(2,2^s)$, $s \geq 1$. In [10], we considered the case where π has dimension 2 over its kernel and in [9] no assumption was made concerning the kernel.

Schaeffer [13] has shown that if a translation plane of even order q^2 whose kernel $\supseteq GF(q)$ admits $SL(2,q)$, then the plane is Desarguesian, Hall or Ott-Schaeffer.

In [9], it is shown that if the assumption on the kernel is dropped, the planes admitting $SL(2,q)$ must have properties quite similar to the Desarguesian, Hall or Ott-Schaeffer planes but it is an open question whether such planes must, in fact, be in one of these three classes.

In the dimension 2 situation, if the plane admits $SL(2,2^s)$ then the plane also admits $GL(2,2^s)$ (see (2,2)). So, in particular, the Hall and Ott-Schaeffer planes of even order q^2 must admit $GL(2,q)$ as a collineation group.

In this paper we observe that this situation can be reversed (see Theorem (2.7)).

We assume the reader is familiar with the papers [9] and [10].

2. THE MAIN THEOREM.

(2.1) NOTE. If π is a translation plane of even order with kern $\supseteq GF(2^s)$ and admitting $SL(2,2^s)$, $s>1$, then π admits $GL(2,2^s)$ as a collineation group.

PROOF. $GL(2,2^s) = SL(2,2^s) \times \mathcal{C}$ where $\mathcal{C}$ is the center of $GL(2,2^s)$. Conversely, if π admits a group $SL(2,2^s) \times K$ where K is cyclic of order 2^s-1, then, by the obvious isomorphism, $SL(2,2^s) \times K$ is isomorphic to $GL(2,2^s)$.

Since the kern homology group K of order 2^s-1 commutes with elements of the linear translation complement and $SL(2,2^s)$ is linear by [10] (see the proof of (2.1)), we have a group $SL(2,2^s) \times K$ (since $SL(2,2^s)$ is simple).

(2.2) COROLLARY. If π is a translation plane of even order $2^{2r} = q^2$ whose kernel contains $GF(q)$ admits $SL(2,2^s)$, $s>1$, then π admits $GL(2,2^s)$.

PROOF. $s \mid r$ by [10] ((2.2) and (2.4)). Thus, there is a cyclic subgroup $\bar{K}$ (of the group of kern homologies) of order 2^s-1. By (2.1), we have the proof to (2.2).

We also note that (2.2) is not valid for translation planes of odd order (see Foulser [4]).

If a translation plane is of dim 2 then the group $\bar{\mathcal{G}}$ in the linear translation complement generated by all Baer involutions is always $SL(2,2^s)$ if there are no elations and the group is nonsolvable (see [10], (3.27)). However, essentially nothing is known concerning the group $\bar{\mathcal{G}}$ without assuming that the Baer subplanes are disjoint (as subspaces).

(2.3) THEOREM. Let π be a translation plane of even order admitting a collineation group $\mathcal{G}$ whose sylow 2-subgroups fix Baer subplanes pointwise.

Let Q be a sylow 2-subgroup of $\mathcal{G}$. Then either

(1) Q is normal in $\mathcal{G}$,

(2) $|Q| = 2$,

or (3) the subgroup generated by the Baer involutions is $SL(2,2^s)$ for $s > 1$.

PROOF. Suppose neither (1) nor (2). Consider $\mathcal{N}_{\mathcal{G}}(Q^x) \cap Q$ (normalizer of Q^x in $\mathcal{G}$) for $x \in \mathcal{G}$.

Suppose $g \in \mathcal{N}_{\mathcal{G}}(Q^x) \cap Q$ so that $Q^{xg} = Q^x$ for $g \in Q$. Since $|g| = 2$ by Foulser ([2], (2.5)), g must centralize some involution h of Q^x. Thus, $\langle g,h \rangle$ is an elementary abelian 2-group and so is contained in a sylow 2-subgroup $\bar{Q}$.

Let $Q^x, Q, \bar{Q}$ respectively fix the Baer subplanes π_1, π_2, π_3 pointwise. Then $g \in Q \cap \bar{Q}$ so g fixes π_2 and π_3 pointwise. Thus, $\pi_2 = \pi_3$ and similarly $\pi_1 = \pi_3$. Thus, $\langle Q^x, Q \rangle$ fixes π_1 pointwise. By Foulser [3] (Theorem 2), $\langle Q^x, Q \rangle$ is a subgroup of a one-dimensional affine group. So, the involutions of $\langle Q^x, Q \rangle$ belong to the same sylow 2-subgroup. Thus $Q^x = Q$ and $x \in \mathcal{N}_{\mathcal{G}}(Q)$.

Thus, $\mathcal{N}_{\mathcal{G}}(Q^x) \cap Q = \langle 1 \rangle$ for $x \in \mathcal{G} - \mathcal{N}_{\mathcal{G}}(Q)$.

We can thus apply Hering's main theorem of [7]. Let S denote the normal closure of Q in $\mathcal{S}$. Then S is isomorphic to $SL(2,2^s)$, $s > 1$, $S_Z(2^s)$, $PSU(3,2^s)$, or $SU(3,2^s)$. But only $SL(2,2^s)$ has elementary abelian sylow 2-subgroups which must be the case by Foulser ([2], (2.5)).

Clearly, S is the group generated by the Baer involutions of $\mathcal{S}$.

(2.4) THEOREM. (Special case of the main theorem of Foulser-Johnson, Ostrom [5].) Let π be a translation plane of order q^2 which admits $GL(2,q)$ and where the p-elements $(p^r = q)$ are elations. Then π is Desarguesian.

(2.5) THEOREM. Let π be a translation plane of order $q^2 \neq 4$ or 9 which admits $GL(2,q)$ where the sylow p-subgroups, $p^r = q$, fix Baer subplanes pointwise. Then π is a Hall plane.

PROOF. If $|\pi|$ is even then by Johnson and Ostrom [9], (4.1), the Baer subplanes fixed pointwise fall into a derivable net $\mathcal{N}$. Thus, the derived plane is Desarguesian by (2.4) since $GL(2,q)$ must fix the net. If $|\pi|$ is odd, the Baer subplanes fall into a derivable net by Foulser [2] if $q \neq 3$ (Theorem (5.1)) and thus (2.4) applies

(2.6) THEOREM. Let π be a translation plane of even order q^2 which admits $GL(2,q)$ where the sylow 2-groups fix subsets of order q that are contained in components. Then π is an Ott-Schaeffer plane.

PROOF. By [9], (4.6), we have orbits on ℓ_∞ of lengths $q+1$, $\frac{1}{2}q(q-1)$, and $\frac{1}{2}q(q-1)$ under $SL(2,q)$.

Let σ be in the center of $GL(2,q)$. Then σ must fix each Baer subplane which is fixed pointwise by an involution and must fix each line of the orbit of length $q+1$.

The $q-1$ Baer subplanes fixed pointwise by elements of a sylow 2-group of $SL(2,q)$ cover the components other than the components in the orbit of length q+1. Let $\prod_{i=1}^{\ell} p_i^{\alpha_i}$ be the prime decomposition of q-1 and let σ_i be an element of order $p_i^{\alpha_i}$. Then, since σ_i fixes each Baer subplane π_j indicated above and π_j shares precisely one component with the orbit of length q+1, σ_i fixes a component from each Baer subplane π_j and so fixes each component of π. Thus, $\prod_{i=1}^{\ell} \sigma_i = \sigma$ fixes each component of π. Therefore, π has dim 2/kern so that π is an Ott-Schaeffer plane by Schaeffer.

We can now state our main theorem.

(2.7) THEOREM. Let π be a translation plane of even order q^2 which admits GL(2,q) as a collineation group. Then the fixed point space of each Sylow 2-subgroup is a component, Baer subplane, or Baer subline and

(i) π is Desarguesian if and only if the Sylow 2-subgroups fix components pointwise,

(ii) π is Hall if and only if the Sylow 2-subgroups fix Baer subplanes pointwise,

(iii) π is Ott-Schaeffer if and only if the Sylow 2-subgroups fix Baer sublines (sublines of order q) pointwise.

PROOF. By (2.3), (2.4), (2.5), and (2.6), it remains to show that the fixed point spaces are as asserted. We can thus assume that the involutions are Baer. We can assume that a Sylow 2-subgroup Q does not pointwise fix a component, Baer suplane, or Baer subline. Let Q fix X pointwise.

Let $\mathcal{Z}$ denote the center of GL(2,q).

Case 1: The center is fixed-point-free.

Let $\mathcal{Z} = \langle\sigma\rangle$. Then, if $P \in X$, $\sigma^i P \in X$. If X intersects, nontrivially, a line fixed by σ then Q fixes $\geq q$ points of a component. So, assume X does not nontrivially intersect any component fixed by σ.

Obviously then X cannot lie on a component of π and is thus a subplane of order 2^s. If $\rho \in Q$ and fixes the Baer subplane π_ρ pointwise then X is a subplane of π_ρ. Thus, $2^s \le \sqrt{q}$ (since $X \ne \pi_\rho$ by assumption). Let $\mathcal{L}$ be a component of X and $\langle\sigma^i\rangle$ the stabilizer of $\mathcal{L}$ in $\langle\sigma\rangle = \mathcal{S}$. Thus, $|\langle\sigma^i\rangle| \le 2^s-1$ since $\mathcal{S}$ is fixed-point-free and the $\mathcal{S}$-orbit of $\mathcal{L}$ has length $\le 2^s+1$ since $\mathcal{S}$ fixes X and X has 2^s+1 components. So, $q-1 \le (s^2-1)(2^s+1) = 2^{2s}-1 \le q-1$ so that X has order $\sqrt{q}$ and the orbit length of $\mathcal{L}$ is $\sqrt{q}+1$.

By Foulser ([3] (Theorem 2)), $Q|\pi_\rho$ has order $\le \sqrt{q}$. Let $\bar{Q}$ be the maximal subgroup of Q which fixes π_ρ pointwise. Then $|\bar{Q}| \ge \sqrt{q}$. If $q=4$ then $|\bar{Q}|=2$. If $\sqrt{q} > 2$, $\langle\bar{Q},\bar{Q}^x\rangle \cong SL(2,\sqrt{q})$ or $SL(2,q)$ by [9](2.1) for some $x \in GL(2,q)$. That is, if $\bar{Q}$ and $\bar{Q}^x$ are in distinct Sylow 2-subgroups then $\langle\bar{Q},\bar{Q}^x\rangle \cong SL(2,2^{\bar{s}})$ for some $\bar{s} \ge 1$. Since $|\bar{Q}| \ge \sqrt{q}$, $2^{\bar{s}} \ge \sqrt{q}$. But, $\langle\bar{Q},\bar{Q}^x\rangle \le SL(2,q)$ so $2^{\bar{s}} = \sqrt{q}$ or q. By [9](2.9), we know that $|\bar{Q}| = \sqrt{q}$.

Let the normalizer of Q in $SL(2,a)$ be $Q\mathcal{C}$. $\bar{Q}$ fixes π_ρ pointwise so there is a subgroup $\bar{\mathcal{C}}$ of $\mathcal{C}$ of order $\sqrt{q}-1$ that fixes π_ρ. Moreover, since $\mathcal{S}$ is transitive on the points of X and $\mathcal{C}$ permutes the points fixed by Q, we have that if $\mathcal{C} = \langle\sigma\rangle$ then there is an element g of $\mathcal{S}$ such that σg fixes a point of X and thus fixes X pointwise.

Let the fixed point subplane of $\langle\sigma g\rangle$ be $\bar{\pi}$.

Now there are $\sqrt{q}+1$ distinct subgroups $\bar{Q}_i$ of Q each fixing X pointwise and also fixing pointwise a Baer subplane π_i. ($\mathcal{C}$ is regular on the involutions of $Q-\langle 1\rangle$.) That is, $Q = \left(\bigcup_{i=1}^{\sqrt{q}+1} \bar{Q}_i - \langle 1\rangle\right) \cup \langle 1\rangle$.

Suppose $\mathcal{L}$ is a component not in X common to π_i and π_j. Then $\langle\bar{Q}_i,\bar{Q}_j\rangle$ fixes $\mathcal{L}$ and fixes X pointwise and thus fixes a Baer subplane pointwise. Hence, π_i and π_j share no components outside of X.

So there are $(\sqrt{q}+1)(q-\sqrt{q})$ components that are permuted by $\mathcal{C} = \langle\sigma\rangle$.

Since $\mathcal{S}$ fixes each subplane π_i and also leaves X invariant, $\mathcal{S}$ fixes $\pi_i - X$ and thus permutes the same set of components.

So, $\langle \sigma g \rangle$ permutes the remaining set $\mathcal{J}$ of $q^2+1-((\sqrt{q}+1)(q-\sqrt{q})+\sqrt{q}+1))$ components $= q\sqrt{q}(\sqrt{q}-1)$.

Let Q^x fix a subplane Y of order $\sqrt{q}$ pointwise. If $X \cap Y \neq \mathcal{O}$ then $X = Y$ since $\mathcal{S}$ is regular on $X - \mathcal{O}$ and fixes X and Y. But, $\langle Q^x, Q \rangle \cong SL(2,q)$ then fixes X pointwise, contrary to [9] (4.3).

If X and Y share a component then they share $\sqrt{q}+1$ components. Since Q fixes X pointwise, Q maps Y onto q other pairwise disjoint subplanes of order $\sqrt{q}$ which are pointwise fixed by the q other Sylow 2-subgroups. (By [9] (4.3), it follows that $GL(2,q)$ acts on these sets of subplanes of order $\sqrt{q}$ as it acts on the Sylow 2-subgroups.) Thus, each of the q+1 subplanes is on the same set of $\sqrt{q}+1$ components. Thus, $\langle Q, Q^x \rangle$ fixes each of these components. $\mathcal{B} \cong SL(2,q)$ fixes each of $\sqrt{q}+1$ components and permutes the remaining set $\mathcal{J}$ of $q^2 - \sqrt{q}$ components. Let $\rho \in \mathcal{B} \ni |\rho|$ is 2-primitive. Then $\langle \rho \rangle$ and $\langle \rho^x \rangle$ cannot fix a common component $\mathcal{L}$, for otherwise $\langle \rho, \rho^x \rangle = \mathcal{B}$ fixes $\mathcal{L}$ and a Sylow 2-subgroup fixes a Baer subplane pointwise. Let ρ fix k components of $\mathcal{J}$ so k is odd. There are $\frac{1}{2}q(q-1)$ conjugates of ρ, none of which can fix any of these k components. So there is an orbit under $\mathcal{B}$ of length $\frac{1}{2}q(q-1)k^*$ where k^* is odd (the normalizer of ρ contains an involution τ which must fix one of these k components). So τ fixes a component $\mathcal{L}$ of $\mathcal{J}$ and fixes a Baer subplane π_τ pointwise. Clearly, the Sylow 2-subgroup containing τ must have a subgroup of order $\sqrt{q}$ fixing π_τ pointwise. Unless, $\sqrt{q} = 2$, we have a contradiction. However, we have considered planes of order 16 in [11] and this possibility does not occur.

Thus, X and Y do not share a component. Thus, $\langle \sigma g \rangle$ fixes Y

($\mathcal{C}$ fixes two subplanes X and Y if $\mathcal{C}$ normalizes Q and Q^x) and fixes X pointwise. So the elements in $\langle\sigma g\rangle$ of prime power order $\Big|\sqrt{q}-1$ must fix a component of Y and thus fix a Baer subplane pointwise $\bar{\pi}$. So $\langle\sigma g\rangle$ must fix $\bar{\pi}$ and by Foulser [3] (Thm. 2) there is a subgroup $\mathcal{C}^*$ of order $\geq\sqrt{q}+1$ fixing $\bar{\pi}$ pointwise. (That is, $\langle\sigma g\rangle$ fixes pointwise a Baer subplane X of $\bar{\pi}$ so $|\langle\sigma g\rangle\big|\bar{\pi}|\leq\sqrt{q}-1$.) Let $\sqrt{q}-1=\prod_{i=1}^{k}p_i^{\alpha_i}$ be the prime decomposition. There exist elements θ_i of order $p_i^{\alpha_i}$ in $\langle\sigma g\rangle$ which fix Baer subplanes ($\langle\sigma g\rangle$ fixes Y) π_{θ_i} pointwise. Each π_{θ_i} is fixed pointwise by a subgroup $\tilde{\mathcal{C}}$ of order $\sqrt{q}+1$ (by uniqueness of subgroups of $\langle\sigma g\rangle$ of a given order). Thus, $\pi_{\theta_i}=\bar{\pi}$. So $\prod_{i=1}^{k}\theta_i$ fixes $\bar{\pi}$ pointwise. Let θ be the element of order $\sqrt{q}+1$. Thus, $\prod_{i=1}^{k}\theta_i\theta=\sigma g$ fixes $\bar{\pi}$ pointwise.

We assert that $Y\subseteq\bar{\pi}$. If not, then $\langle\sigma g\rangle$ fixes Y and acts faithfully on Y. Since $\mathcal{S}$ is regular on $Y-\{\mathcal{O}\}$, no element of $\langle\sigma g\rangle$ can fix a point of Y. Since the elements of order dividing $\sqrt{q}-1$ fix a component of Y, we must have that $\langle\sigma g\rangle$ fixes a component of Y and thus cannot act faithfully on Y. That is, $\bar{\pi}$ and Y must share a component.

Let τ be an involution that maps $X\to Y$. Thus $\bar{\pi}$ and $\overline{\pi\tau}$ both contain X and Y. Since $X\cap Y=\mathcal{O}$, $\bar{\pi}=\overline{\pi\tau}$ (X and Y are r-dim subspaces of $\bar{\pi}$ and $\overline{\pi\tau}$).

Note that $Q\overset{\tau}{\longleftrightarrow}Q^x$ since $X\overset{\tau}{\longleftrightarrow}Y$. Thus, $\langle\tau,\mathcal{C}\rangle$ stabilizes $\{Q,Q^x\}$ and is thus dihedral of order $2(q-1)=D_{q-1}$. Thus $\sigma^\tau=\sigma^{-1}$ and $(\sigma g)^\tau=\sigma^{-1}g$. Since $(\sigma^{-1}g)$ also fixes $\bar{\pi}$ pointwise, then $(\sigma^{-1}g)(\sigma g)=g^2$ fixes $\bar{\pi}$ pointwise. Since $g^2\in\mathcal{S}$, $g^2=1$ and thus $g=1$.

So $\langle\sigma\rangle=\mathcal{C}$ fixes $\bar{\pi}$ pointwise. Let the $q+1$ subplanes of order $\sqrt{q}$ fixed pointwise by the Sylow 2-subgroups be denoted by $X_1,X_2,\cdots,X_{q+1}$. Let π_{X_i,X_j} be the Baer subplane containing X_i,X_j and fixed pointwise by the cyclic subgroup $\mathcal{C}_{i,j}$ of SL(2,q). (Recall that SL(2,q) is 3-transitive on its Sylow 2-subgroups and thus on the subplanes $\{X_i\}$.)

Note that $\bar{\pi}$ contains precisely X and Y, for otherwise $\bar{\pi}$ would contain all q+1 subplanes of order $\sqrt{q}$ as $\mathcal{C}$ is regular on remaining subplanes $\neq$ X or Y.

We assert that π_{X_1,X_2} and π_{X_1,X_j}, $j \neq 2$, share only the components of X_1.

PROOF. If the subplanes share a component $\mathcal{L}$ then $\langle \mathcal{L}, X_1 \rangle$ is a Baer subplane (smallest subplane properly containing X_1 has order $(\sqrt{q})^2$) so $\pi_{X_1,X_2} = \pi_{X_1,X_j}$, $j \neq 2$, but then X_1, X_2, X_j are in π_{X_1,X_2}, contrary to the above.

We also assert that π_{X_1,X_2} and π_{X_3,X_4} share no common component.

PROOF. Let $\mathcal{L}$ be a common component. Then $\langle \mathcal{C}_{1,2}, \mathcal{C}_{3,4} \rangle$ fixes $\mathcal{L}$ since $\mathcal{C}_{i,j}$ fixes π_{X_1,X_2} pointwise. Note that $q > 4$. That is, since X_1 and X_2 share no common components $q+1 \geq 2(\sqrt{q}+1)$. So

$$\mathcal{C}_{1,2} \text{ is regular on } \{X_3, \cdots, X_{q+1}\} = \mathcal{J}_{1,2}$$

$$\mathcal{C}_{3,4} \text{ is regular on } \{X_1, X_2, X_5, \cdots, X_{q+1}\} = \mathcal{J}_{3,4}.$$

Since $q > 4$, $\mathcal{J}_{1,2} \cap \mathcal{J}_{3,4} \neq \emptyset$ so that there is an orbit of length q+1. Thus, $(q+1)(q-1) \,\Big|\, |\langle \mathcal{C}_{1,2}, \mathcal{C}_{3,4} \rangle|$. By examination of the subgroups of SL(2,q), it follows that $\langle \mathcal{C}_{1,2}, \mathcal{C}_{3,4} \rangle \cong SL(2,q)$.

So SL(2,q) fixes $\mathcal{L}$ and acts faithfully on $\mathcal{L}$. Then the Sylow 2-subgroups Q_i fix subspaces of $\mathcal{L}$ pointwise. Thus, X_i intersects $\mathcal{L}$ for each i. But, this is a contradiction.

We consider $\pi_{X_i,X_j} - \{X_i, X_j\}$ and $\pi_{X_k,X_m} - \{X_k, X_m\}$. By the above two statements, there are <u>no</u> common components if $\{i,j\} \neq \{k,m\}$. We thus may count the components of $\bigcup_{i,j} (\pi_{X_i,X_j} - \{X_i, X_j\})$, for the $\frac{q(q+1)}{2}$ unordered pairs (i,j), as $\frac{q(q+1)}{2}(q+1-2(\sqrt{q}+1))$. We also have $\bigcup_{i=1}^{q+1} X_i$ so we must have at least $(q+1)(\sqrt{q}+1)$ additional components. Therefore, $q^2+1 \geq \frac{q(q+1)}{2}(q+1-2(\sqrt{q}+1)+(q+1(\sqrt{q}+1)$ and $q>4$ which is clearly a contradiction unless $q+1-2\sqrt{q}+1<2 \Rightarrow q=2$. So Case 1 is completed.

Case 2: The center $\mathcal{C}$ is not fixed-point-free. Let $\rho \in \mathcal{C}-\langle 1\rangle$ fix pointwise a set of affine points $\mathcal{A} \neq \emptyset$. $\langle\rho\rangle$ is characteristic in $\mathcal{C}$ and $\mathcal{C}$ is normal in $GL(2,q)$. So $SL(2,q)$ fixes $\mathcal{A}$.

Case 2a: $\mathcal{A}$ is a subplane of order 2^s.

So $SL(2,q)$ permutes 2^s+1 components and unless $2^s+1 \geq q+1$, $SL(2,q)$ must fix each component (see (8.23), p. 214 [8]). So, if $2^s+1<q+1$ then $SL(2,q)$ must fix pointwise each set of $2^s<q$ affine points on each of the components of $\mathcal{A}$. That is, $SL(2,q)$ must fix $\mathcal{A}$ pointwise, which is a contradiction to [9] (4.3). So, $2^s+1 \geq q+1$ or $2^s=q$. Thus, $SL(2,q)$ fixes a Baer subplane $\mathcal{A}$ (since some element of $\mathcal{C}$ fixes it pointwise). $SL(2,q)$ is faithful on $\mathcal{A}$ by [9] (4.2) so $\mathcal{A}$ is Desarguesian by Lüneburg [11] Satz 4 and the Sylow 2-subgroups of $SL(2,q)$ act as elations on $\mathcal{A}$. That is, the 2-groups either fix Baer subplanes or Baer sublines pointwise. Thus, we must have

Case 2b: $\mathcal{A}$ is not a subplane so $\mathcal{A}$ is contained in a component.

Assuming $|\rho|$ is prime, ρ fixes exactly two components, say $x=0$ and $y=0$. Thus, $\mathcal{C}$ must fix both $x=0$ and $y=0$ and $GL(2,q)$ must also fix $x=0$ and $y=0$.

Let θ be an element such that $|\theta|$ is a prime 2-primitive divisor of q^2-1 (which exists by [1] unless $q=8$ and since q is assumed to be square this case does not come up). Then θ is irreducible on any component θ fixes.

That is, θ either fixes $\mathcal{L}$ pointwise or is fixed point-free on $\mathcal{L}$

(since θ is completely reducible on $\mathcal{L}$ if θ fixes a subspace $\mathcal{R}$ of $\mathcal{L}$ pointwise then $\mathcal{R} \oplus \overline{\mathcal{R}} = \mathcal{L}$ implies θ fixes $\overline{\mathcal{R}}$ pointwise).

Thus, θ must fix $\mathcal{J}$. So we may take $\mathcal{J}$ to be $x = 0$. Thus, ρ is a $((0), x=0)$-homology.

That is, either $\mathcal{J} = \mathcal{L}$ or $\mathcal{J} \subsetneqq \mathcal{L}$ and every prime 2-primitive divisor element fixes $\mathcal{J}$ pointwise. Since SL(2,q) is simple and fixes $\mathcal{J}$, either SL(2,q) fixes $\mathcal{J}$ pointwise or is faithful on $\mathcal{J}$. The first alternative cannot exist by [9] (4.2). So SL(2,q) is faithful on $\mathcal{J}$ and thus $\mathcal{J} = \mathcal{L}$.

If $\mathcal{G}$ is f.p.f. on $y = 0$ then since SL(2,q) fixes $y = 0$, we may apply our previous argument of Case 1. That is, a Sylow 2-subgroup Q of SL(2,q) must fix a subspace $\hat{X}$ of $\mathcal{L}$ pointwise. Since $\mathcal{G}$ is f.p.f. on $y = 0$ and fixes $\hat{X}$, then $|\hat{X}| \geq q$ contrary to our assumptions.

Thus, our argument shows that the group $\mathcal{G}$ induces on $x = 0$ $(y = 0)$ is semiregular of order $|B|$ $(|A|)$ if $\mathcal{G}$ is represented by $(x,y) \to (xA, yB)$. Thus, $|\mathcal{G}| = q-1 = \mathrm{LCM}(|A|, |B|)$. Since SL(2,q) fixes $x = 0$ and $y = 0$, a Sylow 2-subgroup fixes a subplane π_σ of order 2^s pointwise where $2^s < q$. But, $\mathcal{G}$ fixes $\pi_\sigma \cap (x=0)$ so $|B| \mid 2^s-1$ and similarly $|A| \mid 2^s-1$ so $\mathrm{LCM}(|A|, |B|) \mid 2^s-1$ which implies $q \leq 2^s$.

Thus, we have the proof to our theorem.

ACKNOWLEDGMENT. The author gratefully acknowledges the helpful suggestions of Professor D. A. Foulser in the preparation of this article.

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